Convergence Analysis of ISTA and FISTA for "Strongly + Semi" Convex Programming

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Abstract

The iterative shrinkage/thresholding algorithm (ISTA) and its faster version FISTA have been widely used in the literature. In this paper, we consider general versions of the ISTA and FISTA in the more general "strongly + semi" convex setting, i.e., minimizing the sum of a strongly convex function and a semiconvex function; and conduct convergence analysis for them. The consideration of a semiconvex function makes it possible to consider some noncovex regularization arising often in contemporary sparsity-driven applications such as the well-known smoothly clipped absolute deviation (SCAD) penalty. Meanwhile, the semiconvex function makes the convergence analysis more challenging than the regular "convex + convex" case. We develop a new analytic framework based on the Féjer monotonicity for the convergence analysis. In the "strongly + semi" convex setting, the respective $O(1/k)$ and $O(1/k^2)$ convergence rates of the general ISTA and FISTA are also established, where k is the iteration counter. We apply the general versions of ISTA and FISTA to solve the SCAD-ℓ_2 model and show their efficiency including the apparent acceleration effectiveness of the latter.

Keywords: Iteratively shrinkage/thresholding algorithm, FISTA, semiconvex, nonconvex penalty, SCAD, convergence rate.

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1 Introduction

For many applications, their mathematical models can be represented as minimizing the sum of two functions without coupled variables:

$$\min_{x\in\mathcal{R}^n} \; F(x) := f(x) + g(x), \tag{1.1}$$

where $f : \mathcal{R}^n \to \mathcal{R}$ and $g : \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$. Denote by S the solution set of (1.1) and assume $S \neq \emptyset$. One particularly interesting scenario is where f represents a data-fidelity term (usually continuously differentiable) and g is a regularization term (usually nonsmooth and/or nonconvex). It is a common sense that for concrete applications of the abstract model (1.1), both f and g have their own properties/features; and instead of considering the sum of f and g as a whole, we should treat them separately in algorithmic design so that a splitting algorithm capable of taking full advantage of f and g's properties/features effectively can be developed.

When both f and g are assumed to be convex, designing an algorithm for the "convex + convex" case of (1.1) is relatively easier. For instance, if f is further assumed to be continuously differentiable with Lipschitz continuous gradient, the very classical forward-backward splitting method studied in earlier literature (e.g., [15, 16, 21]) can be immediately applied:

$$x_{k+1} = \arg\min_{x\in\mathcal{R}^n} \left\{ g(x) + \frac{1}{2\alpha}\|x - (x_k - \alpha\nabla f(x_k))\|^2 \right\}, \tag{1.2}$$

in which the step size $\alpha > 0$ should be judiciously chosen (see, e.g., [15, 16]). More concretely, if we consider the popular $\ell_2 - \ell_1$ model:

$$\min_{x\in\mathcal{R}^n} \frac{1}{2}\|Ax - b\|^2 + \tau\|x\|_1, \tag{1.3}$$

where $\|x\|_1 := \sum_{i=1}^n |x_i|$, $A \in \mathcal{R}^{m\times n}$, $b \in \mathcal{R}^n$ and $\tau > 0$, then the scheme (1.2) is specified as

$$x_{k+1} = \arg\min_{x\in\mathcal{R}^n} \left\{ \tau\|x\|_1 + \frac{1}{2\alpha}\|x - (x_k - \alpha A^T(Ax_k - b))\|^2 \right\}. \tag{1.4}$$

It is well-known (see, e.g., [6]) that the solution of (1.4) is explicitly given by the closed-form

$$x_{k+1} = \mathcal{T}_{\alpha\tau}(x_k - \alpha A^T(Ax_k - b)),$$

where $\mathcal{T}_\beta : \mathcal{R}^n \to \mathcal{R}^n$ is the so-called shrinkage operator defined as

$$T_\beta(x)_i := (|x_i| - \beta)_+ \mathrm{sign}(x_i).$$

An interesting scenario of (1.3) is where $m \ll n$ and the $\ell_2 - \ell_1$ model (1.3) accounts for the convex relaxation of the problem of finding the sparest vector x among all solution points of the underdetermined system $Ax = b$. This scenario finds sparsity-driven applications in a wide range of areas and it explains some important applications such as the least absolute shrinkage and selection operator (LASSO) in [24] and basis pursuit denoising problem in [10]. The scheme (1.2) turns out to be a fundamental solver to the model (1.3); with a rich literature available. In the literature, the scheme (1.2) is also known as the iterative shrinkage/thresholding algorithm (ISTA). One particularly important development of (1.2) is the fast iterative shrinkage/thresholding algorithm (FISTA) proposed in [6] that is now a very popular first-order solver for many sparsity-driven applications. The FISTA scheme for (1.1) reads as

$$\begin{cases} x_k = \arg\min_{x\in\mathcal{R}^n} \left\{ g(x) + \frac{1}{2\alpha}\|x - (y_k - \alpha\nabla f(y_k))\|^2 \right\}, \\ t_{k+1} = \frac{1+\sqrt{1+4t_k^2}}{2}, \\ y_{k+1} = x_k + \frac{t_k - 1}{t_{k+1}}(x_k - x_{k-1}), \end{cases} \tag{1.5}$$

with $y_1 = x_0$ and $t_1 = 1$. The respective convergence rates of ISTA and FISTA are proved to be $O(1/k)$ and $O(1/k^2)$ in [6], where k is the iteration counter. In addition to the theoretically derived faster worst-case bound, numerical acceleration of the FISTA over ISTA has been well verified by various literatures.

Despite of its importance as a convex relaxation problem of an originally nonconvex model and its relatively easiness in sense of algorithmic design, the "convex + convex" case of (1.1) fails to explain some important applications — one reason is that a nonconvex regularization is what really appears in the true modelling process of many application problems and using the even convex tight surrogate may cause negative effects such as biasedness in estimates. We thus have to use nonconvex regularization in some sparsity-driven applications to obtain more meaningful results, though it is generally more difficult in mathematical sense. A good example is the well-known smoothly clipped absolute deviation (SCAD) penalty (see its definition in (3.6)-(3.7)) originally proposed in [14], which can result in an estimator with the important properties: unbiasedness, sparsity and continuity. This is an obvious advantage over both the convex ℓ_1 penalty which usually leads to biased estimators and the concave ℓ_p penalty with $0 \le p < 1$ which usually does not satisfy the continuity condition, see the elaboration in [14]. One more example with wide applications in statistics is the the minimax concave plus (MCP) penalty (see its definition in (3.9)-(3.10)) proposed in [26]. We will show in Section 3 that both the SCAD and MCP penalties are nonconvex but semiconvex (see Definition 2.6). Moreover, the semiconvex penalty accounts for the smoothed surrogate (*i.e.,* $\sum_{i=1}^n (|x_i| + \epsilon)^p$ with $\epsilon > 0$) of the widely-considered ℓ_p norm (*i.e.,* $\|x\|_p^p := \sum_{i=1}^n |x_i|^p$ with $0 < p < 1$) in the literature (see, e.g., [11]); the details will also be given in Section 3. Further, as well analyzed in the literature, e.g., [1, 4, 5, 9], an interesting theoretical property is that the proximal operators of semiconvex functions are general enough to produce all separable monotone threshold functions and this property plays a crucial role in constructing a penalty function associated with a given monotone function (see, e.g., [5]).

On the other hand, for some applications the data-fidelity term should be treated more sophisticatedly with some ad hoc purposes from their application perspectives and accordingly the strong convexity may appear. For example, it was suggested in [27] to enhance the ℓ_2 term in the model (1.3) with an additional ℓ_2 norm so as to encourage strongly correlated variables to be in or out of the model at the same time — the so-called "grouping effect" initiated in [27]. The elastic net model replacing the term $\frac{1}{2}\|Ax - b\|^2$ in (1.3) by the ℓ_2-enhanced data-fidelity term

$$\frac{1}{2}\|Ax - b\|^2 + \frac{\lambda}{2}\|x\|^2 \text{ with } \lambda > 0 \tag{1.6}$$

was proposed in [27] to simultaneously do automatic variable selection and continuous shrinkage, with the ability of selecting groups of correlated variables. Obviously, the function in (1.6) is strongly convex with constant $\lambda + \lambda_{\min}(A^T A)$. Statistically, it was shown in [27] that the additional ℓ_2 term helps remove the limitation on the number of selected variables and stabilizes the ℓ_1 regularization path. More elaborated analysis for the application of some strongly convex data-fidelity terms in high-dimension data analysis can be found in the literature. For example, in [7], it was explained that an additional ℓ_2 norm in regression models can help an algorithm perceive the input data as having higher variance, which makes it shrink the weights on features; see also the so-called weight decay regularization [7]. This kind of strongly convex data-fidelity terms have been popularly used in the machine learning community to strengthen the model stability and enhance the generalization ability (see, e.g., [17, 18]).

Some more advanced applications even urge the combination of the mentioned strong convexity in data-fidelity and semiconvexity in regularization; hence the necessity of considering the "strongly + semi" convex setting of (1.1). A concrete example fitting this scenario is the SCAD-ℓ_2 model proposed in [25]; see (7.1) in Section 7 for the details. Roughly speaking, the SCAD-ℓ_2 model suggests using the ℓ_2-enhanced data-fidelity term (1.6) with the SCAD penalty; and so preserves both the "grouping effect" and the "unbiased, sparse and continuous" estimates. Statistically, it was analyzed in [25] that the SCAD-ℓ_2 has better performances in terms of the effectiveness of reducing prediction error, preserving variable selection accuracy, and achieving group effect. For applications of the "strongly + semi" convex setting of (1.1), we also refer to [19] for the joint denoising and sharpening image recovery problem, in which a semiconvex regularization term helps the design of energy functions that describe some desired effects more accurately than purely convex ones.

Because of the mentioned critical concerns in various application perspectives, we are interested in the "strongly + semi" convex setting of (1.1), i.e, f is strongly convex (see Definition 2.5) with constant $\rho_1 \geq 0$ and $g : \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$ is a proper lower semicontinuous semiconvex function (see Definition 2.6) with constant $\rho_2 \geq 0$. Given the popular scenario where f is a quadratic or ℓ_2-enhanced quadratic term representing a data-fidelity term, we also assume that f is continuously differentiable and ∇f is Lipschitz continuous with constant $\sigma > 0$. According to definitions, we know that $\rho_1 \leq \sigma$ holds automatically. We further assume that $\rho_1 \geq \rho_2$ as the existing literature; thus the model (1.1) is still a convex programming problem in analytical sense if we only consider f and g as a whole. But as we have just elaborated, we need to treat the strongly convex and semiconvex functions separately in algorithmic design and considering them together seems not to be attractive from the algorithmic design perspective. Thus, it seems too general to proceed any meaningful discussion in the context of designing splitting algorithms for the generic nonconvex case of (1.1) with $\rho_1 < \rho_2$. Note that the special case where $\rho_1 = \rho_2 = 0$, i.e., f and g are both convex functions, is included in our discussion. Thus, the "strongly + semi" convex setting of (1.1) under our discussion is more general than its regular "convex + convex" case where both f and g are convex.

Given the well-verified efficiency of the ISTA and FISTA for the "convex + convex" case, we want to consider their general versions in the more general "strongly + semi" convex setting of (1.1) and conduct convergence analysis — these are the main purposes of this paper. As we shall see, the semiconvexity makes the convergence analysis more challenging than the "convex + convex" case and analyzing the Fejér monotonicity of the sequence of the general ISTA or FISTA turns out to be crucial for establishing their convergence results. To discuss the general ISTA in the "strongly + semi" convex setting, first of all, (1.2) should be well defined in the sense that the minimization problem in (1.2) has a unique solution. As shown in Theorem 4.1, the restriction $0 < \alpha < 1/\rho_2$ is sufficient to ensure this property and thus it is a necessary condition on the step size of α for our discussion of the convergence of the general ISTA and FISTA in the "strongly + semi" convex setting of (1.1). Since ρ_2 may be 0, we use the convention $1/0 = +\infty$ throughout. Indeed, our analysis will be conducted under some conditions that are sufficient to ensure this necessary condition; see Theorems 4.4, 5.3 and 6.3 for the convergence results. As comparison, for the "convex + convex" case of (1.1), the step size α is required to be $0 < \alpha < 2/\sigma$ to ensure the convergence of the ISTA (1.2) (see, e.g., [12]); and the respective $O(1/k)$ and $O(1/k^2)$ convergence rates of the ISTA and FISTA are established in [6] under the condition $\alpha = 1/\sigma$. Our conditions on the step size α are actually more general in the "strongly + semi" convex setting; see Remarks 4.1, 5.1, and 6.1 for explanations.

The rest of this paper is organized as follows. In Section 2, we summarize some basic definitions and known results for further analysis. In Section 3, we show that the mentioned SCAD and MCP penalties, and the smoothed surrogate of the ℓ_p norm with $0 < p < 1$ are all semiconvex. Then, we prove the convergence for a general version of the ISTA in terms of the iterative sequence in the "strongly + semi" convex setting of (1.1) in Section 4. In Sections 5 and 6, we establish the $O(1/k)$ and $O(1/k^2)$ convergence rates for the general ISTA and FISTA in the "strongly + semi" convex setting of (1.1), respectively. In Section 7, we apply the general ISTA and FISTA to solve the SCAD-ℓ_2 model in [25] and report some preliminary numerical results. Finally, we draw some conclusions in Section 8. An appendix is also included to show the $O(1/k)$ convergence rate of the backtracking version of the general ISTA in the "strongly + semi" convex setting of (1.1).

2 Preliminaries

In this section, we recall some standard definitions and known results for further analysis.

Definition 2.1. [3] Let C be a nonempty subset of $\mathcal{R}^n$ and let $\{x_k\}_{k\in N}$ be a sequence in $\mathcal{R}^n$.

Then $\{x_k\}_{k\in N}$ is Fejér monotone with respect to C if

$$\|x_{k+1} - x\| \leq \|x_k - x\|, \quad \forall x \in C, \forall k \in N.$$

Definition 2.2. [3] Let D be a nonempty subset of $\mathcal{R}^n$ and let $T : D \to \mathcal{R}^n$. Then T is
(i) firmly nonexpansive if

$$\|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 \leq \|x - y\|^2, \quad \forall x, y \in D;$$

(ii) κ-Lipschitz continuous if there exists $\kappa \geq 0$ such that

$$\|Tx - Ty\| \leq \kappa\|x - y\|, \quad \forall x, y \in D;$$

(iii) β-cocoercive if there exists $\beta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \beta\|Tx - Ty\|^2, \quad \forall x, y \in D.$$

Definition 2.3. [23] A mapping $T : \mathcal{R}^n \rightrightarrows \mathcal{R}^n$ is strongly monotone if there exists $\mu > 0$ such that

$$\langle v_1 - v_0, x_1 - x_0\rangle \geq \mu\|x_1 - x_0\|^2, \ \forall v_0 \in T(x_0), \ v_1 \in T(x_1).$$

Definition 2.4. [23] Given a function $f : \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$, the effective domain and epigraph of f are defined respectively by

$$\mathrm{dom} f := \{x \mid f(x) < +\infty\} \text{ and } \mathrm{epi} f := \{(x, \alpha) \in \mathcal{R}^n \times \mathcal{R} : f(x) \leq \alpha\}.$$

We say that the function f is proper (respectively, lower semicontinuous) if $\mathrm{dom} f$ (respectively, $\mathrm{epi} f$) is nonempty (respectively, closed).

Definition 2.5. [23] A function $f : \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$ is strongly convex with constant $\mu > 0$ if for any $x, y \in \mathcal{R}^n$ and for any $\theta \in (0, 1)$, we have

$$f((1-\theta)x + \theta y) \leq (1-\theta)f(x) + \theta f(y) - \frac{\mu\theta(1-\theta)}{2}\|x - y\|^2.$$

Moreover, if the above inequality holds for $\mu = 0$, then we call f convex function.

Definition 2.6. [8] A proper lower semicontinuous function $g : \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$ is called semiconvex with constant $\omega \geq 0$ if the function

$$x \longmapsto g(x) + \frac{\omega}{2}\|x\|^2$$

is convex. Specially, if $\omega = 0$, then g is convex.

Definition 2.7. [23] Consider a function $f : \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$ and a point $\bar{x}$ with $f(\bar{x})$ finite.
(i) The regular subdifferential of f at $\bar{x}$, written $\hat{\partial} f(\bar{x})$, is the set of vectors $x^* \in \mathcal{R}^n$ that satisfy

$$\liminf_{x \neq \bar{x}, x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \geq 0.$$

(ii) The subdifferential of f at $\bar{x}$, written $\partial f(\bar{x})$, is defined as follows:

$$\partial f(\bar{x}) = \{x^* \in \mathcal{R}^n : \exists x_k \to \bar{x}, f(x_k) \to f(\bar{x}), x_k^* \in \hat{\partial} f(x_k), \text{with } x_k^* \to x^*\}.$$

Remark 2.1. *It follows from Definitions 2.6 and 2.7 that the following assertions hold (see, e.g., [8, 23]).*
(i) If $h : \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$ is a proper function and $f : \mathcal{R}^n \to \mathcal{R}$ is continuous differentiable, then $\partial(f+h)(\bar{x}) = \nabla f(\bar{x}) + \partial h(\bar{x})$ for any $\bar{x} \in domh$.
(ii) For any proper convex function $f : \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$ and any point $\bar{x} \in domf$, one has $\partial f(\bar{x}) = \hat{\partial} f(\bar{x}) = \{v | f(x) \geq f(\bar{x}) + \langle v, x - \bar{x}\rangle$ for all $x\}$. If in addition f is differentiable at $\bar{x}$, then $\partial f(\bar{x}) = \{\nabla f(\bar{x})\}$.
(iii) Let $g : \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Then g is semiconvex with constant ω if and only if $g(x_2) \geq g(x_1) + \langle \xi_1, x_2 - x_1\rangle - \frac{\omega}{2}\|x_2 - x_1\|^2$, for all $x_1, x_2 \in \mathcal{R}^n$, $\xi_1 \in \partial g(x_1)$.

Below we quote some lemmas that will be used later. Their proofs can be found in the literature; thus omitted.

Lemma 2.1. [23] For a function $f : \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$ and a scalar $\mu > 0$, the following properties are equivalent:
(i) ∂f is strongly monotone with constant μ;
(ii) f is strongly convex with constant μ;
(iii) $f - \frac{\mu}{2}\|\cdot\|^2$ is convex.

Remark 2.2. *Based on (iii) of Lemma 2.1, we know that f is strongly convex with constant μ if and only if $f(x_2) \geq f(x_1) + \langle \xi_2, x_2 - x_1\rangle + \frac{\mu}{2}\|x_2 - x_1\|^2$, for all $x_1, x_2 \in \mathcal{R}^n$, $\xi_2 \in \partial f(x_1)$.*

Lemma 2.2. [23] A proper lower semicontinuous function $f : \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$ is convex if and only if ∂f is monotone, in which case ∂f is maximal monotone.

Lemma 2.3. [23] Let $T : \mathcal{R}^n \rightrightarrows \mathcal{R}^n$ be monotone and a scalar $\lambda > 0$. Then $(I + \lambda T)^{-1}$ is monotone and nonexpansive. Moreover, T is maximal monotone if and only if $\mathrm{dom}(I + \lambda T)^{-1} = \mathcal{R}^n$. In that case, $(I + \lambda T)^{-1}$ is a single-valued mapping from all of $\mathcal{R}^n$ into itself.

Lemma 2.4. [3] Let $T : \mathcal{R}^n \rightrightarrows \mathcal{R}^n$ be maximally monotone and let $\gamma > 0$. Then $(I + \gamma T)^{-1}$ is firmly nonexpansive.

Lemma 2.5. [2] Let $f : \mathcal{R}^n \to \mathcal{R}$ be convex, continuously differentiable on $\mathcal{R}^n$, and ∇f be σ-Lipschitz continuous for some $\sigma \in (0, +\infty)$. Then ∇f is $\frac{1}{\sigma}$-cocoercive.

Lemma 2.6. [20] Let $f : \mathcal{R}^n \to \mathcal{R}$ be a continuously differentiable function and gradient ∇f is Lipschitz continuous with constant $\sigma > 0$, then for any $x, y \in \mathcal{R}^n$, we have

$$f(x) \leq f(y) + \langle \nabla f(y), x - y\rangle + \frac{\sigma}{2}\|x - y\|^2.$$

3 Semiconvex examples

In this section, we show that some important nonconvex penalties that are being widely used in the literature are semiconvex functions. Examples include the SCAD penalty in [14], the MCP penalty in [26], and the smoothed surrogate of the ℓ_p norm with $0 < p < 1$. We first prove a lemma useful for this purpose, while its conclusion is meaningful on its own.

Lemma 3.1. Let $h : \mathcal{R} \to \mathcal{R}$ be convex in $(-\infty, b]$ and $[b, +\infty)$ for some $b \in \mathcal{R}$, respectively. Suppose that the directional derivatives $h'_-(b)$ and $h'_+(b)$ satisfy $h'_-(b) \leq h'_+(b)$ (which holds automatically when h is convex in a neighborhood $N(b)$ of b), then h is convex in $\mathcal{R}$.

Proof. For any $x, y \in \mathcal{R}$, we need to prove that for any $t \in (0, 1)$, it holds

$$h(tx + (1-t)y) \leq th(x) + (1-t)h(y). \tag{3.1}$$

If both x and y belong to the same interval $(-\infty, b]$ or $[b, +\infty)$, then we know (3.1) holds. Thus, we only need to prove the assertion (3.1) for any $x \in (-\infty, b)$, $y \in (b, +\infty)$ and $t \in (0, 1)$. This is equivalent to show that the segment defined by

$$[(x, h(x)), (y, h(y))] := \{(tx + (1-ty)), (th(x) + (1-t)h(y) : t \in (0,1))\}$$

is included by the epigraph of h, epi$h$. Let d be the vertical coordinate of the intersecting point of the segment $[(x, h(x)), (y, h(y))]$ and the straight line $L := \{(b, y) | y \in \mathcal{R}\}$. Then there exists $\lambda > 0$ such that $b = (1-\lambda)x + \lambda y$ and $d = (1-\lambda)h(x) + \lambda h(y)$. Since h is convex in $[b, +\infty)$, by [22, Theorem 23.1] we have

$$h'_+(b) \leq \frac{h(y) - h(b)}{y - b}. \tag{3.2}$$

Similarly, by the convexity of h in the interval $(-\infty, b]$, we know

$$\frac{h(x) - h(b)}{x - b} \leq h'_-(b). \tag{3.3}$$

Note that $h'_-(b) \leq h'_+(b)$ (If h is convex on $N(b)$, we can get this assertion from [22, Theorem 24.1]). Combining this with (3.2) and (3.3), we get

$$\frac{h(x) - h(b)}{x - b} \leq \frac{h(y) - h(b)}{y - b}. \tag{3.4}$$

Recall $b = (1-\lambda)x + \lambda y$. So (3.4) can be written as

$$h(b) \leq (1-\lambda)h(x) + \lambda h(y) = d.$$

Thus $\{(x, h(x)), (b, d), (y, h(y))\} \subset$ epi$h$. Again, since h is convex in $(-\infty, b]$ and $[b, +\infty)$, h is convex in $\mathcal{R}$ and the proof is complete. □

The assertion in Lemma 3.1 can be easily extended to a general case as stated below. We skip the proof.

Corollary 3.1. Suppose there exist points $b_1, b_2, \cdots, b_{m+1}$ such that $b_1 < b_2 < \cdots < b_{m+1}$. Let $h : \mathcal{R} \to \mathcal{R}$ be convex in all the intervals $(-\infty, b_1]$, $[b_1, b_2]$, $\cdots$, $[b_{m+1}, +\infty)$ individually. If the directional derivatives $h'_-(b_i)$ and $h'_+(b_i)$ satisfy $h'_-(b_i) \leq h'_+(b_i)$ for any $i \in \{1, 2, \cdots, m+1\}$ (which holds automatically when h is convex in a neighborhood $N(b_i)$ of b_i), then h is convex in $\mathcal{R}$.

3.1 SCAD is semiconvex

Now we show that the SCAD penalty proposed in [14] is semiconvex. Let us first recall the definition of a function in [14]:

$$g_\lambda(\theta) := \begin{cases} \lambda|\theta|, & |\theta| \le \lambda, \\ \frac{-\theta^2+2a\lambda|\theta|-\lambda^2}{2(a-1)}, & \lambda < |\theta| \le a\lambda, \\ \frac{(a+1)\lambda^2}{2}, & |\theta| > a\lambda, \end{cases} \tag{3.5}$$

where $a > 2$ and $\lambda > 0$ correspond the knots of the quadratic spline function.

Next, we apply Lemma 3.1 to show that the function defined in (3.5) is semiconvex. Note that this conclusion is essentially implied by Lemma 1 in [25]. Here we provide an explicit and different proof.

Theorem 3.1. For any $a > 2$ and $\lambda > 0$, the function $g_\lambda(\theta)$ defined in (3.5) is semiconvex with constant $\frac{1}{a-1}$.

Proof. When $\theta > 0$, obviously $g'_\lambda(\theta)$ exists and it is given by

$$g'_\lambda(\theta) = \begin{cases} \lambda, \ \theta \le \lambda, \\ \frac{a\lambda-\theta}{a-1}, \ \lambda < \theta \le a\lambda, \\ 0, \ \theta > a\lambda. \end{cases}$$

Thus, we have

$$g'_\lambda(\theta) + \omega\theta = \begin{cases} \omega\theta + \lambda, \ \theta \le \lambda, \\ (\omega - \frac{1}{a-1})\theta + \frac{a\lambda}{a-1}, \ \lambda < \theta \le a\lambda, \\ \omega\theta, \ \theta > a\lambda, \end{cases}$$

which is monotonically nondecreasing if $\omega \ge \frac{1}{a-1}$. Thus, when $\theta > 0$, we have

$$(g_\lambda(\cdot) + \omega|\cdot|^2)'(\theta) = g'_\lambda(\theta) + \omega\theta.$$

It follows from [3, Proposition 8.12] that $g_\lambda(\cdot) + \frac{\omega}{2}|\cdot|^2$ is convex in $(0, +\infty)$ if $\omega \ge \frac{1}{a-1}$. Since $g_\lambda(\cdot) + \frac{\omega}{2}|\cdot|^2$ is even and continuous at 0, we know that $g_\lambda(\cdot) + \frac{\omega}{2}|\cdot|^2$ is convex in the intervals $(-\infty, 0]$ and $[0, +\infty)$. Note that $g_\lambda(\cdot)$ is an absolute value function in a neighborhood of $\theta = 0$. Thus, we conclude that $g_\lambda(\cdot)+\frac{\omega}{2}|\cdot|^2$ is convex in $\mathcal{R}$ in view of Lemma 3.1. The proof is complete. □

For $x \in \mathcal{R}^n$, based on (3.5), the SCAD penalty is given in [14] as

$$\sum_{i=1}^{n} g_\lambda(|x_i|) \tag{3.6}$$

with

$$g_\lambda(|x_i|) := \begin{cases} \lambda|x_i|, & |x_i| \le \lambda, \\ \frac{-x_i^2+2a\lambda|x_i|-\lambda^2}{2(a-1)}, & \lambda < |x_i| \le a\lambda, \\ \frac{(a+1)\lambda^2}{2}, & |x_i| > a\lambda. \end{cases} \tag{3.7}$$

Theorem 3.1 immediately implies that the SCAD penalty $\sum_{i=1}^n g_\lambda(|x_i|)$ defined in (3.6)-(3.7) is a semiconvex function with constant $\frac{1}{a-1}$. Note that in [14] the values of the parameters a and λ were suggested to be chosen pairwise over a two-dimensional grids using some criteria such as the cross-validation; and $a = 3.7$ was suggested therein.

3.2 MCP is semiconvex

Then, we show that the MCP penalty proposed in [26] is also semiconvex. First, recall the seed function of MCP given by

$$P_\gamma(|\theta|;\lambda) := \begin{cases} \lambda|\theta| - \frac{1}{2\gamma}\theta^2, & |\theta| < \lambda\gamma, \\ \frac{\lambda^2\gamma}{2}, & |\theta| \ge \lambda\gamma, \end{cases} \tag{3.8}$$

with $\gamma > 0$ and $\lambda > 0$. Again, based on Lemma 3.1, we can prove that the function defined in (3.8) is semiconvex.

Theorem 3.2. For any $\gamma > 0$ and $\lambda > 0$, the function $P_\gamma(|\theta|;\lambda)$ defined in (3.8) is semiconvex with constant $\frac{1}{\gamma}$.

Proof. Define

$$\tilde{P}_\gamma(\theta;\lambda) := P_\gamma(|\theta|;\lambda) + \frac{\omega}{2}\theta^2 = \begin{cases} \frac{\lambda^2\gamma}{2} + \frac{\omega}{2}\theta^2, & \theta \ge \lambda\gamma, \\ \lambda\theta + (\frac{\omega}{2} - \frac{1}{2\gamma})\theta^2, & 0 \le \theta < \lambda\gamma, \\ -\lambda\theta + (\frac{\omega}{2} - \frac{1}{2\gamma})\theta^2, & -\lambda\gamma < \theta < 0, \\ \frac{\lambda^2\gamma}{2} + \frac{\omega}{2}\theta^2, & \theta \le -\lambda\gamma. \end{cases}$$

Then the function $\tilde{P}_\gamma(\cdot;\lambda)$ is even and continuous at 0. To prove the convexity of $\tilde{P}_\gamma(\cdot;\lambda)$ in $(-\infty, 0]$ and $[0, +\infty)$, we only need to show $\tilde{P}_\gamma(\cdot;\lambda)$ is convex on $(0, +\infty)$. The remaining proof is similar as that of Theorem 3.3, thus omitted. □

For $x \in \mathcal{R}^n$, based on (3.8), the MCP penalty is given in [26] as

$$\sum_{i=1}^n P_\gamma(|x_i|;\lambda) \tag{3.9}$$

with

$$P_\gamma(|x_i|;\lambda) := \begin{cases} \lambda|x_i| - \frac{1}{2\gamma}\theta^2, & |x_i| < \lambda\gamma, \\ \frac{\lambda^2\gamma}{2}, & |x_i| \ge \lambda\gamma. \end{cases} \tag{3.10}$$

It is thus known from Theorem 3.2 that the MCP penalty defined in (3.9)-(3.10) is semiconvex with constant $\frac{1}{a-1}$. In [26], it was suggested to choose $\gamma > 1$; and λ should be larger than the so-called universal penalty level defined in [13]. Statistically, these are the regularization and penalty level parameters, respectively.

3.3 Smoothed surrogate of the ℓ_p norm is semiconvex

Finally, we show that the smoothed surrogate of the ℓ_p norm with $0 < p < 1$ is also semiconvex. Recall that its definition is $\sum_{i=1}^n (|x_i| + \epsilon)^p$ with $0 < p < 1$ and $\epsilon > 0$.

Theorem 3.3. For any $0 < p < 1$ and $\epsilon > 0$, the function $\sum_{i=1}^n (|x_i| + \epsilon)^p$ is semiconvex with constant $p(1-p)\epsilon^{p-2}$.

Proof. We need to prove the convexity of the function $\sum_{i=1}^n (|x_i| + \epsilon)^p + \frac{\omega}{2}\|x\|^2$ with $\omega \geq p(1-p)\epsilon^{p-2}$. Since the function $\sum_{i=1}^n (|x_i| + \epsilon)^p + \frac{\omega}{2}\|x\|^2$ is separable, we only need to show the convexity of the one-dimension function

$$l(x_i) := (|x_i| + \epsilon)^p + \frac{\omega}{2} x_i^2$$

with $\omega \geq p(1-p)\epsilon^{p-2}$. Since $l(\cdot)$ defined above is even and continuous at 0, we only need to show the convexity of $l(\cdot)$ in $(0, +\infty)$ when $\omega \geq p(1-p)\epsilon^{p-2}$ for the purpose of proving the convexity of $l(\cdot)$ in $(-\infty, 0]$ and $[0, +\infty)$. For any $x_i > 0$, we have

$$l'(x_i) = p(x_i + \epsilon)^{p-1} + \omega x_i, \quad l''(x_i) = p(p-1)(x_i + \epsilon)^{p-2} + \omega. \tag{3.11}$$

Moreover, for any $x_i > 0$, since $x_i + \epsilon > \epsilon$, we know

$$p(p-1)(x_i + \epsilon)^{p-2} + \omega > p(p-1)\epsilon^{p-2} + \omega \geq 0,$$

where the second inequality follows from the assumption $\omega \geq p(1-p)\epsilon^{p-2}$. Hence, $l''(x_i) > 0$. It follows from [23, Theorem 2.13] that $l(\cdot)$ is strictly convex (hence convex) in $(0, +\infty)$. Next, we need to show $l'_-(0) \leq l'_+(0)$. By (3.11) we know that $l'_+(0) = p\epsilon^{p-1}$. Thus, $l'_-(0) = -p\epsilon^{p-1}$ because $l(\cdot)$ is an even function. By Lemma 3.1, the proof is complete. □

4 Convergence of the ISTA: "strongly + semi" convex case

In this section, we establish the convergence of a general version of the ISTA (1.2) in the "strongly + semi" convex setting of (1.1). The convergence result is in terms of the iterative sequence $\{x_k\}_{k\in N}$ generated by (1.2). For the "convex + convex" case, the convergence of ISTA is essentially guaranteed by the convergence of the forward-backward splitting method, see, e.g., [12]. Its extension to the "strongly + semi" convex case, however, deserves an elaborated analysis. Note that it was claimed in [4] that the convergence of (1.2) for the "strongly + semi" convex setting of (1.1) was proved for the special case where $\rho_1 = \rho_2 \geq 0$ and $\rho_1 < \sigma$. But the proof of Lemma 8 in [4] is incorrect and thus it invalidates the convergence proof therein. Hence, the convergence of the ISTA (1.2) in the "strongly + semi" convex setting of (1.1) is so far still unknown.

4.1 Some common results

To study the iterative scheme (1.2), hereafter let us use the notation

$$T := (I + \alpha\partial g)^{-1}(I - \alpha\nabla f), \tag{4.1}$$

where ∂g is the subdifferential of g defined in Definition 2.7. Obviously, it is useful to know the relation between the solution set S of (1.1) and the set of fixed point of the operator T defined in (4.1). This relationship was revealed in [4] (see Proposition 10 therein). Here, we give a much simpler proof based on some properties of a maximal monotone operator. A full proof is included for completeness.

Lemma 4.1. For model (1.1), suppose f is strongly convex with constant $\rho_1 \geq 0$, continuously differentiable and ∇f is Lipschitz continuous with constant $\sigma > 0$; g is proper lower semicontinuous semiconvex function with constant $\rho_2 \geq 0$. Suppose $\rho_1 \geq \rho_2$ and $S \neq \emptyset$. Let T be defined in (4.1) with $0 < \alpha < 1/\rho_2$. Then $x^* \in S$ if and only if $x^* = Tx^*$. Moreover,

$$S = \left\{ x^* \in \mathcal{R}^n \mid (I + \frac{\alpha}{1-\alpha\rho_2}\partial\tilde{g})^{-1}(\frac{1-\alpha\rho_1}{1-\alpha\rho_2}I - \frac{\alpha}{1-\alpha\rho_2}\nabla\tilde{f})(x^*) = x^* \right\}, \tag{4.2}$$

where $\tilde{f}(\cdot)$ and $\tilde{g}(\cdot)$ are respectively defined by

$$\tilde{f}(\cdot) := f(\cdot) - \frac{\rho_1}{2}\|\cdot\|^2 \tag{4.3}$$

and

$$\tilde{g}(\cdot) := g(\cdot) + \frac{\rho_2}{2}\|\cdot\|^2. \tag{4.4}$$

Proof. First, we prove the assertion that $x^* \in S$ if and only if $x^* = Tx^*$. It follows from the assertions (i) and (ii) of Remark 2.1 that $x^* \in S$ if and only if

$$0 \in \partial(f+g)(x^*) = \nabla f(x^*) + \partial g(x^*). \tag{4.5}$$

For any $\alpha \neq 0$, (4.5) is equivalent to

$$x^* - \alpha\nabla f(x^*) \in x^* + \alpha\partial g(x^*). \tag{4.6}$$

If $(I + \alpha\partial g)^{-1}$ is single-valued everywhere, then (4.6) is equivalent to $x^* = Tx^*$. Thus, the remaining part of the proof is to show that if $0 < \alpha < 1/\rho_2$, then $(I + \alpha\partial g)^{-1}$ is single-valued everywhere. For $\tilde{g}(\cdot)$ defined in (4.4), since $1 - \alpha\rho_2 \neq 0$, we have

$$(I + \alpha\partial g)^{-1} = ((1-\alpha\rho_2)I + \alpha\partial\tilde{g})^{-1} = (I + \frac{\alpha}{1-\alpha\rho_2}\partial\tilde{g})^{-1}(\frac{1}{1-\alpha\rho_2})I. \tag{4.7}$$

Since $\tilde{g}$ is proper lower semicontinuous convex, it follows from Lemma 2.2 that $\partial\tilde{g}$ is maximal monotone. For $0 < \alpha < 1/\rho_2$, it follows from Lemma 2.3 that $(I + \frac{\alpha}{1-\alpha\rho_2}\partial\tilde{g})^{-1}$ is single-valued everywhere; so is $(I + \alpha\partial g)^{-1}$.

Next, we show (4.2) holds. Since f is strongly convex with constant ρ_1 and continuously differentiable, it follows from Lemma 2.1 that $\tilde{f}$ defined in (4.3) is convex. Hence, we have

$$I - \alpha\nabla f = (1 - \alpha\rho_1)I - \alpha\nabla\tilde{f}. \tag{4.8}$$

Combining this with (4.7), we obtain

$$\begin{aligned} T &= (I + \frac{\alpha}{1-\alpha\rho_2}\partial\tilde{g})^{-1}\frac{1}{1-\alpha\rho_2}(I - \alpha\nabla f) \\ &= (I + \frac{\alpha}{1-\alpha\rho_2}\partial\tilde{g})^{-1}(\frac{1-\alpha\rho_1}{1-\alpha\rho_2}I - \frac{\alpha}{1-\alpha\rho_2}\nabla\tilde{f}). \end{aligned} \tag{4.9}$$

Hence, (4.2) holds and the proof is complete. □

In the following, we represent the ISTA (1.2) in terms of the convex functions $\tilde{f}$ and $\tilde{g}$ defined respectively in (4.3) and (4.4).

Theorem 4.1. The ISTA (1.2) with $0 < \alpha < 1/\rho_2$ is well defined under our "strongly + semi" convex setting of (1.1) and it can be rewritten as

$$x_{k+1} = (I + \frac{\alpha}{1-\alpha\rho_2}\partial\tilde{g})^{-1}(\frac{1-\alpha\rho_1}{1-\alpha\rho_2}I - \frac{\alpha}{1-\alpha\rho_2}\nabla\tilde{f})(x_k), \tag{4.10}$$

where $\tilde{f}$ and $\tilde{g}$ are defined in (4.3) and (4.4), respectively.

Proof. Recall the definition in (4.4). The ISTA (1.2) can be written as

$$x_{k+1} = \arg\min_{x\in\mathcal{R}^n}\left\{\tilde{g}(x) - \frac{\rho_2}{2}\|x\|^2 + \frac{1}{2\alpha}\|x - (x_k - \alpha\nabla f(x_k))\|^2\right\}, \tag{4.11}$$

in which the minimization problem is strongly convex because of $0 < \alpha < 1/\rho_2$. Thus, the ISTA (1.2) is well defined. Moreover, deriving the optimality condition of the minimization problem in (4.11), we have

$$0 \in \partial\tilde{g}(x_{k+1}) - \rho_2 x_{k+1} + \frac{1}{\alpha}(x_{k+1} - x_k + \alpha\nabla f(x_k)). \tag{4.12}$$

According to the proof of Lemma 4.1, we know that $(I + \frac{\alpha}{1-\alpha\rho_2}\partial\tilde{g})^{-1}$ is single-valued when $0 < \alpha < 1/\rho_2$. Thus, combining (4.8) and (4.12), we get (4.10) immediately. The proof is complete. □

We proceed to conduct the convergence analysis for the case of "$\rho_1 \le \sigma$"; note that $\rho_1 = \sigma$ holds when, e.g., $f(x) = \frac{1}{2}\|x\|^2$. It is worthy to mention that the analysis for $\rho_1 = \sigma$ is different from that for $\rho_1 < \sigma$. We thus discuss them separately later on. Our main tool for analysis is the Fejér monotonicity (see Definition 2.1) of the sequence generated by (1.2).

4.2 The case where $\rho_1 < \sigma$

We first prove a lemma and then present the convergence of a general version of the ISTA (1.2) for the case where $\rho_1 < \sigma$.

Lemma 4.2. For model (1.1), suppose $f : \mathcal{R}^n \to \mathcal{R}$ is strongly convex function with constant $\rho_1 \geq 0$, continuously differentiable and ∇f is Lipschitz continuous with constant $\sigma > 0$. Suppose $\rho_1 < \sigma$. Let $\tilde{f}$ be defined in (4.3). Then for any $x, y \in \mathcal{R}^n$, we have

$$\langle \nabla \tilde{f}(x) - \nabla \tilde{f}(y), x - y\rangle \geq \frac{1}{t(\rho_1)} \|\nabla \tilde{f}(x) - \nabla \tilde{f}(y)\|^2, \tag{4.13}$$

where

$$t(\rho_1) := \begin{cases} \sigma, & \text{if} \quad \rho_1 = 0, \\ \sigma - \rho_1, & \text{if} \quad 0 < \rho_1 \leq \frac{\sigma}{2}, \\ \sqrt{\sigma^2 - \rho_1^2}, & \text{if} \quad \frac{\sigma}{2} < \rho_1 < \sigma. \end{cases} \tag{4.14}$$

Proof. Since ∇f is Lipschitz continuous with constant $\sigma > 0$, it follows from Lemma 2.5 that ∇f is $\frac{1}{\sigma}$-cocoercive. That is, for any $x, y \in \mathcal{R}^n$, we have

$$\langle \nabla f(x) - \nabla f(y), x - y\rangle \geq \frac{1}{\sigma} \|\nabla f(x) - \nabla f(y)\|^2. \tag{4.15}$$

Moreover, f is differentiable and strongly convex with constant $\rho_1 \geq 0$. It follows from the assertion (ii) of Remark 2.1 and Lemma 2.1 that ∇f is strongly monotone with constant ρ_1. Thus, for any $x, y \in \mathcal{R}^n$, we have

$$\langle \nabla f(x) - \nabla f(y), x - y\rangle \geq \rho_1 \|x - y\|^2. \tag{4.16}$$

Note that $\nabla f = \nabla \tilde{f} + \rho_1 I$. If $\rho_1 = 0$, then for any $x, y \in \mathcal{R}^n$, it holds that

$$\|\nabla \tilde{f}(x) - \nabla \tilde{f}(y)\| = \|\nabla f(x) - \nabla f(y)\| \leq \sigma \|x - y\|. \tag{4.17}$$

Meanwhile, if $\rho_1 > 0$, then for any $x, y \in \mathcal{R}^n$, we have

$$\|\nabla \tilde{f}(x) - \nabla \tilde{f}(y)\|^2 = \|\nabla f(x) - \nabla f(y)\|^2 - 2\rho_1 \langle \nabla f(x) - \nabla f(y), x - y\rangle + \rho_1^2 \|x - y\|^2. \tag{4.18}$$

Note that the inner-product term of the right-hand side in (4.18) can be amplified by the result in (4.15) or (4.16). Substituting (4.16) into (4.18), then for any $x, y \in \mathcal{R}^n$, we get

$$\|\nabla \tilde{f}(x) - \nabla \tilde{f}(y)\|^2 \leq \|\nabla f(x) - \nabla f(y)\|^2 - \rho_1^2 \|x - y\|^2 \leq (\sigma^2 - \rho_1^2) \|x - y\|^2,$$

where the second inequality follows from the Lipschitz continuity of ∇f. We thus have

$$\|\nabla \tilde{f}(x) - \nabla \tilde{f}(y)\| \leq \sqrt{\sigma^2 - \rho_1^2} \|x - y\|, \ \forall x, y \in \mathcal{R}^n. \tag{4.19}$$

On the other hand, substituting (4.15) into (4.18), for any $x, y \in \mathcal{R}^n$, we obtain

$$\|\nabla \tilde{f}(x) - \nabla \tilde{f}(y)\|^2 \quad \leq \quad \|\nabla f(x) - \nabla f(y)\|^2 - \frac{2\rho_1}{\sigma} \|\nabla f(x) - \nabla f(y)\|^2 + \rho_1^2 \|x - y\|^2$$

$$= \quad (1-\frac{2\rho_1}{\sigma})\|\nabla f(x)-\nabla f(y)\|^2+\rho_1^2\|x-y\|^2. \tag{4.20}$$

For the scenario where $\sigma \geq 2\rho_1$, it follows from the Lipschitz continuity of ∇f and (4.20) that

$$\|\nabla \tilde{f}(x)-\nabla \tilde{f}(y)\|^2 \leq (\sigma-\rho_1)^2\|x-y\|^2,\ \forall x,y\in\mathcal{R}^n,$$

which means

$$\|\nabla \tilde{f}(x)-\nabla \tilde{f}(y)\|n \leq (\sigma-\rho_1)\|x-y\|,\ \forall x,y\in\mathcal{R}^n. \tag{4.21}$$

For the other scenario where $\sigma < 2\rho_1$, notice that (4.20) can be written as

$$\|\nabla \tilde{f}(x)-\nabla \tilde{f}(y)\|^2 \leq (1-\frac{\rho_1}{\sigma})\|\nabla f(x)-\nabla f(y)\|^2+\rho_1^2\|x-y\|^2-\frac{\rho_1}{\sigma}\|\nabla f(x)-\nabla f(y)\|^2. \tag{4.22}$$

Hence, the fact of $\sigma > \rho_1$, the Lipschitz continuity of ∇f and (4.22) enable us to derive that, for any $x, y \in \mathcal{R}^n$, we have

$$\begin{aligned}\|\nabla \tilde{f}(x)-\nabla \tilde{f}(y)\|^2 &\leq (\sigma^2-\rho_1\sigma)\|x-y\|^2+\rho_1^2\|x-y\|^2-\frac{\rho_1}{\sigma}\|\nabla f(x)-\nabla f(y)\|^2\\ &= (\sigma^2-\rho_1\sigma+\rho_1^2)\|x-y\|^2-\frac{\rho_1}{\sigma}\|\nabla f(x)-\nabla f(y)\|^2.\end{aligned}$$

This means for any $x, y \in \mathcal{R}^n$, the following inequality holds:

$$\|\nabla \tilde{f}(x)-\nabla \tilde{f}(y)\| \leq \sqrt{\sigma^2-\rho_1\sigma+\rho_1^2}\|x-y\|. \tag{4.23}$$

With simple algebraic calculation, we know that if $\sigma < 2\rho_1$, then it holds that

$$\sqrt{\sigma^2-\rho_1^2} < \sqrt{\sigma^2-\rho_1\sigma+\rho_1^2}.$$

Regardless of the fact that $\sigma - \rho_1 \leq \sqrt{\sigma^2-\rho_1^2}$ always holds, (4.21) holds only when $\sigma \geq 2\rho_1$. Thus, with the definition of $t(\rho_1)$ in (4.14), it follows from (4.17), (4.19), (4.21) and (4.23) that $\nabla \tilde{f}$ is $t(\rho_1)$-Lipschitz continuous, where $t(\rho_1) > 0$. Since $\tilde{f}$ is convex and continuously differentiable in $\mathcal{R}^n$, it follows from Lemma 2.5 that (4.13) holds. The proof is complete. □

Next, we prove the convergence of a general version of the ISTA (1.2) for the case where $\rho_1 < \sigma$. As we can see, the semiconvexity of $g(x)$ makes the proof more challenging than the "convex + convex" case.

Theorem 4.2. For model (1.1), suppose f is strongly convex with constant $\rho_1 \geq 0$, continuously differentiable and ∇f is Lipschitz continuous with constant $\sigma > 0$; g is proper lower semicontinuous semiconvex function with constant $\rho_2 \geq 0$. Suppose $\sigma > \rho_1 \geq \rho_2$ and $S \neq \emptyset$. If $0 < \alpha < \frac{2}{2\rho_1+t(\rho_1)}$, where $t(\rho_1)$ defined in (4.14), then the sequence $\{x_k\}_{k\in N}$ generated by the ISTA (1.2) converges to a solution point in S.

Proof. First, since $t(\rho_1) > 0$, the conditions $0 < \alpha < \frac{2}{2\rho_1+t(\rho_1)}$ and $\rho_1 \geq \rho_2$ clearly imply that $0 < \alpha < 1/\rho_2$. Thus, it follows from Theorem 4.1 that the ISTA (1.2) is well defined under the assumptions. Then, for any $x^* \in S$, it follows from (4.2) and (4.10) that

$$\|x_{k+1}-x^*\|^2 = \|(I+\frac{\alpha}{1-\alpha\rho_2}\partial\tilde{g})^{-1}(\frac{1-\alpha\rho_1}{1-\alpha\rho_2}I-\frac{\alpha}{1-\alpha\rho_2}\nabla\tilde{f})(x_k)$$

$$
\begin{aligned}
&- (I + \frac{\alpha}{1-\alpha\rho_2}\partial\tilde{g})^{-1}(\frac{1-\alpha\rho_1}{1-\alpha\rho_2}I - \frac{\alpha}{1-\alpha\rho_2}\nabla\tilde{f})(x^*)\|^2 \\
&\le \|(\frac{1-\alpha\rho_1}{1-\alpha\rho_2}I - \frac{\alpha}{1-\alpha\rho_2}\nabla\tilde{f})(x_k) - (\frac{1-\alpha\rho_1}{1-\alpha\rho_2}I - \frac{\alpha}{1-\alpha\rho_2}\nabla\tilde{f})(x^*)\|^2 \\
&- \|x_{k+1} - (\frac{1-\alpha\rho_1}{1-\alpha\rho_2}I - \frac{\alpha}{1-\alpha\rho_2}\nabla\tilde{f})(x_k) - x^* + (\frac{1-\alpha\rho_1}{1-\alpha\rho_2}I - \frac{\alpha}{1-\alpha\rho_2}\nabla\tilde{f})(x^*)\|^2 \\
&= \|(\frac{1-\alpha\rho_1}{1-\alpha\rho_2}I - \frac{\alpha}{1-\alpha\rho_2}\nabla\tilde{f})(x_k) - (\frac{1-\alpha\rho_1}{1-\alpha\rho_2}I - \frac{\alpha}{1-\alpha\rho_2}\nabla\tilde{f})(x^*)\|^2 \\
&- \|(x_{k+1} - x^*) - ((\frac{1-\alpha\rho_1}{1-\alpha\rho_2}I - \frac{\alpha}{1-\alpha\rho_2}\nabla\tilde{f})(x_k) - (\frac{1-\alpha\rho_1}{1-\alpha\rho_2}I - \frac{\alpha}{1-\alpha\rho_2}\nabla\tilde{f})(x^*))\|^2 \\
&= -\|x_{k+1} - x^*\|^2 + \frac{2(1-\alpha\rho_1)}{1-\alpha\rho_2}\langle x_{k+1} - x^*, x_k - x^*\rangle \\
&- \frac{2\alpha}{1-\alpha\rho_2}\langle x_{k+1} - x^*, \nabla\tilde{f}(x_k) - \nabla\tilde{f}(x^*)\rangle \\
&= -\|x_{k+1} - x^*\|^2 + \frac{1-\alpha\rho_1}{1-\alpha\rho_2}(\|x_{k+1} - x^*\|^2 + \|x_k - x^*\|^2 - \|x_{k+1} - x_k\|^2) \\
&- \frac{2\alpha}{1-\alpha\rho_2}\langle x_k - x^*, \nabla\tilde{f}(x_k) - \nabla\tilde{f}(x^*)\rangle + \frac{2\alpha}{1-\alpha\rho_2}\langle x_k - x_{k+1}, \nabla\tilde{f}(x_k) - \nabla\tilde{f}(x^*)\rangle,
\end{aligned}
$$

where the inequality follows from the firm nonexpansiveness of $(I + \frac{\alpha}{1-\alpha\rho_2}\partial\tilde{g})^{-1}$ which is deduced by the maximal monotonicity of $\partial\tilde{g}$ and Lemma 2.4. Rearranging terms, we have

$$
\begin{aligned}
&\frac{1-\alpha\rho_1 + 2\alpha(\rho_1 - \rho_2)}{1-\alpha\rho_2}\|x_{k+1} - x^*\|^2 \\
\le\ &\frac{1-\alpha\rho_1}{1-\alpha\rho_2}\|x_k - x^*\|^2 - \frac{1-\alpha\rho_1}{1-\alpha\rho_2}\|x_{k+1} - x_k\|^2 - \frac{2\alpha}{1-\alpha\rho_2}\langle x_k - x^*, \nabla\tilde{f}(x_k) - \nabla\tilde{f}(x^*)\rangle \\
&+\frac{2\alpha}{1-\alpha\rho_2}\langle x_k - x_{k+1}, \nabla\tilde{f}(x_k) - \nabla\tilde{f}(x^*)\rangle. \qquad (4.24)
\end{aligned}
$$

Since $1 - \alpha\rho_2 \ge 1 - \alpha\rho_1 > 0$, $\rho_1 \ge \rho_2$ and $\alpha > 0$, we have

$$
\frac{1-\alpha\rho_1}{1-\alpha\rho_2} \le \frac{1-\alpha\rho_1 + 2\alpha(\rho_1 - \rho_2)}{1-\alpha\rho_2}.
$$

Substituting the inequality above into the left-hand side of (4.24) and dividing $\frac{1-\alpha\rho_1}{1-\alpha\rho_2}$ on both sides, we get

$$
\begin{aligned}
\|x_{k+1} - x^*\|^2 \quad \le \quad &\|x_k - x^*\|^2 - \|x_{k+1} - x_k\|^2 - \frac{2\alpha}{1-\alpha\rho_1}\langle x_k - x^*, \nabla\tilde{f}(x_k) - \nabla\tilde{f}(x^*)\rangle \\
&+\frac{2\alpha}{1-\alpha\rho_1}\langle x_k - x_{k+1}, \nabla\tilde{f}(x_k) - \nabla\tilde{f}(x^*)\rangle. \qquad (4.25)
\end{aligned}
$$

It follows from Lemma 4.2 that

$$
\langle \nabla\tilde{f}(x_k) - \nabla\tilde{f}(x^*), x_k - x^*\rangle \ge \frac{1}{t(\rho_1)}\|\nabla\tilde{f}(x_k) - \nabla\tilde{f}(x^*)\|^2. \qquad (4.26)
$$

Moreover, using Cauchy-Schwarz inequality, we have

$$
\langle x_k - x_{k+1}, \nabla\tilde{f}(x_k) - \nabla\tilde{f}(x^*)\rangle \le \frac{t(\rho_1)}{4}\|x_{k+1} - x_k\|^2 + \frac{1}{t(\rho_1)}\|\nabla\tilde{f}(x_k) - \nabla\tilde{f}(x^*)\|^2. \qquad (4.27)
$$

Therefore, substituting (4.26) and (4.27) into (4.25), we get

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - (1 - \frac{\alpha t(\rho_1)}{2(1-\alpha\rho_1)})\|x_{k+1} - x_k\|^2. \tag{4.28}$$

With simple algebraic calculation, we know that

$$1 - \frac{\alpha t(\rho_1)}{2(1-\alpha\rho_1)} > 0 \text{ with } \alpha < \frac{2}{2\rho_1 + t(\rho_1)}.$$

Thus, it can be further derived from (4.28) that

$$\|x_{k+1} - x^*\| \leq \|x_k - x^*\|, \tag{4.29}$$

which implies the Fejér monotonicity of the sequence $\{x_k\}_{k\in N}$ with respect to S. This means for any $x^* \in S$, the sequence $\{\|x_k - x^*\|\}_{k\in N}$ is monotonically nonincreasing and $\{x_k\}_{k\in N}$ is bounded (indeed, $\{\|x_k - x^*\|\}_{k\in N}$ is convergent). Again, because of (4.28), we know

$$(1 - \frac{\alpha t(\rho_1)}{2(1-\alpha\rho_1)})\|x_{k+1} - x_k\|^2 \leq \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2.$$

Adding the above inequalities for $k = 0, 1, \ldots, m$, we have

$$\sum_{k=0}^{m}(1 - \frac{\alpha t(\rho_1)}{2(1-\alpha\rho_1)})\|x_{k+1} - x_k\|^2 \leq n\|x_0 - x^*\|^2 - \|x_{m+1} - x^*\|^2 \leq \|x_0 - x^*\|^2.$$

Thus, taking $m \to \infty$, we obtain

$$\sum_{k=0}^{\infty}(1 - \frac{\alpha t(\rho_1)}{2(1-\alpha\rho_1)})\|x_{k+1} - x_k\|^2 \leq \|x_0 - x^*\|^2 < +\infty,$$

and hence

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{4.30}$$

The boundedness of the sequence $\{x_k\}_{k\in N}$ indicates that there exists at least one cluster point. Let $\bar{x}$ be a cluster point of $\{x_k\}_{k\in N}$ and $\{x_{k_j}\}_{j\in N}$ be the subsequence converging to $\bar{x}$. Because of (4.30), we know that $\{x_{k_j+1}\}_{j\in N}$ also converges to $\bar{x}$. Since $\tilde{g}$ is proper lower semicontinuous convex, it follows from Lemmas 2.2 and 2.3 that $(I + \frac{\alpha}{1-\alpha\rho_2}\partial\tilde{g})^{-1}$ with $\frac{\alpha}{1-\alpha\rho_2} > 0$ is nonexpansive and hence continuous. Then the operator

$$(I + \frac{\alpha}{1-\alpha\rho_2}\partial\tilde{g})^{-1}(\frac{1-\alpha\rho_1}{1-\alpha\rho_2}I - \frac{\alpha}{1-\alpha\rho_2}\nabla\tilde{f})$$

is continuous. We take the limit in (4.10) along the subsequence $\{x_{k_j}\}_{j\in N}$ and derive that $\bar{x} = T\bar{x}$. Thus, it follows from Lemma 4.1 that $\bar{x} \in S$. This means that any cluster point of $\{x_k\}_{k\in N}$ is a solution point of model (1.1). We can replace x^* by $\bar{x}$ in (4.29). Then $\{\|x_k - \bar{x}\|\}_{k\in N}$ is convergent. Since $\|x_{k_j} - \bar{x}\| \to 0$, we know that $x_k \to \bar{x}$. The proof is complete. □

4.3 The case where $\rho_1 = \sigma$

The convergence analysis of the ISTA (1.2) for this case is different from that for $\rho_1 < \sigma$ mainly in that $\tilde{f}$ defined in (4.3) is an affine function under this additional assumption; and this fact simplifies the proof of the Fejér monotonicity of the sequence $\{x_k\}_{k\in N}$, compared with the proof of Theorem 4.2. We thus discuss it separately.

We first show the speciality of a function under our assumptions in the following lemma and then present the convergence result.

Lemma 4.3. If $f : \mathcal{R}^n \to \mathcal{R}$ is strongly convex with constant $\rho_1 > 0$, continuously differentiable and ∇f is Lipschitz continuous with $\sigma > 0$. If $\sigma = \rho_1$, then f is quadratic, i.e., there exists $b \in \mathcal{R}^n$ and $c \in \mathcal{R}$ such that $f(x) = \frac{\rho_1}{2}\|x\|^2 + \langle b, x\rangle + c$.

Proof. First, as assumed, f is continuously differentiable and ∇f is Lipschitz continuous with $\sigma > 0$, strongly convex with constant $\rho_1 > 0$, and $\sigma = \rho_1$. Thus, it follows from (4.16) that for any $x, y \in \mathcal{R}^n$, we have

$$\rho_1\|x-y\|^2 \le \langle \nabla f(x) - \nabla f(y), x-y\rangle \le \|\nabla f(x) - \nabla f(y)\| \cdot \|x-y\| \le \rho_1\|x-y\|^2,$$

where the third inequality follows from the Lipschitz continuity of ∇f. Thus, we obtain

$$\langle \nabla f(x) - \nabla f(y), x-y\rangle = \rho_1\|x-y\|^2,\ \forall x, y \in \mathcal{R}^n. \tag{4.31}$$

By means of (4.3) and Lemma 2.1, we know that $\tilde{f}$ is convex and $\nabla f(x) = \nabla \tilde{f}(x) + \rho_1 x$. Thus, it follows from (4.31) that

$$\langle \nabla \tilde{f}(x) - \nabla \tilde{f}(y), x-y\rangle = 0,\ \forall x, y \in \mathcal{R}^n. \tag{4.32}$$

The convexity of $\tilde{f}$ implies that for any $x, y \in \mathcal{R}^n$, we have

$$\tilde{f}(x) \ge \tilde{f}(y) + \langle \nabla \tilde{f}(y), x-y\rangle = \tilde{f}(y) + \langle \nabla \tilde{f}(x), x-y\rangle \ge \tilde{f}(x).$$

Thus, it holds that

$$\tilde{f}(x) = \tilde{f}(y) + \langle \nabla \tilde{f}(y), x-y\rangle,\ \forall y \in \mathcal{R}^n.$$

That is, $\tilde{f}$ must be affine, or, there exist $b \in \mathcal{R}^n$ and $c \in \mathcal{R}$ such that $\tilde{f}(x) = \langle b, x\rangle + c$. The proof is complete. □

Now, we present the convergence of the ISTA (4.33) when $\rho_1 = \sigma$.

Theorem 4.3. For model (1.1), suppose f is strongly convex function with constant $\rho_1 > 0$, continuously differentiable and ∇f is Lipschitz continuous with constant $\sigma > 0$; g is semiconvex with constant $\rho_2 \ge 0$. If $\sigma = \rho_1$ and $S \ne \emptyset$, then the sequence $\{x_k\}_{k\in N}$ generated by the ISTA (1.2) with $0 < \alpha < 1/\sigma$ converges to a solution point in S.

Proof. First, recall $\rho_1 \geq \rho_2$. Thus, we have $1 - \alpha\rho_2 \geq 1 - \alpha\rho_1 > 0$ under the condition $0 < \alpha < 1/\sigma$. Thus, the ISTA (1.2), or (4.10), can be further written as

$$x_{k+1} = (I + \frac{\alpha}{1-\alpha\rho_2}\partial\tilde{g})^{-1}(\frac{1-\alpha\rho_1}{1-\alpha\rho_2}x_k - \frac{\alpha b}{1-\alpha\rho_2}), \tag{4.33}$$

and it is well defined. Since, $\lambda\tilde{f} \equiv b$, for any $x^* \in S$, it follows from (4.25) that

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - \|x_{k+1} - x_k\|^2,$$

which implies the Fejér monotonicity of $\{x_k\}_{k\in N}$ with respect to S. The remaining part of the proof is analogous to that in Theorem 4.2; hence omitted. □

The assertions in Theorems 4.2 and 4.3 can be merged as one theorem, representing the convergence of a general version of the ISTA (1.2) for the "strongly + semi" convex setting of (1.1).

Theorem 4.4. For model (1.1), suppose f is strongly convex function with constant $\rho_1 \geq 0$, continuously differentiable and ∇f is Lipschitz continuous with constant $\sigma > 0$; g is proper lower semicontinuous and semiconvex with constant $\rho_2 \geq 0$. Suppose $\rho_1 \geq \rho_2$ and $S \neq \emptyset$. If $0 < \alpha < \frac{2}{2\rho_1 + s(\rho_1)}$ with

$$s(\rho_1) := \begin{cases} \sigma, & \text{if} \quad \rho_1 = 0, \\ \sigma - \rho_1, & \text{if} \quad 0 < \rho_1 \leq \frac{\sigma}{2}, \\ \sqrt{\sigma^2 - \rho_1^2}, & \text{if} \quad \frac{\sigma}{2} < \rho_1 < \sigma, \\ 0, & \text{if} \quad \rho_1 = \sigma > 0, \end{cases}$$

then the sequence generated by the ISTA (1.2) converges to a solution point in S.

Remark 4.1. *For the special case where $\rho_1 = 0$, by assumption we know $\rho_2 = 0$ and hence f and g are both convex. In this case, the ISTA (1.2) reduces to the application of the classical forward-backward splitting method to model (1.1). Our restriction on α in Theorem 4.4 reduces to $0 < \alpha < \frac{2}{\sigma}$, which coincides with the analysis in the literature, e.g., [15, 16]. Moreover, because of the general rule of choosing the step size α as stated in Theorem 4.4, we call the scheme (1.2) with this rule of step size "the general ISTA".*

5 Convergence rate of ISTA: "strongly + semi" convex case

In this section, we prove the $O(1/k)$ convergence rate for the proposed general ISTA (1.2) in the "strongly + semi" convex setting of (1.1). It is worthy to mention that in general it is more demanding to derive the convergence rate for an algorithm than its convergence; thus for the convergence rate results to be presented, the range of the step size α is generally more restrictive than that in Theorem 4.4 which is only for ensuring the convergence. As before, we present the analysis for the cases of $\rho_1 < \sigma$ and $\rho_1 = \sigma$ separately.

5.1 The case where $\rho_1 < \sigma$

The $O(1/k)$ convergence rate of the ISTA (1.2) for the "convex + convex" setting of (1.1) was derived in [6] for the case where $\alpha = 1/\sigma$. In this subsection, we prove the $O(1/k)$ convergence rate of the general ISTA in the more general "strongly + semi" convex setting with $\rho_1 < \sigma$. We also consider the case where $0 < \alpha \le 1/\sigma$ which is more general than the case of $\alpha = 1/\sigma$ in [6]. It is worthy to note that the semiconvexity in our case makes it more difficult to handle some quadratic terms appearing in the analysis; thus the proof is generally more complicated than that for the "convex + convex" case.

Theorem 5.1. For model (1.1), suppose f is strongly convex with constant $\rho_1 \ge 0$, continuously differentiable and ∇f is Lipschitz continuous with constant $\sigma > 0$; g is proper lower semicontinuous semiconvex function with constant $\rho_2 \ge 0$. Suppose $\sigma > \rho_1 \ge \rho_2$ and $S \ne \emptyset$. Let $\{x_k\}_{k\in N}$ be the sequence generated by the ISTA (1.2) with $0 < \alpha \le 1/\sigma$, then we have

$$F(x_k) - F(x^*) \le \frac{1-\alpha\rho_2}{2\alpha k}\|x^* - x_0\|^2 = O(\frac{1}{k}),\ \forall x^* \in S. \tag{5.1}$$

Proof. Together with $\sigma > \rho_1 \ge \rho_2$, the condition $0 < \alpha \le 1/\sigma$ ensures $1 - \alpha\rho_2 > 0$ and thus the ISTA (1.2) is well defined under the assumptions of this theorem. For the iterate x_n, it follows from the optimality condition of (1.2) that

$$\frac{1}{\alpha}(x_n - x_{n+1}) - \nabla f(x_n) \in \partial g(x_{n+1}). \tag{5.2}$$

Since f is strongly convex and g is semiconvex, by Remark 2.2 and the assertion (iii) of Remark 2.1, we have

$$f(x) \ge f(x_n) + \langle \nabla f(x_n), x - x_n\rangle + \frac{\rho_1}{2}\|x - x_n\|^2,\ \forall x \in \mathcal{R}^n$$

and

$$g(x) \ge g(x_{n+1}) + \langle \frac{1}{\alpha}(x_n - x_{n+1}) - \nabla f(x_n), x - x_{n+1}\rangle - \frac{\rho_2}{2}\|x - x_{n+1}\|^2, \forall x \in \mathcal{R}^n.$$

Adding the above two inequalities, we know that

$$\begin{aligned} F(x) \quad \ge \quad & F(x_{n+1}) + f(x_n) - f(x_{n+1}) + \frac{1}{\alpha}\langle x_n - x_{n+1}, x - x_{n+1}\rangle \\ & -\langle \nabla f(x_n), x_n - x_{n+1}\rangle + \frac{\rho_1}{2}\|x - x_n\|^2 - \frac{\rho_2}{2}\|x - x_{n+1}\|^2,\ \forall x \in \mathcal{R}^n. \end{aligned} \tag{5.3}$$

Since ∇f is Lipschitz continuous with constant σ, it follows from Lemma 2.6 that

$$f(x_{n+1}) \le f(x_n) + \langle \nabla f(x_n), x_{n+1} - x_n\rangle + \frac{\sigma}{2}\|x_{n+1} - x_n\|^2. \tag{5.4}$$

Substituting (5.4) into (5.3), we have

$$F(x) \ge F(x_{n+1}) + \frac{1}{\alpha}\langle x_n - x_{n+1}, x - x_{n+1}\rangle + \frac{\rho_1}{2}\|x - x_n\|^2 - \frac{\rho_2}{2}\|x - x_{n+1}\|^2 - \frac{\sigma}{2}\|x_{n+1} - x_n\|^2,\ \forall x \in \mathcal{R}^n. \tag{5.5}$$

Note the identity

$$\langle x_n - x_{n+1}, x - x_{n+1}\rangle = \frac{1}{2}(\|x_n - x_{n+1}\|^2 + \|x - x_{n+1}\|^2 - \|x - x_n\|^2).$$

We thus know that (5.5) can be written as

$$\begin{aligned} F(x) &\geq F(x_{n+1}) + \frac{1}{2}(\rho_1 - \frac{1}{\alpha})\|x - x_n\|^2 + \frac{1}{2}(\frac{1}{\alpha} - \rho_2)\|x - x_{n+1}\|^2 + \frac{1}{2}(\frac{1}{\alpha} - \sigma)\|x_{n+1} - x_n\|^2 \\ &\geq F(x_{n+1}) + \frac{1}{2}(\rho_2 - \frac{1}{\alpha})\|x - x_n\|^2 + \frac{1}{2}(\frac{1}{\alpha} - \rho_2)\|x - x_{n+1}\|^2, \ \forall x \in \mathcal{R}^n, \end{aligned} \tag{5.6}$$

where the last inequality follows from $\alpha \leq 1/\sigma$ and $\rho_1 \geq \rho_2$. Setting $x = x^* \in S$ in (5.6), we get

$$F(x_{n+1}) - F(x^*) \leq \frac{1 - \alpha\rho_2}{2\alpha}\|x_n - x^*\|^2 - \frac{1 - \alpha\rho_2}{2\alpha}\|x_{n+1} - x^*\|^2. \tag{5.7}$$

We also have

$$\begin{aligned} \sum_{n=0}^{k-1}(F(x_{n+1}) - F(x^*)) &\leq \sum_{n=0}^{k-1}(\frac{1 - \alpha\rho_2}{2\alpha}\|x_n - x^*\|^2 - \frac{1 - \alpha\rho_2}{2\alpha}\|x_{n+1} - x^*\|^2) \\ &= \frac{1 - \alpha\rho_2}{2\alpha}\|x_0 - x^*\|^2 - \frac{1 - \alpha\rho_2}{2\alpha}\|x_k - x^*\|^2 \\ &\leq \frac{1 - \alpha\rho_2}{2\alpha}\|x_0 - x^*\|^2, \end{aligned} \tag{5.8}$$

where the first inequality follows from (5.7). Recall that $1 - \alpha\rho_2 > 0$. Thus, setting $x = x_n$ in (5.6), for all $n \geq 0$, we have

$$F(x_{n+1}) \leq F(x_n). \tag{5.9}$$

Hence, using (5.8) and (5.9), we get

$$k(F(x_k) - F(x^*)) \leq \frac{1 - \alpha\rho_2}{2\alpha}\|x_0 - x^*\|^2. \tag{5.10}$$

The assertion (5.1) holds and the proof is complete. □

Theorem 5.1 shows that the ISTA (1.2) with $0 < \alpha \leq 1/\sigma$ in the "strongly + semi" convex setting under our discussion has the worst-case $O(1/k)$ convergence rate measured by the gap of objection function values between the k-th iterate and an optimal solution point. It is trivial to further polish the rate from $O(1/k)$ to $o(1/k)$; we provide the detail in the following corollary.

Corollary 5.1. Under the assumptions of Theorem 5.1, it holds that

$$F(x_k) - F(x^*) \sim o(1/k), \quad \text{as } k \to \infty. \tag{5.11}$$

Proof. Taking $k \to \infty$ in (5.8), we further have

$$\sum_{n=0}^{\infty}(F(x_{n+1}) - F(x^*)) \leq \frac{1 - \alpha\rho_2}{2\alpha}\|x_0 - x^*\|^2. \tag{5.12}$$

Note that (5.9) implies that the sequence $\{F(x_k)\}_{k\in N}$ is monotonically nonincreasing. Thus we have

$$0 \le F(x_{k+1}) - F(x^*) \le F(x_k) - F(x^*). \tag{5.13}$$

Moreover, it follows from (5.12) that

$$\sum_{i=\lfloor \frac{k}{2} \rfloor}^{k} (F(x_i) - F(x^*)) \to 0, \quad \text{as } k \to \infty,$$

where $\lfloor \frac{k}{2} \rfloor$ is the largest integer no greater than the number $\frac{k}{2}$. Again from (5.13), we have

$$\frac{k}{2}(F(x_k) - F(x^*)) \le \sum_{i=\lfloor \frac{k}{2} \rfloor}^{k} (F(x_i) - F(x^*)) \to 0, \quad \text{as } k \to \infty, \tag{5.14}$$

which implies the assertion (5.11) directly. □

Remark 5.1. *For the special case where $\rho_1 = 0$, by assumption we know $\rho_2 = 0$ and hence f and g are both convex. In this case, Theorem 5.1 with $\alpha = 1/\sigma$ reduces to Theorem 3.1 in [6].*

5.2 The case where $\rho_1 = \sigma$

In the subsection, we derive the $O(1/k)$ convergence rate for the general version of ISTA (1.2) in the "strongly + semi" convex setting of (1.1) for the special case where $\sigma = \rho_1$. Unlike the last subsection in which the step size α is restricted by $0 < \alpha \le 1/\sigma$, we only focus on the restriction of $0 < \alpha < 1/\sigma$ for the step size because the particular case where $\alpha = 1/\sigma$ turns out to be trivial when $\rho_1 = \sigma$. Below we elaborate on the reason. First, recall that (see Lemma 4.3) when $\rho_1 = \sigma$, $f(x)$ is quadratic and we denote it by $f(x) = \frac{\rho_1}{2}\|x\|^2 + \langle b, x\rangle + c$. Hence, $\nabla f(x) = \rho_1 x + b$ and the ISTA (1.2) reduces to

$$x_k = \arg\min_{x\in\mathcal{R}^n} \left\{ g(x) + \frac{1}{2\alpha}\|x - ((1-\alpha\rho_1)x_k - \alpha b)\|^2 \right\}, \tag{5.15}$$

which can be further specified as

$$x_k = \arg\min_{x\in\mathcal{R}^n} \left\{ g(x) + \frac{\rho_1}{2}\|x + \frac{1}{\rho_1}b\|^2 \right\} \tag{5.16}$$

when $\alpha = 1/\rho_1$. Under the assumption $\rho_1 \ge \rho_2$ by default, the problem (5.16) is trivial when either $\rho_1 > \rho_2$ or $\rho_1 = \rho_2$. To see it, if $\rho_1 > \rho_2$, then the minimization problem in (5.16) is strongly convex and its solution is independent of k. Thus, $\{x_k\}_{k\in N}$ generated by (5.16) is a constant sequence with each iterate being a solution point. On the other hand, if $\rho_1 = \rho_2$, then the minimization problem in (5.16) is not necessarily well defined. For example, taking $g(x) = \langle a_1, x\rangle - \frac{\rho_1}{2}\|x\|^2$ with $a_1 + b \ne \mathbf{0}$, then the objective function $g(x) + \frac{\rho_1}{2}\|x + \frac{1}{\rho_1}b\|^2$ is unbounded below. Therefore, we do not include the case where $\alpha = 1/\sigma$ when discussing the convergence rate of the ISTA (1.2) with $\rho_1 = \sigma$.

Theorem 5.2. For model (1.1), suppose f is strongly convex with constant $\rho_1 > 0$, continuously differentiable and ∇f is Lipschitz continuous with constant $\sigma > 0$; g is semiconvex with constant $\rho_2 \geq 0$. Suppose $\sigma = \rho_1 \geq \rho_2$ and $S \neq \emptyset$. Let $\{x_k\}_{k\in N}$ be the sequence generated by the ISTA (1.2) with $0 < \alpha < 1/\sigma$, then we have

$$F(x_k) - F(x^*) \leq \frac{1-\alpha\rho_2}{2\alpha k}\|x^* - x_0\|^2 = O(\frac{1}{k}),\ \forall x^* \in S.$$

Proof. Similar as Theorem 4.3, the conditions of this theorem ensure that the ISTA (1.2) is well defined. Then, by Lemma 4.3 we have $f(x) = \frac{\rho_1}{2}\|x\|^2 + \langle b, x\rangle + c$ and hence $\nabla f(x) = \rho_1 x + b$. Then, (5.2) can be written as

$$\frac{1}{\alpha}(x_n - x_{n+1}) - (\rho_1 x_n + b) \in \partial g(x_{n+1}).$$

Observe that

$$f(x) = f(x_{n+1}) + \langle \rho_1 x_{n+1} + b, x - x_{n+1}\rangle + \frac{\rho_1}{2}\|x - x_{n+1}\|^2,\quad \forall x \in \mathcal{R}^n.$$

Since g is semiconvex, it follows from the assertion (iii) of Remark 2.1 that

$$g(x) \geq g(x_{n+1}) + \langle \frac{1}{\alpha}(x_n - x_{n+1}) - (\rho_1 x_n + b), x - x_{n+1}\rangle - \frac{\rho_2}{2}\|x - x_{n+1}\|^2,\quad \forall x \in \mathcal{R}^n.$$

Adding the above two relations, for all $x \in \mathcal{R}^n$, we have

$$F(x) \geq F(x_{n+1}) + (\frac{1}{\alpha} - \rho_1)\langle x_n - x_{n+1}, x - x_{n+1}\rangle + \frac{\rho_1}{2}\|x - x_{n+1}\|^2 - \frac{\rho_2}{2}\|x - x_{n+1}\|^2. \quad (5.17)$$

Notice the identity

$$\langle x_n - x_{n+1}, x - x_{n+1}\rangle = \frac{1}{2}(\|x_n - x_{n+1}\|^2 + \|x - x_{n+1}\|^2 - \|x - x_n\|^2).$$

Thus, it follows from (5.17) that

$$\begin{aligned} F(x) \quad &\geq \quad F(x_{n+1}) + \frac{\rho_1}{2}\|x - x_{n+1}\|^2 - \frac{\rho_2}{2}\|x - x_{n+1}\|^2, \\ &\quad + \frac{1}{2}(\frac{1}{\alpha} - \rho_1)(\|x_n - x_{n+1}\|^2 + \|x - x_{n+1}\|^2 - \|x - x_n\|^2) \\ &= \quad F(x_{n+1}) + \frac{1}{2}(\frac{1}{\alpha} - \rho_2)\|x - x_{n+1}\|^2 - \frac{1}{2}(\frac{1}{\alpha} - \rho_1)\|x - x_n\|^2 \\ &\quad + \frac{1}{2}(\frac{1}{\alpha} - \rho_1)\|x_n - x_{n+1}\|^2,\quad \forall x \in \mathcal{R}^n. \end{aligned} \quad (5.18)$$

Set $x = x^* \in S$ in (5.18) and recall $0 < \alpha < 1/\sigma$ and $\rho_1 \geq \rho_2$. We thus have

$$F(x_{n+1}) - F(x^*) \leq \frac{1-\alpha\rho_2}{2\alpha}\|x_n - x^*\|^2 - \frac{1-\alpha\rho_2}{2\alpha}\|x_{n+1} - x^*\|^2.$$

The rest of the proof is similar to that in Theorem 5.1; thus omitted. □

Remark 5.2. *Similarly as Corollary 5.1, we can easily polish the result in Theorem 5.2 to a $o(1/k)$ convergence rate. For succinctness, we skip the detail.*

Finally, despite of their different proofs, Theorems 5.1 and 5.2 can be merged as the following theorem representing the convergence rate result of a general version of the ISTA (1.2) in the "strongly + semi" convex setting of (1.1).

Theorem 5.3. For model (1.1), suppose f is strongly convex with constant $\rho_1 \geq 0$, continuously differentiable and ∇f is Lipschitz continuous with constant $\sigma > 0$; g is proper lower semicontinuous and semiconvex with constant $\rho_2 \geq 0$. Let $\{x_k\}_{k\in N}$ be the sequence generated by the ISTA (1.2) and $x^* \in S$, then the following estimations holds:
(i) if $\sigma > \rho_1 \geq \rho_2$ and $0 < \alpha \leq 1/\sigma$, then $F(x_k) - F(x^*) \sim O(1/k)$ (or $o(1/k)$), as $k \to \infty$;
(ii) if $\sigma = \rho_1 \geq \rho_2$ and $0 < \alpha < 1/\sigma$, then $F(x_k) - F(x^*) \sim O(1/k)$ (or $o(1/k)$), as $k \to \infty$.

6 Convergence rate of FISTA: "strongly + semi" convex case

In this section, we establish the $O(1/k^2)$ convergence rate for a general version of the FISTA (1.5) in the "strongly + semi" convex setting of (1.1). A useful result is that for the sequence $\{t_k\}_{k\in N}$ updated according to (1.5), we have $t_k \geq \frac{k+1}{2}$ for all $k \geq 1$ as proved in Lemma 4.3 in [6]. As before, the proofs for the cases of $\rho_1 < \sigma$ and $\rho_1 = \sigma$ are different and will be presented in separate subsections. But first of all, we prove a lemma that will be used in both subsections.

Lemma 6.1. For model (1.1), suppose f is strongly convex with constant $\rho_1 \geq 0$, continuously differentiable and ∇f is Lipschitz continuous with constant $\sigma > 0$; g is proper lower semicontinuous and semiconvex with constant $\rho_2 \geq 0$. Suppose $\rho_1 \geq \rho_2$. Let $\{x_k, y_k\}_{k\in N}$ be the sequence generated by the FISTA (1.5) with $0 < \alpha \leq 1/\sigma$, then we have

$$F(x) \geq F(x_k) + \frac{1}{2}(\frac{1}{\alpha} - \rho_2)\|x - x_k\|^2 + \frac{1}{2}(\rho_1 - \frac{1}{\alpha})\|x - y_k\|^2, \ \ \forall x \in \mathcal{R}^n. \tag{6.1}$$

Proof. Recall $\sigma \geq \rho_1$ holds automatically. The conditions $\rho_1 \geq \rho_2$ and $0 < \alpha \leq 1/\sigma$ ensure that $1 - \alpha\rho_2 > 0$ and thus the FISTA (1.5) is well defined under the assumptions. Then, for the iterate x_k, it follows from the optimality condition of the minimization problem in the first equation of (1.5) that

$$-\nabla f(y_k) + \frac{1}{\alpha}(y_k - x_k) \in \partial g(x_k).$$

Since f is strongly convex and g is semiconvex, by Remark 2.2 and the assertion (iii) of Remark 2.1 we have

$$f(x) \geq f(y_k) + \langle \nabla f(y_k), x - y_k\rangle + \frac{\rho_1}{2}\|x - y_k\|^2, \ \forall x \in \mathcal{R}^n \tag{6.2}$$

and

$$g(x) \geq g(x_k) + \langle -\nabla f(y_k) + \frac{1}{\alpha}(y_k - x_k), x - x_k\rangle - \frac{\rho_2}{2}\|x - x_k\|^2, \ \forall x \in \mathcal{R}^n. \tag{6.3}$$

Adding (6.2) and (6.3), we know

$$\begin{aligned} F(x) \quad \geq \quad & F(x_k) + f(y_k) - f(x_k) + \langle \nabla f(y_k), x_k - y_k \rangle \\ & + \frac{1}{\alpha} \langle y_k - x_k, x - x_k \rangle + \frac{\rho_1}{2} \|x - y_k\|^2 - \frac{\rho_2}{2} \|x - x_k\|^2, \ \forall x \in \mathcal{R}^n. \end{aligned} \tag{6.4}$$

Setting $x = x_k$ and $y = y_k$, using Lemma 2.6, we obtain

$$f(x_k) \leq f(y_k) + \langle \nabla f(y_k), x_k - y_k \rangle + \frac{\sigma}{2} \|x_k - y_k\|^2. \tag{6.5}$$

Then, combining (6.4) and (6.5), we have

$$F(x) \geq F(x_k) + \frac{1}{\alpha} \langle y_k - x_k, x - x_k \rangle - \frac{\sigma}{2} \|x_k - y_k\|^2 + \frac{\rho_1}{2} \|x - y_k\|^2 - \frac{\rho_2}{2} \|x - x_k\|^2, \ \forall x \in \mathcal{R}^n. \tag{6.6}$$

Observe that

$$\langle y_k - x_k, x - x_k \rangle = \frac{1}{2} (\|y_k - x_k\|^2 + \|x - x_k\|^2 - \|x - y_k\|^2). \tag{6.7}$$

Substituting (6.7) into (6.6) and using the condition $\alpha \leq 1/\sigma$, we obtain (6.1) immediately. The proof is complete. □

6.1 The case where $\rho_1 < \sigma$

Recall that the $O(1/k^2)$ convergence rate of FISTA was established in [6] in the "convex + convex" setting of (1.1) for the case where $\alpha = 1/\sigma$. Here we prove the $O(1/k^2)$ convergence rate of FISTA in the more general "strongly + semi" convex setting of (1.1) with $\rho_1 < \sigma$ for the more general case where $0 < \alpha \leq 1/\sigma$. Due to the semiconvexity of one function, our proof is more complicated even though the roadmap still follows that in [6].

Theorem 6.1. For model (1.1), suppose f is strongly convex with constant $\rho_1 \geq 0$, continuously differentiable and ∇f is Lipschitz continuous with constant $\sigma > 0$; g is proper lower semicontinuous and semiconvex with constant $\rho_2 \geq 0$. Suppose $\sigma > \rho_1 \geq \rho_2$ and $S \neq \emptyset$. Let $\{x_k, y_k\}_{k \in N}$ be the sequence generated by the FISTA (1.5) with $0 < \alpha \leq 1/\sigma$, then we have

$$F(x_k) - F(x^*) \leq \frac{2(1 - \alpha\rho_2)}{\alpha(k+1)^2} \|x^* - x_0\|^2 = O(\frac{1}{k^2}), \ \forall x^* \in S. \tag{6.8}$$

Proof. First, the conditions $\sigma > \rho_1 \geq \rho_2$ and $0 < \alpha \leq 1/\sigma$ ensure that the FISTA (1.5) is well defined. Then, for any $x^* \in S$, let us set

$$\hat{x}_n := \frac{1}{t_{n+1}} x^* + (1 - \frac{1}{t_{n+1}}) x_n. \tag{6.9}$$

Since $t_n \geq \frac{n+1}{2}$, for all $n \geq 1$, it holds that $\frac{1}{t_{n+1}} \in (0, 1]$ for any $n \geq 0$. Because of $\rho_1 \geq \rho_2$, it follows from the convexity of F that

$$F(\hat{x}_n) \leq \frac{1}{t_{n+1}} F(x^*) + (1 - \frac{1}{t_{n+1}}) F(x_n). \tag{6.10}$$

Multiplying t_{n+1}^2 on both sides of (6.10), we have

$$t_{n+1}^2 F(\hat{x}_n) \leq t_{n+1} F(x^*) + (t_{n+1}^2 - t_{n+1}) F(x_n). \tag{6.11}$$

Since $t_{n+1}^2 - t_{n+1} = t_n^2$, it follows from (6.11) that

$$t_{n+1}^2 F(\hat{x}_n) \leq t_{n+1}^2 F(x^*) - t_n^2 F(x^*) + t_n^2 F(x_n). \tag{6.12}$$

Adding $t_{n+1}^2 F(x_{n+1})$ on both sides of (6.12) and rearranging the terms, we obtain

$$t_{n+1}^2 (F(x_{n+1}) - F(x^*)) - t_n^2 (F(x_n) - F(x^*)) \leq t_{n+1}^2 (F(x_{n+1}) - F(\hat{x}_n)). \tag{6.13}$$

Next, we estimate the term $F(x_{n+1}) - F(\hat{x}_n)$. Actually, set $k = n+1$ in (6.1) we have

$$F(x) \geq F(x_{n+1}) + \frac{1}{2}(\frac{1}{\alpha} - \rho_2)\|x - x_{n+1}\|^2 + \frac{1}{2}(\rho_1 - \frac{1}{\alpha})\|x - y_{n+1}\|^2, \ \forall x \in \mathcal{R}^n. \tag{6.14}$$

Specifically, setting $x = \hat{x}_n$ in (6.14), we get

$$F(x_{n+1}) - F(\hat{x}_n) \leq \frac{1 - \alpha\rho_1}{2\alpha}\|\hat{x}_n - y_{n+1}\|^2 - \frac{1 - \alpha\rho_2}{2\alpha}\|\hat{x}_n - x_{n+1}\|^2. \tag{6.15}$$

Let us use the notation

$$u_n := x_{n-1} + t_n(x_n - x_{n-1}),$$

where we set $u_0 := x_{-1} \equiv x_0$. Then, it follows from (1.5) and (6.9) that

$$\begin{aligned} \hat{x}_n - y_{n+1} &= \frac{1}{t_{n+1}}x^* + (1 - \frac{1}{t_{n+1}})x_n - x_n - \frac{t_n - 1}{t_{n+1}}(x_n - x_{n-1}) \\ &= \frac{1}{t_{n+1}}(x^* - x_{n-1} + t_n(x_{n-1} - x_n)) \\ &= \frac{1}{t_{k+1}}(x^* - u_n) \end{aligned} \tag{6.16}$$

and

$$\hat{x}_n - x_{n+1} = \frac{1}{t_{n+1}}x^* + (1 - \frac{1}{t_{n+1}})x_n - x_{n+1} = \frac{1}{t_{n+1}}(x^* - x_n + t_{n+1}(x_n - x_{n+1})) = \frac{1}{t_{n+1}}(x^* - u_{n+1}). \tag{6.17}$$

Thus, substituting (6.15), (6.16) and (6.17) into (6.13), we get

$$t_{n+1}^2 (F(x_{n+1}) - F(x^*)) - t_n^2 (F(x_n) - F(x^*)) \leq \frac{1 - \alpha\rho_1}{2\alpha}\|x^* - u_n\|^2 - \frac{1 - \alpha\rho_2}{2\alpha}\|x^* - u_{n+1}\|^2.$$

Notice that $\sigma > \rho_1 \geq \rho_2$ and $0 < \alpha \leq 1/\sigma$ imply $0 < 1 - \alpha\rho_1 \leq 1 - \alpha\rho_2$. Thus, we have

$$t_{n+1}^2 (F(x_{n+1}) - F(x^*)) - t_n^2 (F(x_n) - F(x^*)) \leq \frac{1 - \alpha\rho_2}{2\alpha}\|x^* - u_n\|^2 - \frac{1 - \alpha\rho_2}{2\alpha}\|x^* - u_{n+1}\|^2.$$

With $t_0 = 0$, adding the above inequality from $n = 0, 1, \cdots, k-1$, we obtain

$$t_k^2 (F(x_k) - F(x^*)) \leq \frac{1 - \alpha\rho_2}{2\alpha}\|x^* - u_0\|^2.$$

By assumption we have $u_0 = x_0$. Again since $t_k \geq \frac{k+1}{2}$, we deduce (6.8) from the above inequality immediately. The proof is complete. □

Remark 6.1. *For the special case where $\rho_1 = 0$, by assumption we know $\rho_2 = 0$ and hence f and g are both convex. In this case, Theorem 6.1 with $\alpha = 1/\sigma$ reduces to Theorem 4.4 in [6].*

6.2 The case where $\rho_1 = \sigma$

Now we prove the $O(1/k^2)$ convergence rate of the general FISTA in the more general "strongly + semi" convex setting of (1.1) with $\rho_1 = \sigma$ for the case where $0 < \alpha < 1/\sigma$. Similar as Section 5.2, we do not consider the step size of $\alpha = 1/\sigma$ when $\rho_1 = \sigma$. The proof of the following theorem is analogous to that of Theorem 6.1; thus omitted.

Theorem 6.2. For model (1.1), suppose f is strongly convex with constant $\rho_1 > 0$, continuously differentiable and ∇f is Lipschitz continuous with constant $\sigma > 0$; g is semiconvex with constant $\rho_2 \geq 0$. Suppose $\sigma = \rho_1 \geq \rho_2$ and $S \neq \emptyset$. Let $\{x_k, y_k\}_{k\in N}$ be the sequence generated by the FISTA (1.5) with $0 < \alpha < 1/\sigma$, then we have

$$F(x_k) - F(x^*) \leq \frac{2(1-\alpha\rho_2)}{\alpha(k+1)^2}\|x^* - x_0\|^2 = O(\frac{1}{k^2}),\ \forall x^* \in S. \tag{6.18}$$

Last, the assertions in Theorems 6.1 and 6.2 can be merged as the following theorem representing the convergence rate result of a general version of the FISTA (1.5) in the "strongly + semi" convex setting of (1.1).

Theorem 6.3. For model (1.1), suppose f is strongly convex function with constant $\rho_1 \geq 0$, continuously differentiable and ∇f is Lipschitz continuous with constant $\sigma > 0$, g is proper lower semicontinuous and semiconvex with constant $\rho_2 \geq 0$. Suppose $\rho_1 \geq \rho_2$ and $S \neq \emptyset$. Let $\{x_k, y_k\}_{k\in N}$ be the sequence generated by the FISTA (1.5) and $x^* \in S$, then the following estimations holds:
(i) if $\sigma > \rho_1 \geq \rho_2$ and $0 < \alpha \leq 1/\sigma$, then $F(x_k) - F(x^*) \sim O(1/k^2),$ as $k \to \infty$;
(ii) if $\sigma = \rho_1 \geq \rho_2$ and $0 < \alpha < 1/\sigma$, then $F(x_k) - F(x^*) \sim O(1/k^2),$ as $k \to \infty$.

7 Numerical results

In this section, we apply the proposed general ISTA and FISTA to solve a particular application of the "strongly + semi" convex setting of (1.1): the SCAD-ℓ_2 model in [25]. By this specific application, we show the efficiency of both the ISTA and FISTA, and the acceleration effectiveness of the latter. All the codes were written by MATLAB R2016a and all experiments were performed on a MacBook Pro with OS X Yosemite system and an Intel(R) Core(TM) i5 CPU processor (2.7GH) with a 8GB memory.

7.1 Iterative schemes of ISTA and FISTA for SCAD-ℓ_2

First, let us recall the SCAD-ℓ_2 model proposed in [25]:

$$\min_{x\in\mathcal{R}^n} \frac{1}{2}\|Ax - b\|^2 + \frac{\mu}{2}\|x\|^2 + \sum_{i=1}^{n} g_\lambda(|x_i|), \tag{7.1}$$

where $A \in \mathcal{R}^{m\times n}$, $b \in \mathcal{R}^m$, $\mu > 0$ and the SCAD penalty $g_\lambda(|x_i|)$ is defined in (3.6)-(3.7). Note that their individual statistical meanings of the three function components in (7.1) require us to group the data-fidelity terms $\frac{1}{2}\|Ax-b\|^2$ and $\frac{\mu}{2}\|x\|^2$ together while leave the SCAD penalty $\sum_{i=1}^n g_\lambda(|x_i|)$ as the other component, as analyzed in, e.g., [14, 25]. In [25], it was suggested to take $\mu > \frac{1}{a-1}$ to ensure the grouping effect as the elastic net model in [27]. Hence, the SCAD-ℓ_2 model (7.1) corresponds to the special "strongly + semi" convex setting of (1.1) with $f(x) = \frac{1}{2}\|Ax-b\|^2 + \frac{\mu}{2}\|x\|^2$ and $g(x) = \sum_{i=1}^n g_\lambda(|x_i|)$ and $\rho_1 > \rho_2$; recall that $\frac{1}{a-1}$ is the semiconvex constant of the SCAD penalty $\sum_{i=1}^n g_\lambda(|x_i|)$.

The specific implementation of the ISTA (1.2) for the "strongly + semi" convex programming problem (7.1) reads as

$$\begin{cases} z_k = x_k - \alpha\left(A^T(Ax_k - b) + \mu x_k\right), \\ x_{k+1} = \arg\min_{x\in\mathcal{R}^n}\left\{\alpha\sum_{i=1}^n g_\lambda(|x_i|) + \frac{1}{2}\|x-z_k\|^2\right\} \end{cases} \tag{7.2}$$

and that of the FISTA is

$$\begin{cases} z_k = y_k - \alpha\left(A^T(Ay_k - b) + \mu y_k\right), \\ x_k = \arg\min_{x\in\mathcal{R}^n}\left\{\alpha\sum_{i=1}^n g_\lambda(|x_i|) + \frac{1}{2}\|x-z_k\|^2\right\}, \\ t_{k+1} = \dfrac{1+\sqrt{1+4t_k^2}}{2}, \\ y_{k+1} = x_k + \dfrac{t_k - 1}{t_{k+1}}(x_k - x_{k-1}). \end{cases} \tag{7.3}$$

Notice that the solution of the subproblem

$$\operatorname*{argmin}_{x\in\mathcal{R}^n}\left\{\alpha\sum_{i=1}^n g_\lambda(|x_i|) + \frac{1}{2}\|x-z\|^2\right\} \tag{7.4}$$

is given explicitly by

$$x_i = \begin{cases} \mathrm{sign}(z_i)(|z_i| - \alpha\lambda)_+, & |z_i| \le (1+\alpha)\lambda, \\ \frac{(a-1)z_i - \mathrm{sign}(z_i)\alpha a\lambda}{a-1-\alpha}, & (1+\alpha)\lambda < z \le a\lambda, \\ z_i, & a\lambda < |z_i|. \end{cases} \tag{7.5}$$

Thus, the implementation of both the schemes (7.2) and (7.3) is easy.

7.2 Experiments set-up

We test two datasets: the "Leukemia" dataset obtained from the site `http://mldata.org/repository/data/viewslug/leukemia/` and the "News20" dataset from `http://qwone.com/`

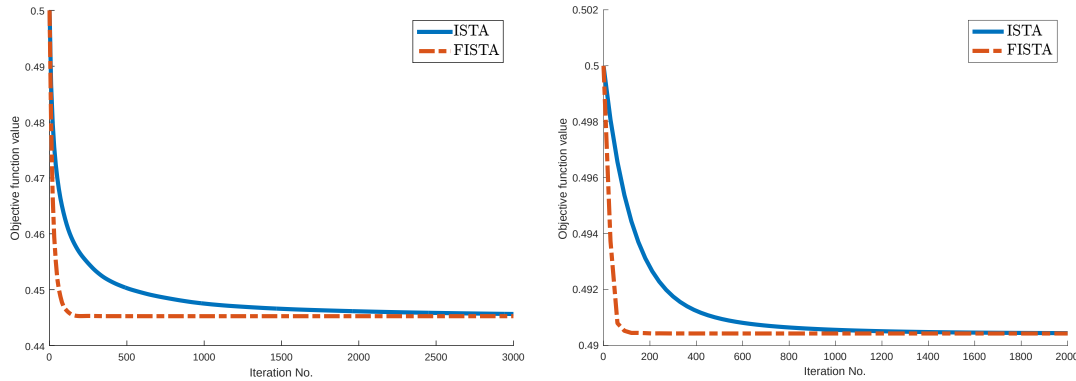
~jason/20Newsgroups/. We have ($A \in \mathcal{R}^{72\times 7129}$, $b \in \mathcal{R}^{72}$) and ($A \in \mathcal{R}^{11269\times 53975}$, $b \in \mathcal{R}^{11269}$) for these two datasets, respectively. The latter is a collection of approximately 20,000 newsgroup documents, partitioned (nearly) evenly across 20 different newsgroups, each corresponding to a different topic. The matrices A and the vector b are both normalized columnwise in our experiments.

As analyzed in [14, 25], empirically the parameter μ and λ could be better chosen through the cross-validation or generalized cross-validation technique by some given samples. Here, however, for the sole purpose of showing the numerical efficiency, we simply fix them as constants and do not discuss how to find their statistically best choices. More specifically, as mentioned, the parameter a in SCAD's definition was suggested in [14] to be $a = 3.7$. Thus $\frac{1}{a-1} = \frac{1}{2.7}$. We set $\lambda = 0.1\sqrt{2\log n}$ inspired by [14], where n is the column number of A. Since $n = 7129$ and 53975 for the leukemia and News20 datasets, respectively, we have $\lambda = 0.1\sqrt{2\log 7129} \approx 0.4212$ and $0.1\sqrt{2\log 53975} \approx 0.4668$ for these two datasets, respectively. For the matrices A of the datasets to be tested, $\lambda_{\min}(A^TA) = 0$ and thus the strong convexity constant in (7.1) is $\rho_1 = \mu$. For the parameter μ, as $\mu > \frac{1}{a-1}$ is required in [25], we simply set $\mu = 1.1 \times \frac{1}{a-1} \approx 0.4074$ in our experiments. Then, it holds that $\rho_1 = \mu > \rho_2 = \frac{1}{a-1}$. Moreover, with the given dataset, the Lipschitz constant σ of the gradient of the strongly convex component in (7.1) can be calculated by $\sigma = \lambda_{\max}(A^TA)+\mu$, which is approximately 1609.3 and 525.8314 for the leukemia and News20 datasets, respectively. Obviously we have $\rho_1 \leq \frac{\sigma}{2}$ for both datasets; and as proved in Theorem 4.4, the convergence of the ISTA and FISTA can be guaranteed if the stepsize α is restricted in the interval $(0, \frac{2}{\sigma+\rho_1})$. By calculation, we know that the value of $\frac{2}{\sigma+\rho_1}$ is approximately 0.012 and 0.038 for the leukemia and News20 datasets, respectively. The initial iterate is chosen as $x_0 = \mathbf{0}$ for both the ISTA and FISTA; and $y_1 = x_0$, $t_1 = 1$ for the latter.

7.3 Results

Recall that the convergence rate analysis in [6] for the original ISTA and FISTA in the "convex + convex" case of (1.1) is conducted for the case where $\alpha = 1/\sigma$; and we extend the restriction to $\alpha \in (0, 1/\sigma]$ when $\rho_1 < \sigma$. Thus, we first show the comparison of the proposed general ISTA and FISTA with the same step size of $\alpha = 1/\sigma$, i.e., the original ISTA and FISTA in [6], when they are applied to solve the SCAD-ℓ_2 model (7.1). The evolutions of the objective function values with respect to the iteration numbers are plotted in Figure 1. To reflect their difference more clearly, we terminate the iterations when the difference of objective function value of two consecutive iterates is less than a tolerance of 10^{-6}; and in Table 1 we report the comparison in terms of iteration number ("It.") and computing time in seconds ("CPU") when the same levels of objective function value ("Obj") are attained. These results show that both the ISTA and FISTA can be applied to the "strongly +semi" convex problem (7.1). The FISTA accelerates ISTA very significantly for the specific SCAD-ℓ_2 model; with 12 and 8 times faster accelerations for these two tested datasets, respectively.

On the other hand, it follows from the fact $\rho_1 \leq \frac{\sigma}{2}$, the definition of $t(\rho_1)$ in (4.14) and



(a) leukemia dataset

(b) News20 dataset

Figure 1: Objective values w.r.t iteration number for ISTA and FISTA with $\alpha = 1/\sigma$

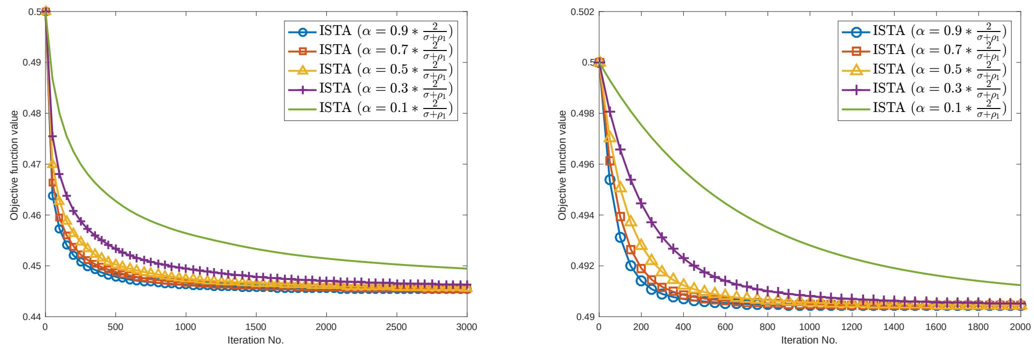
Theorem 4.2 that the proposed general ISTA is convergent with step size $\alpha \in (0, \frac{2}{\sigma+\rho_1})$. We thus also test the performances of the general ISTA with some representative step sizes: $0.1 * \frac{2}{\sigma+\rho_1}$, $0.3 * \frac{2}{\sigma+\rho_1}$, $0.5 * \frac{2}{\sigma+\rho_1}$, $0.7 * \frac{2}{\sigma+\rho_1}$ and $0.9 * \frac{2}{\sigma+\rho_1}$. The results are reported in Figure 2 and Table 2. The theoretically established convergence of the proposed general ISTA with step size $\alpha \in (0, \frac{2}{\sigma+\rho_1})$ is numerically verified. As well-known that larger step sizes are preferred as long as the convergence is theoretically ensured, larger step sizes such as $\alpha = 0.9 * \frac{2}{\sigma+\rho_1}$ are also recommended to numerically implement the general ISTA.

Table 1: Numerical results of ISTA and FISTA with $\alpha = 1/\sigma$ for SCAD-ℓ_2

	leukemia dataset			News20 dataset		
	It.	CPU	Obj	It.	CPU	Obj
ISTA	2527	1.6	0.4458	844	4.4	0.4906
FISTA	198	0.11	0.4452	108	0.67	0.4904

Table 2: Numerical results of ISTA with different step sizes for SCAD-ℓ_2

	leukemia dataset			News20 dataset		
	It.	CPU	Obj	It.	CPU	Obj
ISTA ($\alpha = 0.9 * \frac{2}{\sigma+\rho_1}$)	1831	1.2	0.4455	588	3.2	0.4905
ISTA ($\alpha = 0.7 * \frac{2}{\sigma+\rho_1}$)	2117	1.3	0.4456	687	3.6	0.4905
ISTA ($\alpha = 0.5 * \frac{2}{\sigma+\rho_1}$)	2527	1.6	0.4458	844	4.4	0.4906
ISTA ($\alpha = 0.3 * \frac{2}{\sigma+\rho_1}$)	3209	2.1	0.4462	1157	6.1	0.4907
ISTA ($\alpha = 0.1 * \frac{2}{\sigma+\rho_1}$)	5619	3.6	0.4472	2323	12.0	0.4911



(a) leukemia dataset (b) News20 dataset

Figure 2: Objective values w.r.t iteration number for ISTA with different step sizes

8 Conclusions

We extend the well-known iterative shrinkage/thresholding algorithm (ISTA) and its fast version (FISTA) to the more general convex setting of minimizing the sum of a strongly convex function and a semiconvex function, which is the mathematical model of some important sparsity-driven applications arising in a wide range of areas. For the “strongly + semi” convex setting; the respective worst-case $O(1/k)$ and $O(1/k^2)$ convergence rates of the general ISTA and FISTA are established. We also apply the general ISTA and FISTA to the SCAD-ℓ_2 model and their efficiency is numerically verified. In particular, for this application, the acceleration effectiveness of FISTA is extremely apparent. Our theoretical results seem to considerably enlarge the application range of ISTA and FISTA, especially for some sparsity-driven problems with non-convex penalties arising in data science. The convergence results proved in this paper justify that it is also promising to apply the ISTA and FISTA to solve such “strongly + semi” convex programming problems. Our analysis is also valid for the backtracking versions of ISTA and FISTA in the “strongly + semi” convex setting, which assume that the Lipschitz continuity constant of the gradient of the smooth function is unknown and a searching procedure should be applied to find an estimated value of the constant. For succinctness, we only include the detail of estimating the $O(1/k)$ convergence rate of the ISTA in the appendix, and skipped that of the FISTA. Last, for the general ISTA in the “strongly + semi” convex setting, we also derive the convergence in terms of the iterative sequence; while as well known, the same convergence of the original FISTA remains unknown even for the “convex + convex” setting. Thus, it is interesting to investigate this piece of convergence result for the FISTA in both the “convex + convex” and “strongly + semi” convex cases in the future.

Appendix: $O(1/k)$ convergence rate of the backtracking version of ISTA in the "strongly + semi" convex setting of (1.1)

In [6], the backtracking versions of ISTA and FISTA are also analyzed for the "convex + convex" setting of (1.1) when the Lipschitz continuity constant σ of ∇f is not unknown. Both the backtracking versions can be extended to the "strongly + semi" convex setting of (1.1) as well. For succinctness, we only elaborate on the analysis of the $O(1/k)$ convergence rate for the backtracking version of the ISTA (1.2) with $\rho_1 < \sigma$; extensions to other cases are similar.

First, recall that in [6], for any $L > 0$, the function

$$p_L(y) := \arg\min_{x \in \mathcal{R}^n} \left\{ g(x) + \frac{L}{2}\|x - (y - \frac{1}{L}\nabla f(y))\|^2 \right\} \tag{8.1}$$

is defined. Then, the backtracking version of ISTA without knowing σ in [6] for the "convex + convex" setting of (1.1) is presented below.

Backtracking version of ISTA

Step 0. Take $L_0 > \rho_2$, some $\eta > 1$ and $x_0 \in \mathcal{R}^n$.

Step k. ($k \geq 1$) Find the smallest nonnegative integers i_k such that $\bar{L} = \eta^{i_k} L_{k-1}$

$$f(p_{\bar{L}}(x_{k-1})) \leq f(x_{k-1}) + \langle \nabla f(x_{k-1}), p_{\bar{L}}(x_{k-1}) - x_{k-1} \rangle + \frac{\bar{L}}{2}\|x_{k-1} - p_{\bar{L}}(x_{k-1})\|^2. \tag{8.2}$$

Set $L_k = \eta^{i_k} L_{k-1}$ and compute

$$x_k = p_{L_k}(x_{k-1}). \tag{8.3}$$

In view of Theorem 4.1, we know that (8.1) is well defined if $L > \rho_2$. Also, the inequality (8.2) is satisfied with $\bar{L} \geq \sigma$, according to Lemma 2.6. Thus, $L_k \leq \eta\sigma$ for every $k \geq 1$. In the following theorem, we derive the $O(1/k)$ convergence rate for the backtracking version of ISTA in the "strongly + semi" convex setting of (1.1). The proof is similar to that of Theorem 5.1; but technically more tedious.

Theorem 8.1. For model (1.1), suppose f is strongly convex with constant $\rho_1 \geq 0$, continuously differentiable and ∇f is Lipschitz continuous with constant $\sigma > 0$ but σ is unknown; g is proper lower semicontinuous semiconvex function with constant $\rho_2 \geq 0$. Suppose $\sigma > \rho_1 \geq \rho_2$ and $S \neq \emptyset$. Let $\{x_k\}_{k \in N}$ be the sequence generated by the backtracking version of ISTA (8.2)-(8.3) with $L_k > \rho_2$ and $\eta > 1$, then we have

$$F(x_k) - F(x^*) \leq \frac{\eta\sigma - \rho_2}{2k}\|x_0 - x^*\|^2 = O(\frac{1}{k}), \ \forall x^* \in S. \tag{8.4}$$

Proof. First, as mentioned, the backtracking version of ISTA (8.2)-(8.3) is well defined with the assumptions. For the iterate x_n, it follows from the optimality condition of (8.3) that

$$-\nabla f(x_{n-1}) + L_n(x_{n-1} - x_n) \in \partial g(x_n).$$

Since f is strongly convex and g is semiconvex, by Remark 2.2 and the assertion (iii) of Remark 2.1 we have

$$f(x) \geq f(x_{n-1}) + \langle \nabla f(x_{n-1}), x - x_{n-1} \rangle + \frac{\rho_1}{2}\|x - x_{n-1}\|^2, \quad \forall x \in \mathcal{R}^n$$

and

$$g(x) \geq g(x_n) + \langle -\nabla f(x_{n-1}) + L_n(x_{n-1} - x_n), x - x_n \rangle - \frac{\rho_2}{2}\|x - x_n\|^2, \quad \forall x \in \mathcal{R}^n.$$

Adding the above two inequalities, we obtain

$$\begin{aligned} F(x) \quad \geq \quad & F(x_n) + f(x_{n-1}) - f(x_n) + L_n\langle x_{n-1} - x_n, x - x_n \rangle \\ & - \langle \nabla f(x_{n-1}), x_{n-1} - x_n \rangle + \frac{\rho_1}{2}\|x - x_{n-1}\|^2 - \frac{\rho_2}{2}\|x - x_n\|^2, \quad \forall x \in \mathcal{R}^n. \end{aligned} \tag{8.5}$$

It follows from (8.2) that

$$f(x_n) \leq f(x_{n-1}) + \langle \nabla f(x_{n-1}), x_n - x_{n-1} \rangle + \frac{L_n}{2}\|x_{n-1} - x_n\|^2. \tag{8.6}$$

Then, substituting (8.6) into (8.5), we have

$$\begin{aligned} F(x) \quad \geq \quad & F(x_n) + L_n\langle x_{n-1} - x_n, x - x_n \rangle \\ & + \frac{\rho_1}{2}\|x - x_{n-1}\|^2 - \frac{\rho_2}{2}\|x - x_n\|^2 - \frac{L_n}{2}\|x_{n-1} - x_n\|^2, \quad \forall x \in \mathcal{R}^n. \end{aligned} \tag{8.7}$$

Note the identity

$$\langle x_{n-1} - x_n, x - x_n \rangle = \frac{1}{2}(\|x_{n-1} - x_n\|^2 + \|x - x_n\|^2 - \|x - x_{n-1}\|^2).$$

We know that (8.7) can be written as

$$F(x) \geq F(x_n) + \frac{1}{2}(\rho_1 - L_n)\|x - x_{n-1}\|^2 + \frac{1}{2}(L_n - \rho_2)\|x - x_n\|^2, \quad \forall x \in \mathcal{R}^n. \tag{8.8}$$

Since $\rho_1 \geq \rho_2$ and $L_n > \rho_2$, we have

$$\frac{2}{L_n - \rho_2}(F(x) - F(x_n)) \geq \|x_n - x\|^2 - \|x_{n-1} - x\|^2, \quad \forall x \in \mathcal{R}^n. \tag{8.9}$$

Setting $x = x^* \in S$ in (8.9) and using the fact $F(x^*) - F(x_n) \leq 0$ and $L_n \leq \eta\sigma$, we get

$$F(x_n) - F(x^*) \leq \frac{\eta\sigma - \rho_2}{2}\|x_{n-1} - x^*\|^2 - \frac{\eta\sigma - \rho_2}{2}\|x_n - x^*\|^2. \tag{8.10}$$

Also, we have

$$\begin{aligned} \sum_{n=1}^{k}(F(x_n) - F(x^*)) \quad & \leq \quad \sum_{n=1}^{k}\left(\frac{\eta\sigma - \rho_2}{2}\|x_{n-1} - x^*\|^2 - \frac{\eta\sigma - \rho_2}{2}\|x_n - x^*\|^2\right) \\ & = \quad \frac{\eta\sigma - \rho_2}{2}\|x_0 - x^*\|^2 - \frac{\eta\sigma - \rho_2}{2}\|x_k - x^*\|^2 \end{aligned}$$

$$\leq \quad \frac{\eta\sigma - \rho_2}{2}\|x_0 - x^*\|^2, \tag{8.11}$$

where the first inequality follows from (8.10). Hence, taking $x = x_{n-1}$ in (8.8) and again using $L_n > \rho_2$, we know that, for all $n \geq 1$, it holds

$$F(x_n) \leq F(x_{n-1}). \tag{8.12}$$

Thus, using (8.11) and (8.12), we have

$$k(F(x_k) - F(x^*)) \leq \frac{\eta\sigma - \rho_2}{2}\|x_0 - x^*\|^2.$$

Therefore, the assertion (8.4) is proved. □

Remark 8.1. *For the special case where f and g are both convex, i.e., $\rho_1 = \rho_2 = 0$, the assumption $L_k > \rho_2$ holds automatically; and for this case, Theorem 8.1 reduces to Theorem 3.1 in [6]. Moreover, similar as Corollary 5.1, the $O(1/k)$ convergence rate established in Theorem 8.1 can be improved to $o(1/k)$. We skip the proof for succinctness.*

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