A bound on the Carathéodory number

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Abstract

The Carathéodory number $\kappa(\mathcal{K})$ of a pointed closed convex cone $\mathcal{K}$ is the minimum among all the κ for which every element of $\mathcal{K}$ can be written as a nonnegative linear combination of at most κ elements belonging to extreme rays. Carathéodory's Theorem gives the bound $\kappa(\mathcal{K}) \leq \dim \mathcal{K}$. In this work we observe that this bound can be sharpened to $\kappa(\mathcal{K}) \leq \ell_{\mathcal{K}} - 1$, where $\ell_{\mathcal{K}}$ is the length of the longest chain of nonempty faces contained in $\mathcal{K}$, thus tying the Carathéodory number with a key quantity that appears in the analysis of facial reduction algorithms. We show that this bound is tight for several families of cones, which include symmetric cones and the so-called smooth cones. We also give a family of examples showing that this bound can also fail to be sharp. In addition, we furnish a new proof of a result by Güler and Tunçel which states that the Carathéodory number of a symmetric cone is equal to its rank. Finally, we connect our discussion to the notion of *cp-rank* for completely positive matrices.

1 Introduction

Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a closed convex cone which is pointed, *i.e.*, $\mathcal{K} \cap -\mathcal{K} = \{0\}$. The *Carathéodory number* of $x \in \mathcal{K}$ is the smallest nonnegative integer $\kappa(x)$ for which

$$x = d_1 + \ldots + d_{\kappa(x)},$$

where each d_i belongs to an extreme ray of $\mathcal{K}$. We then define the *Carathéodory number of* $\mathcal{K}$ as

$$\kappa(\mathcal{K}) = \max\{\kappa(x) \mid x \in \mathcal{K}\}.$$

The Carathéodory number is a key geometric quantity and has a few surprising connections. For instance, Güller and Tunçel showed that $\kappa(\mathcal{K})$ is a lower bound for the optimal barrier parameter for self-concordant barriers when $\mathcal{K}$ is an homogeneous cone, see Proposition 4.1 in [11]. When $\mathcal{K}$ is, in fact, a symmetric cone, the inequality turns into an equality, see also the work by Tunçel and Truong [1] and the related article by Tunçel and Xu [21]. Recently, in an article by Naldi [15], $\kappa(\mathcal{K})$ was studied in the context of the so-called Hilbert cones.

The well-known Carathéodory Theorem tells us that the *dimension of* $\mathcal{K}$ is an upper bound for $\kappa(\mathcal{K})$. In this note, we will show the bound

$$\kappa(\mathcal{K}) \leq \ell_{\mathcal{K}} - 1, \tag{1}$$

where $\ell_{\mathcal{K}}$ is the length of the longest chain of faces in $\mathcal{K}$. We remark that $\ell_{\mathcal{K}}$ is an important quantity that appears in the analysis of *facial reduction algorithms* (FRAs) [6]. Namely, $\ell_{\mathcal{K}} - 1$ is an upper

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bound for the minimum number of steps before a problem over $\mathcal{K}$ is fully regularized. See [22, 17] for a detailed discussion on facial reduction.

Of course, $\ell_{\mathcal{K}} - 1$ is itself bounded by $\dim \mathcal{K}$, but here we will discuss several cases for which the former is strictly smaller than the latter, see Table 1.

The only extra assumption we will make is that $\mathcal{K}$ must be a *pointed cone*, that is, $\mathcal{K} \cap -\mathcal{K} = \{0\}$. This is done to ensure that $\mathcal{K}$ has extreme rays.[1]

We now present a summary of the results. The bound (1) is proven in Section 3. In fact, a slightly more general statement is proven, namely, that given $x \in \mathcal{K}$, we have

$$\kappa(x) \leq \ell_{\mathcal{F}(x,\mathcal{K})} - 1,$$

where $\mathcal{F}(x, \mathcal{K})$ is the minimal face of $\mathcal{K}$ containing x. We will also show a family of cones for which the inequality in (1) is strict.

Given $\mathcal{K}$, there is a compact convex set C such that $\mathcal{K}$ is generated by $\{1\} \times C$. This process can also be reversed, so that given C, the cone generated by $\{1\} \times C$ is closed and pointed. Moreover, there is a correspondence between extreme points of C and extreme rays of $\mathcal{K}$. Therefore, the bound on $\kappa(\mathcal{K})$ also induces a bound on $\kappa(C)$, namely

$$\kappa(C) \leq \ell_C.$$

This time, for $x \in C$, $\kappa(x)$ is the smallest integer for which we can write x as a convex combination of $\kappa(x)$ extreme *points*. As before, ℓ_C is the length of the longest chain of faces in C. This is discussed in Section 3.

The other contribution of this article is a discussion of several examples in which (1) turns into an equality. A part of Theorem 4.2 of [21] shown by Tunçel and Xu asserts that a pointed polyhedral homogeneous convex n-dimensional cone P satisfies

$$\kappa(P) = n.$$

In this article, we will slightly generalize this fact so that we have

$$\kappa(P) = \dim P = \ell_P - 1$$

for any pointed polyhedral cone without the homogeneous hypothesis. Moreover, we will strength the result and show that whenever the set of extreme rays of $\mathcal{K}$ is *countable*, Equation (1) turns into an equality.

In [13], Liu and Pataki defined a *smooth cone* as a pointed, full-dimensional cone $\mathcal{K}$ for which all faces distinct from $\{0\}$ and $\mathcal{K}$ are extreme rays. For those cones, we will show in Section 4 that (1) holds with equality as well.

If $\mathcal{K}$ is a symmetric cone, Güller and Tunçel showed in Lemma 4.1 of [11] that

$$\kappa(\mathcal{K}) = \operatorname{rank} \mathcal{K}.$$

However, the proof of Lemma 4.1 consists of a case-by-case analysis using the classification of Euclidean Jordan Algebras. There is also a proof in [1] via the theory of homogeneous cones, see Theorem 8 therein. We will give a new proof which we hope is simpler, using only elementary properties of Jordan Algebras. Moreover, we will also show that $\operatorname{rank} \mathcal{K} = \ell_{\mathcal{K}} - 1$, which is a new result, as far as we know. These results are discussed in Section 5. We recall that among the symmetric cones we have the second order cone, the positive semidefinite cone and direct products of them.

Finally, in Section 6, we discuss what is currently known about $\kappa(\mathcal{K})$ and $\ell_{\mathcal{K}}$ for three families of cones that are of great interest recently: the copositive cone, the completely positive cone and the doubly nonnegative cone. We also observe that the Carathéodory number of copositive matrices coincides with the so-called *cp-rank*.

[1]Note that this is not a restrictive assumption. Letting $\operatorname{lin} \mathcal{K} = \mathcal{K} \cap -\mathcal{K}$, we have $\mathcal{K} = \mathcal{K} \cap (\operatorname{lin} \mathcal{K}^{\perp}) + \operatorname{lin} \mathcal{K}$ and $\mathcal{K} \cap (\operatorname{lin} \mathcal{K}^{\perp})$ is a pointed cone, for which our results apply.

2 Preliminaries

Let $\mathcal{K}$ be a closed convex cone contained in $\mathbb{R}^n$. We denote its dual by $\mathcal{K}^* = \{x \in \mathbb{R}^n \mid \langle x, y\rangle \geq 0, \forall y \in \mathcal{K}\}$, where $\langle \cdot, \cdot \rangle$ is some inner product in $\mathbb{R}^n$. Given a closed convex set C, we will denote its dimension, interior, recession cone, relative interior and relative boundary by $\dim C, \operatorname{int} C, \operatorname{rec} C, \operatorname{ri} C, \operatorname{bd} C$, respectively. Recall that we have $\operatorname{bd} C = C \setminus \operatorname{ri} C$. If $\mathcal{F} \subseteq C$ is a convex set, we say that $\mathcal{F}$ is a *face* if the following condition holds: if $x, y \in C$ and $\alpha x + (1-\alpha)y \in \mathcal{F}$ for some $0 < \alpha < 1$, then $x, y \in \mathcal{F}$.

A face consisting of a single point is called an *extreme point*. If $\mathcal{K}$ is a pointed closed convex cone, the only extreme point is zero. We refer to an one-dimensional face of $\mathcal{K}$ as an *extreme ray*.

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two faces of C, then $\operatorname{ri} \mathcal{F}_1 \cap \operatorname{ri} \mathcal{F}_2 \neq \emptyset$ if and only if $\mathcal{F}_1 = \mathcal{F}_2$, see Proposition 2.2 in [16]. Now, suppose that $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Then, a necessary and sufficient condition for the inclusion to be proper (*i.e.*, $\mathcal{F}_1 \subsetneq \mathcal{F}_2$) is that $\dim \mathcal{F}_1 < \dim \mathcal{F}_2$. See Corollaries 18.1.2 and 18.1.3 in [19]. The following result will also be useful.

Lemma 1 (Theorem 6.4 in [19])**.** *Let $\mathcal{K}$ be a convex cone, $w \in \operatorname{ri} \mathcal{K}$ and $v \in \mathcal{K}$. Then, there is $\alpha > 1$ such that $\alpha w + (1-\alpha)v \in \mathcal{K}$.*

Let $d \in \mathcal{K}$, then the generalized eigenvalue function of $\mathcal{K}$ with respect to d is

$$\lambda_{\mathcal{K}}^d(x) = \inf\{t \mid x - td \notin \mathcal{K}\}.$$

This function was introduced by Renegar in [18]. We remark that when $\mathcal{K} = \mathcal{S}^n_+$ is the cone of positive semidefinite matrices and $d = I_n$ is the identity matrix, then $\lambda_{\mathcal{S}^n_+}^{I_n}$ is the usual minimum eigenvalue function. While in [18] the reference point d is always a relative interior of $\mathcal{K}$, an important twist here is that we will allow d to be a relative boundary point of $\mathcal{K}$.

We now collect a few properties of $\lambda_{\mathcal{K}}^d(x)$.

Lemma 2. *Let $d \in \mathcal{K}$ with $d \neq 0$ and $x \in \mathbb{R}^n$. The following assertions hold.*

(i) If $x \in \mathcal{K}$ then $\lambda_{\mathcal{K}}^d(x) < +\infty$.

(ii) If $x \in \operatorname{ri} \mathcal{K}$ then $\lambda_{\mathcal{K}}^d(x) > 0$.

(iii) For $\alpha \in \mathbb{R}$, we have $\lambda_{\mathcal{K}}^d(x + \alpha d) = \lambda_{\mathcal{K}}^d(x) + \alpha$.

(iv) If $x \in \mathcal{K}$, then $x - \lambda_{\mathcal{K}}^d(x)d \in \mathcal{K} \setminus \operatorname{ri} \mathcal{K}$.

Proof. (*i*) Suppose that $\lambda_{\mathcal{K}}^d(x) = +\infty$. Then $x - td \in \mathcal{K}$ for all $t \in \mathbb{R}$, which implies that $d, -d \in \mathcal{K}$. Since $\mathcal{K}$ is pointed, we have $d = 0$, a contradiction.

(*ii*) If $x \in \operatorname{ri} \mathcal{K}$, by Lemma 1, there exists $\alpha > 1$ such that $\alpha x + (1-\alpha)d \in \mathcal{K}$. This means that $x - \frac{(\alpha-1)}{\alpha} d \in \mathcal{K}$, which implies that $\lambda_{\mathcal{K}}^d(x) \geq \frac{(\alpha-1)}{\alpha} > 0$.

(*iii*) It is clear from the definition of $\lambda_{\mathcal{K}}^d(x)$. Note that we use the convention that $+\infty + \alpha = +\infty$ and $-\infty + \alpha = -\infty$.

(*iv*) Due to items (*i*) and (*iii*), we have $\lambda_{\mathcal{K}}^d(x - \lambda_{\mathcal{K}}^d(x)d) = 0$. By item (*ii*), $x - \lambda_{\mathcal{K}}^d(x)d \notin \operatorname{ri} \mathcal{K}$. Due to the definition of $\lambda_{\mathcal{K}}^d(x)$, for every $\epsilon > 0$,

$$x - (\lambda_{\mathcal{K}}^d(x) - \epsilon)d \in \mathcal{K},$$

so that $x - \lambda_{\mathcal{K}}^d(x)d \in \mathcal{K}$.

□

The next lemma is a classical result about the existence of extreme rays. We include the proof for the sake of self-containment.

Lemma 3. *Let $\mathcal{K}$ be a closed, pointed convex cone with* $\dim \mathcal{K} \geq 1$. *Then $\mathcal{K}$ contains at least one extreme ray, i.e., an one dimensional face.*

Proof. The first step is to pick $e^* \in \operatorname{ri} \mathcal{K}^*$ and show that

$$C = \{x \in \mathcal{K} \mid \langle x, e^* \rangle = 1\}$$

is compact. Since C is a closed convex set, it suffices to show that its recession cone satisfies $\operatorname{rec} C = \{0\}$. We have

$$\operatorname{rec} C = \{x \in \mathcal{K} \mid \langle x, e^* \rangle = 0\}.$$

However, $x \in \operatorname{rec} C$ if and only if $x \in (\mathcal{K}^*)^\perp$, due to the choice of e^*[2]. As $(\mathcal{K}^*)^\perp \subseteq \mathcal{K} \cap -\mathcal{K}$ (they are equal, in fact), we have $x = 0$.

Finally, we invoke the Krein-Milman Theorem, which implies that a nonempty compact convex set has at least one extreme point z. Then, one can verify that the half-line $h_z = \{\alpha z \mid \alpha \geq 0\}$ is an extreme ray of $\mathcal{K}$. □

3 Main result and discussion

In what follows, we will denote by $\mathcal{F}(S, \mathcal{K})$ the minimal face of $\mathcal{K}$ that contains $S\,(\subset \mathcal{K})$. We also write $\mathcal{F}(x, \mathcal{K})$ when $S = \{x\}$. Given a face $\mathcal{F}$, we have $\mathcal{F} = \mathcal{F}(x, \mathcal{K})$ if and only if $x \in \operatorname{ri} \mathcal{F}$, see Proposition 2.2 in [16].

A *chain of faces of* $\mathcal{K}$ is a finite sequence of faces of $\mathcal{K}$ such that each face properly contains the next. If we have a chain $\mathcal{F}_1 \supsetneq \ldots \supsetneq \mathcal{F}_\ell$, we define its *length* as the number of faces, which in this case is ℓ. We will denote by $\ell_{\mathcal{K}}$, the length of the longest chain of faces of $\mathcal{K}$.

Theorem 4. *Let $\mathcal{K}$ be a pointed closed convex cone and $x \in \mathcal{K}$. Then*

$$\kappa(x) \leq \ell_{\mathcal{F}(x,\mathcal{K})} - 1.$$

In particular, $\kappa(\mathcal{K}) \leq \ell_{\mathcal{K}} - 1$.

Proof. Let $x \in \mathcal{K}$. If $\dim \mathcal{F}(x, \mathcal{K}) \leq 1$, we are done. Otherwise, due to Lemma 3, $\mathcal{F}(x, \mathcal{K})$ contains an extreme ray $\{\alpha d_1 \mid \alpha \geq 0\}$. In particular, $d_1 \in \mathcal{K} \setminus \{0\}$. Now, let

$$x_1 = x - \lambda^{d_1}_{\mathcal{F}(x,\mathcal{K})}(x) d_1$$
$$\mathcal{F}_1 = \mathcal{F}(x_1, \mathcal{K}).$$

Due to item (iv) of Lemma 2, $x_1 \notin \operatorname{ri} \mathcal{F}(x, \mathcal{K})$, therefore $\mathcal{F}_1 \subsetneq \mathcal{F}(x, \mathcal{K})$. We then proceed by induction, defining

$$x_i = x_{i-1} - \lambda^{d_i}_{\mathcal{F}_{i-1}}(x_{i-1}) d_i,$$
$$\mathcal{F}_i = \mathcal{F}(x_i, \mathcal{K}),$$

where $\{\lambda d_i \mid \lambda \geq 0\}$ is an extreme ray of $\mathcal{F}_{i-1}$, which exists as long as $\dim \mathcal{F}_{i-1} \geq 1$. Similarly, $x_i \notin \operatorname{ri} \mathcal{F}_{i-1}$, so that $\mathcal{F}_{i-1} \supsetneq \mathcal{F}_i$. Because we are in a finite dimensional space, there is an index ℓ for which $\mathcal{F}_\ell = \{0\}$, that is, $x_{\ell-1} - \lambda^{d_\ell}_{\mathcal{F}_{\ell-1}}(x_{\ell-1}) d_\ell = 0$. Unwinding the recursion, we can express x as a positive linear combination of ℓ points belonging to extreme rays and, at the same time, we obtain a chain of faces

$$\mathcal{F}(x, \mathcal{K}) \supsetneq \mathcal{F}_1 \supsetneq \ldots \supsetneq \mathcal{F}_\ell = \{0\}.$$

So that $\ell + 1 \leq \ell_{\mathcal{F}(x,\mathcal{K})}$. As any chain of faces of $\mathcal{F}(x, \mathcal{K})$ is also a chain of $\mathcal{K}$, we also obtain $\ell + 1 \leq \ell_{\mathcal{K}}$. □

[2]This is a general fact. Suppose that $\mathcal{K}$ is a convex cone, $w \in \operatorname{ri} \mathcal{K}$ and $z \in \mathcal{K}^*$. Then $z \in \mathcal{K}^\perp$ if and only if $\langle w, z \rangle = 0$. To see that, suppose that $\langle w, z \rangle = 0$. Then, given $v \in \mathcal{K}$ we have $\alpha w + (1 - \alpha) v \in \mathcal{K}$ for some $\alpha > 1$, by Lemma 1. Taking the inner product with z, we see that $\langle w, z \rangle$ must be zero.

Now, let C be a nonempty compact convex set. Similarly, given $x \in C$, we may define the Carathéodory number of x as the minimum number $\kappa(x)$ necessary to express x a convex combination of $\kappa(x)$ *extreme points*. Using Theorem 4, we can also say something about the Carathéodory number for compact convex sets thanks to the following well-known result. We include a proof in Appendix A.

Proposition 5. *Let $C \subseteq \mathbb{R}^n$ be a nonempty compact convex set. Let*

$$\mathcal{K} = \{(\alpha, \alpha x) \mid \alpha \geq 0, x \in C\}.$$

Then

(i) *$\mathcal{K}$ is a pointed closed convex cone.*

(ii) *Let $\mathcal{F}$ be a face of $\mathcal{K}$ that is not $\{0\}$, then*

$$\mathcal{F}_C = \{x \in C \mid (1, x) \in \mathcal{F}\}$$

is a face of C. Moreover, $\dim \mathcal{F}_C = \dim \mathcal{F} - 1$.

(iii) *Let $\mathcal{F}_C$ be a face of C, then*

$$\mathcal{F} = \{(\alpha, \alpha x) \mid \alpha \geq 0, x \in \mathcal{F}_C\}$$

is a face of $\mathcal{K}$. Moreover, $\dim \mathcal{F} = \dim \mathcal{F}_C + 1$.

In a similar fashion, we will define ℓ_C as the length of the longest chain of faces in C.

Theorem 6. *Let C be a nonempty compact convex set and $x \in C$. Then,*

$$\kappa(x) \leq \ell_C.$$

Proof. Let $\mathcal{K}$ be as in Proposition 5, then the first step is showing that

$$\kappa(x) = \kappa((1, x)),$$

where it is understood that $\kappa(1, x)$ is computed with respect to $\mathcal{K}$. Suppose that

$$(1, x) = (\alpha_1, \alpha_1 x_1) + \ldots + (\alpha_\ell, \alpha_\ell x_\ell) \tag{2}$$

where each $(\alpha_i, \alpha_i x_i)$ lies in an extreme ray of $\mathcal{K}$ and the α_i are positive. Due to item *(ii)* of Proposition 5, it must be the case that the x_i are extreme points of C. So Equation (2) also expresses x as a convex combination of ℓ extreme points. Conversely, if x is expressed as a convex combination of ℓ extreme points, it is also possible to express $(1, x)$ as a sum of ℓ extreme rays as in Equation (2).

If we show that $\ell_{\mathcal{K}} = \ell_C + 1$, then the result will follow by Theorem 4. To show that this is indeed the case, consider an arbitrary chain of faces of C

$$\mathcal{F}_C^1 \supsetneq \ldots \supsetneq \mathcal{F}_C^\ell.$$

Then, following Proposition 5, we also obtain a chain of faces of $\mathcal{K}$ and we can enlarge the chain by adding the zero face.

$$\mathcal{F}^1 \supsetneq \ldots \supsetneq \mathcal{F}^\ell \supsetneq \{0\}.$$

This process can be reversed and any chain of faces of $\mathcal{K}$ that does not contain the zero face also gives rise to a chain of faces in C. □

4 Tightness of the bound

The main result of this section gives a few conditions ensuring that $\kappa(\mathcal{K}) = \ell_{\mathcal{K}} - 1$. We will also furnish an example where the bound fails to be tight.

A pointed closed convex cone $\mathcal{K}$ is said to be *strictly convex* if $\operatorname{int} \mathcal{K} \neq \emptyset$ and we have $\mathcal{F}(\{x, y\}, \mathcal{K}) = \mathcal{K}$ for every linearly independent $x, y \in \operatorname{bd} \mathcal{K}$, see [2] and Definition 2.A.4 in [3]. An equivalent concept is the notion of *smooth cones*, which was considered in [13]: $\mathcal{K}$ is a smooth cone if $\operatorname{int} \mathcal{K} \neq \emptyset$ and every face of $\mathcal{K}$ different from $\{0\}$ and $\mathcal{K}$ is an extreme ray.[3] It is known that every strictly convex cone $\mathcal{K}$ with $\dim \mathcal{K} \geq 2$ satisfies $\kappa(\mathcal{K}) = 2$, see Lemma 4.1 in [21]. In what follows, we give a new proof and we will point out the connection to $\ell_{\mathcal{K}}$.

Theorem 7. *Let $\mathcal{K}$ be a pointed closed convex cone.*

(*i*) *If the set of extreme rays of $\mathcal{K}$ is countable, then we have*

$$\kappa(\mathcal{K}) = \ell_{\mathcal{K}} - 1 = \dim \mathcal{K}. \tag{3}$$

In particular, (3) *holds when $\mathcal{K}$ is polyhedral.*

(*ii*) *If $\mathcal{K}$ is a strictly convex cone with $\dim \mathcal{K} \geq 2$, then we have*

$$\kappa(\mathcal{K}) = \ell_{\mathcal{K}} - 1 = 2.$$

Proof. (*i*) In view of the definition of $\kappa(\mathcal{K})$, we can write

$$\mathcal{K} = \bigcup \{\operatorname{cone}(\{d_1, \ldots, d_{\kappa(\mathcal{K})}\}) \mid d_i \text{ belongs to an extreme ray of } \mathcal{K},\ \|d_i\| = 1\}$$

where $\operatorname{cone}(\{d_1, \ldots, d_{\kappa(\mathcal{K})}\})$ denotes the smallest convex cone containing $\{d_1, \ldots, d_{\kappa(\mathcal{K})}\}$. Since the set of extreme rays is countable, it follows that $\mathcal{K}$ is a countable union of cones of dimension at most $\kappa(\mathcal{K})$.

This forces that $\kappa(\mathcal{K}) \geq \dim \mathcal{K}$ since a finite dimensional convex cone cannot be covered by a countable union of convex subsets with strictly smaller dimmension[4]. Therefore, Theorem 4 concludes that $\kappa(\mathcal{K}) = \ell_{\mathcal{K}} - 1 = \dim \mathcal{K}$.

As an immediate consequence, polyhedral cones satisfy (3) because the set of extreme rays of a polyhedral cone is finite (see, *e.g.*, [19, Theorem 19.1]).

(*ii*) The definition of smooth cones implies $\ell_{\mathcal{K}} \leq 3$ since every chain of faces of $\mathcal{K}$ cannot be longer than the one of the form $\mathcal{K} \supsetneq \mathcal{F} \supsetneq \{0\}$ where $\mathcal{F}$ is an extreme ray. Furthermore, it is clear that $\kappa(\mathcal{K}) > 1$ due to $\dim \mathcal{K} \geq 2$. Hence, Theorem 4 concludes that $\kappa(\mathcal{K}) = \ell_{\mathcal{K}} - 1 = 2$.

□

We now give a family of examples where $\kappa(\mathcal{K}) < \ell_{\mathcal{K}} - 1$.

Example 8. *Consider the cone*

$$\mathcal{K} = \left\{ z = (x, t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; t \geq \sqrt{x_1^2 + \cdots + x_n^2}, x_2 \geq 0, \ldots, x_n \geq 0 \right\} \subset \mathbb{R}^{n+1}.$$

(Notice that the component x_1 is allowed to be negative.) Then, for $n \geq 1$ we have

$$\kappa(\mathcal{K}) = 2, \quad \ell_{\mathcal{K}} = n + 2.$$

[3] In fact, if $\mathcal{K}$ is a strictly convex cone, then any face $\mathcal{F}$ with $\{0\} \subsetneq \mathcal{F} \subsetneq \mathcal{K}$ is one dimensional because every $x, y \in \mathcal{F}$ cannot be linearly independent, since $\mathcal{F} \subset \operatorname{bd} \mathcal{K}$. If $\mathcal{K}$ is a smooth cone, on the other hand, every linearly independent $x, y \in \operatorname{bd} \mathcal{K}$ must satisfy $\mathcal{F}(\{x, y\}, \mathcal{K}) = \mathcal{K}$ since we have $\dim \mathcal{F}(\{x, y\}, \mathcal{K}) \geq 2$.

[4] Consider the linear span of $\mathcal{K}$ endowed with the Lebesgue measure μ, which is a vector space of dimension $\dim \mathcal{K}$. Also, recall that convex sets are Lebesgue measurable. As $\mathcal{K}$ contains balls of dimension $\dim \mathcal{K}$, we have $\mu(\mathcal{K}) > 0$. If $\mathcal{K}$ is covered by a countable collection $\{V_i\}$ of convex sets with dimension strictly smaller than $\dim \mathcal{K}$, we arrive at an contradiction $\mu(\mathcal{K}) \leq \sum_i \mu(V_i) = 0$, due to the countable subadditivity of μ. See [12], for an algebraic discussion in a more general context.

Proof. We remark that $\mathcal{K}$ is the intersection of the second order cone $\mathcal{Q}^{n+1} = \{(x,t) \mid t \geq \sqrt{x_1^2 + \cdots x_n^2}\}$ and the cone $\mathcal{K}' := \{(x,t) \mid x_2 \geq 0, \ldots, x_n \geq 0\}$.

Let us show that $\kappa(z) \leq 2$ for any $z \in \mathcal{K}$. For $z = (x,t) \in \mathcal{K}$, define

$$\lambda_\pm := \pm\left(-x_1 + \sqrt{t^2 - (x_2^2 + \cdots + x_n^2)}\right)$$

so that

$$\sqrt{(x_1 \pm \lambda_\pm)^2 + x_2^2 + \cdots + x_n^2} = t. \tag{4}$$

Now, let e_1 denote the unit vector along the first coordinate. Equation (4) implies that the points $d_\pm := z \pm \lambda_\pm e_1$ belong to both $\mathcal{K}$ and the boundary of $\mathcal{Q}^{n+1}$. Since every boundary point of $\mathcal{Q}^{n+1}$ lies in an extreme ray of $\mathcal{Q}^{n+1}$ (see Example 2.6 in [16]), we conclude that the points $d_\pm$ must belong to extreme rays of $\mathcal{K}$ as well. Since z lies on the segment between d_- and d_+, we have $\kappa(z) \leq 2$. As $\mathcal{K}$ is not a single extreme ray, we must have $\kappa(\mathcal{K}) = 2$.

Finally, we verify $\ell_{\mathcal{K}} = n+2$ as follows. Define the faces $\{\mathcal{F}_i\}_{i=1}^{n+2}$ of $\mathcal{K}$ by $\mathcal{F}_1 := \mathcal{K}$,

$$\begin{aligned}
\mathcal{F}_{i+1} &:= \{(x,t) \in \mathcal{K}' \mid x_2 = \cdots = x_{i+1} = 0\} \cap \mathcal{Q}^{n+1}, \quad i = 1, \ldots, n-1, \\
\mathcal{F}_{n+1} &:= \{(x,t) \in \mathcal{K}' \mid x_2 = \cdots = x_n = 0\} \cap \{(x,t) \in \mathcal{Q}^{n+1} \mid x_1 = t \geq 0\},
\end{aligned}$$

and $\mathcal{F}_{n+2} := \{0\}$, which gives a chain of faces of $\mathcal{K}$ of length $n+2$. Note that these are indeed faces of $\mathcal{K}$, since they arise as intersections of faces of $\mathcal{K}'$ and $\mathcal{Q}^{n+1}$. Since $\mathcal{K}$ is contained in a space of dimension $n+1$, it must be indeed the largest possible chain. □

5 The symmetric cone case

We say that $\mathcal{K}$ is a *symmetric cone* if $\mathcal{K} = \mathcal{K}^*$, $\operatorname{int} \mathcal{K} \neq \emptyset$ and for every pair of elements x, y in the interior of $\mathcal{K}$ there is an invertible linear transformation T such that $T(\mathcal{K}) = \mathcal{K}$ and $T(x) = T(y)$. The theory of symmetric cones is strongly connected with the study of *Euclidean Jordan Algebras*. The default reference is the book by Faraut and Korányi [8] but there are many introductory accounts in the context of optimization, see [20, 10]. In this section, we will furnish in Theorem 19 another proof that $\kappa(x) = \operatorname{rank} x$. Moreover in Theorem 20, we will show that $\ell_{\mathcal{K}} = \operatorname{rank} \mathcal{K} + 1$, which is a new result as far as we know. The reader who is already familiar with the theory of Jordan Algebras can skip to Section 5.2.

5.1 Preliminaries

Let $\mathcal{E}$ be a finite dimensional real vector space equipped with a bilinear form $\circ : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$. Write x^2 for $x \circ x$. Now, suppose that $\circ$ satisfies the following properties for all $x, y \in \mathcal{E}$:

1. $x \circ y = y \circ x$,
2. $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$.

Then, $(\mathcal{E}, \circ)$ is said to be a *Jordan Algebra* and $\circ$ is said to be a *Jordan product*.

Furthermore, suppose that $\mathcal{E}$ is equipped with an inner product $\langle \cdot, \cdot \rangle$ such that for all $x, y, z \in \mathcal{E}$ we have

$$\langle x \circ y, z \rangle = \langle y, x \circ z \rangle.$$

Then, $(\mathcal{E}, \circ)$ is said to be an *Euclidean Jordan Algebra*. Given a Jordan algebra, we define its cone of squares as

$$\mathcal{K} = \{x \circ x \mid x \in \mathcal{E}\}.$$

Example 9. *Let $\mathcal{S}^n$ denote the space of $n \times n$ symmetric matrices equipped with the inner product such that $\langle A, B\rangle = \operatorname{trace}(AB)$, for $A, B \in \mathcal{S}^n$. Then, $\mathcal{S}^n$ is an Euclidean Jordan Algebra with the following product*

$$A \circ B = \frac{AB + BA}{2}.$$

The corresponding cone of squares is the cone of positive semidefinite matrices $\mathcal{S}^n_+$.

Now, consider $\mathbb{R}^{n+1}$ equipped with the usual Euclidean inner product. Given $x \in \mathbb{R}^n$, write $x = (x_0, \overline{x})$, where $x_0 \in \mathbb{R}$ and $\overline{x} \in \mathbb{R}^n$. Consider the following Jordan Products in $\mathbb{R}^{n+1}$

$$x \circ y = (\langle x, y\rangle, x_0\overline{y} + y_0\overline{x}).$$

The corresponding cone of squares is the second order cone $\mathcal{Q}^{n+1} = \{(x_0, \overline{x}) \mid x_0 \geq \sqrt{\langle \overline{x}, \overline{x}\rangle}\}$.

It is known that $\mathcal{K}$ is a symmetric cone if and only if it arises as the cone of squares induced by some Euclidean Jordan Algebra $(\mathcal{E}, \circ)$, see Theorems III.2.1 and III.3.1 in [8]. Moreover, the construction in Theorem III.3.1 shows that we can safely assume that $(\mathcal{E}, \circ)$ has an identity element e. That is, $x \circ e = x$ for all $x \in \mathcal{E}$.

One of the beautiful aspects of the theory of Jordan Algebras is that it allows the definition of objects such as eigenvalues, determinant, trace and rank in a very general context. In particular, we also have a version of the Spectral Theorem for Jordan Algebras. In what follows, we will say that c is *idempotent* if $c \circ c = c$. Morover, c is *primitive* if it is nonzero and there is no way of writing

$$c = a + b,$$

with a and b nonzero idempotent elements satisfying $a \circ b = 0$.

Theorem 10 (Spectral theorem, see Theorem III.1.2 in [8])**.** *Let $(\mathcal{E}, \circ)$ be an Euclidean Jordan Algebra and let $x \in \mathcal{E}$. Then there are:*

1. *primitive idempotents $c_1, \ldots, c_r$ satisfying*

$$c_i \circ c_j = 0, \qquad \text{for } i \neq j \tag{5}$$
$$c_i \circ c_i = c_i, \qquad i = 1, \ldots, r \tag{6}$$
$$c_1 + \ldots + c_r = e, \qquad i = 1, \ldots, r \tag{7}$$

2. *unique real numbers $\lambda_1, \ldots, \lambda_r$ satisfying*

$$x = \sum_{i=1}^{r} \lambda_i c_i. \tag{8}$$

We say that the $c_1, \ldots, c_r$ in Theorem 10 form a *Jordan Frame* for x. The $\lambda_1, \ldots, \lambda_r$ are the eigenvalues of x. We remark that r only depends on the algebra $\mathcal{E}$. Given $x \in \mathcal{E}$, we define its trace by

$$\operatorname{trace}(x) = \lambda_1 + \ldots + \lambda_r,$$

where $\lambda_1, \ldots, \lambda_r$ are the eigenvalues of x. As in the case of matrices, it turns out that the trace function is linear, see Proposition II.4.3 in [8].

For an element $x \in \mathcal{E}$, we define the *rank* of x as the number of nonzero λ_i that appear in the Equation (8). Then, the rank of $\mathcal{K}$ is

$$\operatorname{rank} \mathcal{K} = \max\{\operatorname{rank} x \mid x \in \mathcal{K}\} = \operatorname{trace}(e).$$

For the next theorem, we need the following notation. Given $x \in \mathcal{E}$ and $a \in \mathbb{R}$, we write

$$V(x, a) = \{z \in \mathcal{E} \mid x \circ z = az\}.$$

Theorem 11 (Peirce decomposition, see Proposition IV.1.1 in [8])**.** *Let $c \in \mathcal{E}$ be an idempotent. Then $\mathcal{E}$ decomposes itself as an orthogonal direct sum as follows.*

$$\mathcal{E} = V(c,1) \bigoplus V\left(c, \frac{1}{2}\right) \bigoplus V(c,0).$$

In addition, $V(c,1)$ and $V(c,0)$ are Euclidean Jordan Algebras satisfying $V(c,1) \circ V(c,0) = \{0\}$.

The Peirce decomposition can be interpreted as follows. Given an element $x \in \mathcal{E}$, we have the linear transformation $L_x : \mathcal{E} \to \mathcal{E}$ given by

$$L_x(y) = x \circ y.$$

The fact that the algebra is Euclidean implies that L_x is self-adjoint, therefore $\mathcal{E}$ decomposes as a direct sum of the eigenspaces. Moreover, if x is an idempotent, it can be shown that the only possible eigenvalues are $0, 1, \frac{1}{2}$. So the decomposition in Theorem 11 is unsurprising. The remarkable part is the statement that $V(c,1)$ and $V(c,0)$ are also algebras and that they are orthogonal with respect to the Jordan product as well.

Lemma 12. *Let $(\mathcal{E}, \circ)$ be an Euclidean Jordan Algebra and let $\mathcal{K}$ be the cone of squares*

$$\mathcal{K} = \{z \circ z \mid z \in \mathcal{E}\}.$$

Let $x \in \mathcal{E}$ and denote its eigenvalues by $\lambda_1, \ldots, \lambda_r$. Then $x \in \mathcal{K}$ if and only if $\lambda_i \geq 0$ for all i.

Proof. ($\Rightarrow$) Let $c_1, \ldots, c_r$ be a Jordan Frame for x such that

$$x = \sum_{i=1}^{r} \lambda_i c_i.$$

Since the c_i are idempotent, they all belong to $\mathcal{K}$. Therefore, if the λ_i are nonzero, it is clear that x belongs to $\mathcal{K}$ as well.

($\Leftarrow$) Since $x \in \mathcal{K}$, we have $x = z \circ z$ for some $z \in \mathcal{E}$. Take a Jordan Frame for z:

$$z = \sum_{i=1}^{r} \mu_i d_i.$$

Item 1. of Theorem 4 allows us to conclude that

$$x = \sum_{i=1}^{r} \mu_i^2 d_i.$$

Then, uniqueness implies that each λ_i must be among the μ_j^2. In particular, all the λ_i are nonnegative. □

Lemma 13 (Exercise 3 in Chapter III of [8])**.** *Let $x, y \in \mathcal{K}$. Then $x \circ y = 0$ if and only if $\langle x, y \rangle = 0$.*

Proof. ($\Rightarrow$) We have

$$0 = \langle e, x \circ y \rangle = \langle e \circ x, y \rangle = \langle x, y \rangle,$$

due to the fact that the Jordan Algebra is Euclidean.

($\Leftarrow$) First we show an auxiliary fact. Suppose that $\langle x, c \circ c \rangle = 0$. Consider the function

$$f(z) = \frac{1}{2} \langle x, z \circ z \rangle.$$

Since $x \in \mathcal{K}$ and $\mathcal{K} = \mathcal{K}^*$, f is nonnegative everywhere. It follows that c is a local minimum of f, therefore

$$\nabla f(c) = x \circ c = 0.$$

Now take a Jordan frame $c_1, \ldots, c_r$ for y. We can write

$$y = \sum_{i=1}^{\operatorname{rank} y} \lambda_i c_i,$$

where we suppose that only the first $\operatorname{rank} y$ eigenvalues are nonzero and, therefore, positive. The fact that $\langle x, y \rangle = 0$, implies that $\langle x, c_i \rangle = 0$ for $i = 1, \ldots, \operatorname{rank} y$.

Since the c_i are idempotent, we also have $\langle x, c_i \circ c_i \rangle = 0$. By what we have just shown, we have $x \circ c_i = 0$, for $i = 1, \ldots, \operatorname{rank} y$. This implies that $x \circ y = 0$. □

Lemma 14. *Let $\mathcal{K}$ be a closed convex cone and let $x \in \mathcal{K}$. Then $x \notin \operatorname{ri} \mathcal{K}$ if and only if $\{x\}^\perp \cap (\mathcal{K}^* \setminus \mathcal{K}^\perp) \neq \emptyset$.*

Proof. Note that x does not belong to $\operatorname{ri} \mathcal{K}$ if and only if x and $\mathcal{K}$ can be *properly separated*, see Theorem 11.3 in [19]. This means that there is a hyperplane $H = \{z \mid \langle z, s \rangle = \alpha\}$ such that x and $\mathcal{K}$ belong to opposite closed half-spaces and at least one of them is not entirely contained in $\mathcal{K}$. We may assume that

$$\langle x, s \rangle \leq \alpha \leq \langle y, s \rangle,$$

for all $y \in \mathcal{K}$. In order for the inequality to hold, we must have $s \in \mathcal{K}^*$. Furthermore, since $x \in \mathcal{K}$ and $0 \in \mathcal{K}$, we conclude that $\langle x, s \rangle = 0$ and $\alpha = 0$. So that $x \in H$ and $H = \{s\}^\perp$. As the separation is proper we have $\mathcal{K} \not\subset \{s\}^\perp$. Therefore, $s \in \{x\}^\perp \cap (\mathcal{K}^* \setminus \mathcal{K}^\perp)$.

Reciprocally, by definition, the existence of $s \in \{x\}^\perp \cap (\mathcal{K}^* \setminus \mathcal{K}^\perp) \neq \emptyset$ ensures that $\{s\}^\perp$ properly separates x and $\mathcal{K}$. □

The following lemma is well-known and can be derived from various propositions that appear in [8], such as Proposition III.2.2. It also follows from Equation (10) in [20], but it appears there without proof. For the sake of self-containment, we include a short proof below.

Proposition 15. *Let $\mathcal{K}$ be a symmetric cone of rank r. The following are equivalent.*

- (i) $x \in \operatorname{ri} \mathcal{K}$
- (ii) $x \in \mathcal{K}$ *and* $\operatorname{rank} x = r$.
- (iii) *all the eigenvalues of x are positive.*

Proof. $(i) \Rightarrow (ii)$ Since $x \in \operatorname{ri} \mathcal{K}$, x clearly belongs to $\mathcal{K}$. Write the Jordan decomposition for x.

$$x = \sum_{i=1}^{r} \lambda_i c_i.$$

Note that if $\lambda_i = 0$, then $\langle x, c_i \rangle = 0$. As $\mathcal{K}$ is self-dual and c_i is idempotent, we have that $c_i \in \{x\}^\perp \cap (\mathcal{K}^* \setminus \mathcal{K}^\perp)$, which according to Lemma 14, implies that $x \notin \operatorname{ri} \mathcal{K}$.

$(ii) \Rightarrow (iii)$ If $x \in \mathcal{K}$, the eigenvalues of x must be nonnegative, due to Lemma 12. Since we are assuming that the rank of x is r, they must be positive.

$(iii) \Rightarrow (i)$ Take a Jordan decomposition for x

$$x = \sum_{i=1}^{r} \lambda_i c_i.$$

Then, clearly, $x \in \mathcal{K}$. Suppose that $x \notin \operatorname{ri} \mathcal{K}$. Then, Lemma 14 tells us the existence of an $s \in \mathcal{K}$ such that $s \notin \mathcal{K}^\perp$ and $\langle s, x \rangle = 0$. Therefore, $\langle s, c_i \rangle = 0$, for every i. Which implies that

$$\langle s, c_1 + \ldots + c_r \rangle = \langle s, e \rangle = 0.$$

According to Lemma 13, we have $s \circ e = 0$, which implies $s = 0$. This is a contradiction. □

5.2 The Carathéodory number of a symmetric cone

In order to compute $\kappa(\mathcal{K})$, we need to know the facial structure of $\mathcal{K}$. The next result can be derived from Theorem 2 in [9], due to Faybusovich. We give a different presentation here. In what follows, recall that $\mathcal{F}(x, \mathcal{K})$ indicates the minimal face of $\mathcal{K}$ which contains x.

Proposition 16. *Let $\mathcal{K}$ be a symmetric cone of rank r and $x \in \mathcal{K}$. Furthermore, let $c_1, \ldots, c_r$ be a Jordan frame for x, ordered in such a way that*

$$x = \sum_{i=1}^{\operatorname{rank} x} \lambda_i c_i$$

and $\lambda_1, \ldots, \lambda_{\operatorname{rank} x}$ are positive. Then

(i) $\mathcal{F}(x, \mathcal{K}) = \mathcal{K} \cap \{c_{\operatorname{rank} x+1} + \ldots + c_r\}^\perp$ *and* $\mathcal{F}(x, \mathcal{K})$ *is the cone of squares of* $V(c_1 + \ldots + c_{\operatorname{rank} x}, 1)$,

(ii) $\operatorname{rank} \mathcal{F}(x, \mathcal{K}) = \operatorname{rank} x$.

In addition, $\mathcal{F}(x, \mathcal{K})$ is properly contained in $\mathcal{K}$ if and only if $\operatorname{rank} x < r$.

Proof. Let $s = \operatorname{rank} x$ and

$$c = c_1 + \ldots + c_s$$
$$w = c_{s+1} + \ldots + c_r.$$

According to Theorem 11, $V(c, 1)$ is an Euclidean Jordan Algebra. Let $\tilde{\mathcal{F}}$ denote the cone of squares of $V(c, 1)$. Note that since $\mathcal{K}$ is self-dual, $\{w\}^\perp$ is a supporting hyperplane of $\mathcal{K}$. Therefore, $\mathcal{K} \cap \{w\}^\perp$ is a face of $\mathcal{K}$. Our first step is to show that $\tilde{\mathcal{F}} = \mathcal{K} \cap \{w\}^\perp$.

$\boxed{\tilde{\mathcal{F}} \subseteq \mathcal{K} \cap \{w\}^\perp}$ Let $y \in \tilde{\mathcal{F}}$ and pick a Jordan Frame for y by seeing it as an element of $V(c, 1)$. Then,

$$y = \lambda_1 d_1 + \ldots + \lambda_s d_s,$$

where $d_1 + \ldots + d_s = c$, since c is the identity in $V(c, 1)$. Moreover, due to Lemma 12, the λ_i are all nonnegative. Since $c \circ w = 0$, we have $\langle c, w \rangle = 0$, by Lemma 13. As each d_i belongs to $\mathcal{K}$, we also have $\langle d_i, w \rangle = 0$, which implies that $y \in w^\perp$.

$\boxed{\tilde{\mathcal{F}} \supseteq \mathcal{K} \cap \{w\}^\perp}$ Let $y \in \mathcal{K} \cap \{w\}^\perp$, and following Theorem 11, decompose y as

$$y = y_1 + y_2 + y_3,$$

with $y_1 \in V(c, 1), y_2 \in V\left(c, \frac{1}{2}\right), y_3 \in V(c, 0)$. Because $y \in \{w\}^\perp$, we have $y \circ w = 0$, by Lemma 13. Therefore,

$$\begin{aligned} y \circ w &= (y_1 + y_2 + y_3) \circ (e - c) \\ &= y_1 + y_2 + y_3 - y_1 - \frac{1}{2} y_2 \\ &= \frac{y_2}{2} + y_3 \\ &= 0 \end{aligned}$$

Since y_2 and y_3 are orthogonal, we conclude that $y_2 = y_3 = 0$. So that $y = y_1$. Because $y \in \mathcal{K}$, all its eigenvalues are nonnegative, due to Lemma 12. We can also compute the eigenvalues of y, by seeing it as an element of $V(c, 1)$. Note that a Jordan Frame for y in $V(c, 1)$ can be extended to a Jordan frame for y in $\mathcal{E}$ by adding the remaining $c_{s+1}, \ldots, c_r$. Due to uniqueness, it follows that the eigenvalues of y in $V(c, 1)$ are also nonnegative. Therefore, $y \in \tilde{\mathcal{F}}$.

We then conclude that $\tilde{\mathcal{F}} = \mathcal{K} \cap \{w\}^\perp$ and, therefore, $\tilde{\mathcal{F}}$ is a face of $\mathcal{K}$. Theorem 10 guarantees that no element in $\mathcal{K} \cap \{w\}^\perp$ has rank bigger than s. As $c \in \mathcal{K} \cap \{w\}^\perp$, the rank of $\mathcal{K} \cap \{w\}^\perp$ is indeed s. This proves item (ii). Moreover, due to item (iii) of Proposition 15, $x \in \operatorname{ri}(\mathcal{K} \cap \{w\}^\perp)$. Therefore, $\mathcal{F}(x, \mathcal{K}) = \mathcal{K} \cap \{w\}^\perp$.

Finally, note that if $s = r$, then $V(c, 1) = V(e, 1) = \mathcal{E}$, so that $\mathcal{F}(x, \mathcal{K}) = \mathcal{K}$. Therefore, if $\mathcal{F}(x, \mathcal{K})$ is a proper face, then $s < r$. Conversely, if $s < r$, it is clear that $\mathcal{F}(x, \mathcal{K})$ must be proper, since it does not contain e. □

Note that if $\mathcal{F}$ is an arbitrary face of $\mathcal{K}$, then $\mathcal{F}(x, \mathcal{K}) = \mathcal{F}$, for all $x \in \operatorname{ri} \mathcal{F}$. So Proposition 16 applies to all faces of $\mathcal{K}$.

Before we proceed we need the following observation, which is a corollary to the Jordan decomposition.

Corollary 17. *Let c be a primitive idempotent, then*

$$V(c, 1) = \{\beta c \mid \beta \in \mathbb{R}\}.$$

Proof. $V(c, 1)$ is an Euclidean Jordan Algebra and, in fact, c is the identity element in $V(c, 1)$. Let $x \in V(c, 1)$ and consider a Jordan frame $d_1, \ldots, d_r$ for x. Because

$$d_1 + \ldots + d_r = c,$$

it must be the case that $r = 1$, since c is primitive. Therefore, $x = \beta c$. □

The next result was proved for simple Jordan Algebras in [8]. Here, we give a more general statement.

Corollary 18. *Let $\mathcal{K}$ be a symmetric cone and $x \in \mathcal{K}$ with $x \neq 0$. The following are equivalent.*

(i) *x belongs to an extreme ray.*

(ii) *x has rank 1, i.e., $x = \alpha c$ with $\alpha > 0$ and c primitive idempotent.*

Proof. $(i) \Rightarrow (ii)$ Consider a Jordan Frame for x and write

$$x = \sum_{i=1}^{r} \lambda_i c_i.$$

Due to Lemma 12, we have $\lambda_i \geq 0$ for all i.

Let $\mathcal{F}$ be the extreme ray of $\mathcal{K}$ that contains x. Because $\mathcal{F}$ is a face, if $\lambda_i > 0$, then $c_i \in \mathcal{F}$. Since $\mathcal{F}$ has dimension one and the c_i are orthogonal, exactly one of the λ_i is positive while all the others are zero.

$(ii) \Rightarrow (i)$ Let $\mathcal{F} = \mathcal{F}(x, \mathcal{K})$. Due to Proposition 16, $\mathcal{F}$ has rank one and is the cone of squares of $V(c, 1)$, where c is a primitive idempotent. Due to Corollary 17, both $V(c, 1)$ and $\mathcal{F}$ are one-dimensional. □

The Jordan decomposition together with Corollary 18 shows that given $x \in \mathcal{K}$ we can write it as a sum of at most $\operatorname{rank} x$ elements that live in extreme rays. That is,

$$\kappa(x) \leq \operatorname{rank} x.$$

The caveat is that the decomposition given by the Spectral Theorem requires that the elements be orthogonal to each other, while in the definition of κ there is no such requirement.

The next result shows that, in fact, $\kappa(x) = \operatorname{rank} x$. This has been proven before by Güller and Tunçel [11], but the exposition given here is, perhaps, more elementary and does not rely on the classification of Euclidean Jordan Algebras neither on the theory of homogeneous cones as in [1].

Theorem 19. *Let $\mathcal{K}$ be a symmetric cone and $x \in \mathcal{K}$. We have*

$$\kappa(x) = \operatorname{rank} x.$$

Proof. The first observation is that we may assume that $x \in \operatorname{ri} \mathcal{K}$. If not, we pass to the minimal face $\mathcal{F}$ of $\mathcal{K}$ containing x. Then, $x \in \operatorname{ri} \mathcal{F}$ and $\mathcal{F}$ is a symmetric cone inside some Euclidean Jordan algebra, due to Proposition 16.

Next, since $\mathcal{K}$ is homogeneous, there is a bijective linear transformation T that maps x to the identity e and satisfies $T(\mathcal{K}) = \mathcal{K}$. Since T maps extreme rays to extreme rays, we have that $\kappa(x) = \kappa(e)$.

We will now show that $\kappa(e) = \operatorname{rank}(e)$. Suppose that

$$e = z_1 + \ldots + z_{\kappa(e)},$$

where each z_i is nonzero and belongs to an extreme ray. Due to Corollary 18, we may assume that

$$e = \alpha_1 d_1 + \ldots \alpha_{\kappa(e)} d_{\kappa(e)}, \tag{9}$$

where the α_i are positive and the d_i are primitive idempotents.

Recall that if we have any Jordan frame, since the sum of idempotents is equal to e, the eigenvalues of e are all equal to one. Applying the trace map at both sides of Equation (9), we conclude that

$$\operatorname{rank}(e) = \alpha_1 + \ldots + \alpha_{\kappa(e)}. \tag{10}$$

We now examine the following expression.

$$(1 - \alpha_i) d_i = (e - \alpha_i d_i) \circ d_i.$$

We take the inner product with d_i:

$$\begin{aligned}
(1 - \alpha_i)\langle d_1, d_1 \rangle &= \langle (e - \alpha_i d_i) \circ d_i, d_i \rangle \\
&= \langle e - \alpha_i d_i, d_i \circ d_i \rangle \\
&= \langle e - \alpha_i d_i, d_i \rangle \\
&\geq 0.
\end{aligned}$$

The second equality follows from the fact that the algebra is Euclidean. The last inequality stems from Equation (9), which implies that $e - \alpha_i d_i \in \mathcal{K}$. Since $\langle d_i, d_i \rangle > 0$, we must have $1 \geq \alpha_i$, for every i. In view of Equation (10), we obtain $\operatorname{rank}(e) \leq \kappa(e)$.

Since we already know that $\kappa(e) \leq \operatorname{rank}(e)$, we have $\operatorname{rank}(e) = \kappa(e)$.

□

5.3 The longest chain of faces of a symmetric cone

Theorem 20. *Let $\mathcal{K}$ be a symmetric cone. We have*

$$\ell_{\mathcal{K}} = \operatorname{rank} \mathcal{K} + 1.$$

Proof. First, we construct a chain of faces that has length $\operatorname{rank} \mathcal{K} + 1$. Let e be the identity element and $c_1, \ldots, c_r$ a Jordan frame for e, with $r = \operatorname{rank} \mathcal{K}$. Then, from Proposition 16, we have

$$\mathcal{K} \supsetneq \mathcal{F}(c_1 + \ldots + c_{r-1}, \mathcal{K}) \supsetneq \ldots \supsetneq \mathcal{F}(c_1, \mathcal{K}) \supsetneq \{0\}.$$

Note that the inclusions are indeed strict, since $c_i \in \mathcal{F}(c_1 + \ldots + c_i, \mathcal{K})$ but $c_i \notin \mathcal{F}(c_1 + \ldots + c_{i-1}, \mathcal{K})$. This shows that there is at least one chain of length $\operatorname{rank} \mathcal{K} + 1$.

Now suppose that we have an arbitrary chain of faces

$$\mathcal{F}_1 \supsetneq \ldots \supsetneq \mathcal{F}_\ell.$$

We can select ℓ points such that $x_i \in \operatorname{ri} \mathcal{F}_i$ for all i. With that choice, we have $\mathcal{F}_i = \mathcal{F}(x_i, \mathcal{K})$. Due to Proposition 16, the only way that those inclusions can be strict is if $\operatorname{rank} x_i > \operatorname{rank} x_{i+1}$ for all i. Since $r \geq \operatorname{rank} x_1$, we conclude that ℓ can be at most $r+1$.

□

The upshot of this section is that for symmetric cones we have

$$\kappa(\mathcal{K}) = \operatorname{rank} \mathcal{K} = \ell_{\mathcal{K}} - 1,$$

so the bound in Theorem 4 is tight.

6 Comments on three other classes of cones

To conclude this work, we will make a few comments about some cones of matrices. Denote by $\mathcal{CP}_n$ the cone of $n \times n$ *completely positive matrices*. Recall that a symmetric matrix X is said to be *completely positive* if there is an $n \times r$ matrix V such that $X = VV^\top$ and all the entries of V are nonnegative. The smallest r for which this decomposition is possible is called the *cp-rank* of X.

Due to a result by Berman (see [5] and also Theorem 4.2 in [7]), Y belongs to an extreme ray of $\mathcal{CP}_n$ if and only if $Y = xx^\top$ for some nonzero x such that all its entries are nonnegative. This means that the *cp-rank* of X coincides with $\kappa(X)$ computed with respect to $\mathcal{CP}_n$.

Translating to our terminology, one of the open problems described in [4] is to find a nontrivial upper bound to $\kappa(\mathcal{CP}_n)$. It is known that $\kappa(\mathcal{CP}_n) = n$, for $n \leq 4$ and that $\kappa(\mathcal{CP}_5) = 6$. For $n \geq 6$, the current best result is that

$$\kappa(\mathcal{CP}_n) \leq \frac{n(n+1)}{2} - 4,$$

see Section 4.2 in [4] for more information on those results. We cannot help but speculate whether computing $\ell_{\mathcal{CP}_n}$ could help lower this bound. It seems that this might be an unexplored route. In low dimension the bound might fail to be tight, but it is said that the geometry of $\mathcal{CP}_n$ changes heavily when n increases.

Now, let $\mathcal{D}^n$ denote the cone of *symmetric doubly nonnegative matrices*. A symmetric matrix X belongs to $\mathcal{D}^n$ if it is positive semidefinite and all its entries are nonnegative. The importance of $\mathcal{D}^n$ is that it can be used to relax problems over $\mathcal{CP}_n$ and, in fact, for $n \leq 4$, we have $\mathcal{D}^n = \mathcal{CP}_n$. Unfortunately, for $\mathcal{D}^n$, Theorem 4 does not shed much light on $\kappa(\mathcal{D}^n)$, since it was shown in Proposition 26 of [14] that $\ell_{\mathcal{D}^n} = \frac{n(n+1)}{2} + 1$. In low dimension we know that the bound is not tight, since we have $\kappa(\mathcal{D}^n) = \kappa(\mathcal{CP}_n) = n$ for $n \leq 4$. However, $\kappa(\mathcal{D}^n)$ seems to be unknown for large n. We have, nevertheless, the following easy lower bound.

Proposition 21. *For the cone of $n \times n$ doubly nonnegative matrices we have*

$$\kappa(\mathcal{D}^n) \geq n.$$

Proof. First, note that if $\mathcal{F}$ is a face of some cone $\mathcal{K}$, then $\kappa(\mathcal{K}) \geq \kappa(\mathcal{F})$. We will proceed by showing the existence of a face of $\mathcal{D}^n$ whose Carathéodory number is equal to n.

Note that $\mathcal{D}^n = \mathcal{S}^n_+ \cap \mathcal{N}^n$, where $\mathcal{N}^n$ is the cone of symmetric matrices with nonnegative entries. Let T_n denote the cone of diagonal matrices with nonnegative entries. Note that T_n is a face of $\mathcal{N}^n$ and satisfies $T_n = \mathcal{S}^n_+ \cap T_n$. As T_n is the intersection of a face of $\mathcal{S}^n_+$ with a face of $\mathcal{N}^n$, we conclude that T_n is a face of $\mathcal{D}^n$. As T_n is polyhedral and $\dim T_n = n$, we obtain $\kappa(T_n) = n$, by item *i.* of Theorem 7. It follows that $\kappa(\mathcal{D}^n) \geq n$. □

We also do not have much of an idea of what happens with $\ell_{\mathcal{K}}$ and $\kappa(\mathcal{K})$ when $\mathcal{K}$ is the cone of copositive matrices $\mathcal{COP}_n$. Recall that a symmetric matrix X is said to be copositive if $v^\top X v \geq 0$ for all v with nonnegative entries.

Table 1 summarizes what is known about the Carathéodory number and the size of the longest chain of faces for a few families of cones.

$\mathcal{K}$	$\kappa(\mathcal{K})$	$\ell_{\mathcal{K}}$
polyhedral cone of dimension k	k	$k+1$
symmetric cone of rank r	r	$r+1$
smooth cone	2	3
$n \times n$ doubly nonnegative cone	$\geq n$	$\frac{n(n+1)}{2}+1$
$n \times n$ completely positive cone	nontrivial bounds are known [4]	?
$n \times n$ copositive cone	?	?

Table 1: Values of $\ell_{\mathcal{K}}$ and $\kappa(\mathcal{K})$

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A Proof of Proposition 5

Proposition. *Let $C \subseteq \mathbb{R}^n$ be a nonempty compact convex set. Let*

$$\mathcal{K} = \{(\alpha, \alpha x) \mid \alpha \geq 0, x \in C\}.$$

Then

(*i*) *$\mathcal{K}$ is a pointed closed convex cone.*

(*ii*) *Let $\mathcal{F}$ be a face of $\mathcal{K}$ that is not $\{0\}$, then*

$$\mathcal{F}_C = \{x \in C \mid (1, x) \in \mathcal{F}\}$$

is a face of C. Moreover, $\dim \mathcal{F}_C = \dim \mathcal{F} - 1$.

(*iii*) *Let $\mathcal{F}_C$ be a face of C, then*

$$\mathcal{F} = \{(\alpha, \alpha x) \mid \alpha \geq 0, x \in \mathcal{F}_C\}$$

is a face of $\mathcal{K}$. Moreover, $\dim \mathcal{F} = \dim \mathcal{F}_C + 1$.

Proof. (*i*) Here we only show the closedness of $\mathcal{K}$. Let

$$\{(\alpha_k, \alpha_k x_k)\}_{k=1}^{+\infty} \subset \mathcal{K}$$

be a sequence converging to (α^*, z), with $x_k \in C$ for all k. When $\alpha^* = 0$, the compactness of C leads to $z = \lim_{k\to+\infty} \alpha_k x_k = 0$ concluding that $(\alpha^*, z) \in \mathcal{K}$. In the case $\alpha^* > 0$, we have $\alpha_k > 0$ for all sufficiently large k. Then, we see that $z/\alpha^* \in C$ since $\{(1, x_k)\} = \{\frac{1}{\alpha_k}(\alpha_k, \alpha_k x_k)\}$ converges to $\frac{1}{\alpha^*}(\alpha^*, z) = (1, z/\alpha^*)$ and C is closed. Therefore, $(\alpha^*, z) \in \mathcal{K}$.

(*ii*) First of all, $\mathcal{F}_C$ is a convex set, since it is a projection on $\mathbb{R}^n$ of the intersection between $\mathcal{K}$ and the hyperplane

$$\{(\alpha, x) \mid \alpha = 1\}.$$

Now, let $x, y \in C$ be such that

$$\gamma x + (1 - \gamma)y \in \mathcal{F}_C,$$

for some $0 < \gamma < 1$. Therefore,

$$\gamma(1, x) + (1 - \gamma)(1, y) \in \mathcal{F}.$$

As $\mathcal{F}$ is a face of $\mathcal{K}$, we conclude that $(1, x), (1, y) \in \mathcal{F}$ and that $x, y \in \mathcal{F}_C$. Hence, $\mathcal{F}_C$ is a face of C. Let $s = \dim \mathcal{F}_C$ and take an affinely independent subset $\{x_0, \ldots, x_s\}$ of $\mathcal{F}_C$. Then, the implications

$$\sum_{i=0}^{s} \gamma_i(1, x_i) = 0 \;\Rightarrow\; \sum_{i=0}^{s} \gamma_i = 0$$

$$\sum_{i=0}^{s} \gamma_i x_i = 0 \;\Rightarrow\; \gamma_0 = \cdots = \gamma_s = 0$$

show that $\{(1, x_i)\}_{i=0}^{s} \subset \mathcal{F}$ are *linearly* independent. This means that $\dim \mathcal{F}_C + 1 = s + 1 \leq \dim \mathcal{F}$. Conversely, let $t = \dim \mathcal{F}$ and $\{(\alpha_i, \alpha_i x_i)\}_{i=1}^{t} \subset \mathcal{F}$ be linearly independent. Then we have $\alpha_i \neq 0$ so that $\{x_i\}_{i=1}^{t} \subset \mathcal{F}_C$ follows and its affine independence can be shown in a similar manner. This yields that $\dim \mathcal{F} - 1 = t - 1 \leq \dim \mathcal{F}_C$ and therefore $\dim \mathcal{F}_C = \dim \mathcal{F} - 1$ holds.

(*iii*) It is straightforward to check that $\mathcal{F}$ is a subset of $\mathcal{K}$ that is a convex cone. We will check that it is indeed a face. Suppose that $(\alpha_1, \alpha_1 x_1), (\alpha_2, \alpha_2 x_2) \in \mathcal{K}$ are such that

$$(\alpha_1 + \alpha_2, \alpha_1 x_1 + \alpha_2 x_2) \in \mathcal{F}.$$

Furthermore, suppose that both α_1 and α_2 are greater than zero. By definition, we have

$$\alpha_1 x_1 + \alpha_2 x_2 = (\alpha_1 + \alpha_2) z,$$

for some $z \in \mathcal{F}_C$. This means that

$$\frac{\alpha_1}{\alpha_1 + \alpha_2} x_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2} x_2 = z,$$

so that x_1 and x_2 belong to $\mathcal{F}_C$. Therefore, both $(\alpha_1, \alpha_1 x_1)$ and $(\alpha_2, \alpha_2 x_2)$ belong to $\mathcal{F}$. Thus, $\mathcal{F}$ is a face of $\mathcal{K}$.

Finally, notice that the face

$$\{x \in C \mid (1, x) \in \mathcal{F}\}$$

coincides with $\mathcal{F}_C$. Hence, from assertion (*ii*) we obtain $\dim \mathcal{F} = \dim \mathcal{F}_C + 1$.

□