Complete Description of Matching Polytopes with One Linearized Quadratic Term for Bipartite Graphs

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Abstract

We consider, for complete bipartite graphs, the convex hulls of characteristic vectors of matchings, extended by a binary number indicating whether the matching contains two specific edges. This polytope is associated to the quadratic matching problem with a single linearized quadratic term. We provide a complete irredundant inequality description, which settles a conjecture by Klein (Ph.D. thesis, TU Dortmund, 2015).

1 Introduction

Let $K_{m,n} = (A \cup B, E)$ be the complete bipartite graph with the node partition $A \cup B$, $|A| = m$ and $|B| = n$ for $m, n \geq 2$. The *maximum weight matching problem* is to maximize the sum $w(M) := \sum_{e \in M} w_e$ over all matchings M in $K_{m,n}$ for given edge weights $w \in \mathbb{Q}^E$. Note that we generally abbreviate $\sum_{j \in J} v_j$ as $v(J)$ for vectors v and subsets J of their index sets.
Following the usual approach in polyhedral combinatorics, we identify the matchings M with their *characteristic vectors* $\chi(M) \in \{0,1\}^E$, which satisfy $\chi(M)_e = 1$ if and only if $e \in M$ holds. The maximum weight matching problem is then equivalent to the problem of maximizing the linear objective w over the *matching polytope*, i.e., the convex hull of all characteristic vectors of matchings. In order to use linear programming techniques, one requires a description of that so-called polytope in terms of linear inequalities. Such a description is well-known and consists of the following constraints [1]:

$$x_e \geq 0 \qquad \text{for all } e \in E \tag{1}$$

$$x(\delta(v)) \leq 1 \qquad \text{for all } v \in A \cup B \tag{2}$$

For general (nonbipartite) graphs, Edmonds [4, 5] proved that adding the following *Blossom Inequalities* is sufficient to describe the matching polytope:

$$x(E[S]) \leq \frac{1}{2}(|S| - 1) \qquad \text{for all } S \subseteq V,\ |S| \text{ odd}$$

His result was based on a primal-dual approximation algorithm, which also proved that the weighted matching problem can be solved in polynomial time. Later, Schrijver [12] gave a direct (and more geometric) proof of the polyhedral result. Note that one also often considers the special case of *perfect* matchings, which are those matchings that match every node of the graph. The associated *perfect matching polytope* is the face of the matching polytope obtained by requiring that all Inequalities (2) are satisfied with equality:

$$x(\delta(v)) = 1 \qquad \text{for all } v \in A \cup B. \tag{3}$$

For more background on matchings and the matching polytopes we refer to Schrijver's book [12]. For a basic introduction into polytopes and linear programmg we recommend to read [13].
In this paper, we consider the more general *quadratic matching problem* for which we have, in addition to w, a set $\mathcal{Q} \subseteq \binom{E}{2}$ and weights $p : \mathcal{Q} \to \mathbb{Q}$ for the edge-pairs in $\mathcal{Q}$. The objective is now to maximize $w(M) + \sum_{q \in \mathcal{Q}, q \subseteq M} p_q$, again over all matchings M. A special case of the problem (by

requiring the matchings to be perfect) is the *quadratic assignment problem*, a problem that is not just NP-hard [11], but also hard to solve in practice (see [10] for a survey).

A common strategy is then to linearize this quadratic objective function by introducing additional variables $y_{e,f} = x_e \cdot x_f$ for all $\{e,f\} \in \mathcal{Q}$. Usually, the straight-forward linearization of this product equation is very weak, and one seeks to find (strong) inequalities that are valid for the associated polytope.
One way of finding such inequalities, recently suggested by Buchheim and Klein [2], is the so-called *one term linearization technique*. The idea is to consider the special case of $|\mathcal{Q}|$ in which the optimization problem is still polynomially solvable. By the "equivalence of separation and optimization" [8], one can thus hope to characterize all (irredundant) valid inequalities and develop separation algorithms. These inequalities remain valid when more than one monomial are present, and hence one can use the results of this special case in the more general setting. Buchheim and Klein suggested this for the quadratic spanning-tree problem and conjectured a complete description of the associated polytope. This conjecture was later confirmed by Fischer and Fischer [6] and Buchheim and Klein [3]. Fischer et al. [7] recently generalized this result to matroids and multiple monomials, which must be nested in a certain way. In her dissertation [9], Klein considered several other combinatorial polytopes, in particular the quadratic assignment polytope that corresponds to the quadratic perfect matching problem in bipartite graphs. We extend this to the non-perfect matchings and our setup is as follows:

Consider two disjoint edges $e_1 = \{a_1, b_1\}$ and $e_2 = \{a_2, b_2\}$ (with $a_i \in A$ and $b_i \in B$ for $i = 1,2$) in $K_{m,n}$ and denote by $V^\star := \{a_1, a_2, b_1, b_2\}$ the union of their node sets. Our polytopes of interest are the convex hulls of all vectors $(\chi(M), y)$ for which M is a matching in G, $y \in \{0,1\}$ and one of the relationships between M and y holds:

- $P^{1Q\downarrow}_{\text{match}} := P^{1Q\downarrow}_{\text{match}}(K_{m,n}, e_1, e_2)$: $y = 1$ implies $e_1, e_2 \in M$.
- $P^{1Q\uparrow}_{\text{match}} := P^{1Q\uparrow}_{\text{match}}(K_{m,n}, e_1, e_2)$: $y = 0$ implies $e_1 \notin M$ or $e_2 \notin M$.
- $P^{1Q}_{\text{match}} := P^{1Q}_{\text{match}}(K_{m,n}, e_1, e_2)$: $y = 1$ if and only if $e_1, e_2 \in M$.

Note that $P^{1Q\downarrow}_{\text{match}}$ (resp. $P^{1Q\uparrow}_{\text{match}}$) is the *downward* (resp. *upward*) monotonization of P^{1Q}_{match} with respect to the y-variable, and that

$$P^{1Q}_{\text{match}} = \operatorname{conv}(P^{1Q\downarrow}_{\text{match}} \cap P^{1Q\uparrow}_{\text{match}} \cap (\mathbb{Z}^E \times \mathbb{Z}))$$

holds. Clearly, Constraints (1) and (2) as well as the bound constraints

$$0 \le y \le 1 \tag{4}$$

are valid for all three polytopes. Additionally, the two inequalities

$$y \le x_{e_i} \qquad i = 1,2 \tag{5}$$

are also valid for P^{1Q}_{match} and $P^{1Q\downarrow}_{\text{match}}$ (and belong to the standard linearization of $y = x_{e_1} \cdot x_{e_2}$). Klein [9] introduced two more inequality classes, and proved them to be facet-defining (see Theorems 6.2.2 and 6.2.3 in [9]). They read

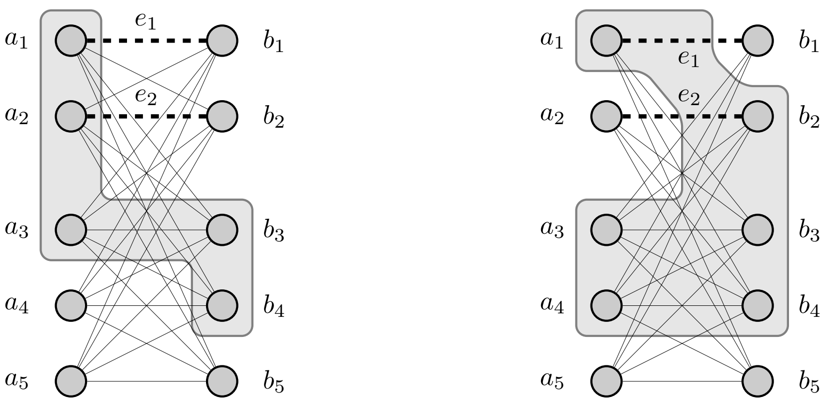
$$x(E[S]) + y \le \frac{1}{2}(|S| - 1) \qquad \text{for all } S \in \mathcal{S}^\downarrow \text{ and} \tag{6}$$

$$x(E[S]) + x_{e_1} + x_{e_2} - y \le \frac{1}{2}|S| \qquad \text{for all } S \in \mathcal{S}^\uparrow, \tag{7}$$

where the index sets (see Figure 1) are defined as follows:

$$\begin{aligned}\mathcal{S}^\downarrow := \{S \subseteq A \cup B \mid\ & |S| \text{ odd and either} \\ & S \cap V^\star = \{a_1, a_2\} \text{ and } |S \cap A| = |S \cap B| + 1 \text{ or} \\ & S \cap V^\star = \{b_1, b_2\} \text{ and } |S \cap B| = |S \cap A| + 1 \text{ holds}\}.\end{aligned}$$

$$\mathcal{S}^\uparrow := \{S \subseteq A \cup B \mid |S \cap A| = |S \cap B| \text{ and either } S \cap V^\star = \{a_1, b_2\} \text{ or } S \cap V^\star = \{a_2, b_1\}\}.$$



(a) A set $S \in \mathcal{S}^{\downarrow}$ indexing Inequality (6).

(b) A set $S \in \mathcal{S}^{\uparrow}$ indexing Inequality (7).

Figure 1: Node Sets Indexing Additional Facets.

Klein [9] even conjectured, that Constraints (1)–(7), where Inequalities (2) are replaced by equations, completely describe the mentioned face of $P^{\mathrm{1Q}}_{\mathrm{match}}$. We will confirm this conjecture in Corollary 8.
In contrast to the two proofs for the one-quadratic-term spanning-tree polytopes [6, 3], our proof technique is not based on linear programming duality. In fact, the two additional inequality families presented above introduce two sets of dual multipliers, which seem to make this proof strategy hard, or at least quite technical. Instead, we were heavily inspired by Schrijver's direct proof [12] for the matching polytope.

The paper is structured as follows: In Section 2 we present our main results together with their proofs, which are based on two key lemmas, one for $P^{\mathrm{1Q}\downarrow}_{\mathrm{match}}$ and one for $P^{\mathrm{1Q}\uparrow}_{\mathrm{match}}$. Then, in Section 3 we focus on perfect matchings and establish the corresponding results for this special case. The proofs for the two key lemmas are similiar with respect to the general strategy, but are still quite different due to the specific constructions they depend on. Hence, each lemma has its own dedicated section. Although Klein already proved that the new inequalities are facet-defining, she only did so for the case of perfect matchings. Hence, for the sake of completeness, we do the same for the general case in Section 6. We conclude this paper with a small discussion on our proof strategy and on a property of $P^{\mathrm{1Q}}_{\mathrm{match}}$.

2 Main Results

We will prove our result using two key lemmas, each of which is proved within its own section.

Lemma 1. *Let* $(\hat{x}, \hat{y}) \in \mathbb{Q}^E \times \mathbb{Q}$ *satisfy Constraints* (1), (2), (4), (5) *and* (6). *Let furthermore* $(\hat{x}, \hat{y})$ *satisfy at least one of the Inequalities* (5) *for* $i^* \in \{1, 2\}$ *or* (6) *for a set* $S^* \in \mathcal{S}^{\downarrow}$ *with equality. Then* $(\hat{x}, \hat{y})$ *is a convex combination of vertices of* P^{1Q}_{match}.

Lemma 1 will be proved in Section 4.

Lemma 2. *Let* $(\hat{x}, \hat{y}) \in \mathbb{Q}^E \times \mathbb{Q}$ *satisfy Constraints* (1), (2), (4), *and Inequalities* (7) *for all* $S \in \mathcal{S}^{\uparrow}$. *Let furthermore* $(\hat{x}, \hat{y})$ *satisfy at least one of the Inequalities*(7) *for a set* $S^* \in \mathcal{S}^{\uparrow}$ *with equality. Then* $(\hat{x}, \hat{y})$ *is a convex combination of vertices of* P^{1Q}_{match}.

Lemma 2 will be proved in Section 5. We continue with the consequences of the two lemmas.

Theorem 3. $P^{1Q\downarrow}_{match}$ *is equal to the set of* $(x, y) \in \mathbb{R}^E \times \mathbb{R}$ *that satisfy Constraints* (1), (2), (4), (5) *and* (6).

Proof. Let P be the polytope defined by Contraints (1)–(6). We first show $P^{\mathrm{1Q}\downarrow}_{\mathrm{match}} \subseteq P$ by showing $(\chi(M), y) \in P$ for all feasible pairs $(\chi(M), y)$, i.e., matchings M in G and suitable y. Clearly, $\chi(M)$ satisfies Constraints (1) and (2), and y satisfies (4).

Let $S \in \mathcal{S}^{\downarrow}$, define $\bar{S} := S \setminus \{a_1, a_2, b_1, b_2\}$, and observe that $|\bar{S}|$ is odd. If $y = 1$, then $e_1, e_2 \in M$ holds, i.e., the Constraint (5) is satisfied. Hence, only nodes in $\bar{S}$ can be matched to other nodes in S, and there are at most $\lfloor |\bar{S}|/2 \rfloor = (|S| - 3)/2$ of them. If $y = 0$ holds, then the validity follows from the fact that S has odd cardinality. This shows that Constraint (6) is always satisfied.

To show $P \subseteq P^{1Q\downarrow}_{\text{match}}$, we consider a vertex $(\hat{x}, \hat{y})$ of P. Note that since P is rational we have $(\hat{x}, \hat{y}) \in \mathbb{Q}^E \times \mathbb{Q}$. If it satisfies at least one of the Inequalities (5) for some $i^* \in \{1, 2\}$ or (6) for some $S^* \in \mathcal{S}^{\downarrow}$, Lemma 1 yields that $(\hat{x}, \hat{y})$ is a convex combination of vertices of P^{1Q}_{match}, which are vertices of $P^{1Q\downarrow}_{\text{match}}$.
Hence, $(\hat{x}, \hat{y})$ is even a vertex of the polytope defined only by the Constraints (1), (2) and (4). Thus, $\hat{y} \in \{0, 1\}$ and $\hat{x} = \chi(M)$ holds for some matching M in $K_{m,n}$. Since Inequalities (5) are *strictly* satisfied, we must have $\hat{y} = 0$, which concludes the proof. □

Theorem 4. *$P^{1Q\uparrow}_{match}$ is equal to the set of $(x, y) \in \mathbb{R}^E \times \mathbb{R}$ that satisfy Constraints* (1), (2), (4) *and* (7).

Proof. Let P be the polytope defined by Contraints (1)–(4) and (7). We first show $P^{1Q\uparrow}_{\text{match}} \subseteq P$ by showing $(\chi(M), y) \in P$ for all feasible pairs $(\chi(M), y)$, i.e., matchings M in G and suitable y. Clearly, $\chi(M)$ satisfies Constraints (1) and (2), and y satisfies (4).
For $S \in \mathcal{S}^{\uparrow}$, M contains at most $\frac{1}{2}(|S \cup e_1 \cup e_2|) = \frac{1}{2}|S| + 1$ edges in $E[S] \cup e_1 \cup e_2$. Thus, if $y = 1$ holds, Constraint (7) is satisfied. If $y = 0$ holds and $e_1, e_2 \notin M$, then it is trivially satisfied. Otherwise, i.e., if $y = 0$ holds and M contains exactly one of the two edges, we can assume w.l.o.g. that $e_1 \in \delta(M)$ and $e_2 \notin \delta(M)$ hold. Since $S \setminus e_1$ has odd cardinality and at most $|S|/2 - 1$ edges of M can have both endnodes in S, the constraint is also satisfied in this case.

To show $P \subseteq P^{1Q\uparrow}_{\text{match}}$, we consider a vertex $(\hat{x}, \hat{y})$ of P. Note that since P is rational we have $(\hat{x}, \hat{y}) \in \mathbb{Q}^E \times \mathbb{Q}$. If it satisfies at least one of the Inequalities(7) for some $S^* \in \mathcal{S}^{\uparrow}$, Lemma 2 yields that $(\hat{x}, \hat{y})$ is a convex combination of vertices of P^{1Q}_{match}, which are vertices of $P^{1Q\uparrow}_{\text{match}}$.
Hence, $(\hat{x}, \hat{y})$ is even a vertex of the polytope defined only by the Constraints (1), (2) and (4). Thus, $\hat{y} \in \{0, 1\}$ and $\hat{x} = \chi(M)$ holds for some matching M in $K_{m,n}$. If $\hat{y} = 0$ holds, then Inequality (7) for $S = \{a_1, b_2\}$ reads $\hat{x}_{e_1} + \hat{x}_{a_1, b_2} + \hat{x}_{e_2} \leq 1$, and thus implies $e_1 \notin M$ or $e_2 \notin M$, which concludes the proof. □

Theorem 5. *P^{1Q}_{match} is equal to the set of $(x, y) \in \mathbb{R}^E \times \mathbb{R}$ that satisfy Constraints* (1), (2), (4), (5), (6) *and* (7), *i.e., $P^{1Q}_{match} = P^{1Q\downarrow}_{match} \cap P^{1Q\uparrow}_{match}$.*

Proof. Let P be the polytope defined by Contraints (1)–(7). By Theorems (3) and (4) we have $P^{1Q}_{\text{match}} \subseteq P^{1Q\downarrow}_{\text{match}} \cap P^{1Q\uparrow}_{\text{match}} = P$.

To show $P \subseteq P^{1Q}_{\text{match}}$, we consider a vertex $(\hat{x}, \hat{y})$ of P. Note that since P is rational we have $(\hat{x}, \hat{y}) \in \mathbb{Q}^E \times \mathbb{Q}$. If it satisfies at least one of the Inequalities (5) for some $i^* \in \{1, 2\}$ (6) for some $S^* \in \mathcal{S}^{\downarrow}$, Lemma 1 yields that $(\hat{x}, \hat{y})$ is a convex combination of vertices of P^{1Q}_{match}. If it satisfies at least one of the Inequalities(7) for some $S^* \in \mathcal{S}^{\uparrow}$, Lemma 2 yields that $(\hat{x}, \hat{y})$ is a convex combination of vertices of P^{1Q}_{match}.
Hence, $(\hat{x}, \hat{y})$ is even a vertex of the polytope defined only by the Constraints (1), (2) and (4). Thus, $\hat{y} \in \{0, 1\}$ and $\hat{x} = \chi(M)$ holds for some matching M in $K_{m,n}$. Inequalities (5) and Inequality (7) for $S = \{a_1, b_2\}$ imply that $y = 1$ holds if and only if $e_1, e_2 \in M$ holds, which concludes the proof. □

3 Perfect Matchings

Throughout this section we assume $m = n$, since otherwise, $K_{m,n}$ does not contain perfect matchings. Since the formulations for perfect matchings are obtained by replacing Inequalities (2) by Equations (3), the corresponding polytopes are faces of the ones defined in the Section 1, and we immediately obtain the following results from the corresponding theorems in Section 2:

Corollary 6. *The convex hull of all $(\chi(M), y) \in \{0, 1\}^E \times \{0, 1\}$, for which M is a perfect matching M in $K_{n,n}$ and $y = 1$ implies $e_1, e_2 \in M$, is equal to the set of $(x, y) \in \mathbb{R}^E \times \mathbb{R}$ that satisfy Constraints* (1), (4), (5), (6) *and* (3).

Corollary 7. *The convex hull of all $(\chi(M), y) \in \{0,1\}^E \times \{0,1\}$, for which M is a perfect matching M in $K_{n,n}$ and $y = 0$ implies $e_1 \in M$ or $e_2 \in M$, is equal to the set of $(x, y) \in \mathbb{R}^E \times \mathbb{R}$ that satisfy Constraints* (1), (4), (7) *and* (3).

Corollary 8. *The convex hull of all $(\chi(M), y) \in \{0,1\}^E \times \{0,1\}$, for which M is a perfect matching M in $K_{n,n}$ and $y = 1$ holds if and only if $e_1, e_2 \in M$, is equal to the set of $(x, y) \in \mathbb{R}^E \times \mathbb{R}$ that satisfy Constraints* (1), (4), (5), (6), (7) *and* (3).

4 Proofs for the Downward Monotonization

This section contains the proof of Lemma 1. We first introduce relevant objects which are fixed for the rest of this section, and then present the main proof. To improve readability, the proofs of several claims are deferred to the end of this section.
Let $(\hat{x}, \hat{y}) \in \mathbb{Q}^E \times \mathbb{Q}$ be as stated in the lemma. Let $\bar{G} = (A \cup B, \bar{E})$ be the graph $K_{m,n}$ with the additional edges $e_a := \{a_1, a_2\}$ and $e_b := \{b_1, b_2\}$, i.e., $\bar{E} := E \cup \{e_a, e_b\}$. Define the vector $\bar{x} \in \mathbb{Q}^{\bar{E}}$ as follows (see Figure 2):

- $\bar{x}_e := \hat{x}_e$ for all $e \in E \setminus \{e_1, e_2\}$.
- $\bar{x}_{e_i} := \hat{x}_{e_i} - \hat{y}$ for $i = 1, 2$.
- $\bar{x}_{e_a} := \bar{x}_{e_b} := \hat{y}$.

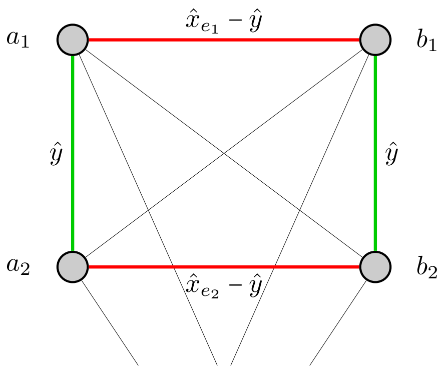


Figure 2: Graph $\bar{G}$ and vector $\bar{x}$ in the Proof of Lemma 1.

Claim 9. *$\bar{x}$ is in the matching polytope of $\bar{G}$.*

By Claim 9, and since $\bar{x}$ is rational, it can be written as a convex combination of characteristic vectors of matchings using only rational multipliers. Multiplying with a sufficiently large integer k, we obtain that $\bar{x} = \frac{1}{k}\sum_{j=1}^{k} \chi(\bar{M}_j)$ holds for matchings $\bar{M}_1, \ldots, \bar{M}_k$ in G, where matchings may occur multiple times. Let $J_a := \{j \in [k] \mid e_a \in \bar{M}_j\}$ and $J_b := \{j \in [k] \mid e_b \in \bar{M}_j\}$, and observe that $|J_a| = \hat{y}k = |J_b|$ holds. We may assume that the convex combination is chosen such that $|J_a \setminus J_b|$ is minimum.

Claim 10. *The convex combination satisfies $J_a = J_b$.*

By Claim 10 we can write $J := J_a = J_b$. We construct matchings $\hat{M}_j$ for $j \in [k]$ that are related to the corresponding $\bar{M}_j$. To this end, let $C := \{e_1, e_2, e_a, e_b\}$ and define $\hat{M}_j := \bar{M}_j \Delta C$ for all $j \in J$ and $\hat{M}_j := \bar{M}_j$ for all $j \in [k] \setminus J$. All $\hat{M}_j$ are matchings in $\bar{G}$ since for all $j \in J$, the matchings $\bar{M}_j$ contain both edges e_a and e_b. In fact, none of the $\hat{M}_j$ contains these edges, and hence they are even matchings in $K_{m,n}$.

Claim 11. *We have $\hat{x} = \frac{1}{k}\sum_{j=1}^{k} \chi(\hat{M}_j)$.*

Together with $\hat{y}k = |J|$, Claim 11 yields

$$(\hat{x}, \hat{y}) = \frac{1}{k}\left(\sum_{j\in J}(\chi(\hat{M}_j), 1) + \sum_{j\in[k]\smallsetminus J}(\chi(\hat{M}_j), 0)\right),$$

and it remains to prove that all participating vectors are actually feasible for P^{1Q}_{match}. For the first sum, this is easy to see, since for all $j \in J$, the matchings $\hat{M}_j$ contain both edges e_1 and e_2 by construction. The matchings in the second sum are considered in two claims, depending $(\hat{x}, \hat{y})$.

Claim 12. *Let $(\hat{x}, \hat{y})$ satisfy Inequality (5) for some $\hat{\imath} \in \{1, 2\}$ with equality. Then $\hat{M}_j$ contains at most one of the two edges e_1, e_2 for all $j \in [k] \smallsetminus J$.*

Claim 13. *Let $(\hat{x}, \hat{y})$ satisfy Inequality (6) for some $\hat{S} \in \mathcal{S}^{\downarrow}$ with equality. Then $\hat{M}_j$ contains at most one of the two edges e_1, e_2 for all $j \in [k] \smallsetminus J$.*

Since, by the assumptions of Lemma 1, the premise of at least one of the Claims 12 or 13 is satisfied, $(\hat{x}, \hat{y})$ is indeed a convex combination of vertices of P^{1Q}_{match}, which concludes the proof of Lemma 1. □

Before actually proving the claims of this section, we list further valid inequalities.

Proposition 14. *Let $(\hat{x}, \hat{y})$ satisfy Constraints (1)–(6) and define*

$$\mathcal{S}^{\downarrow}_{ext} := \{S \subseteq A \cup B \mid |S| \text{ is odd and } S \cap V^* \in \{\{a_1, a_2\}, \{b_1, b_2\}\}\}.$$

Then $(\hat{x}, \hat{y})$ even satisfies Inequality (6) for all $S \in \mathcal{S}^{\downarrow}_{ext}$.

Proof of Proposition 14. We only have to prove the statement for $S \in \mathcal{S}^{\downarrow}_{\text{ext}} \smallsetminus \mathcal{S}^{\downarrow}$. W.l.o.g. we assume that $S \cap V^* = \{a_1, a_2\}$, since the other case is similar. Let $A' := S \cap A$ and $B' := S \cap B$.
If $|A'| < |B'| + 1$ holds, we have $|A'| \leq |B'| - 1$ since $|S|$ is odd. Then the sum of $x(\delta(a)) \leq 1$ for all $a \in A'$ plus the sum of $-x_e \leq 0$ for all $e \in \delta(A') \smallsetminus (E[S] \cup \{e_1\})$ reads $x(E[S]) + x_{e_1} \leq |A'| \leq \frac{1}{2}(|S| - 1)$. Adding $y \leq x_{e_1}$ yields the desired inequality.
If $|A'| > |B'| + 1$ holds, we have $|A'| \geq |B'| + 3$ since $|S|$ is odd. Then the sum of $x(\delta(b)) \leq 1$ for all $b \in B'$ plus the sum of $-x_e \leq 0$ for all $e \in \delta(B') \smallsetminus E[S]$ reads $x(E[S]) \leq |B'| \leq \frac{1}{2}(|S| - 1)$. Adding $y \leq 1$ yields the desired inequality, which concludes the proof. □

Proof of Claim 9. From $\hat{x} \geq \mathbb{O}$ and (5) we obtain that also $\bar{x} \geq \mathbb{O}$ holds. By construction and since $\hat{x}$ satisfies (2), $\bar{x}(\delta(v)) \leq 1$ holds for every node $v \in A \cup B$.
Suppose, for the sake of contradiction, that $\bar{x}(E[S]) > \frac{1}{2}(|S| - 1)$ holds for some odd-cardinality set $S \subseteq A \cup B$. From $\hat{x}(E[S]) \leq \frac{1}{2}(|S| - 1)$ we deduce $\bar{x}(E[S]) > \bar{x}(E[S])$, i.e., $E[S]$ contains at least one of the edges $\{e_a, e_b\}$, since only for these edges the $\bar{x}$-value is strictly greater than the corresponding $\hat{x}$-value. Observe that $E[S]$ also must contain at most one of these edges, since otherwise it would also contain the two edges edges e_1, e_2, which yielded $\bar{x}(E[S]) = \hat{x}(E[S]) \leq \frac{1}{2}(|S| - 1)$. Hence, we have $S \in \mathcal{S}^{\downarrow}_{\text{ext}}$, and thus $\bar{x}(E[S]) = \hat{x}(E[S]) + \hat{y} \leq \frac{1}{2}(|S| - 1)$ by Proposition 14. This proves that $\bar{x}$ is in the matching polytope of $\bar{G}$. □

Proof of Claim 10. Suppose, for the sake of contradiction, that $J_a \neq J_b$ holds. Let $j_a \in J_a \smallsetminus J_b$ and let $j_b \in J_b \smallsetminus J_a$. Consider the matchings $\bar{M}_{j_a}$ and $\bar{M}_{j_b}$ and note that $\bar{M}_{j_a} \Delta \bar{M}_{j_b}$ contains both edges e_a and e_b. Let C_a and C_b be (the edge sets of) the connected components of $\bar{M}_{j_a} \Delta \bar{M}_{j_b}$ that contain e_a and e_b, respectively.
We claim that C_a and C_b are not the same component. Assume, for the sake of contradiction, that $C := C_a = C_b$ is a connected component (i.e., an alternating cycle or path) of $M \Delta M'$ that contains a and b. Consider a path $P \subseteq C \smallsetminus \{e_a, e_b\}$ that connects an endnode of e_a with an endnode of e_b (if C is an alternating cycle, there exist two such paths and we pick one arbitrarily). On the one hand, $(A \cup B, \bar{E} \smallsetminus \{e_a, e_b\})$ is bipartite and thus P must have odd length. On the other hand, $e_a \in \bar{M}_{j_a}$ and $e_b \in \bar{M}_{j_b}$ hold, and hence P must have even length, yielding a contradiction.
Define two new matchings $\bar{M}'_{j_a} := \bar{M}_{j_a} \Delta C_a$ and $\bar{M}'_{j_b} := \bar{M}_{j_b} \Delta C_a$, and note that $\chi(\bar{M}_{j_a}) + \chi(\bar{M}_{j_b}) = \chi(\bar{M}'_{j_a}) + \chi(\bar{M}'_{j_b})$ holds, i.e., we can replace $\bar{M}_{j_a}$ and $\bar{M}_{j_b}$ by $\bar{M}'_{j_a}$ and $\bar{M}'_{j_b}$ in the convex combination. The fact that $\bar{M}'_{j_a}$ contains none of the two edges e_a and e_b, while $\bar{M}'_{j_b}$ contains both, yields a contradiction to the assumption that $|J_a \smallsetminus J_b|$ is minimum. □

Proof of Claim 11. Consider the vector $d := \sum_{j=1}^{k}(\chi(\hat{M}_j) - \chi(\bar{M}_j))$. By construction of the $\hat{M}_j$, we have $d_e = 0$ for all $e \notin C$ and $d_{e_1} = d_{e_2} = -d_{e_a} = -d_{e_b} = |J| = k\hat{y}$. A simple comparison with the construction of $\bar{x}$ from $\hat{x}$ concludes the proof. □

Proof of Claim 12. From $\hat{x}_{e_{i^*}} = \hat{y}$ we obtain that $\bar{x}_{e_{i^*}} = 0$ and hence $e_{i^*} \notin \bar{M}_j$ holds, which concludes the proof. □

Proof of Claim 13. From $\hat{x}(E[S^*]) + \hat{y} = \frac{1}{2}(|S^*|-1)$ and the construction of $\bar{x}$ we obtain $\bar{x}(E[S^*]) = \frac{1}{2}(|S^*|-1)$. But since $x(E[S^*]) \le \frac{1}{2}(|S^*|-1)$ is valid for all $\chi(\bar{M}_j)$, equality must hold for all $j \in [k]$. Thus, $|\bar{M}_j \cap \{e_1, e_2\}| \le |\bar{M}_j \cap \delta(S^*)| \le 1$ holds for all j, which concludes the proof. □

5 Proofs for the Upward Monotonization

This section contains the proof of Lemma 2. The setup is similar to that of the previous section, starting with the relevant objects.

Let $(\hat{x}, \hat{y}) \in \mathbb{Q}^E \times \mathbb{Q}$ be as stated in the lemma. Let $\bar{G} = (\bar{V}, \bar{E})$ be the graph $K_{m,n}$ with two additional nodes u and v, i.e., $\bar{V} = A \cup B \cup \{u, v\}$, and edge set $\bar{E} := E \cup \{\{u,v\}, \{a_1,u\}, \{a_2,v\}, \{b_1,v\}, \{b_2,u\}\}$. Define two vectors $\tilde{x}, \bar{x} \in \mathbb{R}^{\bar{E}}$ as follows (see Figure 3):

- $\tilde{x}_e := \hat{x}_e$ and $\bar{x}_e := \hat{x}_e$ for all $e \in E \setminus \{e_1, e_2\}$.
- $\tilde{x}_{e_i} := \hat{x}_{e_i}$ and $\bar{x}_{e_i} := \frac{1}{2}\hat{y}$ for $i = 1, 2$.
- $\tilde{x}_{\{u,v\}} := 1$ and $\bar{x}_{\{u,v\}} := 1 - \hat{x}_{e_1} - \hat{x}_{e_2} + \hat{y}$.
- $\tilde{x}_{a_1,u} := \tilde{x}_{b_1,v} := 0$ and $\bar{x}_{\{a_1,u\}} := \bar{x}_{\{b_1,v\}} := \hat{x}_{e_1} - \frac{1}{2}\hat{y}$.
- $\tilde{x}_{a_2,v} := \tilde{x}_{b_2,u} := 0$ and $\bar{x}_{\{a_2,v\}} := \bar{x}_{\{b_2,u\}} := \hat{x}_{e_2} - \frac{1}{2}\hat{y}$.

The vector $\tilde{x}$ is essentially a trivial lifting of $\hat{x}$ into $\mathbb{R}^{\bar{E}}$ by setting the value for edge $\{u,v\}$ to 1 and the values for the other new edges to 0. It is easy to see that $\tilde{x}$ is in the matching polytope of $\bar{G}$. The vector $\bar{x}$ is a modification of $\tilde{x}$ on the edges of the following two cycles:

$$C_1 := \{\{a_1,u\}, \{u,v\}, \{v,b_1\}, \{b_1,a_1\}\},$$
$$C_2 := \{\{a_2,v\}, \{v,u\}, \{u,b_2\}, \{b_2,a_2\}\}.$$

The values on the two opposite (in C_1) edges $\{a_1,b_1\}$ and $\{u,v\}$ are decreased by $\hat{x}_{e_1} - \frac{1}{2}\hat{y}$, and increased by the same value on the other two edges. Similarly, the values on the edges $\{a_2,b_2\}$ and $\{u,v\}$ are decreased by $\hat{x}_{e_2} - \frac{1}{2}\hat{y}$, while they are increased by the same value on the other two edges of C_2.

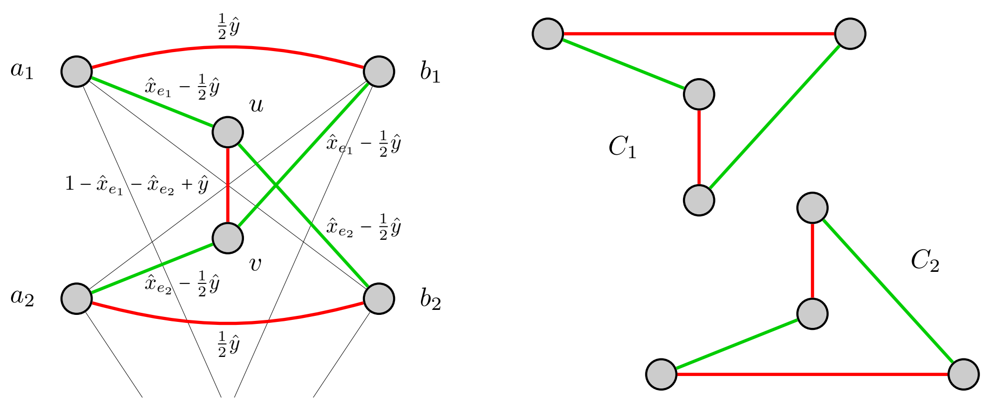


Figure 3: Graph $\bar{G}$, vector $\bar{x}$ and cycles C_1 and C_2 in the Proof of Lemma 2.

Claim 15. *$\bar{x}$ is in the matching polytope of $\bar{G}$.*

By Claim 15, and since $\bar{x}$ is rational, it can be written as a convex combination of characteristic vectors of matchings using only rational multipliers. Multiplying with a sufficiently large integer k, we obtain that $\bar{x} = \frac{1}{k}\sum_{j=1}^{k}\chi(\bar{M}_j)$ holds for matchings $\bar{M}_1,\dots,\bar{M}_k$ in G, where matchings may occur multiple times. We define the index sets

$$\begin{aligned}
J_a &:= \left\{j \in [k] \mid \{a_1,u\},\{a_2,v\} \in \bar{M}_j\right\}, & J_b &:= \left\{j \in [k] \mid \{b_1,v\},\{b_2,u\} \in \bar{M}_j\right\},\\
J_1 &:= \left\{j \in [k] \mid \{a_1,u\},\{b_1,v\} \in \bar{M}_j\right\}, & J_2 &:= \left\{j \in [k] \mid \{a_2,v\},\{b_2,u\} \in \bar{M}_j\right\},\\
J_1' &:= \left\{j \in [k] \mid \{a_2,b_2\} \in \bar{M}_j\right\}, & J_2' &:= \left\{j \in [k] \mid \{a_1,b_1\} \in \bar{M}_j\right\} \text{ and}\\
N &:= \left\{j \in [k] \mid \{u,v\} \in \bar{M}_j\right\}.
\end{aligned}$$

We assume that the convex combination is chosen such that $|J_a| + |J_b|$ is minimum.
Using the assumption from the lemma, that $(\hat{x},\hat{y})$ satisfies Inequality (7) for some set $S^* \in \mathcal{S}^\uparrow$ with equality, we can derive the following statement.

Claim 16. *Let $j \in [k]$. The matching $\bar{M}_j$ contains at most one of the edges e_1, e_2, or $\{u,v\}$. It furthermore matches u and v (not necessarily to each other).*

Claim 17. *The convex combination satisfies $J_a = J_b = \emptyset$, and $J_1 \cup J_2 \cup N$ is a partitioning of $[k]$.*

Claim 18. *We have $J_i' \subseteq J_1$ for $i = 1,2$ and thus J_1' and J_2' are disjoint.*

Claim 19. *We have $|J_1' \cup J_2'| = \hat{y}k$.*

We construct matchings $\tilde{M}_j$ and $\hat{M}_j$ for $j \in [k]$ that are related to the corresponding $\bar{M}_j$. Define $\tilde{M}_j := \bar{M}_j \Delta C_1$ for all $j \in J_1$, $\tilde{M}_j := \bar{M}_j \Delta C_2$ for all $j \in J_2$. By Claim 17, all remaining indices are the $j \in N$, and for those we define $\tilde{M}_j := \bar{M}_j$. All $\tilde{M}_j$ are matchings in $\bar{G}$ since for all $j \in J_i$ $(i = 1,2)$ the cycle C_i is an $\bar{M}_j$-alternating cycle. We define $\hat{M}_j := \tilde{M}_j \setminus \{u,v\}$ for all $j \in [k]$, which are matchings in $K_{m,n}$ since $\{u,v\} \in \tilde{M}_j$ holds for all $j \in [k]$,

Claim 20. *We have $\tilde{x} = \frac{1}{k}\sum_{j=1}^{k}\chi(\tilde{M}_j)$ and $\hat{x} = \frac{1}{k}\sum_{j=1}^{k}\chi(\hat{M}_j)$.*

Claims 19 and 20 yield

$$(\hat{x},\hat{y}) = \frac{1}{k}\left(\sum_{j\in J_1'}(\chi(\hat{M}_j),1) + \sum_{j\in J_2'}(\chi(\hat{M}_j),1) + \sum_{j\in[k]\setminus(J_1'\cup J_2')}(\chi(\hat{M}_j),0)\right),$$

and it remains to prove that all participating vectors are actually feasible for P^{1Q}_{match}.
To this end, let $j \in J_1'$ and observe that $\{a_2,b_2\} \in \bar{M}_j$ and, by Claim 18, $\{a_1,u\},\{b_1,v\} \in \bar{M}_j$ holds. Thus, the symmetric difference with C_1 yields $\{a_1,b_1\},\{a_2,b_2\} \in \hat{M}_j$. Similarly, $\{a_1,b_1\},\{a_2,b_2\} \in \hat{M}_j$ holds for all $j \in J_2'$. Let $j \in [k] \setminus (J_1' \cup J_2')$. First, $\bar{M}_j$ contains none of the edges $\{a_1,b_1\}$, $\{a_2,b_2\}$. Second, the construction of $\hat{M}_j$ from $\bar{M}_j$ adds at most one of the two edges $\{a_1,b_1\}$, $\{a_2,b_2\}$, which proves that $\hat{M}_j$ does not contains both of them. This concludes the proof. □

Before actually proving the claims of this section, we list further valid inequalities.

Proposition 21. *Let $(\hat{x},\hat{y})$ satisfy Constraints (1)–(4) and (7). Then $(\hat{x},\hat{y})$ satisfies the following inequalities:*

(a) $x_{e_1} + x_{e_2} - y \le 1$.

(b) $x(E[S]) + x_{e_i} - \frac{1}{2}\hat{y} \le \frac{1}{2}|S|$ *for $i \in \{1,2\}$, $S \subseteq A \cup B$ with $|S|$ even and $e_i \in \delta(S)$.*

(c) Inequality (7) is satisfied even for all $S \in \mathcal{S}^\uparrow_{ext} \supseteq \mathcal{S}^\uparrow$, defined as follows:

$$\mathcal{S}^\uparrow_{ext} := \{S \subseteq A \cup B \mid |S| \text{ is even and } S \cap V^* \in \{\{a_1,b_2\},\{a_2,b_1\}\}\}.$$

If $(\hat{x},\hat{y})$ also satisfies Inequality (7) for some $S^* \in \mathcal{S}^\uparrow$ with equality, then the following inequalities hold as well:

(d) $y \le x_{e_1}$ *and* $y \le x_{e_2}$.

Proof of Proposition 21. We prove validity for each inequality individually:

(a) The inequality is the sum of Inequality (7) for $S = \{a_1, b_2\}$ and $-x_{\{a_1,b_2\}} \leq 0$.

(b) Since $|S \cup e_i|$ is odd (and since $K_{m,n}$ is bipartite), the usual Blossom Inequality $x(E[S \cup e_i]) \leq \frac{1}{2}|S|$ is implied by Constraints (1) and (2). Adding $-x_e \leq 0$ for all $e \in E[S \cup e_1] \setminus \{e_1\}$ and $-\frac{1}{2}y \leq 0$ yields the desired inequality.

(c) We only have to prove the statement for $S \in \mathcal{S}^{\uparrow}_{\text{ext}} \setminus \mathcal{S}^{\uparrow}$. W.l.o.g. we assume that $S \cap V^* = \{a_1, b_2\}$ and $|S \cap A| < |S \cap B|$, since the other cases are similar. Let $A' := S \cap A$ and $B' := S \cap B$ and observe that $|A'| \leq |B'| - 2$ holds because $|S|$ is even. Then the sum of $x(\delta(a)) \leq 1$ for all $a \in A'$ plus the sum of $-x_e \leq 0$ for all $e \in \delta(A') \setminus (E[S] \cup \{e_1\})$ reads $x(E[S]) + x_{e_1} \leq |A'| = \frac{1}{2}|S| - 1$. Adding $x_{e_2} \leq 1$ and $-y \leq 0$ yields the desired inequality, which concludes the proof.

(d) Let $i \in \{1, 2\}$ and $j := 3 - i$. Similar to the proof of (b) we have that $x(E[S^*]) + x_{e_j} \leq \frac{1}{2}|S^*|$ is implied by Constraints (1) and (2). Subtracting this from the equation $x(E[S^*]) + x_{e_1} + x_{e_2} - y = \frac{1}{2}|S^*|$ yields $x_{e_i} - y \geq 0$, which concludes the proof.

□

Proof of Claim 15. Since $(\hat{x}, \hat{y}) \geq \mathbb{O}$ holds, Parts (a) and (d) of Propositon 21 yield $\bar{x} \geq \mathbb{O}$. The degree constraints are also satisfied, since $\bar{x}(\delta(v)) = \hat{x}(\delta(v))$ holds for the nodes $v \in V^*$ and since $\bar{x}(\delta(u)) = \bar{x}(\delta(v)) = 1$ holds.
Suppose, for the sake of contradiction, that $\bar{x}(E[\bar{S}]) > \frac{1}{2}(\bar{S}| - 1)$ holds for some odd-cardinality set $\bar{S} \subseteq \bar{V}$. Clearly, $\tilde{x}(E[\bar{S}]) \leq \frac{1}{2}(|\bar{S}| - 1)$ holds, i.e., $(\bar{x} - \tilde{x})(E[\bar{S}]) > 0$. This implies that $E[\bar{S}]$ must intersect some C_i $(i = 1, 2)$ in such a way that the sum of the respective modifications (increase or decrease by $\hat{x}_{e_i} - \frac{1}{2}\hat{y}$) is positive. Similar to the proof of Claim 9, we conclude that $\bar{S}$ must touch one of the cycles in precisely two nodes, whose connecting edge e satisfies $\bar{x}_e > \tilde{x}_e$. Hence, (at least) one of the following four conditions must be satisfied:

$$(1)\ \bar{S} \cap \{a_1, b_1, u, v\} = \{a_1, u\}, \qquad (2)\ \bar{S} \cap \{a_1, b_1, u, v\} = \{b_1, v\},$$
$$(3)\ \bar{S} \cap \{a_2, b_2, u, v\} = \{a_2, v\}, \qquad (4)\ \bar{S} \cap \{a_2, b_2, u, v\} = \{b_2, u\}.$$

We define $\bar{V}^* := \{a_1, a_2, b_1, b_2, u, v\}$ and $S := \bar{S} \setminus \{u, v\}$. Note that we always have $|S| = |\bar{S}| - 1$ since each of the four conditions implies that either u or v is contained in $\bar{S}$. We now make a case distinction, based on $\bar{S} \cap \bar{V}^*$:

Case 1: $\bar{S} \cap \bar{V}^*$ is equal to $\{a_1, u\}$, $\{a_1, u, a_2\}$, $\{a_1, u, a_2, b_2\}$, $\{b_1, v\}$, $\{b_1, v, b_2\}$ or $\{b_1, v, b_2, a_2\}$. In this case $\bar{x}(E[\bar{S}]) = \tilde{x}(E[\bar{S}]) + \hat{x}_{e_1} - \frac{1}{2}\hat{y} = \hat{x}(E[S]) + \hat{x}_{e_1} - \frac{1}{2}\hat{y} \leq \frac{1}{2}|S| = \frac{1}{2}(|\bar{S}| - 1)$ holds by Proposition 21 (b), which yields a contradiction.

Case 2: $\bar{S} \cap \bar{V}^*$ is equal to $\{a_2, v\}$, $\{a_2, v, a_1\}$, $\{a_2, v, a_1, b_1\}$, $\{b_2, u\}$, $\{b_2, u, b_1\}$, $\{b_2, u, b_1, a_1\}$. In this case $\bar{x}(E[\bar{S}]) = \tilde{x}(E[\bar{S}]) + \hat{x}_{e_2} - \frac{1}{2}\hat{y} = \hat{x}(E[S]) + \hat{x}_{e_2} - \frac{1}{2}\hat{y} \leq \frac{1}{2}|S| = \frac{1}{2}(\bar{S}| - 1)$ holds by Proposition 21 (b), which yields a contradiction.

Case 3: $\bar{S} \cap \bar{V}^*$ is equal to $\{a_1, u, b_2\}$ or $\{a_2, v, b_1\}$.
In this case $\bar{x}(E[\bar{S}]) = \tilde{x}(E[\bar{S}]) + \hat{x}_{e_1} + \hat{x}_{e_2} - \hat{y} = \hat{x}(E[S]) + \hat{x}_{e_1} + \hat{x}_{e_2} - \hat{y} \leq \frac{1}{2}|S| = \frac{1}{2}(|\bar{S}| - 1)$ holds by Proposition 21 (c), which yields a contradiction. □

Proof of Claim 16. Let $S^* \in \mathcal{S}^{\uparrow}$ be such that $(\hat{x}, \hat{y})$ satisfies Inequality (7) with equality. If $a_1, b_2 \in S^*$ holds, then we define $\bar{S} := S^* \cup \{u\}$, and otherwise $\bar{S} := S^* \cup \{v\}$. A simple calculation shows that $\bar{x}(E[\bar{S}]) = \hat{x}(E[S^*]) + \hat{x}_{e_1} + \hat{x}_{e_2} - \hat{y} = \frac{1}{2}|S^*| = \frac{1}{2}(|\bar{S}| - 1)$ holds, i.e., $\bar{x}$ satisfies the Blossom Inequality induced by $\bar{S}$ with equality. Furthermore, $\bar{x}$ satisfies the degree inequalities (2) for nodes u and v with equality. This implies that all characteristic vectors $\chi(\bar{M}_j)$ satisfy these three inequalities with equality, i.e., we have $|\bar{M}_j \cap E[\bar{S}]| = \frac{1}{2}(|\bar{S}| - 1)$ and $|\bar{M}_j \cap \delta(u)| = |\bar{M}_j \cap \delta(v)| = 1$. From the first equation we derive $|\bar{M}_j \cap \delta(\bar{S})| \leq 1$. This, together with the second equation proves the claimed properties. □

Proof of Claim 17. First, the set J_a, J_b, J_1, J_2 and N are disjoint since the indexed matchings all match nodes u and v in different ways. Second, their union is equal to $[k]$ due to the second part of Claim 16. From this we obtain $|J_a| + |J_1| = \bar{x}_{a_1,u}k = \left(\hat{x}_{e_1} - \frac{1}{2}y\right)k = \bar{x}_{b_1,v}k = |J_b| + |J_1|$, and conclude that $|J_a| = |J_b|$ holds.

Now suppose, for the sake contradiction, that $J_a \neq \varnothing$ (and thus $|J_b| = |J_a| \geq 1$) holds. Let $j \in J_a$ and $j' \in J_b$ and let C be the connected component (i.e., an alternating cycle or path) of $\bar{M}_j \Delta \bar{M}_{j'}$ that contains $\{a_2, v\}$.
We claim that $\{a_1, u\} \notin C$ holds. Assuming the contrary, there must exist an odd-length (alternating) path in $K_{m,n}$ that connects either a_1 with a_2 or b_1 with b_2 or there must exist an even-length (alternating) path in $K_{,n}$ that connects either a_1 with b_1 or a_2 with b_2. Since $K_{m,n}$ is bipartite, none of these paths exist, which proves $\{a_1, u\} \notin C$.
Define two new matchings $\bar{M}'_j := \bar{M}_j \Delta C$ and $\bar{M}'_{j'} := \bar{M}_{j'} \Delta C$, and note that $\chi(\bar{M}_j) + \chi(\bar{M}_{j'}) = \chi(\bar{M}'_j) + \chi(\bar{M}'_{j'})$ holds, i.e., we can replace $\bar{M}_j$ and $\bar{M}_{j'}$ by $\bar{M}'_j$ and $\bar{M}'_{j'}$ in the convex combination. The fact that $\bar{M}'_j$ contains the edges $\{a_1, u\}$ and $\{b_1, v\}$ and that $\bar{M}'_{j'}$ contains the edges $\{b_2, u\}$ and $\{a_2, v\}$ contradicts the assumption that the convex combination was chosen with minimum $|J_a| + |J_b|$. Hence, $J_a = J_b = \varnothing$ holds. □

Proof of Claim 18. Let $j \in J'_1$. Using $\{a_2, b_2\} \in \bar{M}_j$, Claim 16 shows that $\{u, v\} \notin \bar{M}_j$, and thus (since a_2 and b_2 are already matched to each other) that $\bar{M}_j$ contains the two edges $\{a_1, u\}$ and $\{b_1, v\}$, i.e., $j \in J_1$ holds. The proof for J'_2 is similar.
From Claim 17 we have $J_1 \cap J_2 = \varnothing$, and hence $J'_1 \cap J'_2 = \varnothing$ holds as well. □

Proof of Claim 19. By Claim 18, we have $J'_1 \cap J'_2 = \varnothing$. The statement now follows from the fact that $\bar{x}_{e_1} = \bar{x}_{e_2} = \frac{1}{2}\hat{y}$ holds. □

Proof of Claim 20. Similar to the proof of Claim 11, we consider the vector $d := \sum_{j=1}^{k}(\chi(\tilde{M}_j) - \chi(\bar{M}_j))$. By construction of the $\tilde{M}_j$, we have

- $d_e = 0$ for all $e \in C_1 \cup C_2$,
- $d_{e_1} = -d_{\{a_1,u\}} = -d_{\{b_1,v\}} = |J_1| = (\hat{x}_{e_1} - \frac{1}{2}\hat{y})k$,
- $d_{e_2} = -d_{\{a_2,v\}} = -d_{\{b_2,u\}} = |J_2| = (\hat{x}_{e_2} - \frac{1}{2}\hat{y})k$, and
- $d_{\{u,v\}} = d_{e_1} + d_{e_2} = (\hat{x}_{e_1} + \hat{x}_{e_2} - \hat{y})k$.

A simple comparison with the construction of $\tilde{x}$ and $\bar{x}$ from $\hat{x}$ proves the first part.
The construction of $\hat{M}_j$ from $\tilde{M}_j$ by removing edge $\{u, v\}$ corresponds to the fact that $\hat{x}$ is the orthogonal projection of $\tilde{x}$ onto $\mathbb{R}^E$, which proves that second part. □

6 Facet Proofs

Proposition 22. *The polytopes P^{1Q}_{match}, $P^{1Q\downarrow}_{match}$ and $P^{1Q\uparrow}_{match}$ are full-dimensional.*

Proof. The point $(\chi(\varnothing), 0)$, the points $(\chi(\{e\}), 0)$ for all $e \in E$ and the point $(\chi(\{e_1, e_2\}), 1)$ are $|E| + 2$ affinely independent points that are contained in all three polytopes. This proves the statement. □

Proposition 23. *Let $e^* \in E$. Then Inequality* (1) *defines a facet for $P^{1Q\uparrow}_{match}$. Furthermore, it defines a facet for P^{1Q}_{match} (and thus for $P^{1Q\downarrow}_{match}$) if and only if $e^* \notin \{e_1, e_2\}$ holds.*

Proof. Consider the point $(x', 0) \in \mathbb{R}^E \times \mathbb{R}$ defined via $x'_{e^*} := -1$ and $x'_e := 0$ for all $e \in E \setminus \{e^*\}$. Clearly, $(x', 0)$ satisfies Inequalities (1) for $e \neq e^*$, but satisfies it for $e = e^*$. Since x_{e^*} appears in the $\leq$-Inequalities (2), (4), (6) and (7) with a nonnegative coefficient, and since $\mathbb{O} \in P^{1Q}_{\text{match}}$ holds, all these are satisfied by $(x', 0)$.
If $e^* \notin \{e_1, e_2\}$ holds, then the above also holds for Inequalities (5), which proves the statement for this case and for $P^{1Q\uparrow}_{\text{match}}$, since there Inequalities (5) are not present.
If $e^* = e_i$ holds for some $i \in \{1, 2\}$, then $x_{e^*} \geq 0$ is clearly implied by $0 \leq y$ and $y \leq x_{e_i}$, proving that it is not facet-defining for P^{1Q}_{match} and $P^{1Q\downarrow}_{\text{match}}$. This concludes the proof. □

Proposition 24. *Let $v^* \in A \cup B$ and let $k := |\delta(v^*)|$. Then Inequality* (2) *is facet-defining for*

- *$P^{1Q\uparrow}_{match}$ in any case, for*
- *$P^{1Q\downarrow}_{match}$ if and only if $k \geq 3$ or $v^* \in V^*$ holds, and for*

- P^{1Q}_{match} *if and only if* $k \geq 3$ *holds.*

Proof. First note that $k = m$ or $k = n$ holds, since we consider the complete bipartite graph, and thus $k \geq 2$ holds. If $k = 2$ and $v^* \notin V^*$ holds, then let $S := \bigcup \delta(v^*)$, i.e., S contains v^* and its neighbors. Inequality (2) is the sum of Inequality (6) for S and $-y \leq 0$. Both inequalities are valid for $P^{1Q\downarrow}_{\text{match}}$ and P^{1Q}_{match}, hence Inequality (2) for $v = v^*$ cannot be facet-defining for those polytopes. If $k = 2$ and $v^* \in V^*$ holds, then let S be equal to $\{a_1, b_2\}$ or equal to $\{a_2, b_1\}$ such that $v^* \in S$ is satisfied. Inequality (2) is the sum of Inequality (7) for S and Inequality (5) for $i \in \{1,2\}$ such that $e_i \in \delta(S)$ holds. Both inequalities are valid for P^{1Q}_{match}, hence Inequality (2) for $v = v^*$ cannot be facet-defining for this polytope.
We now prove non-redundancy of the inequality in the remaining cases. Let $\varepsilon > 0$ be such that $\varepsilon k + 2\varepsilon \leq 1$ holds. We consider the two points $(x', 0), (x', 2\varepsilon) \in \mathbb{R}^E \times \mathbb{R}$ defined via $x'_e := \frac{1}{k} + \varepsilon$ for all $e \in \delta(v^*)$ and $x'_e := 0$ for all $e \in E \setminus \delta(v^*)$. We will prove that $(x', 0)$ satisfies all inequalities, except for the one in question, that are valid for P^{1Q}_{match} and $P^{1Q\downarrow}_{\text{match}}$ if the respective conditions on k and v^* are satisfied. Furthermore, we will prove that $(x', 2\varepsilon)$ satisfies all inequalities, except for the one in question, that are valid for $P^{1Q\uparrow}_{\text{match}}$.
Both points clearly satisfy Constraints (1) and (4) and violate Inequality (2) for $v = v^*$. They furthermore satisfy Inequalities (2) for $v \neq v^*$ since at most one of the edges in the support (in the x-space) of such an inequality can be in $\delta(v^*)$. The first point $(x', 0)$ also clearly satisfies (5).
Suppose Inequality (6) is violated by $(x', 0)$ for a set $S \in \mathcal{S}^{\downarrow}$. Since $|x'|_1 = 1 + k\varepsilon \leq 2$ holds, we must have $|S| = 3$ and the right-hand side of the inequality equals 1. Thus, in order to be violated, $\delta(v^*) \subseteq E[S]$ must hold. It is easy to see that in this case, we have $k = 2$ and v^* must be the unique node in $S \setminus V^*$.
Suppose Inequality (7) is violated by $(x', 0)$ or $(x', 2\varepsilon)$ for a set $S \in \mathcal{S}^{\uparrow}$. Since $|x'|_1 + 2\varepsilon \leq 2$ holds, we must have $|S| = 2$, i.e., $S = \{a_1, b_2\}$ or $S = \{a_2, b_1\}$, and the right-hand side is equal to 1. In order to be violated, $\delta(v^*) \cup \{e_1, e_2\} \subseteq E[S]$ must hold, which implies $v^* \in S \subseteq V^*$. We prove that $(x', 2\varepsilon)$ is not violated and assume that, w.l.o.g., $S = \{a_1, b_2\}$ and $v^* = b_2$ holds. Now observe that $x'(E[S]) = x'_{a_1,b_2}$ and we have $x'(E[S]) + x'_{e_1} + x'_{e_2} - 2\varepsilon = (\frac{1}{2} + \varepsilon) + (\frac{1}{2} + \varepsilon) + 0 - 2\varepsilon = 1$. This concludes the proof. ☐

Proposition 25. *The inequality* $y \geq 0$ *is facet-defining for* $P^{1Q\uparrow}_{match}$, $P^{1Q\downarrow}_{match}$ *and* P^{1Q}_{match}, *while* $y \leq 1$ *is facet-defining for* $P^{1Q\uparrow}_{match}$, *but not for* $P^{1Q\downarrow}_{match}$ *and* P^{1Q}_{match}.

Proof. For fixed value $k \in \{0,1\}$, The point $(\chi(\varnothing), k)$ and the points $(\chi(\{e\}), k)$ for all $e \in E$ are $|E| + 1$ affinely independent points. For $k = 0$, they are contained in all three polytopes and satisfy $y \geq 0$ with equality, which proves the first statement. For $k = 1$, they are contained in $P^{1Q\uparrow}_{\text{match}}$ and satisfy $y \leq 1$ with equality, which proves one direction of the second statement. For the reverse direction, observe that $y \leq 1$ is the sum of Inequality (5) for $i \in \{1,2\}$ and $x_{e_i} \leq 1$, which concludes the proof. ☐

Proposition 26. *For* $i^* = 1, 2$, *Inequalities* (5) *define facets for* P^{1Q}_{match} *and* $P^{1Q\downarrow}_{match}$.

Proof. Let $i^* \in \{1,2\}$. We consider the point (x', y') defined via $x'_{e_{i^*}} := y' := \frac{1}{2}$ and $x'_e := 0$ for $e \in E \setminus \{e_{i^*}\}$ Since $|x'|_1 + y' \leq 1$ holds, the point clearly satisfies Constraints (1), (2), (4), (6), (7), and Inequality (5) for $i \neq i^*$, but violates the latter for $i = i^*$. This concludes the proof. ☐

For the remaining two proofs we will consider a set $S^* \subseteq A \cup B$ of nodes and denote by $A^* := S^* \cap A$ and $B^* := S^* \cap B$ the induced sides of the bipartition. For a matching M in $K_{m,n}$ we denote by $y(M) \in \{0,1\}$ its corresponding y-value, i.e., $y(M) = 1$ if and only if $e_1, e_2 \in M$. Note that this implies $(\chi(M), y(M)) \in P^{1Q}_{\text{match}}$. Another concept from matching theory also turns out to be useful: We say that a matching is *near-perfect* in a set of nodes if it matches all nodes but one of this set.

Proposition 27. *For all* $S^* \in \mathcal{S}^{\downarrow}$, *Inequalities* (6) *define facets for* P^{1Q}_{match} *and* $P^{1Q\downarrow}_{match}$.

Proof. Let $S^* \in \mathcal{S}^{\downarrow}$. We will assume w.l.o.g. that $|A^*| + 1 = |B^*|$ holds, since the proof for $|B^*| = |A^*| + 1$ is similar. Let $\mathcal{M}$ denote the set of matchings M in $K_{m,n}$ that are either near-perfect in S^* or are near-perfect in $S^* \setminus V^*$ and contain edges e_1 and e_2. In the first case we have $|M \cap E[S^*]| = \frac{1}{2}(|S^*| - 1)$ and $y(M) = 0$, and in the second case we have $|M \cap E[S^*]| = \frac{1}{2}(|S^*| - 3)$ and $y(M) = 1$. Hence, for all $M \in \mathcal{M}$, the vector $(\chi(M), y(M))$ satisfies Inequality (6) with equality.

Let $\langle c, x\rangle + \gamma y \le \delta$ dominate Inequality (6) for $S = S^*$, i.e., it is valid for $P^{1Q\downarrow}_{\text{match}}$ and the vectors $(\chi(M), y(M))$ and for all $M \in \mathcal{M}$ we have $\langle c, \chi(M)\rangle + \gamma y(M) = \delta$. We now analyze the coefficients and the right-hand side of the inequality.

- Let $e \in E \setminus E[S^*]$. If e intersects S^*, then let $v \in e \cap S^*$ be its endnode in S^*, otherwise let $v \in S^*$ be arbitrary. If $v \in A^*$, then let M_1 be a perfect matching in $S^* \setminus \{v\}$ (which exists due to $|A^* \setminus \{v\}| = |B^*|$). Otherwise, let M' be a perfect matching in $S^* \setminus \{v, a_1, a_2\}$ (which exists due to $|A^* \setminus \{a_1, a_2\}| = |B^* \setminus \{v\}|$), and extend it to the matching $M_1 := M' \cup \{e_1, e_2\}$. Then e does not intersect M_1 and thus $M_2 := M_1 \uplus \{e\}$ is also a matching that satisfies $y(M_1) = y(M_2)$. By construction we have $M_1, M_2 \in \mathcal{M}$, and hence $\langle c, \chi(M_1)\rangle + \gamma y(M_1) = \delta = \langle c, \chi(M_2)\rangle + \gamma y(M_2)$ holds. This proves $c_e = 0$.
- Let $u \in B^*$ and let $e = \{u, v\}$ and $f = \{u, w\}$ be two incident edges with endnodes $v, w \in S^*$. Let M_1 be a perfect matching in $S^* \setminus \{v\}$ that uses edge f. Then $M_2 := (M_1 \setminus \{f\}) \uplus \{e\}$ is perfect in $S^* \setminus \{w\}$. Clearly, $M_1, M_2 \in \mathcal{M}$ holds by construction. Thus, $\langle c, \chi(M_1)\rangle + \gamma y(M_1) = \delta = \langle c, \chi(M_2)\rangle + \gamma y(M_2)$ holds, i.e., $c_f = c_e$ holds.
- If $|B^*| \ge 2$, then also $|A^*| \ge 3$ holds. Let $v, w \in B^*$ be two nodes, let $u \in A^* \setminus \{a_1, a_2\}$, and let $e := \{u, v\}$ and $f := \{u, w\}$. Let M' be a perfect matching in $S^* \setminus \{a_1, a_2, u, v, w\}$ (which exists due to $|A^* \setminus \{a_1, a_2, u\}| = |B^* \setminus \{v, w\}|$). Define matchings $M_1 := M' \uplus \{e, e_1, e_2\}$ and $M_2 := M' \uplus \{f, e_1, e_2\}$ and observe that $M_1, M_2 \in \mathcal{M}$ and $y(M_1) = 1 = y(M_2)$ holds. Thus, $\langle c, \chi(M_1)\rangle + \gamma y(M_1) = \delta = \langle c, \chi(M_2)\rangle + \gamma y(M_2)$ holds, i.e., $c_e = c_f$ holds.
- Let M_1 be a perfect matching in $S^* \setminus \{a_1\}$ and let $e \in M_1$ be the edges that matches a_2. Define matching $M_2 := (M_1 \setminus \{e\}) \cup \{e_1, e_2\}$, and note that $M_1, M_2 \in \mathcal{M}$ holds. Again, $\langle c, \chi(M_1)\rangle + \gamma y(M_1) = \delta = \langle c, \chi(M_2)\rangle + \gamma y(M_2)$ holds, i.e., $c_e = \gamma$ holds.

The arguments above already fix (c, γ) up to multiplication with a scalar. Hence we can assume that $\gamma = 1$ holds, which proves that (c, γ) is equal to the coefficient vector of Inequality (6) for $S = S^*$. Since there always exists a near-perfect matching M in S^*, and since such a matching has cardinality $|M| = \frac{1}{2}(|S^*| - 1)$, we derive $\delta = \frac{1}{2}(|S^*| - 1)$. This concludes the proof. ☐

Proposition 28. *For all $S \in \mathcal{S}^\uparrow$, Inequalities* (7) *define facets for P^{1Q}_{match} and $P^{1Q\uparrow}_{match}$.*

Proof. Let $S^* \in \mathcal{S}^\uparrow$. We will assume w.l.o.g. that $a_1, b_2 \in S^*$ holds, since the proof for $a_2, b_1 \in S^*$ is similar. Let $\mathcal{M}$ denote the set of matchings M in $K_{m,n}$ that are either perfect in S^* or contain exactly one edge $e \in \{e_1, e_2\}$ and are near-perfect in $S^* \setminus e$ or contain both, e_1 and e_2, and are perfect in $S^* \setminus \{a_1, b_2\}$. In the first two cases we have $|M \cap (E[S^*] \cup \{e_1, e_2\})| = \frac{1}{2}|S^*|$ and $y(M) = 0$, and in the third case we have $|M \cap (E[S^*] \cup \{e_1, e_2\})| = \frac{1}{2}(|S^*|) + 1$ and $y(M) = 1$. Hence, for all $M \in \mathcal{M}$, the vector $(\chi(M), y(M))$ satisfies Inequality (7) with equality.
Let $\langle c, x\rangle + \gamma y \le \delta$ dominate Inequality (7) for $S = S^*$, i.e., it is valid for $P^{1Q\uparrow}_{\text{match}}$ and the vectors $(\chi(M), y(M))$ and for all $M \in \mathcal{M}$ we have $\langle c, \chi(M)\rangle + \gamma y(M) = \delta$. We now analyze the coefficients and the right-hand side of the inequality.

- Let $e \in E \setminus (E[S^*] \cup \{e_1, e_2\})$. If e intersects S^*, then let $v \in e \cap S^*$ be its endnode in S^*, otherwise let $v \in S^*$ be arbitrary. If $v \in A^*$, then let M' be a perfect matching in $S^* \setminus \{v, b_2\}$ (which exists due to $|A^* \setminus \{v\}| = |B^* \setminus \{b_2\}|$), and extend it to the matching $M_1 := M' \cup \{e_2\}$. Otherwise, let M' be a perfect matching in $S^* \setminus \{v, a_1\}$ (which exists due to $|A^* \setminus \{a_1\}| = |B^* \setminus \{v\}|$), and extend it to the matching $M_1 := M' \cup \{e_1\}$. Then e does not intersect M_1 and thus $M_2 := M_1 \uplus \{e\}$ is also a matching that satisfies $y(M_1) = 0 = y(M_2)$. By construction we have $M_1, M_2 \in \mathcal{M}$, and hence $\langle c, \chi(M_1)\rangle + \gamma y(M_1) = \delta = \langle c, \chi(M_2)\rangle + \gamma y(M_2)$ holds. This proves $c_e = 0$.
- Let $u \in S^* \setminus \{a_1, b_2\}$ and let $e = \{u, v\}$ and $f = \{u, w\}$ be two incident edges with endnodes $v, w \in S^*$. W.l.o.g. we can assume $u \in A^*$, since the case of $u \in B^*$ is similar. Let M' be a perfect matching in $S^* \setminus \{a_1, u, v, w\}$ (which exists due to $|A^* \setminus \{a_1, u\}| = |B^* \setminus \{v, w\}|$). Define the two matchings $M_1 := M' \uplus \{e_1, e\}$ and $M_2 := M' \uplus \{e_1, f\}$, and observe that $M_1, M_2 \in \mathcal{M}$ and $y(M_1) = 0 = y(M_2)$ holds. From $\langle c, \chi(M_1)\rangle + \gamma y(M_1) = \delta = \langle c, \chi(M_2)\rangle + \gamma y(M_2)$ we obtain that $c_e = c_f$ holds.

- Let M' be a perfect matching in $S^* \setminus \{a_1, b_2\}$. Define the matchings $M_1 := M' \cup \{a_1, b_2\}$, $M_2 := M' \cup \{e_1\}$, $M_3 := M' \cup \{e_1\}$ and $M_4 := M' \cup \{e_1, e_2\}$. By construction we have $M_1, M_2, M_3, M_4 \in \mathcal{M}$, $y(M_1) = y(M_2) = y(M_3) = 0$ and $y(M_4) = 1$. Thus, $\langle c, \chi(M_i)\rangle + \gamma y(M_i) = \delta$ holds for $i = 1, 2, 3, 4$, which proves $c_{\{a_1,b_2\}} = c_{e_1} = c_{e_2} = c_{e_1} + c_{e_2} - \gamma$.

The arguments above already fix (c, γ) up to multiplication with a scalar. Hence we can assume that $\gamma = 1$ holds, which proves that (c, γ) is equal to the coefficient vector of Inequality (7) for $S = S^*$. Since there always exists a perfect matching M in S^*, and since such a matching has cardinality $|M| = \frac{1}{2}|S^*|$, we derive $\delta = \frac{1}{2}|S^*|$. This concludes the proof. □

7 Discussion

The observation from Section 2 that $P^{1Q}_{\text{match}} = P^{1Q\downarrow}_{\text{match}} \cap P^{1Q\uparrow}_{\text{match}}$ holds, is not specific to matching polytopes. In fact, this is a property of convex sets:

Proposition 29. *Let $C \subseteq \mathbb{R}^n$ be a convex set and let $C^{\uparrow} := \{x + \lambda \mathbb{e}_1 \mid x \in C, \lambda \geq 0\}$ and $C^{\downarrow} := \{x - \lambda \mathbb{e}_1 \mid x \in C, \lambda \geq 0\}$ its respective up- and downward monotonization of the first variable. Then $C = C^{\uparrow} \cap C^{\downarrow}$ holds.*

Proof. Clearly, $C \subseteq C^{\uparrow}, C^{\downarrow}$ and thus $C \subseteq C^{\uparrow} \cap C^{\downarrow}$ holds. In order to prove the reverse direction, let $x \in C^{\uparrow} \cap C^{\downarrow}$. By definition, there exist $x^{(1)}, x^{(2)} \in C$ and $\lambda_1, \lambda_2 \geq 0$ such that $x^{(1)} + \lambda_1 \mathbb{e}_1 = x = x^{(2)} - \lambda_2 \mathbb{e}_1$ holds. If λ_1 holds, then $x = x^{(1)} \in C$ holds, and we are done. Otherwise, the equation

$$\frac{\lambda_2}{\lambda_1 + \lambda_2} x^{(1)} + \frac{\lambda_1}{\lambda_1 + \lambda_2} x^{(2)} = \frac{\lambda_2(x - \lambda_1 \mathbb{e}_1) + \lambda_1(x + \lambda_2 \mathbb{e}_1)}{\lambda_1 + \lambda_2} = x$$

proves that x is a convex combination of two points in C, i.e., $x \in C$ holds, which concludes the proof. □

In the case of matching polytopes we intersect the up- and downward monotonizations with the 0/1-cube, but this does not interfere with the arguments provided above. In fact Proposition 29 does not generalize to the simultaneous monotonization of *several* variables. To see this, consider $P = \operatorname{conv}\{(0,0)^\intercal, (1,1)^\intercal\}$. Its upward-monotonization w.r.t. two both variables is $P + \mathbb{R}^2_+ = \mathbb{R}^2_+$, its downward-monotonization is $P - \mathbb{R}^2_+ = (1,1)^\intercal + \mathbb{R}^2_-$, but their intersection is equal to $[0,1]^2 \neq P$. Hence, this is a purely one-dimensional phenomenon.

Descriptions of monotonizations. A second property is specific, at least to polytopes arising from one-term linearizations: we can obtain the complete description for $P^{1Q\downarrow}_{\text{match}}$ from the one for P^{1Q}_{match} by omitting the $\leq$-inequalities that have a negative y-coefficient. Similarly, we obtain the complete description for $P^{1Q\uparrow}_{\text{match}}$ from the one for P^{1Q}_{match} by omitting the $\leq$-inequalities that have a positive y-coefficient and adding $y \leq 1$ (which is not facet-defining for P^{1Q}_{match}, see Proposition 25). The reason turns out to be that all facets of the projection of P^{1Q}_{match} onto the x-variables are projections of facets of P^{1Q}_{match}. The arguments for the upward-monotonization are as follows:
Let $P \subseteq \mathbb{R}^{n+1}$ be a polytope. After normalizing, we can write its outer description as

$$P = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} \mid Ax \leq b,\ Bx + \mathbb{1}y \leq c,\ Cx - \mathbb{1}y \leq d\}.$$

We assume that P's projection onto the x-variables is the polytope defined by $Ax \leq b$ only. P's upward-monotonization can be obtained by projecting the extended formulation

$$\{(x,y,y') \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \mid (x,y) \in P,\ y - y' \leq 0\}$$

onto the (x, y')-variables. Fourier-Motzkin elimination yields that this projection is described by $Ax \leq b$, $Cx - \mathbb{1}y' \leq d$ and inequalities that are the sum of an inequality from $Bx + \mathbb{1}y \leq c$ and an inequality from $Cx - \mathbb{1}y \leq d$. Since the last type of inequalities are already valid for P's projection onto the x-variables, these are already present in $Ax \leq b$.

Proof technique. The technique we applied in Sections 4 and 5 in order to prove our results can be summarized as follows:

1. Consider a fractional (extreme) point (of the polytope P in question) that satisfies a certain inequality with equality.
2. Modify that point in a way such that it lies in a face F of a polytope Q that we have under control.
3. Write the modified point as a convex combination of vertices of F and derive structural properties that are implied by the fact that they are vertices of that face.
4. Revert the modifications by replacing some of the vertices in the convex combination by others.

Clearly, this technique does not work for arbitrary polytopes. In fact it heavily depends on the fact that P is very related to Q, e.g., a subpolytope. Clearly, the more complicated the modifications are, the more involved the proof will probably be. Thus, on the one hand we believe that the applicability of the technique is quite limited. On the other hand, it does not require LP duality, and hence it could be useful when duality-based methods become unattractive because of many inequality classes.

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