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On cone based decompositions of proper Pareto optimality

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Abstract In recent years, the research focus in multi-objective optimization has shifted from approximating the Pareto optimal front in its entirety to identifying solutions that are well-balanced among their objectives. Proper Pareto optimality is an established concept for eliminating Pareto optimal solutions that exhibit unbounded tradeoffs. Imposing a strict tradeoff bound allows specifying how many units of one objective one is willing to trade in for obtaining one unit of another objective. This notion can be translated to a dominance relation, which we denote by *M-domination*. The mathematical properties of M-domination are thoroughly analyzed in this paper yielding key insights into its applicability as decision making aid. We complement our work by providing four different geometrical descriptions of the M-dominated space given by a union of polyhedral cones. A geometrical description does not only yield a greater understanding of the underlying tradeoff concept, but also allows a quantification of the space dominated by a particular solution or an entire set of solutions. These insights enable us to formulate volume-based approaches for finding approximations of the Pareto front that emphasize regions that are well-balanced among their tradeoffs in subsequent works.

Keywords multi-objective optimization · proper Pareto optimality · cone domination · polyhedral cones · hypervolume

Nomenclature

A	$l \times m$ matrix where $l \in \mathbb{N}$
A^o_σ	Matrix of the ordered objectives approach using permutation σ

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A^s_{ij}	Matrix of the simplified ordered objectives approach with maximum objective difference i and minimum difference j
A^{min}_i	Matrix of the minimum matrix approach of objective i
$C(A)$	Polyhedral cone induced by matrix A
$\mathbf{d}$	Vector in $\mathbb{R}^m$ used to model directions as difference between two vectors
dist	A function for calculating the distance between two elements in the vector space $\mathbb{R}^m$
$\succ_{C(A)}$	Cone dominance relation according to the cone C induced by matrix A
$\succ_M$	M-domination relation according to the tradeoff level M
$\succ_p$	Pareto dominance relation
$\mathbf{f}$	Vector of objective functions
I	Set of objective indices, i.e. $I = \{1, \ldots, m\}$
M	Variable denoting a maximum allowed tradeoff bound
m	Number of objectives
n	Number of decision variables
$\mathbf{r}$	A reference point for calculating the size of the M-dominated space of an element in $\mathbb{R}^m$
$D_M(\mathbf{u})$	Subspace of $\mathbb{R}^m$ that is M-dominated by $\mathbf{u}$
$P_M(\mathbf{u})$	Subspace of $\mathbb{R}^m$ that M-dominates $\mathbf{u}$
$N_M(\mathbf{u})$	Subspace of $\mathbb{R}^m$ that is non-M-dominated to $\mathbf{u}$
$\mathbf{u}, \mathbf{v}, \mathbf{w}$	Variables depicting values in $\mathbb{R}^m$
$\mathbf{x}, \mathbf{y}, \mathbf{z}$	Variables depicting values in $\mathbb{R}^n$
X	Search space
X_p	Set of Pareto optimal solutions
$\mathcal{Y}$	Objective space
$\mathcal{Y}_p$	Image of the set of Pareto optimal points in the objective space

1 Introduction and motivation

Multi-objective optimization aims at finding solutions to problems that feature multiple conflicting goals. In general, these problems exhibit no single solution that optimizes all objectives at the same time. Instead, we obtain a set of Pareto optimal solutions that forms a so-called Pareto optimal front in the objective space. A Pareto optimal solution can only be improved in one goal by deteriorating another objective at the same time [9,8,14,21].

However, not all Pareto optimal solutions are equally attractive. Some solutions might only excel in one aim and suffer from highly undesirable objective values in other goals. For this reason, many different notions have been proposed in the literature to identify preferable subsets of the Pareto front [6, 4,10,7,24,16]. The concept of proper Pareto optimality [15] uses tradeoffs to describe the desirability of a solution. A tradeoff specifies how much it costs in quantities of one objective to attain a certain gain in another goal. It was suggested in [23] to impose a maximum allowed deterioration-improvement-ratio to crop the set of Pareto optimal solutions, i.e. if attaining an improvement in one objective exceeds trading in a given quantity in some other goal, this particular solution is discarded. Subsequent works [25] have applied this notion to define a dominance relation, namely *U-domination*, by imposing a deterioration-improvement-ratio of strictly one, i.e. a solution U-dominates another solution, if its maximum gain exceeds the largest absolute loss compared over all objectives. In this paper, we relax the strict exchange rate allowing larger tradeoff bounds for defining the notion of *M-domination*.

The concept of M-domination allows us to focus our search efforts from the beginning on preferred subsets of the Pareto optimal front. We can define a threshold for obtaining solutions that only present equitable options and thereby save computational resources. Limiting the search on subsets of the Pareto front has a very practical reason. Pareto fronts of most real-world problems are hypersurfaces. Attaining a closed form description is often arduous and sometimes even infeasible [13, 17, 26, 28]. For this reason, multi-objective optimization algorithms try to approximate the Pareto front by a finite set of points. In practical applications however, a decision maker is usually required to implement only a single solution. Having a multitude of possible options at hand burdens the decision making process. This problem can be alleviated by cropping the Pareto front to a subset of solutions that all adhere to some desirability criterion [5]

The question remains how the remaining subset of the Pareto front should be optimally approximated, since these subsets may become rather small compared to the complete front or form disconnected patches [7]. For this reason, traditional techniques such as crowding distance [12], nearest neighbor selection [33], decomposition methods [31] or reference point models [11] appear to be inappropriate. Previous approaches [27, 7] have treated all solutions below the threshold equally. We want to measure the space M-dominated by a set of solutions in the objective space instead. Volume-based methods for computing optimal distributions of solutions on the Pareto front have been successfully applied and seen increasing research interest in recent years [2, 1]. A distribution of points that maximizes the volume of the M-dominated space would constitute a highly desirable approximation of the Pareto front according to the M-domination notion. Although we propose no such algorithm in this paper, we lay the theoretical foundation for implementing it in future works.

We choose to decompose the M-dominated space into polyhedral cones. Polyhedral cones are a well-known concept in multi-objective optimization to describe preferences that generalize Pareto domination [30]. Quantifying the space M-dominated by a set of points is a non-trivial task. Dividing the M-dominated space into polyhedral cones, however, allows us to reuse existing algorithms that are founded on the Pareto domination principle [3, 29]. We propose four different decomposition techniques in this work and present an extensive discussion about their applicability in practical algorithms.

We list the key contributions of our work:

- we propose a new tradeoff-based dominance concept in M-domination that is founded on the notion of proper Pareto optimality. It is shown that M-domination is a generalization of Pareto domination;
- an in-depth analysis of the mathematical properties of M-domination is provided that elaborate on its applicability in a practical context;
- we provide four cone based decompositions of M-domination. Thereby we gain a deeper understanding of the underlying preference notion and obtain the possibility to quantify the space M-dominated by a particular point. This allows the conception of volume-based optimization algorithms in future works.

The remainder of the paper is structured as follows. In the next section, we provide a primer on some key multi-objective optimization concepts. Afterwards, we present the notion of M-domination and provide an in-depth analysis of its mathematical properties. Cone based decompositions of M-domination are explored in the subsequent section. We conclude our work by some final remarks and an outlook on future work.

2 Multi-objective optimization concepts

Without loss of generality, we restrict our analysis to minimization problems. We consider the minimization of m objective functions $\mathbf{f}(\mathbf{x}) := (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ using n decision variables $\mathbf{x} := (x_1, \ldots, x_n)$.

Definition 1 (Multi-objective optimization problem [9, 8, 14, 21]) Let $\mathbf{f}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$ and $X \subseteq \mathbb{R}^n$ be given. The multi-objective optimization problem is defined as follows:

$$\min_{\mathbf{x}} \mathbf{f}(\mathbf{x}) := (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x})) \qquad \text{s.t. } \mathbf{x} \in X. \tag{1}$$

The set X (in Definition 1) is the feasible set and may be described by any number of inequalities, equalities or box constraints. Our analysis is not affected by the composition of X. $\mathcal{Y} = \mathbf{f}(X)$ denotes the objective space.

Definition 2 (Pareto Domination [9, 8, 14, 21]) Let the points $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ be given. $\mathbf{u}$ *dominates* $\mathbf{v}$, expressed as $\mathbf{u} \succ_p \mathbf{v}$, if $u_i \leq v_i$ for all $i = 1, \ldots m$ with strict inequality for at least one i.

Note that Pareto domination as of Definition 2 is defined independently of the optimization context. In order to check, if a solution $\mathbf{x}$ dominates a solution $\mathbf{y}$, we need to compare their objective values $\mathbf{f}(\mathbf{x})$ and $\mathbf{f}(\mathbf{y})$. Pareto domination is a key concept in multi-objective optimization. A dominated solution should never be preferred over a dominating solution independent of user preferences, as it is equal to or worse in all objectives in comparison. Solutions that are not dominated by any other element in X are called non-dominated or Pareto optimal.

Definition 3 (Pareto optimality [22]) A solution $\mathbf{x} \in X$ is called *Pareto optimal* if no $\mathbf{y} \in X$ exists so that $f_i(\mathbf{x}) \leq f_i(\mathbf{y})$ for all $i = 1, \ldots, m$ with strict inequality for at least one i.

We denote the set of all Pareto optimal solutions by X_p and its image by $\mathcal{Y}_p = \mathbf{f}(X_p)$. Next, we introduce the notion of polyhedral cones.

Definition 4 (Polyhedral cone [30]) Let $A \in \mathbb{R}^{l \times m}$ be a matrix with $l \in \mathbb{N}$. The polyhedral cone $C(A) \subset \mathbb{R}^m$ induced by A is defined by

$$C(A) := \{\mathbf{d} \in \mathbb{R}^m | (A\mathbf{d})_i > 0, \ \forall i = 1, \ldots, l\}. \tag{2}$$

Cones may be used as well to describe dominance relations. A point $\mathbf{u}$ dominates another point $\mathbf{v}$, if the direction $\mathbf{d} = \mathbf{v} - \mathbf{u}$ lies in the polyhedral cone induced by the matrix A.

Definition 5 (Cone domination) Let the points $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ and the matrix $A \in \mathbb{R}^{l \times m}$ be given. Then, $\mathbf{u}$ cone-dominates $\mathbf{v}$, expressed as $\mathbf{u} \succ_{C(A)} \mathbf{v}$, if (and only if) $\mathbf{v} - \mathbf{u} \in C(A)$.

A solution $\mathbf{x}$ dominates a solution $\mathbf{y}$ according to the polyhedral cone induced by matrix A, if $\mathbf{f}(\mathbf{y})$ lies in the cone $C(A)$ that is shifted from the origin to $\mathbf{f}(\mathbf{x})$. Consequently, $\mathbf{d}$ can be expressed as $\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})$. Note that the cone itself is independent of $\mathcal{Y}$ as it is defined in $\mathbb{R}^m$. We need to distinguish between the *set of solutions* in X that are dominated by $\mathbf{x}$ and the *space* that is dominated by $\mathbf{f}(\mathbf{x})$ in $\mathbb{R}^m$. Our analysis is mostly concerned with the dominated space, as the image of the dominated set is only defined in the realm of $\mathcal{Y}$. Cones can be utilized to describe preferences that generalize the concept of Pareto domination. The cone described by the identity matrix in m dimensions, for example, is equivalent to the Pareto domination principle in Definition 2, if all inequalities are strict.

3 A proper Pareto optimality based dominance relation

Proper Pareto optimality was first described in [15] to eliminate Pareto optimal solutions that exhibit an unbounded tradeoff in their objective values to other solutions. Subsequent works [23] were able to simplify the original definition and defined the tradeoff threshold to be a fixed number instead of being bound by a finite real number.

Definition 6 (Proper Pareto optimality [15, 23]) Let $M \in \mathbb{R}_+$ be given. Then, a point $\mathbf{x} \in X$ is called *proper Pareto optimal* if $\mathbf{x} \in X_p$ and if for all i and $\mathbf{y} \in X_p$ satisfying $f_i(\mathbf{y}) < f_i(\mathbf{x})$, there exists an index j such that $f_j(\mathbf{x}) < f_j(\mathbf{y})$ and moreover

$$\frac{f_i(\mathbf{x}) - f_i(\mathbf{y})}{f_j(\mathbf{y}) - f_j(\mathbf{x})} \leq M. \tag{3}$$

Definition 6 states that $\mathbf{x}$ being proper Pareto optimal requires for every solution $\mathbf{y}$ and objective i, in which $\mathbf{x}$ possesses a greater objective value than $\mathbf{y}$, the existence of another objective j, for which $\mathbf{x}$ has a smaller objective value. Additionally, the ratio of differences of function values of objective i by j must be smaller than the given number M. In other words, M specifies how many units in any objective a solution is allowed to give up for obtaining one unit of another objective.

The notion of proper Pareto optimality can be translated to a binary relation for describing the relationship between two solutions $\mathbf{x}, \mathbf{y} \in X$. We denote this relation by *M-domination.* $\mathbf{x}$ is said to M-dominate $\mathbf{y}$, if $\mathbf{y}$ is not proper Pareto optimal with respect to the set $X = \{\mathbf{x}, \mathbf{y}\}$. Consequently, $\mathbf{x}$ is proper Pareto optimal, if it is not M-dominated by any other point in X. In order to provide a general foundation for out geometrical analysis of M-domination, we define the relation in $\mathbb{R}^m$ independent of the optimization context.

Definition 7 (M-Domination) Let the two points $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ and $M \in \mathbb{R}_+$ be given. The index sets I, $I_<(\mathbf{u}, \mathbf{v})$ and $I_>(\mathbf{u}, \mathbf{v})$ are defined by

$$I := \{1, 2, \ldots, m\}, \tag{4}$$

$$I_<(\mathbf{u}, \mathbf{v}) := \{i \in I | u_i < v_i\}, \tag{5}$$

$$I_>(\mathbf{u}, \mathbf{v}) := \{i \in I | u_i > v_i\}. \tag{6}$$

Then, $\mathbf{u}$ is said to M-dominate $\mathbf{v}$, denoted by $\mathbf{u} \succ_M \mathbf{v}$, if either $\mathbf{u}$ Pareto dominates $\mathbf{v}$ or if $\mathbf{u}$ and $\mathbf{v}$ are non-dominated to each other and additionally

$$\max_{i \in I_<(\mathbf{u}, \mathbf{v})} \min_{j \in I_>(\mathbf{u}, \mathbf{v})} \frac{v_i - u_i}{u_j - v_j} > M \tag{7}$$

holds.

Definition 7 identifies the largest tradeoff $\mathbf{v}$ exhibits with respect to $\mathbf{u}$ and checks whether it falls underneath the threshold set by M. The calculation scheme outlined in (7) suggests computing all tradeoffs of $\mathbf{v}$ to $\mathbf{u}$ and then determining the respective minima and their maximum. We propose an alternative formulation that only compares the largest and smallest difference in objective values. Additionally, the proposed simplification allows the omission of the Pareto domination check as prerequisite.

Proposition 1 *Let the two points* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ *and* $M \in \mathbb{R}_+$ *be given.* $\mathbf{u} \succ_M \mathbf{v}$ *is equivalent to*

$$\max_{i \in I}(v_i - u_i) + M \cdot \min_{i \in I}(v_i - u_i) > 0. \tag{8}$$

Proof Inequality 8 is not preceded by an additional domination check. **u** Pareto dominates **v**, if $v_i - u_i \geq 0$ for all $i = 1, \ldots, m$ with strict inequality for at least one i. This implies that (8) is positive and consequently that $\mathbf{u} \succ_M \mathbf{v}$. Conversely, **u** is Pareto dominated by **v**, if $v_i - u_i \leq 0$ for all $i = 1, \ldots, m$ with strict inequality for at least one i. In this case, (8) is negative and therefore **u** does not M-dominate **v**. Next, we analyze the case where **u** and **v** are non-dominated to each other. (7) can be reformulated to

$$\frac{\max_{i \in I}(v_i - u_i)}{\max_{i \in I}(u_i - v_i)} > M \quad \Leftrightarrow \quad \max_{i \in I}(v_i - u_i) - M \cdot \max_{i \in I}(u_i - v_i) > 0, \qquad (9)$$

which is equivalent to (8). □

Remark 1 Proposition 1 reduces the complexity of calculating an M-domination check between two solutions. The scheme presented in (7) possesses a worst case complexity of $O(m^2)$, if $I_<(\mathbf{u}, \mathbf{v})$ and $I_>(\mathbf{u}, \mathbf{v})$ are about the same size. Instead, (8) is always evaluated in $O(n)$.

Binary relations over sets can be characterized by certain properties. These properties provide important insights into the practical applicability of the underlying binary relation. Transitivity is of special interest in the light of vector sorting techniques. Non-dominated sorting [12], for example, uses the Pareto domination principle to rank a set of solutions into different non-dominated fronts. The members of each rank are non-dominated to each other, while every member of a successive rank is dominated by at least one element of the precursor rank. Intransitive relations do not allow ranking a solution set into different tiers per se, since it could occur that a non-dominated rank does not exist. As we demonstrate in Proposition 2, M-domination is not always transitive in three and higher dimensions.

Proposition 2 *The M-domination relation is intransitive for $m \geq 3$ and $M \in (0, 2)$.*

Proof Let us consider the three points $\mathbf{u} = (0, \ldots, 0)$, $\mathbf{v} = (2, -1, \ldots, -1)$ and $\mathbf{w} = (1, -2, 1, \ldots, 1)$ for an arbitrary $m \geq 3$. We observe that $\mathbf{u} \succeq_M \mathbf{v}$, $\mathbf{v} \succeq_M \mathbf{w}$ and at the same time $\mathbf{w} \succeq_M \mathbf{u}$ for $M \in (0, 2)$. This implies that the M-domination relation is intransitive. □

Remark 2 Although we only provide a counterexample for $M \in (0, 2)$, we conjecture that the result holds for any $M > 0$. Proposition 2 also shows that the notion of U-domination [25] is intransitive for $m \geq 3$.[1]

Antisymmetry is a property that plays an important role in the practical applicability of binary relations as decision rule for choosing a solution to a multi-objective optimization problem. A decision rule should be able to rank

[1] U-domination and M-domination are equivalent for $M = 1$.

all elements of the solution space or grade them in different tiers of desirability. The absence of antisymmetry prevents any ranking mechanism, since two solutions with distinct objective values could dominate each other. In this case, the decision rule implicated by the binary relation would be contradictory. Proposition 3 reveals that M-domination is not antisymmetric for all choices of M. However, the range of values for which M-domination is not antisymmetric bears little to no meaning in practical applications as we explain in Remark 3.

Proposition 3 *The M-domination relation is antisymmetric for $M \geq 1$ and not antisymmetric for $M \in (0, 1)$.*

Proof Let us consider the two arbitrary points $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$. Let $t_{\mathbf{uv}} = \max_{i \in I}(v_i - u_i)/\max_{i \in I}(u_i - v_i)$ denote the tradeoff of $\mathbf{u}$ to $\mathbf{v}$. We know from the proof of Proposition 1 that $\mathbf{u} \succ_M \mathbf{v}$ is equivalent to $tr_{\mathbf{uv}} > M$. Conversely, $\mathbf{v} \succ_M \mathbf{u}$ is the same as $\max_{i \in I}(u_i - v_i)/\max_{i \in I}(v_i - u_i) > M$, with the left-hand side of the inequality being the inverse of $t_{\mathbf{uv}}$. Without loss of generality let us assume $\mathbf{u} \succ_M \mathbf{v}$ holds. Then, M-domination is antisymmetric if and only if $1/t_{\mathbf{uv}} \leq M$. This is exactly the case for $M \geq 1$. If $M \in (0, 1)$ both $t_{\mathbf{uv}}$ and its inverse may be greater than M. □

Remark 3 Proposition 3 demonstrates that M-domination is a coherent ranking mechanism for $M \geq 1$. Choosing $M \in (0, 1)$ makes little sense in a practical context if all goals are treated with equal importance. Selecting $M = 1/2$, for example, would correspond to being only willing to give up one unit of any objective for obtaining two more units of another objective. Still, we can gain important insights by analyzing M-domination for values for M smaller than one.

We continue our analysis by characterizing the space $\mathbb{R}^m$ from the perspective of a single point $\mathbf{u}$. Any dominance concept induces a dominated, non-dominated and preference subset of $\mathbb{R}^m$ with respect to $\mathbf{u}$. A formal definition of the M-dominated space enables us to obtain its geometrical description by polyhedral cones.

Definition 8 Let $\mathbf{u} \in \mathbb{R}^m$ and $M \in \mathbb{R}_+$ be given. Depending on the threshold level M we characterize the points in $\mathbb{R}^m$ as

1. M-dominated space:

$$D_M(\mathbf{u}) := \left\{ \mathbf{v} \in \mathbb{R}^m \,\middle|\, \max_{i \in I}(v_i - u_i) + M \cdot \min_{i \in I}(v_i - u_i) > 0 \right\}, \tag{10}$$

2. M-preference space:

$$P_M(\mathbf{u}) = -D_M(\mathbf{u}), \tag{11}$$

3. M-non-dominated space:

$$N_M(\mathbf{u}) = \mathbb{R}^m \setminus \{P_M(\mathbf{u}) \cup D_M(\mathbf{u})\}. \tag{12}$$

Figure 3 depicts the surfaces of both $D_M(\mathbf{u})$ and $P_M(\mathbf{u})$ at $\mathbf{u} = (0, 0, 0)$ for $M = 5$ in three dimensions. We can observe that $P_M(\mathbf{u})$ is the reflection of $D_M(\mathbf{u})$ at $\mathbf{u}$ as indicated by (11). Those points that lie in between both spaces form part of $N_M(\mathbf{u})$.

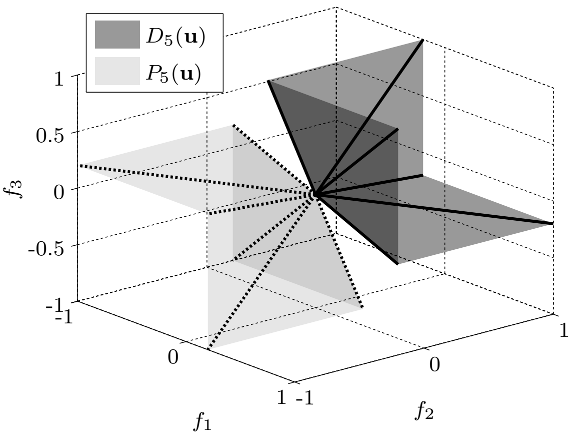

Fig. 1 Comparison of the M-dominated space (dark gray) versus the M-preference space (light gray) for $M = 5$, $m = 3$ and $\mathbf{u} = (0, 0, 0)$. The highlighted areas form the surface of the corresponding spaces. The dotted and straight lines are the generators of the cones $D_5(\mathbf{u})$ and $P_5(\mathbf{u})$, respectively.

Given a multi-objective optimization problem, we can obtain the set of points in X, whose image is not M-dominated by any other solution in X.

Definition 9 (M-set) The set of solutions that are not M-dominated by any solution in $\mathcal{Y}$ is given by

$$X_M := \{\mathbf{x} \in X \mid \mathbf{f}(\mathbf{y}) \not\prec_M \mathbf{f}(\mathbf{x}) \quad \forall\, \mathbf{y} \in X\}\,. \tag{13}$$

Proposition 4 *The set X_M can be empty.*

Proof Proposition 4 is a direct consequence of Proposition 2 and Proposition 3. □

Next, we take a closer look on the geometric properties of the M-dominated space.

Proposition 5 *The M-dominated space is not convex for $m \geq 3$ and for $m = 2$ it is not convex for $M < 1$*

Proof Without loss of generality, let us consider the M-dominated space of the origin $\mathbf{u} = (0, \ldots, 0)$ and the two points $\mathbf{v}^1 = (M + \epsilon, -1, \ldots, -1)$ and $\mathbf{v}^2 = (-1, M+\epsilon, -1, \ldots, -1)$ with a given but arbitrary $M > 0$ and $\epsilon \in (0, M]$.

We observe that $\mathbf{v}^1, \mathbf{v}^2 \in D_M(\mathbf{u})$. If $D_M(\mathbf{u})$ was convex, then $\lambda\mathbf{v}^1 + (1 - \lambda)\mathbf{v}^2 \in D_M(\mathbf{u})$ for all $\lambda \in [0,1]$. If we select $\lambda = 1/2$, we obtain the point $\mathbf{w} = ((M + \epsilon - 1)/2, (M + \epsilon - 1)/2, -1, \ldots, -1)$. If $m \geq 3$, we obtain that $\mathbf{w} \notin D_M(\mathbf{u})$. If $m = 2$ and we additionally require that $M < -1$, $\mathbf{w}$ M-dominates $\mathbf{u}$. □

In the last part of this section, we analyze the behavior of the M-dominated space as M converges towards the extreme values of 0 and ∞.

Theorem 1 (Convergence of M-domination) *Let $\mathbf{u} \in \mathbb{R}^m$ be given. Then, $\mathrm{cl}(D_{M\to\infty}(\mathbf{u})) = \{\mathbf{u}\} + \mathbb{R}^m_+$, i.e. the the closure of the M-dominated space of $\mathbf{u}$ convergences to the positive orthant of $\mathbf{u}$ for $M \to \infty$. Conversely, $\mathrm{cl}(D_{M\to 0}(\mathbf{u})) = \{\mathbf{u}\} + \mathbb{R}^m \setminus (-\mathbb{R}^m)$, i.e. for $M \to 0$, $\mathbf{u}$ M-dominates all points that do not lie in its negative orthant.*

Proof We show the convergence of both sets utilizing the notion of Kuratowski convergence [18]. A sequence of compact subsets A_k in $\mathbb{R}^m$ converges in the Kuratowski sense for $k \to \infty$, if the Kuratowski limit inferior $\mathrm{Li}_{k\to\infty} A_k$ agrees with the Kuratowski limit superior $\mathrm{Ls}_{k\to\infty} A_k$ with

$$\operatorname*{Li}_{k\to\infty} A_k = \left\{ \mathbf{w} \in \mathbb{R}^m \,\middle|\, \limsup_{k\to\infty} \mathrm{dist}(\mathbf{w}, A_k) = 0 \right\}, \tag{14}$$

$$\operatorname*{Ls}_{k\to\infty} A_k = \left\{ \mathbf{w} \in \mathbb{R}^m \,\middle|\, \liminf_{k\to\infty} \mathrm{dist}(\mathbf{w}, A_k) = 0 \right\} \tag{15}$$

and some distance function dist. As the closure of the M-dominated space converges to ∞ and 0, they can be expressed as a sequence of k.

$$\mathrm{cl}\left(D_k^0(\mathbf{u})\right) := \left\{ \mathbf{v} \in \mathbb{R}^m \,\middle|\, \frac{1}{k} \cdot \max_{i\in I}(v_i - u_i) + \min_{i\in I}(v_i - u_i) \geq 0 \right\}, \tag{16}$$

$$\mathrm{cl}\left(D_k^\infty(\mathbf{u})\right) := \left\{ \mathbf{v} \in \mathbb{R}^m \,\middle|\, \max_{i\in I}(v_i - u_i) + \frac{1}{k} \cdot \min_{i\in I}(v_i - u_i) \geq 0 \right\}. \tag{17}$$

As $k \to \infty$, the maximum in Equation 16 and the minimum in Equation 17 disappear. Consequently, limit inferior and limit superior coincide.

$$\operatorname*{Li}_{k\to\infty} \mathrm{cl}\left(D_k^0(\mathbf{u})\right) = \operatorname*{Ls}_{k\to\infty} \mathrm{cl}\left(D_k^0(\mathbf{u})\right) = \left\{ \mathbf{v} \in \mathbb{R}^m \,\middle|\, \min_{i\in I} v_i - u_i \geq 0 \right\}, \tag{18}$$

$$= \{\mathbf{u}\} + \mathbb{R}^m_+ \tag{19}$$

$$\operatorname*{Li}_{k\to\infty} \mathrm{cl}\left(D_k^\infty(\mathbf{u})\right) = \operatorname*{Ls}_{k\to\infty} \mathrm{cl}\left(D_k^\infty(\mathbf{u})\right) = \left\{ \mathbf{v} \in X \,\middle|\, \max_{i\in I} v_i - u_i \leq 0 \right\}. \tag{20}$$

$$= \{\mathbf{u}\} + \left(\mathbb{R}^m \setminus \mathbb{R}^m_-\right) \tag{21}$$

□

Remark 4 Theorem 1 provides us with some interesting insights. As M converges to infinity, M-domination becomes equivalent to Pareto domination. In this sense, M-domination is a generalization of Pareto domination and thereby constitutes an extension of a fundamental concept in multi-objective optimization. Although it makes little practical sense to choose values $M < 1$, it is interesting to note that selecting $M = 0$ results in $\mathbf{u}$ M-dominating all other elements in $\mathbb{R}^m$ besides those, by which it is Pareto dominated. This result also illustrates that M-domination does not violate the Pareto principle. Figure 2 provides a graphical illustration of the findings of Theorem 1.

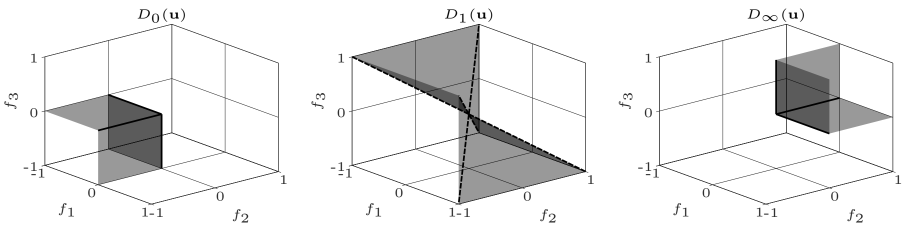

Fig. 2 Illustration of the M-dominated space at the origin in the box $[-1, 1]^3$ using extreme values.

4 A cone-based description of M-domination

This section is dedicated to analyzing the geometric structure of the M-dominated space. We first consider the two-dimensional case, which presents itself as a rather simple exercise. In the subsequent subsections, we discuss four different approaches of describing the M-dominated space by a set of polyhedral cones for an arbitrary number of dimensions. In the final part of this section we derive a statement about the minimum number of cones for describing the M-dominated space and discuss the applicability of each approach for calculating the volume of the M-dominated space. During our entire analysis, we consider the M-dominated space at the arbitrary point $\mathbf{u} \in \mathbb{R}^m$.

In the bi-criteria case, there only exists a single tradeoff, namely between objective one and two. In order to decide, which points $\mathbf{v} \in \mathbb{R}^m$ are M-dominated by $\mathbf{u}$, the tradeoff needs to be bound in both directions.

Proposition 6 (M-domination cone in two dimensions) *Let $m = 2$ and $\mathbf{u} \in \mathbb{R}^2$ be given. Then, the following holds:*

$$D_M(\mathbf{u}) := \{\mathbf{u}\} + C(A), \text{ with } A = \begin{pmatrix} 1 & M \\ M & 1 \end{pmatrix}. \qquad (22)$$

Proof Let us first consider $d_1 > d_2$. In this case, $\mathbf{u} \succ_M \mathbf{v}$ if $d_1 + M \cdot d_2 > 0$. This is the same as $(1, M) \cdot (d_1, d_2)^T > 0$. If $d_1 < d_2$ we get that $(M, 1) \cdot (d_1, d_2)^T > 0$. Since the tradeoff must be bound in both directions, combining the two results we obtain the matrix A. □

4.1 Ordered objectives approach

The situation presents itself in a more complex manner for three and more objectives. As an introductory example, let us assume we find ourselves in three dimensions. If $u_1 > v_1$, the tradeoff of objective one may either be bound by objective two, three or even both at the same time. The challenge lies in representing all these feasible cases in systems of linear inequalities. In order to structure our analysis, we consider the vector $\mathbf{d} = \mathbf{v} - \mathbf{u}$. We can order the elements in $\mathbf{d}$ from largest to smallest. Knowing the largest and smallest difference, we can directly assess, whether the tradeoff from $\mathbf{u}$ to $\mathbf{v}$ is bounded by M. The order of the elements in $\mathbf{d}$ can be represented by a series of linear inequalities. We provide an example in three dimensions. Table 1 lists the necessary linear inequalities for the case where $d_1 \geq d_2 \geq d_3$ and (23) yields the corresponding matrix.

Table 1 Example of the ordered objectives approach.

$d_1 \geq d_2 \geq d_3$
$d_1 + M \cdot d_3 > 0$
$d_1 - d_2 \geq 0$
$d_2 - d_3 \geq 0$

$$A^o_{123} = \begin{pmatrix} 1 & 0 & M \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \tag{23}$$

We acknowledge that the system in Table 1 does not exactly adhere to Definition 4, since the second and third inequality are not strict. This, however, is only a minor nuisance. The cone described by the system in Table 1 is partially open and closed. It is open towards the boundary of the M-dominated space and closed towards the interior of the M-dominated space. For quantifying the M-dominated space of $\mathbf{u}$, it is irrelevant if the cone is open or closed as we may always consider its closure.

The example we have discussed covers only one feasible ordering. There exist six different permutations for three elements, so there are six cases to consider. Each case can be translated to a matrix that in turn represents a polyhedral cone. The union of these cones constitutes the M-dominated space of $\mathbf{u}$. We therefore denote this notion by *ordered objectives approach.*

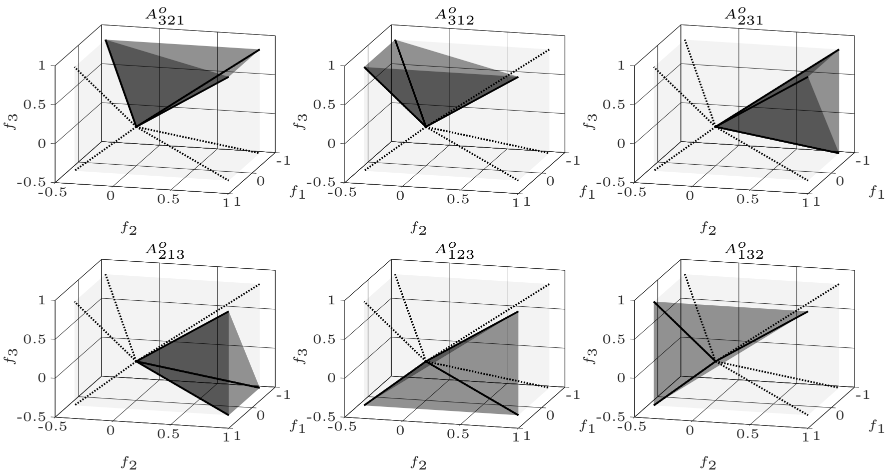

Fig. 3 Division of the M-dominated space of the origin in the box $[-1, 1]^3$ for $M = 3$ using the ordered objective approach.

A visualization of the six different cones in three dimensions is provided in Figure 3.

For an arbitrary number of dimensions, we consider all permutations of orderings of objective differences. This leads to a total of $m!$ matrices in m dimensions. We propose the following notation scheme to formally define the ordered objectives approach for m objectives. The function $\mathbf{a}^M(i,j)$ produces a vector, whose entries are all zero with the exception of index i and j. Entry a_i is one and the value at index j equals to M.

$$\mathbf{a}^M(i,j) = \left(a_1^M, \dots, a_m^M\right) \quad \text{with } a_k^M = \begin{cases} 1 & k = i \\ M & k = j \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

In the same spirit, we define a function $\mathbf{a}^-(i,j)$, which produces a vector whose entries are all zero, however entry a_i equals to one and a_j has the value -1.

$$\mathbf{a}^-(i,j) = \left(a_1^-, \dots, a_m^-\right) \quad \text{with } a_k^- = \begin{cases} 1 & k = i \\ -1 & k = j \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

Let Ω denote the set of all permutations over the index set $I := \{1, \dots, m\}$ and let σ denote a single permutation in the set Ω. Furthermore, $\sigma(i)$ identifies the element at the i-th position in the permuted vector.

Theorem 2 (Ordered objectives approach) *The M-dominated space $D_M(\mathbf{u})$ of a point $\mathbf{u} \in \mathbb{R}^m$ may be expressed as*

$$D_M(\mathbf{u}) = \{\mathbf{u}\} + \bigcup_{\sigma \in \Omega} C(A^o_\sigma) \tag{26}$$

with

$$A^o_\sigma = \begin{pmatrix} \mathbf{a}^M(\sigma(1), \sigma(m))^{1\times m} \\ \mathbf{a}^-(\sigma(1), \sigma(2))^{1\times m} \\ \mathbf{a}^-(\sigma(2), \sigma(3))^{1\times m} \\ \vdots \\ \mathbf{a}^-(\sigma(m-1), \sigma(m))^{1\times m} \end{pmatrix}. \tag{27}$$

Proof Let A_σ denote the matrix that represents the system of linear inequalities induced by the permutation σ.[2] We know that $\bigcup_{\sigma\in\Omega} C(A_\sigma) = \mathbb{R}^m$. $C(A^o_\sigma)$ contains exactly those elements of $C(A_\sigma)$ that are M-dominated by $\mathbf{u}$. We conclude that the union of all $C(A^o_\sigma)$ is equivalent to the non-dominated space.

□

4.2 Min/max ordered objectives approach

The ordered objective approach serves well as a conceptual illustration, however proves to be impractical for volume calculation in higher dimensions, since the number of matrices used for describing the M-dominated space grows factorially. A closer analysis of the ordered objectives approach, however, reveals a possibility for reducing the number of matrices required. The exact order of the entries in $\mathbf{d}$ is irrelevant for checking for M-domination. Instead, it is decisive that the maximum and minimum of $\mathbf{d}$ are known, in order to determine if (8) holds. The other objectives are exclusively required to lie between the minimum and the maximum element of $\mathbf{d}$.

Let us assume that i is the index of the maximum element in $\mathbf{d}$ and j the index of its minimum element. In order to formally represent the system of linear inequalities that ensures that all other entries in $\mathbf{d}$ lie between d_j and d_i, we propose the following notation scheme. The function $B^+(i,j)$ generates a matrix that represents the system of linear inequalities, which states that d_i is larger than all other values in $\mathbf{d}$ with the exception of d_j. This matrix is a combination of the negative identity matrix $-I_{m-2}$, a column vector of ones 1_{m-2} and a column vector of zeros 0_{m-2}. The resulting matrix $B^+(i,j)$ is of size $m-2\times m$. $B^+(i,j)$ is generated by inserting 1_{m-2} and 0_{m-2} into $-I^{m-2}$, such that both vectors are located at the column indexes i and j in the resulting matrix $B^+(i,j)$. A formal summary is given in (28):

[2] A_σ is equivalent to A^o_σ with the exception of the first row, which states the M-domination condition.

$$B^{+}(i,j) \quad \text{with } b^{+}_{kl} = \begin{cases} 1 \; k = i \\ -1 \begin{cases} (k = l) \wedge (k < i, j) \\ (k = l-1) \wedge ((k > i) \otimes (k > j)) \\ (k = l-2) \wedge (k > i, j) \end{cases} \\ 0 \text{ otherwise,} \end{cases} \tag{28}$$

where $\otimes$ describes the XOR operator. Conversely, a matrix for expressing the system of linear inequalities stating that all elements in $\mathbf{d}$ are greater than d_j with the exception of d_i may be described correspondingly. We denote this matrix by $B^{-}(i,j)$. $B^{-}(i,j)$ is generated by inserting 0_{m-2} and a vector of negative ones -1_{m-2} into the positive identity matrix I_{m-2}, such that 0_{m-2} resides at index i and -1_{m-2} at index j in the result matrix.

$$B^{-}(i,j) \quad \text{with } b^{-}_{kl} = \begin{cases} -1 \; k = j \\ 1 \begin{cases} (k = l) \wedge (k < i, j) \\ (k = l-1) \wedge ((k > i) \otimes (k > j)) \\ (k = l-2) \wedge (k > i, j) \end{cases} \\ 0 \text{ otherwise.} \end{cases} \tag{29}$$

The definition of $B^{+}(i,j)$ and $B^{-}(i,j)$ allows us to formally define the *min/max ordered objectives* approach.

Theorem 3 (Min/max ordered objectives approach) *The M-dominated space $D_M(\mathbf{u})$ of a point $\mathbf{u} \in \mathbb{R}^m$ may be expressed as*

$$D_M(\mathbf{u}) = \{\mathbf{u}\} + \bigcup_{i \in I} \bigcup_{j \in I \setminus \{i\}} C(A^{s}_{ij}) \tag{30}$$

with

$$A^{mm}_{ij} = \begin{pmatrix} \mathbf{a}^{M}(i,j)_{1 \times m} \\ B^{+}(i,j)_{m-2 \times m} \\ B^{-}(i,j)_{m-2 \times m} \end{pmatrix}. \tag{31}$$

Proof Let A_{ij} denote the matrix describing the linear system, for which d_i is the maximum and d_j the minimum direction while all other elements of $\mathbf{d}$ are bound between those two directions.[3] We know that $\bigcup_{i \in I} \bigcup_{j \in I \setminus i} A_{ij} = \mathbb{R}^m$. Consequently, $C(A^{mm}_{ij})$ contains all elements of $C(A_{ij})$ that are M-dominated by $\mathbf{u}$. We conclude that the union of all $C(A^{mm}_{ij})$ is equivalent to the M-dominated space by $\mathbf{u}$. □

[3] A_{ij} is equivalent to A^{s}_{ij} with the exception of the first row, which states the M-domination condition.

In order to illustrate the min/max ordered objectives approach, let us consider the matrix A_{21}^{mm} for $m = 4$:

$$A_{21}^{s} = \begin{pmatrix} M & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}. \tag{32}$$

The min/max ordered objectives approach increases the number of linear inequalities required for describing a single order from $m-1$ to $2m-2$. This in turn implies that the number of matrix rows is increased from m to $2m-3$. However, this disadvantage is counteracted by reducing the number of total cones. Since we only consider the largest and smallest difference, we deal with a 2-permutation of m objectives, which is equal to $P(m,2) = m(m-1)$.

4.3 Maximum element approach

The next approach we present further relaxes the prerequisite of having a particular order of objectives for checking the M-domination condition. Let us therefore first consider the case in three dimensions, where the objective differences are ordered according to their magnitude in the following way: $d_1 \geq d_2 \geq d_3$. We know that $\mathbf{u} \succ_M \mathbf{v}$ if $d_1 + M \cdot d_3 > 0$. If this inequality holds, we automatically obtain by transitivity that $d_1 + M \cdot d_2 > 0$ is true as well. We conclude that the exact order of objective differences is irrelevant, as long as we can assert that d_1 is the largest element in $\mathbf{d}$ and the tradeoff of all other objectives is bound to $\mathbf{d}_1$.

In order to simplify the notation of the maximum ordered objectives approach, we propose the following functions. $B^+(i)$ represents the matrix describing the system of linear inequalities stating that all other elements in $\mathbf{d}$ are smaller than d_i. The function is generated by inserting the column vector of ones 1_{m-1} in the negative indentity matrix $-I_{m-1}$ at index i. A formal definition is provided in (33).

$$B^+(i) \quad \text{with } b^+_{kl} = \begin{cases} 1 \; k = i \\ -1 \begin{cases} (k = l) \wedge (k < i) \\ (k = l-1) \wedge (k > i) \end{cases} \\ 0 \text{ otherwise.} \end{cases} \tag{33}$$

The function $B^M(i)$ is created by inserting column vector 1_{m-1} at position i in the product of the identity matrix I_{m-1} by M. This matrix states the requirement for all objectives being bound by objective i:

$$B^M(i) \quad \text{with } b^+_{kl} = \begin{cases} 1 \; k = i \\ M \begin{cases} (k = l) \wedge (k < i) \\ (k = l-1) \wedge (k > i) \end{cases} \\ 0 \text{ otherwise.} \end{cases} \tag{34}$$

Theorem 4 (Maximum element approach) *The M-dominated space $D_M(\mathbf{u})$ of a point $\mathbf{u} \in \mathbb{R}^m$ may be expressed as*

$$D_M(\mathbf{u}) = \{\mathbf{u}\} + \bigcup_{i \in I} C(A_i^{max}) \tag{35}$$

with

$$A_i^{max} = \begin{pmatrix} B^M(i)_{m-1\times m} \\ B^+(i)_{m-1\times m} \end{pmatrix}. \tag{36}$$

Proof We know that $\bigcup_{i\in I} C(B^M(i)) = \mathbb{R}^m$. $C(A_i^{max})$ contains exactly those points of $C(B^M(i))$ that are M-dominated by $\mathbf{u}$. We conclude that $D_M(\mathbf{u}) = \bigcup_{i\in I} C(A_i^{max})$. □

The *maximum element approach* requires only m cones for describing the M-dominated space. Establishing that d_i is the largest element of $\mathbf{d}$ needs m inequalities. Bounding the tradeoff of all goals to objective j takes the same number of inequalities. This results in $2m-2$ rows per matrix.

4.4 Minimum matrix approach

The *minimum matrix approach* is a subsequent relaxation of the maximum element approach. The maximum element approach explicitly states by implementing the block matrix $B^+(i)$ that d_i is the largest element in $\mathbf{d}$. This requirement can also be met by bounding the tradeoff of objective i to any other objective $j \in I \setminus \{i\}$.

Theorem 5 (Minimum matrix approach) *The M-dominated space $D_M(\mathbf{u})$ of a point $\mathbf{u} \in \mathbb{R}^m$ may be expressed as*

$$D_M(\mathbf{u}) = \{\mathbf{u}\} + \bigcup_{i \in I} C(A_i^{min}) \tag{37}$$

with

$$A_i^{min} = \begin{pmatrix} B^M(i)_{m-1\times m} \\ \mathbf{a}^M((i \mod m)+1, i)_{1\times m} \end{pmatrix}. \tag{38}$$

Proof It is sufficient to show that $C(A_i^{max}) \subseteq C(A_i^{min})$ and that $C(A_i^{min})$ only includes M-dominated elements. If $d_i = \arg\max_{j\in I} d_j$, then $Md_i + d_k > 0$ for $k \in I$, if $B^+(i)_j > 0$. This implies that $C(A_i^{max})$ is a subset of $C(A_i^{min})$. If d_i is not the maximum element of $\mathbf{d}$, then $Md_i + d_k > 0$ only holds if $\arg\max_{j\in I}(d_j) + M \arg\min_{j\in I}(d_j) > 0$, which is equivalent to $\mathbf{u} \succ_M \mathbf{v}$. □

The minimum matrix approach requires only m matrices each having m rows. The linear inequalities of the underlying system translate directly to the matrix as of Definition 4, since they are all strict in comparison to the other approaches. Still, the minimum matrix approach features one distinct disadvantage. The cones $C(A_i^{min})$ are overlapping. This is because a single tradeoff may be bound by multiple objectives. The issue is illustrated in Figure 4, which shows the different partitions of the minimum matrix approach for $M = 3$. The issue of overlapping partitions is discussed in the next subsection.

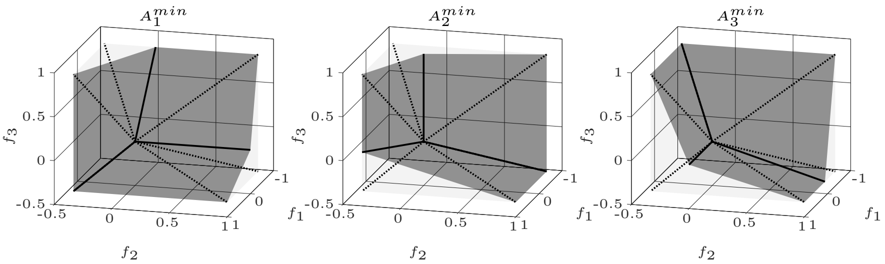

Fig. 4 Division of the M-dominated space of the origin in $[-1, 1]^3$ for $M = 3$ using the minimum matrix approach.

4.5 Minimum number of cones and summary

One might wonder if it possible to even further reduce the number of polyhedral cones for describing the M-dominated space. Theorem 6 shows that m serves as lower bound for the number of convex cones in m dimensions.

Theorem 6 *Describing the M-dominated space $D_M(\mathbf{u})$ of a point $\mathbf{u} \in \mathbb{R}^m$ by a union of convex cones requires at least m cones for $m > 2$.*

Proof Without loss of generality let us consider the M-dominated space of the origin $\mathbf{u} = (0, \dots, 0)$ and let an arbitrary but fixed M be given. Let us consider the point $\mathbf{v}^i$ with $i \in I$, whose entries are all -1 with the exception of entry i, which equals $M + \epsilon$ with $\epsilon \in (0, M]$. From the proof of Proposition 5 we obtain that $\mathbf{v}^1$ and $\mathbf{v}^2$ cannot lie in the same convex partition of $D_M(\mathbf{u})$. This result holds for any $\mathbf{v}^j$ and $\mathbf{v}^k$ with $j, k \in I$, $j \neq k$. Since $|I| = m$, we require at least m convex partitions of $D_M(\mathbf{u})$.

□

The four different approaches we have presented in the previous subsections can all be utilized to calculate the volume of the M-dominated space with respect to a given reference point $\mathbf{r}$. Computing the volume of $\mathbf{u}$ with respect to the partition matrix A and $\mathbf{r}$ is done in the following way. We calculate the volume of the orthotope defined by the vertices $A \cdot \mathbf{u}$ and $A \cdot \mathbf{r}$. The orthotope volume is then divided by the determinant of A if the matrix is square or the root of $det(A^T A)$ if A is rectangular to obtain the partition volume of $\mathbf{u}$ according to A and $\mathbf{r}$. Summing up all partition volumes yields the overall volume of the M-dominated space with respect to $\mathbf{r}$. Using the transformation scheme presented in the previous paragraph allows us to reuse existing hypervolume calculation techniques [3, 29] based on the Pareto domination principle.

Calculating the joint hypervolume of a set of solutions is a non-trivial task, as individual domination cones are usually overlapping. The runtime of state-of-the-art methods increases exponentially in the number of objectives. We have therefore identified three criteria that determine the effectiveness of the partitioning approaches presented in this work. The total number of cones determines how many translations and single volume calculations need to be conducted. The matrix size of each cone decideds upon the complexity of the volume calculation, as the number of rows determines the dimensionality of the transformed coordinate system. The number of columns is identical for all approaches and has the value m. Finally, overlapping cones prohibit the computation of the exact volume, as overlapping subvolumes are included multiple times.

Table 2 summarizes our findings with respect to these three performance criteria. The number of cones grows factorially in the ordered objectives approach, which makes it only suitable for application in lower dimensions. However, it is the only approach that retains the dimensionality of the original coordinate system that is non-overlapping. The min/max ordered objectives approach requires only a quadratic number of cones, however the dimensionality of the transformed coordination system increases almost by a factor of two in comparison. The maximum element approach, in comparison, requires just m cones and needs only one additional matrix row. Although both approaches are non-dominated to each other, we can argue that trading in one matrix row for practically squaring the number of matrices represents an unacceptable tradeoff. The minimum matrix approach features the minimum number of cones and rows per matrix, however the cones it describes are overlapping. This property does not necessarily imply that the minimum matrix approach is ill-suited for finding volume-maximizing distributions. We conjecture, however, that an algorithmic application could lead to an overly strong crowding of solutions in certain regions that does not concur with the notion of M-domination. We observe that no approach is optimal in each criterion. However, for an algorithmic application we suggest further investigating either the maximum element or minimum matrix approach, as they appear to be the only viable options in higher dimensions.

5 Conclusion

In this paper, we have shown how the notion of proper Pareto optimality may be translated to the dominance relation M-domination. We have thoroughly discussed the mathematical properties of M-domination yielding important in-

Table 2 Summary of the approaches for describing the M-dominated space by a set of polyhedral cones.

	# of cones	Rows per matrix	Overlapping
Ordered objectives	$m!$	m	No
Min/max ordered objectives	$m(m-1)$	$2(m-2)+1$	No
Maximum element	m	$2(m-1)$	No
Minimum matrix	m	m	Yes

sights into its algorithmic and practical applicability. By analyzing the convergence behavior, we have shown that M-domination is a generalization of Pareto domination, which places this notion well in the canon of multi-objective optimization literature. We have placed a large focus on providing an analytical and graphical description of the M-dominated space. By showing that this space can be decomposed into a union of polyhedral cones, we have opened up new application scenarios for M-domination in volume based multi-objective optimization algorithms.

Future research may incorporate M-domination directly into the selection mechanism of an optimization algorithm utilizing a volume maximizing approach. Hypervolume [34] is a performance indicator for grading an approximation of the Pareto front with respect to convergence and diversity. The indicator calculates the space Pareto-dominated by a population in the objective space in relation to a given reference point. Algorithms such as IBEA [32], SMS-EMOA [2], HypE [1] and their subsequent enhancements [19, 20] stand testimony to the success of this approach in obtaining approximations of the Pareto front. The decomposition of the M-dominated space in polyhedral cones allows us to quantify the volume M-dominated by a solution. An M-hypervolume maximizing algorithm would not only identify solutions below the threshold M, but also generate more solutions in regions that show a balanced trade-off behavior.

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