# Perturbation Analysis of Singular Semidefinite Program and Its Application to a Control Problem

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We consider the sensitivity of semidefinite programs (SDPs) under perturbations. It is well known that the optimal value changes continuously under perturbations on the right hand side in the case where the Slater condition holds in the primal problems. In this manuscript, we observe by investigating a concrete SDP that the optimal value can be discontinuous if the dual problem is not strictly feasible and one perturbs the SDP with coefficient matrices. We show that the optimal value of such an SDP changes continuously if the perturbations preserve the rank of the space spanned by submatrices of the coefficient matrices and do not change the minimal face which is obtained by facial reduction algorithm. In addition, we determine the kinds of perturbations that make minimal faces invariant. Our results allow us to classify change of the minimal face of an SDP obtained from a control problem under linear perturbations which preserve matrix structures that appear in the associated dynamical systems.

**Keywords:** Semidefinite programming, sensitivity, the facial reduction, the minimal face, H-infinity feedback control problem

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## 1. Introduction

### 1.1. A singular SDP and its perturbation

A semidefinite program (SDP) is the problem of minimizing a linear objective function over the intersection of a positive semidefinite cone and an affine space over symmetric matrices.

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The primal SDP (P) and its dual (D) are mathematically formulated as follows:

$$\sup_{y,Z}\left\{b^Ty : A_0 - \sum_{k\in K} y_k A_k = Z, y\in\mathbb{R}^m,\ Z\in\mathbb{S}^n_+\right\}, \tag{P}$$

$$\inf_{X}\left\{A_0\bullet X : A_k\bullet X = b_k\ (k\in K), X\in\mathbb{S}^n_+\right\}, \tag{D}$$

where $A_k$ are symmetric matrices, $b\in\mathbb{R}^m$, $K=\{1,\ldots,m\}$, $A\bullet B=\sum_{i,j=1}^n A_{ij}B_{ij}$ for every $n\times n$ symmetric matrices $A$ and $B$, and $\mathbb{S}^n_+$ stands for the cone of $n\times n$ positive semidefinite matrices.

It is well known that the strong duality for (P) and (D) holds when both are feasible and the Slater condition holds in at least one of (P) or (D). An SDP is *nonsingular* if the Slater condition holds in both of (P) and (D); otherwise, it is *singular*. Interior-point methods for nonsingular SDP are guaranteed to converge to an optimal solution theoretically. Various variants of interior-point methods are implemented in software for solving SDPs, such as SDPA [8], SDPT3 [26] and SeDuMi [25].

The $H_\infty$ control problem is one of the successful applications of SDP, and is the problem for designing a controller that achieves stabilization with some guaranteed performance. This is reformulated as an SDP and can be solved by SDP software. See *e.g.*, [13, 24]. Various applications of SDP into robust control problems have been studied successfully, See *e.g.*, [24].

Our research interest is in the behavior of the optimal value of the following singular SDP under some perturbations, which are obtained from the $H_\infty$ state feedback control problem:

$$\sup_{x_1,\ldots,x_6}\left\{-x_6 : \begin{array}{l}\begin{pmatrix} 2x_1+2x_2 & & & & & \\ -x_1+x_2+x_3-x_4 & -2x_2-2x_5 & & & & \\ -2x_1+x_2-2x_4 & -2x_2+x_3-2x_5 & x_6 & & & \\ x_1-2x_2+x_4 & x_2-2x_3+x_5 & 0 & x_6 & & \\ 1 & 1 & 1 & 1 & x_6 & \\ 1 & 0 & 0 & 0 & 0 & x_6\end{pmatrix}\in\mathbb{S}^6_+, \\ \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3\end{pmatrix}\in\mathbb{S}^2_+, x_4,x_5,x_6\in\mathbb{R}\end{array}\right\}, \tag{1}$$

where blanks in matrices stand for the transpose of the lower triangular block part. We denote the first block of coefficient matrices in (1) corresponding to $x_i$ by $A_{1,i}$. For instance, $A_{1,1}$ and $A_{1,5}$ are

$$A_{1,1}=\begin{pmatrix} 2 & & & & & \\ -1 & 0 & & & & \\ -2 & 0 & 0 & & & \\ 1 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix} \text{ and } A_{1,5}=\begin{pmatrix} 0 & & & & & \\ 0 & -2 & & & & \\ 0 & -2 & 0 & & & \\ 0 & 1 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}.$$

(1) is obtained from the $H_\infty$ state feedback control problem for the following generalized plant:

$$\left\{\begin{array}{lcl} \dot{x} & = & \begin{pmatrix} -1 & -1 \\ 1 & 0\end{pmatrix}x + \begin{pmatrix} -1 & -1 \\ -1 & 0\end{pmatrix}w + \begin{pmatrix} 0 \\ 1\end{pmatrix}u \\ z & = & \begin{pmatrix} 2 & -1 \\ -1 & 2\end{pmatrix}x + \begin{pmatrix} -1 & 0 \\ -1 & 0\end{pmatrix}w + \begin{pmatrix} 2 \\ -1\end{pmatrix}u \end{array}\right. \tag{2}$$

It follows from [31, Theorems 3.3 and 3.5] that the Slater condition holds in (1), but fails in its dual, and thus we can say that the SDP is singular.

We compare computational results on (1) with the following three perturbed SDPs for (1): For $\epsilon$ =1.0e-16,

(P1) SDP obtained by perturbing the $(2,2)$nd element of $A_{1,5}$ in (1) into $-2(1+\epsilon)$,

(P2) SDP obtained by perturbing the $(2,3)$rd and $(3,2)$nd elements in $A_{1,5}$ of (1) into $-2(1+\epsilon)$.

(P3) SDP obtained by perturbing the $(2,4)$th and $(4,2)$nd elements of $A_{1,5}$ in (1) into $1+\epsilon$.

Since *e.g.* [28] that the standard floating point computation may provide wrong results for singular SDPs, we apply SDPA-GMP [8] to solve (1) with stopping tolerances $\delta$ ($\delta$=1.0e-10, 1.0e-30 and 1.0e-50) and set the floating point computation in to approximately 300 significant digits; otherwise one may encounter strange behavior for SDP software. See [10, 15, 28, 30] for the more details. We provide other parameters used for SDPA in Table 1. See [8] for more details on parameters. Table 2 shows the numerical results.

We observe from Table 2:

- The computed values of (1) are almost the same for all $\delta$, whereas the values for perturbed problems (P1), (P2) and (P3) are different. In fact, we can prove that the optimal values of (1) and (P1) are $-\sqrt{5}$ and $-\sqrt{2}$, respectively. We show the proof in Appendices A and B. This significant difference implies that one needs to choose suitable tolerances $\delta$ in order to use the floating point computation with longer significant digits for singular SDPs.

- The optimal value of (1) is $-\sqrt{5}$, while the optimal values of the perturbed problems are $-\sqrt{2}$ and $-2$. These differences show that a small perturbation of coefficient matrices $A_k$ in (P) may yield a significant change of the optimal value of (1).

Table 1: Information on parameters used for solving (1) and its perturbed problems

| parameter | value | parameter | value | parameter | value |
|---|---|---|---|---|---|
| maxIteration | 10000 | lowerBound | -1.0e+5 | gammaStar | 0.5 |
| epsilonStar | $\delta$ | upperBound | 1.0e+5 | epsilonDash | $\delta$ |
| lambdaStar | 1.0e+4 | betaStar | 0.5 | precision | 1024 |
| omegaStar | 2.0 | betaBar | 0.5 | | |

Table 2: Computed values for (1) and its perturbed problems (P1), (P2) and (P3) by SDPA-GMP

| Problem | $\delta$ =1.0e-10 | $\delta$ =1.0e-30 | $\delta$ =1.0e-50 |
|---|---|---|---|
| (1) | -2.2360679775444764 | -2.2360679774997897 | -2.2360679774997897 |
| (P1) | -2.2360072694172072 | -2.1078335768712432 | -1.4142135623730950 |
| (P2) | -2.2360072694172055 | -2.0000000000000000 | -2.0000000000000000 |
| (P3) | -2.2360072665294605 | -1.4142135623730950 | -1.4142135623730950 |

In summary, perturbations of coefficient matrices $A_k$ in (P) may significantly change the optimal value of perturbed SDP when the original SDP is singular. The motivation of this manuscript is to determine the kinds of perturbations of $A_{1,i}$ in (1) that do not change the optimal values significantly.

### 1.2. Contribution and literature

The main contribution of this manuscript is to determine and analyze the kinds of perturbations of $A_{1,i}$ in (1) that do not change the optimal values significantly and to provide its generalization to (P) and (D). To this end, we apply any perturbations of coefficient matrices $A_k$ and $b_k$ in (P) and (D) which preserve the feasibility of the perturbed SDPs. More precisely, we analyze the following perturbed SDP and its dual.

$$\sup_{y,Z} \left\{ b(t)^T y : \sum_{k \in K} y_k A_k(t) + Z = A_0(t), y \in \mathbb{R}^m,\ Z \in \mathbb{S}^n_+ \right\}, \tag{$P_t$}$$

$$\inf_{X} \left\{ A_0(t) \bullet X : A_k(t) \bullet X = b_k(t)\ (k \in K), X \in \mathbb{S}^n_+ \right\}, \tag{$D_t$}$$

where $t \geq 0$, $A_k(t) \in \mathbb{S}^n,\ b(t) \in \mathbb{R}^m$ are continuous at $t = 0$ and $A_k(0) = A_k,\ b(0) = b$.

First, we show the continuity of the set of optimal solutions of $(P_t)$ and $(D_t)$ for nonsingular SDP, *i.e.*, that the Slater condition holds in both (P) and (D). This has been shown by Gol'šhteĭn for general convex programs, however, here we give a new proof for this case, and extend the result on continuity of the optimal value to the case of singular SDPs. Similar results for nonsingular SDPs can be obtained using the inf-compactness condition [2]. Although an individual optimal solution is rarely continuous under perturbations as in the case of linear programming, we can extract sufficient conditions for continuity of an optimal solution as in Alizadeh, Haeberly and Overton [1]. Namely, suppose that $(X, Z)$ is a pair of optimal solutions for (D) and (P) respectively. Then $(X, Z)$ moves continuously if the Slater condition holds in both (D) and (P), $(X, Z)$ satisfies the strict complementarity condition, $(X, Z)$ is nondegenerate and positive eigenvalues of $X$ and $Z$ are all distinct. It is well known in [4] from the general theory in Convex Analysis that the optimal value changes continuously if one of (P) and (D) satisfies the Slater condition in the case of perturbing only $A_0(t)$. Cheung and Wolkowicz [5] provide a more detailed analysis of this fact, *i.e.*, an order of how a small perturbation in only matrix $A_0(t)$ for the right-hand side of a linear matrix inequality in (P) changes continuously for an optimal value for the SDP under the assumption that the Slater condition holds in (D), but fails in (P). On the other hand, we deal with any perturbation in $A_0$, $A_k$ and $b_k$ in both (P) and (D). It changes the behavior of the optimal value enormously. In fact, the continuity of the optimal value is not guaranteed when the Slater condition holds in only one of (P) and (D), as presented in Table 2.

The second contribution is to provide a result on the continuity of the set of optimal solutions of (P) and (D) by any perturbation for singular SDPs by which $(D_t)$ is feasible. We show that the optimal value of (P) does not significantly change by a perturbation of (P) if $(D_t)$ is feasible and it changes continuously *the minimal face* to (D). Here the minimal face is the intersection of all faces of $\mathbb{S}^n_+$ that contains the feasible region of (D). This proof is based on the first contribution, *i.e.*, the continuity of the set of optimal solutions of (P) and (D) by any perturbation for nonsingular SDPs.

The third contribution is to give conditions on perturbations for the minimal face not to change when the perturbations are restricted to be linear. Using these conditions, we show that the minimal face of the dual of (1) does not change or changes into the full-dimensional cone under perturbations which preserve matrix structures that appear in the $H_\infty$ state feedback control problem obtained for dynamical systems (2).

On the other hand, we see a significant change among their duals of SDPs (1), (P1), (P2) and (P3). The reason for (P1) and (P2) is that their perturbations with any $\epsilon > 0$ change largely the minimal faces of their duals. In fact, we prove in Appendix C that the minimal face of the dual of its perturbed SDP (P1) is smaller than that of the dual of the original SDP (1). It follows from this fact and Table 2 that the optimal value of (P1) has a big jump at $\epsilon = 0$. We show in Appendix C that *facial reduction*, which is a procedure to find the minimal face, for the dual of (1) requires one iteration to obtain the original SDP, whereas facial reduction for the dual of its perturbed SDP (P1) requires two iterations. The number of iterations for the facial reduction is called *the degree of singularity*. The degrees of singularity for the original SDP (1), the dual of (1), and duals of (P1) are 0, 1 and 2, respectively. Cheung and Wolkowicz [5] prove that the difference of the optimal values between a singular SDP and its SDP obtained by linearly perturbing $A_0$ depends on the degree of singularity of the original SDP. This fact implies that a perturbation for coefficient matrices in SDP (P) may change the degree of singularity and make the original SDP more difficult to solve.

The organization of this manuscript is as follows: preliminaries on the minimal face and facial reduction, which finds the minimal face, are introduced in Section 2. In Section 3.2, we show the main results on the continuity of the set of optimal solutions of (P) and (D) for nonsingular SDPs. Singular SDPs are discussed in Section 3.3. In Section 4, we give conditions on perturbations under which the minimal face does not change.

## 2. Preliminaries on face, minimal face and facial reduction

We give a brief introduction to define a face for convex sets and the minimal face for SDPs. These definitions are described in [5, 17, 21] in detail below.

For a convex set $C \subseteq \mathbb{R}^n$, a convex subset $F$ of $C$, we say that $F$ is a *face* of $C$ if $x_1, x_2 \in C$. The intersection of the open line segment $(x_1, x_2)$ and $F$ is nonempty implies that $x_1, x_2$ are both in $F$. For a nonempty convex subset $S$ of $C$, the *minimal face* of $C$ containing $S$ is defined as the intersection of all faces of $C$ that contain $S$.

The following results on facial structure of $\mathbb{S}^n_+$ are known in *e.g.* [16, 18].

**Lemma 2.1.** *(F1) Any face of $\mathbb{S}^n_+$ is either $\{0\}$, $\mathbb{S}^n_+$ or*

$$\left\{X \in \mathbb{S}^n : X = Q \begin{pmatrix} M & O_{r\times(n-r)} \\ O_{(n-r)\times r} & O_{r\times r} \end{pmatrix} Q^T, M \in \mathbb{S}^r_+ \right\}, \tag{3}$$

*where $Q$ is an $n \times n$ orthogonal matrix. It follows from this property that the set $\mathbb{S}^n_+ \cap \{W\}^\perp$ is a face of $\mathbb{S}^n_+$, where $\{W\}^\perp$ stands for the orthogonal complement of the subspace spanned by $W$, i.e., $\{W\}^\perp = \{X \in \mathbb{S}^n : X \bullet W = 0\}$.*

*(F2) When a face is of form (3), the relative interior is of form*

$$\left\{X \in \mathbb{S}^n : X = Q \begin{pmatrix} M & O_{r\times(n-r)} \\ O_{(n-r)\times r} & O_{r\times r} \end{pmatrix} Q^T, M \in \mathbb{S}^r_{++} \right\},$$

*where $\mathbb{S}^r_{++}$ stands for the cone of $r \times r$ positive definite matrices.*

*(F3) For $W_1, \ldots, W_r \in \mathbb{S}^n_+$, we have*

$$\mathbb{S}^n_+ \cap \bigcap_{i=1}^{r} \{W_i\}^\perp = \mathbb{S}^n_+ \cap \{W_1 + \cdots + W_r\}^\perp ,$$

*(F4) The set $\mathbb{S}^n_+ + F^\perp$ is closed for all faces $F$ of $\mathbb{S}^n_+$, where $F^\perp$ stands for the set $\{Z \in \mathbb{S}^n : Z \bullet X = 0\ (\forall X \in F)\}$ and $F^*$ is the dual cone of $F$, i.e., $F^* = \{Z \in \mathbb{S}^n : Z \bullet X \geq 0\ (\forall X \in F)\}$. This property is called the niceness. The niceness property implies that $F^* = \mathbb{S}^n_+ + F^\perp$ for all faces $F$ of $\mathbb{S}^n_+$.*

We define the minimal face and facial reduction for only (D) because the the dual of (1), which is the motivation of this manuscript, is singular and (1) is nonsingular. One can discuss them for (P) in a similar manner. The minimal face for (D) is defined as the minimal face of $\mathbb{S}^n_+$ containing the feasible region $F_D$ of (D). We denote the minimal cone by $F_{\min}$. The following results on the minimal face is known as:

**Lemma 2.2.** *[19, SDP version of Lemma 28.4] For (D), assume that (D) is feasible and, let $F$ be a face of $\mathbb{S}^n_+$ that contains $F_{\min}$. Then $F \neq F_{\min}$ if and only if there exists $(y, W) \in \mathbb{R}^m \times \mathbb{S}^n$ such that*

$$b^T y = 0, W = -\sum_{k \in K} y_k A_k \in F^* \setminus F^\perp. \tag{4}$$

*Moreover, we have $F \cap \{W\}^\perp$ is a face of $\mathbb{S}^n_+$ and $F_{\min} \subseteq F \cap \{W\}^\perp \subsetneq F$.*

We call the above system (4) the *discriminant system* for facial reduction for (D) and a solution $(y, W)$ a *reducing certificate*. We remark that since the positive semidefinite cone has the niceness property in (F4) of Lemma 2.1, (4) can be rewritten as follows:

$$b^T y = 0, W = -\sum_{k \in K} y_k A_k \in (\mathbb{S}^n_+ + F^\perp) \setminus F^\perp.$$

Facial reduction in [3, 20, 17, 19] is based on Lemma 2.2 and is a procedure to generate a sequence $\{F_i\}_{i=0}^s$ of faces of $\mathbb{S}^n_+$ that contains the feasible region, where $F_0 = \mathbb{S}^n_+$ and $F_s = F_{\min}$ and the sequence satisfies $F_{i+1} \subsetneq F_i$ for all $i = 0, \ldots, s-1$. At the $i$th iteration of facial reduction, if a face $F_i$ is not the minimal face $F_{\min}$, then $F_{i+1} = F_i \cap \{W\}^\perp$, where $W$ is a reducing certificate $(y, W)$ in (4). More precisely, the process can be represented as

$$(D) \quad \mathbb{S}^n_+ \overset{(y^1,W^1)}{\longrightarrow} F_1 \overset{(y^2,W^2)}{\longrightarrow} F_2 \overset{(y^3,W^3)}{\longrightarrow} \cdots \overset{(y^s,W^s)}{\longrightarrow} F_s = F^D_{\min}.$$

Here we call $(y^1, W^1), \ldots, (y^s, W^s)$ *a facial reduction sequence* for (D). Facial reduction terminates in at most $n - 1$ iterations or detects the infeasibility of (D). See [19, 27, 29] for the details. We describe facial reduction for (D) in Algorithm 1.

**Algorithm 1:** Facial reduction for (D)

**Input**: Feasible SDP (D)
**Output**: Minimal face $F_{\min}$ of (D)
$F \longleftarrow \mathbb{S}^n_+$;
**while** $\exists$ *reducing certificate* $(y, W)$ *that satisfies (4)* **do**
  $F \longleftarrow F \cap \{W\}^\perp$;
**end**
**return** $F$;

We remark that a solution of the discriminant system (4) is not unique even if (4) is solvable. Cheung and Wolkowicz [5, Proposition B.1] prove that any two finite sequences of reducing certificates for (D) must be of the same length. The length is equal to the number of iteration to find the minimal face $F^D_{\min}$ of (D) and is called *the degree of singularity for (D).*

One of the numerical difficulties in facial reduction is to find reducing certificates $(y, W)$ numerically. A straightforward computation of $(y, W)$ is to convert (4) into an SDP. This, however, may cause the numerical instability if the SDP problem or its dual is not strictly feasible. Instead of solving the SDP problems, partial but robust facial reductions are proposed by using properties and structures in the original problems. For instance, see [33, 32] for semidefinite programming relaxation of combinatorial optimization problems, [14] for Euclidean distance matrix completion problems, [29] for sum-of-square problems and [31] for $H_\infty$ state feedback control problems. Facial reductions are executed in their work without solving any SDP problems to find reducing certificates numerically.

Finally, we give a description of facial reduction for SDP (D) which has multiple linear matrix inequalities. For simplicity, we deal with SDP (1). Since this SDP can be reformulated as follows, we see that it has two linear matrix inequalities.

$$\sup_{y, Z_1, Z_2} \left\{ b^T y : \sum_{k=1}^{6} y_k A_{i,k} + Z_i = A_{i,0} \ (i = 1, 2), y \in \mathbb{R}^6, \ (Z_1, Z_2) \in \mathbb{S}^6_+ \times \mathbb{S}^2_+ \right\}. \tag{5}$$

We denote coefficient matrices $A_k$ for (5) as follows:

$$A_k = \begin{pmatrix} A_{1,k} & O_{6,2} \\ O_{2,6} & A_{2,k} \end{pmatrix} \ (k = 0, 1, \ldots, 6).$$

We denote this matrix by $(A_{1,k}, A_{2,k})$ in this manuscript for simplicity. Then (5) can be equivalently reformulated as

$$\sup_{y, Z} \left\{ b^T y : \sum_{k=1}^{6} y_k A_k + Z = A_i, y \in \mathbb{R}^6, \ Z_1 \in \mathbb{S}^8_+ \right\}.$$

The discriminant system (4) with $F = \mathbb{S}^8_+$ is also reformulated as

$$b^T y = 0, W = -\sum_{k=1}^{6} y_k A_k \in \mathbb{S}^8_+ \setminus \{O_{8\times 8}\}.$$

This is equivalent to

$$b^T y = 0, W_1 = -\sum_{k=1}^{6} y_k A_{1,k} \in \mathbb{S}^6_+ \setminus \{O_{6\times 6}\}, W_2 = -\sum_{k=1}^{6} y_k A_{2,k} \in \mathbb{S}^2_+ \setminus \{O_{2\times 2}\}.$$

Then facial reduction generates a face $F_1 = \mathbb{S}^6_+ \cap \{W_1\}^\perp$ of $\mathbb{S}^6_+$ and a face $F_2 = \mathbb{S}^2_+ \cap \{W_2\}^\perp$ of $\mathbb{S}^2_+$. We say in this manuscript that facial reduction generates a face $F_1 \times F_2$ of the positive semidefinite cone $\mathbb{S}^6_+ \times \mathbb{S}^2_+$.

# 3. Main results

## 3.1. Continuity of Slater points

In this subsection, we discuss the continuity of Slater points. Investigations on behavior of feasible points have a long history and they are often studied via the concept of metric regularity under the condition called the Robinson's constraint qualification [22]. For recent developments, see *e.g.* [7, 11, 12]. In this instance, we need to consider the case that the Robinson's constraint qualification fails and that the theory of metric regularity cannot be applied. On the other hand, the constraints of SDP have linear structures and thus we can use pseudo-inverse operator to analyze the behavior of feasible points.

For $A = (a_{ij})_{1\le i,j\le n} \in \mathbb{S}^n$, $\mathrm{vec}(A)$ is defined as the vectorization of $A$, *i.e.*,

$$\mathrm{vec}(A) = (a_{11}, a_{12}, \dots, a_{1n}, a_{21}, a_{22}, \dots, a_{n1}, \dots, a_{nn})^T.$$

We define the subspace spanned by $A_1, \dots, A_m$ by $\langle A_1, \dots, A_m\rangle$. The dimension of the subspace is denoted by $\mathrm{rank}\langle A_1, \dots, A_m\rangle$. We remark that this is equal to the rank of the matrix $(\mathrm{vec}(A_1), \dots, \mathrm{vec}(A_m))$.

**Lemma 3.1.** *For a matrix $A$ and vector $b$, consider the equation*

$$Ax = b. \tag{Eq}$$

*Suppose $x_0$ is a solution to (Eq) and matrices $E \to 0$, vectors $e \to 0$ satisfy the following conditions: $b + e \in \mathrm{Im}(A + E)$ and $\mathrm{rank}(A + E) = \mathrm{rank}\, A$. Then there exist $\{x_E\}$ which are solutions to*

$$(A + E)x = b + e \tag{$Eq_E$}$$

*such that $x_E \to x_0$ as $(E, e) \to (O, 0)$.*

*Proof.* For the pseudoinverse $A^\dagger$ of $A$, we can write

$$x_0 = x_0 - A^\dagger b + A^\dagger b = (I - A^\dagger A)x_0 + A^\dagger b.$$

Here we put $x_E = (I - (A + E)^\dagger(A + E))x_0 + (A + E)^\dagger(b + e)$. Then we have

$$\begin{aligned}(A + E)x_E &= (A + E)x_0 - (A + E)(A + E)^\dagger(A + E)x_0 + (A + E)(A + E)^\dagger(b + e)\\ &= (A + E)x_0 - (A + E)x_0 + b + e = b + e\end{aligned}$$

and hence $x_E$ is a solution to ($Eq_E$). Since $\mathrm{rank}(A + E) = \mathrm{rank}\, A$, we have $(A + E)^\dagger \to A^\dagger$ by [23, Theorem 5.2]. Therefore

$$\begin{aligned}\|x_E - x_0\| &\le \|(I - (A + E)^\dagger(A + E))x_0 - (I - A^\dagger A)x_0\| + \|(A + E)^\dagger(b + e) - A^\dagger b\|\\ &\le \|(A + E)^\dagger(A + E) - A^\dagger A\|\|x_0\| + \|(A + E)^\dagger - A^\dagger\|\|b\| + \|(A + E)^\dagger\|\|e\| \to 0,\end{aligned}$$

as $(E, e) \to (O, 0)$. □

**Theorem 3.2.** *Suppose $X_0$ is a strictly feasible point of (D). If for each $t \in [0, \delta]$, ($D_t$) is feasible and $\mathrm{rank}\langle A_1(t), \dots, A_m(t)\rangle = \mathrm{rank}\langle A_1, \dots, A_m\rangle$, then there exist strictly feasible points $\{X_t\}$ of ($D_t$) with $X_t \to X_0$.*

*Proof.* Define $S(t) = \begin{pmatrix}\mathrm{vec}(A_1(t)) & \cdots & \mathrm{vec}(A_m(t))\end{pmatrix}$. Then the constraints of $(D_t)$ are defined as $S(t)^T \mathrm{vec}(X) = b(t)$ and $X \in \mathbb{S}^n_+$. Let $X_0$ be a strictly feasible point of (D). Then we have $S(0)^T \mathrm{vec}(X_0) = b$ and $X_0 \in \mathbb{S}^n_{++}$. Since the feasibility of $(D_t)$ ensures $b(t) \in \mathrm{Im}\, S(t)^T$ and the rank condition is satisfied, we can apply Lemma 3.1. There exist feasible points $\{X_t\}$ of $(D_t)$ with $X_t \to X_0$, therefore $X_t$ is strictly feasible for sufficiently small $t$. □

**Theorem 3.3.** *Suppose $(y_0, Z_0)$ be a strictly feasible point of (P). Then there exist strictly feasible points $\{(y_t, Z_t)\}$ of $(P_t)$ such that $(y_t, Z_t) \to (y_0, Z_0)$.*

*Proof.* For $S(t)$ in the proof of Theorem 3.2, the feasible set of $(P_t)$ is the set of $(y, Z)$ which satisfies the system

$$\begin{pmatrix} S(t) & I_{n^2} \end{pmatrix} \begin{pmatrix} y \\ \mathrm{vec}(Z) \end{pmatrix} = \mathrm{vec}(A_0(t)), \;\; Z \in \mathbb{S}^n_+.$$

Here the coefficient matrix has a full row rank, the conditions in Lemma 3.1 are satisfied and hence the desired strictly feasible points exist. □

## 3.2. Main results I : Stability of nonsingular SDPs

In this section, we show that the optimal value of nonsingular SDPs changes continuously. We impose the following conditions:

*Condition* 1.

1. Both (P) and (D) are strictly feasible;
2. $A_1, \dots, A_m$ are linearly independent.

**Theorem 3.4.** *Under Condition 1, the optimal value of $(D_t)$ varies continuously.*

To prove this theorem, we use the following lemmas.

**Lemma 3.5.** *Under Condition 1, let $(X_0, y_0, Z_0)$ be strict feasible points of (D) and (P). Then there exist strictly feasible points $(D_t)$ and $(P_t)$ such that $(X_t, Z_t) \to (X_0, Z_0)$.*

*Proof.* By strict feasibility, $(P_t)$ and $(D_t)$ are feasible for sufficiently small $t$. In addition, the rank condition in Theorem 3.2 is satisfied since $\{A_k\}_{k\in K}$ is linearly independent. Therefore Theorems 3.3 and 3.2 ensure the conclusion. □

Let $U(t)$ be the set of optimal solutions of $(D_t)$ and

$$V(t) = \{Z \in \mathbb{S}^n \mid (y, Z) \text{ is optimal in } (P_t)\}.$$

**Lemma 3.6.** *Suppose that there exist strictly feasible points $(X_t, Z_t)$ of $(D_t)$ and $(P_t)$ such that $(X_t, Z_t) \to (X_0, Z_0)$. Then both sets $U(t)$ and $V(t)$ are nonempty and uniformly bounded, i.e., there exist $\delta > 0$ and compact sets $C_1, C_2$ such that*

$$U(t) \subset C_1, \;\; V(t) \subset C_2 \quad (0 \le t \le \delta).$$

*Proof.* Since $(D_t)$ and $(P_t)$ have strictly feasible points, the strong duality theorem ensures that $U(t)$ and $V(t)$ are nonempty. Let $X$ and $(y, Z)$ be arbitrary optimal solutions to $(D_t)$ and $(P_t)$. Since they are feasible, we have

$$A_k(t) \bullet (X - X_t) = 0,\ \sum_{k\in K}(y_k - y_{t,k})A_k(t) + Z - Z_t = 0.$$

Then it implies that $(X - X_t) \bullet (Z - Z_t) = 0$ and hence $X \bullet Z_t + X_t \bullet Z = X_t \bullet Z_t$. Moreover, positive semidefiniteness guarantees that $X \bullet Z_t \leq X_t \bullet Z_t$. Thus, by Lemma 3.5, there exists $\epsilon > 0$ such that for sufficiently small $t$,

$$\|X\| \leq \frac{X_t \bullet Z_t}{\lambda_{\min}(Z_t)} < \frac{X_0 \bullet Z_0 + \epsilon}{\lambda_{\min}(Z_0) - \epsilon}.$$

Therefore $U(t)$ is uniformly bounded for small $t$. Similar arguments are applied to $V(t)$. □

**Lemma 3.7.** *Suppose that $(D_t)$ has the same optimal value as $(P_t)$ and both have optimal solutions. We define the function $L : \mathbb{S}^n \times \mathbb{R}^m \times [0, +\infty) \to \mathbb{R}$ as follows:*

$$L(X, y, t) = A_0(t) \bullet X + \sum_{k\in K} y_k(A_k(t) \bullet X - b_k(t)).$$

*Then $(X(t), y(t), A_0(t) - \sum_k y_k(t)A_k(t))$ are optimal solutions of $(D_t)$ and $(P_t)$ if and only if*

$$L(X(t), y, t) \leq L(X(t), y(t), t) \leq L(X, y(t), t),\ \forall (X, y) \in \mathbb{S}^n_+ \times \mathbb{R}^m.$$

*Proof.* We can prove by direct computation. □

In the following, we denote the unit ball in $\mathbb{S}^n$ by $\mathbb{B}$. For $X \in \mathbb{S}^n$ and $C \subset \mathbb{S}^n$, we define

$$d(X, C) = \inf\{\|X - Y\| : Y \in C\}.$$

**Proposition 3.8.** *If both $U(t)$ and $V(t)$ are nonempty and uniformly bounded, and $\{A_k\}$ is linearly independent, then for any $\epsilon > 0$ there exists $\eta > 0$ such that*

$$U(t) \subset U(0) + \epsilon\mathbb{B},\ V(t) \subset V(0) + \epsilon\mathbb{B} \quad (0 \leq t \leq \eta).$$

*Proof.* Suppose that the conclusion is false. Then there exist $\epsilon > 0$, $t_j \to 0$ and $X(t_j) \in U(t_j)$, $Z(t_j) \in V(t_j)$ such that

$$d\left(X(t_j), U(0)\right),\ d\left(Z(t_j), V(0)\right) \geq \epsilon. \tag{6}$$

Since $\{A_k\}_{k\in K}$ is linearly independent, there exists a unique $y(t_j)$ such that $(y(t_j), Z(t_j))$ is optimal in $(P_{t_j})$ for each $j$. By Lemma 3.7, we have

$$L(X(t_j), y, t_j) \leq L(X(t_j), y(t_j), t_j) \leq L(X, y(t_j), t_j),\ \forall (X, y) \in \mathbb{S}^n_+ \times \mathbb{R}^m.$$

Since $\{(X(t_j), y(t_j), Z(t_j))\}$ is uniformly bounded, we may assume that $(X(t_j), y(t_j), Z(t_j)) \to (\widetilde{X}, \tilde{y}, \widetilde{Z})$ as $j \to \infty$. Thus we have

$$L(\widetilde{X}, y, 0) \leq L(\widetilde{X}, \tilde{y}, 0) \leq L(X, \tilde{y}, 0),\ \forall (X, y) \in \mathbb{S}^n_+ \times \mathbb{R}^m.$$

By applying Lemma 3.7 again, $(\widetilde{X}, \tilde{y}, \widetilde{Z})$ is a optimal of (P) and (D). This contradicts the inequalities (6). □

*Proof of Theorem 3.4.* By Proposition 3.8, for any $\epsilon > 0$, $X(t) \in U(t)$, there exist $\eta > 0$ and $X^t \in U(0)$ such that for $t \in (0, \eta)$,

$$|A_0(t) \bullet X(t) - A_0 \bullet X^t| \leq k_1\|X(t) - X^t\| + k_2\|A_0(t) - A_0(0)\| < \epsilon,$$

for some $k_1, k_2 > 0$. This completes the proof of Theorem 3.4. □

### 3.3. Main results II : Stability of singular SDPs

In this subsection, we consider continuity of the optimal value of singular SDPs. We impose the following conditions on a SDP:

*Condition* 2.

1. (D) is feasible and (P) is strictly feasible;

2. $A_1, \dots, A_m$ are linearly independent.

Then by applying the facial reduction in Algorithm 1 to (D), there exists orthogonal matrix $Q$ and $r \in \mathbb{N}$ such that

$$\inf_{X_3} \left\{ Q^T A_0 Q \bullet \begin{pmatrix} O & O \\ O & X_3 \end{pmatrix} : \begin{array}{l} Q^T A_k Q \bullet \begin{pmatrix} O & O \\ O & X_3 \end{pmatrix} = b_k \ (k \in K), \\ X_3 \in \mathbb{S}^r_+ \end{array} \right\} \tag{$F(D)$}$$

has the same optimal value as (D) and $F(D)$ is strictly feasible.

Here for $n \times n$ matrix $M$, we denote by $M_3$ the right bottom block of the partitioning

$$M = \begin{pmatrix} M_1 & M_2^T \\ M_2 & M_3 \end{pmatrix}, \tag{7}$$

where $M_1 \in \mathbb{S}^{n-r}, M_2 \in \mathbb{R}^{r \times (n-r)}, M_3 \in \mathbb{S}^r$. We note that the minimal face of (D) determines this partitioning uniquely. Then we can rewrite this problem as follows:

$$\inf_X \left\{ (Q^T A_0 Q)_3 \bullet X : (Q^T A_k Q)_3 \bullet X = b_k(t) \ (k \in K), X \in \mathbb{S}^r_+ \right\}. \tag{$F(D)$}$$

**Theorem 3.9.** *Under Condition 2, suppose that the minimal face of (D) can be written as*

$$\left\{ Q \begin{pmatrix} O & O \\ O & X \end{pmatrix} Q^T \;\middle|\; X \in \mathbb{S}^r_+ \right\}$$

*for some orthogonal matrix $Q$ and $r \in \mathbb{N}$. In addition if $\{A(t), b(t) \mid 0 \le t \le \delta\}$ satisfy the following assumptions for some $\delta > 0$:*

1. *($D_t$) is feasible for $t \in [0, \delta]$;*

2. *For any $t \in [0, \delta]$, there exist $Q(t)$ such that $\lim_{t \to 0} Q(t) = Q$ and the minimal face of ($D_t$) can be written as*

$$\left\{ Q(t) \begin{pmatrix} O & O \\ O & X \end{pmatrix} Q(t)^T \;\middle|\; X \in \mathbb{S}^r_+ \right\};$$

3. $\operatorname{rank} \left\langle (Q(t)^T A_1(t) Q(t))_3, \dots, (Q(t)^T A_m(t) Q(t))_3 \right\rangle$
$= \operatorname{rank} \left\langle (Q^T A_1 Q)_3, \dots, (Q^T A_m Q)_3 \right\rangle, \ 0 \le t \le \delta,$

*then the optimal value of ($D_t$) varies continuously at $t = 0$.*

*Proof.* By the assumptions 1 and 2, the optimal value of $(D_t)$ is equal to

$$\inf_X \left\{ (Q(t)^T A_0(t) Q(t))_3 \bullet X : (Q(t)^T A_k(t) Q(t))_3 \bullet X = b_k(t) \ (k \in K), X \in \mathbb{S}^r_+ \right\} \qquad (F(D_t))$$

for each $t \in [0, \delta]$. Thus if the continuity of the optimal values of $F(D_t)$ at $t = 0$ are shown, then that of the optimal values of $(D_t)$ is also shown. Now the dual of $F(D_t)'$ is

$$\sup_{y,Z} \left\{ b(t)^T y : \sum_{k \in K} y_k (Q(t)^T A_k(t) Q(t))_3 + Z = (Q(t)^T A_0 Q(t))_3, Z \in \mathbb{S}^r_+ \right\}. \qquad (F(D_t)')$$

Then $F(D_t)$ has the same optimal value of $F(D_t)'$. In addition, $F(D_t)$ and $F(D_t)'$ have strictly feasible points $X_0$ and $(y_0, Z_0)$ respectively by the properties of the facial reduction algorithm.

By the assumptions 1 and 3, we can apply Theorem 3.2 and hence there exist strictly feasible points $\{X_t\}$ of $F(D_t)$ with $X_t \to X_0$. We also have strictly feasible points $\{(y_t, Z_t)\}$ of $F(D_t)'$ with $(y_t, Z_t) \to (y_0, Z_0)$ by Theorem 3.3. Thus Lemma 3.6 implies that

$$M_t = \{(X, Z) \mid X \text{ is an optimal of } F(D_t),\ (y, Z) \text{ is an optimal of } F(D_t)'\}$$

is uniformly bounded for $t \in [0, \delta']$ for some $\delta' > 0$. For any $t_j \to 0$, let $(X_{t_j}, Z_{t_j}) \in M_{t_j}$ and

$$S_Q(t) = \begin{pmatrix} \mathrm{vec}(Q(t)^T A_1(t) Q(t))_3) & \cdots & \mathrm{vec}(Q(t)^T A_m(t) Q(t))_3) \end{pmatrix},$$

for $0 \le t \le \delta'$. Then $(X(t_j), S_Q(t_j)^\dagger (A_0(t_j) - Z(t_j)), Z(t_j))$ is a pair of optimal solutions of $F(D_{t_j})$ and $F(D_{t_j})'$. By the assumption 3, $S_Q(t_j)^\dagger \to S_Q(0)^\dagger$ and hence $\{(X(t_j), S_Q(t_j)^\dagger(A_0(t_j) - Z(t_j)), Z(t_j))\}$ is bounded. Therefore we may assume $(X(t_j), S_Q(t_j)^\dagger(A_0(t_j) - Z(t_j)), Z(t_j)) \to (\widetilde{X}, \tilde{y}, \widetilde{Z})$. Thus by the same arguments in Proposition 3.8, for any $\epsilon > 0$ there exists $\eta > 0$ such that $M_t \subset M_0 + \epsilon\mathbb{B} \quad (0 \le t \le \eta)$. This implies the continuity of the optimal value of $F(D_t)$ at $t = 0$. □

**Corollary 3.10.** *Suppose that there exist $\delta > 0$ such that for all $t \in [0, \delta]$, $(D_t)$ is feasible and has the same minimal face as (D), and*

$$\mathrm{rank} \left\langle (Q^T A_1(t) Q)_3, \ldots, (Q^T A_m(t) Q)_3 \right\rangle = \mathrm{rank} \left\langle (Q^T A_1 Q)_3, \ldots, (Q^T A_m Q)_3 \right\rangle$$

*for $0 \le t \le \delta$ . Then the optimal value of $(D_t)$ varies continuously at $t = 0$.*

*Example* 3.11. We present an example of perturbations which preserve the minimal face but do not satisfy the rank condition. Let the coefficient matrices in SDP (1) corresponding to $x_k$ be $A_k$, then $A_1 = (A_{1,1}, A_{1,2})$, $A_2 = (A_{2,1}, A_{2,2})$, where

$$A_{1,1} = \begin{pmatrix} 2 & & & & & \\ -1 & 0 & & & & \\ -2 & 0 & 0 & & & \\ 1 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ A_{1,2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$A_{2,1} = \begin{pmatrix} 2 & & & & & \\ 1 & -2 & & & & \\ 1 & -2 & 0 & & & \\ -2 & 1 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ A_{2,2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and others are chosen similarly.

For the dual of (1), the following $(y, W)$ is the reducing certificate obtained at the first iteration of facial reduction:

$$y = (1,0,0,-1,0,0)^T, W = (W_1, W_2), W_1 = \begin{pmatrix} 2 & & & & & \\ 0 & 0 & & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, W_2 = \begin{pmatrix} 1 & \\ 0 & 0 \end{pmatrix}. \quad (8)$$

The resulting faces are

$$\begin{aligned} F_1^1 &= \mathbb{S}_+^6 \cap \{W_1\}^\perp = \left\{ X \in \mathbb{S}^6 : X = \begin{pmatrix} 0 & 0 \\ 0 & X_1 \end{pmatrix}, X_1 \in \mathbb{S}_+^5 \right\}, \text{ and} \\ F_2^1 &= \mathbb{S}_+^2 \cap \{W_2\}^\perp = \left\{ X \in \mathbb{S}^2 : X = \begin{pmatrix} 0 & 0 \\ 0 & X_2 \end{pmatrix}, X_2 \geq 0 \right\}. \end{aligned} \quad (9)$$

At the second iteration, we need to find a solution $(y, W)$ such that

$$\begin{aligned} & y_6 \leq 0, \begin{pmatrix} y_1 & \\ y_2 & y_3 \end{pmatrix} \in (F_2^1)^*, \\ & \begin{pmatrix} 2y_1 + 2y_2 & & & & & \\ -y_1 + y_2 + y_3 - y_4 & -2y_2 - 2y_5 & & & & \\ -2y_1 + y_2 - 2y_4 & -2y_2 + y_3 - 2y_5 & y_6 & & & \\ y_1 - 2y_2 + y_4 & y_2 - 2y_3 + y_5 & 0 & y_6 & & \\ 0 & 0 & 0 & 0 & y_6 & \\ 0 & 0 & 0 & 0 & 0 & y_6 \end{pmatrix} \in (F_1^1)^*. \end{aligned} \quad (10)$$

From this conic system, we obtain $y_6 = y_3 = y_2 + y_5 = 0$, and thus any solutions $(y, W)$ of this conic system do not change $F_1^1$ and $F_2^1$. Therefore we obtain them the minimal face $F_1^1 \times F_2^1$ with one iteration. This implies that the degree of singularity of the dual of (1) is one. In contrast, we see in Appendix C the degree of the dual of (P1) is two.

We consider perturbed SDP (P3) where the coefficient matrices are denoted by $A_1(\epsilon), \ldots, A_6(\epsilon)$. The pair $(y, W)$ in (8) is also a reducing certificate of the first iteration of facial reduction for (P3). At the second iteration, we obtain the conic system where $(2,4)$th and $(4,2)$nd elements of (10) are replaced with $y_2 - 2y_3 + (1+\epsilon)y_5$. Since we have $y_6 = y_3 = y_2 + y_5 = 0$ again, the minimal face is the same as one of the original (1). However, the optimal value changes discontinuously as in Table 2 because the rank condition is not satisfied. In fact, $\langle (A_1)_3, \ldots, (A_6)_3 \rangle$ has a basis

$$\left( \begin{pmatrix} -2 & & & & \\ -2 & 0 & & & \\ 1 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, (0) \right), \left( \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ -2 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, (1) \right), \left( \begin{pmatrix} 0 & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, (0) \right)$$

and thus the rank is 3. However, for $\epsilon > 0$, a basis of $\langle (A_1(\epsilon))_3, \ldots, (A_6(\epsilon))_3 \rangle$ needs to have additionally

$$(A_5(\epsilon))_3 = \left( \begin{pmatrix} -2 & & & & \\ -2 & 0 & & & \\ 1+\epsilon & 0 & 0 & & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \left(0\right) \right)$$

and hence the rank is 4.

## 4. Behavior of minimal faces under perturbations

In this section, we determine the kind of perturbations that make minimal faces invariant. This results is provided in Proposition 4.2. In addition, we deal with the matrix-wise perturbation for (1). In fact, (1) is originally constructed from the coefficients in the control system (2). We provide a result what kind of perturbations on these coefficient matrices in (2) make minimal faces invariant, which is presented in Example 4.8.

We consider the perturbed problem as the following:

$$\inf_X \left\{ A_0 \bullet X : (A_k + tE_k) \bullet X = b_k \ (k \in K), X \in \mathbb{S}^n_+ \right\}. \qquad (D_t)$$

Throughout this section, we consider the following conditions:

*Condition* 3.

1. (D) is feasible and (P) is strictly feasible;
2. $A_1, \ldots, A_m$ are linearly independent;
3. $(D_t)$ is feasible for sufficiently small $t > 0$.

In addition, we say that $(D_t)$ satisfies the *rank condition* if for an orthogonal matrix $Q$ associated to the minimal face of (D),

$$\begin{aligned} &\operatorname{rank} \left\langle (Q^T(A_1 + tE_1)Q)_3, \ldots, (Q^T(A_m + tE_m)Q)_3 \right\rangle \\ &\qquad\qquad = \operatorname{rank} \left\langle (Q^T A_1 Q)_3, \ldots, (Q^T A_m Q)_3 \right\rangle, \ 0 \le t \le \delta, \end{aligned}$$

for some $\delta > 0$, where the submatrix $M_3$ for $M \in \mathbb{S}^n_+$ is determined by the minimal face of (D) as in (7). The following lemma is a key observation. Namely, under the rank condition, we can show that infeasibility of the last discriminant system is preserved under linear perturbations.

**Lemma 4.1.** *Let $F_{\min}$ be the minimal face of (D). Suppose that $(D_t)$ satisfies the rank condition. Then*

$$\sum_{k \in K} y_k A_k \in (\mathbb{S}^n_+ + F_{\min}^\perp) \setminus F_{\min}^\perp, \ b^T y = 0$$

*is infeasible if and only if*

$$\sum_{k \in K} y_k \left(A_k + tE_k\right) \in (\mathbb{S}^n_+ + F_{\min}^\perp) \setminus F_{\min}^\perp, \ b^T y = 0$$

*is infeasible for sufficiently small $t > 0$.*

*Proof.* By the minimality of $F_{\min}$, the discriminant system

$$\sum_{k\in K} y_k A_k \in (\mathbb{S}^n_+ + F^{\perp}_{\min}) \setminus F^{\perp}_{\min},\ b^T y = 0$$

is infeasible and hence the reduced problem $F(D)$ of (D) has a strictly feasible point which solves the discriminant system

$$\left(Q^T A_k Q\right)_3 \bullet X = b_k, X \in \mathbb{S}^r_{++}.$$

This fact follows from (F2) in Lemma 2.1. Since the rank condition holds for (D), Theorem 3.2 implies the existence of a solution to

$$\left(Q^T \left(A_k + tE_k\right) Q\right)_3 \bullet X = b_k, X \in \mathbb{S}^r_{++}.$$

This is equivalent to solving

$$(A_k + tE_k) \bullet X = b_k,\ \ X \in \mathrm{rint}(F_{\min}),$$

where $\mathrm{rint}(F_{\min})$ is the relative interior of the minimal face $F_{\min}$. By the properties of facial reduction algorithm, it is also equivalent to the infeasibility of

$$\sum_{k\in K} y_k \left(A_k + tE_k\right) \in (\mathbb{S}^n_+ + F^{\perp}_{\min}) \setminus F^{\perp}_{\min}, b^T y = 0.$$

□

**Proposition 4.2.** *For a facial reduction sequence* $(\hat{y}^1, \widehat{W}^1), \ldots, (\hat{y}^s, \widehat{W}^s)$ *of (D), let the minimal face of (D) be* $F_{\min}$ *and* $\hat{K} = \{k : \hat{y}^i_k = 0\ (\forall i = 1, \ldots, s)\}$*. Suppose that* $(D_t)$ *satisfies Condition 3, the rank condition and* $E_k = 0$ $(k \notin \hat{K})$*. Then the minimal face of* $(D_t)$ *is equal to* $F_{\min}$ *for sufficiently small* $t > 0$*.*

*Proof.* Suppose that the facial reduction for $(D_t)$ generates faces as

$$(D)\quad \mathbb{S}^n_+ \overset{(\hat{y}^1,\widehat{W}^1)}{\longrightarrow} F_1 \overset{(\hat{y}^2,\widehat{W}^2)}{\longrightarrow} F_2 \overset{(\hat{y}^3,\widehat{W}^3)}{\longrightarrow} \cdots \overset{(\hat{y}^s,\widehat{W}^s)}{\longrightarrow} F_s = F_{\min}.$$

Now the $(s+1)$st discriminant system of (D) is

$$\sum_{k\in K} y_k A_k \in (\mathbb{S}^n_+ + F^{\perp}_s) \setminus F^{\perp}_s,\ b^T y = 0,$$

and is infeasible.

By the assumption 1, it is obvious that $(\hat{y}, \widehat{W}^1), \ldots, (\hat{y}, \widehat{W}^s)$ are reducing certificates up to the $s$-th loop of the facial reduction for $(D_t)$ and they generate the same faces. It is summarized as

$$(D_t)\quad \mathbb{S}^n_+ \overset{(\hat{y}^1,\widehat{W}^1)}{\longrightarrow} F_1 \overset{(\hat{y}^2,\widehat{W}^2)}{\longrightarrow} F_2 \overset{(\hat{y}^3,\widehat{W}^3)}{\longrightarrow} \cdots \overset{(\hat{y}^s,\widehat{W}^s)}{\longrightarrow} F_s.$$

By Lemma 4.1, the $(s+1)$st discriminant system of $(D_t)$

$$\sum_{k\in K} y_k \left(A_k + tE_k\right) \in (\mathbb{S}^n_+ + F^{\perp}_s) \setminus F^{\perp}_s,\ b^T y = 0.$$

is infeasible for sufficiently small $t > 0$ and hence $F_s$ is the minimal of $(D_t)$. □

As a corollary, we obtain simple geometric conditions, which are also easier to be verified.

**Corollary 4.3.** *For a facial reduction sequence $(\hat{y}^1, \widehat{W}^1), \dots, (\hat{y}^s, \widehat{W}^s)$ of (D), let the minimal face of (D) be $F_{\min}$ and $\hat{K} = \{k : \hat{y}^i_k = 0\ (\forall i = 1, \dots, s)\}$. If Condition 3 holds and*

1. $E_k = 0\ (k \notin \hat{K})$;
2. $E_k \in F_{\min}^{\perp}\ (k \in \hat{K})$,

*then the minimal face of $(D_t)$ is equal to $F_{\min}$ for sufficiently small $t > 0$.*

*Proof.* It suffices to show that the $(s+1)$st discriminant system of $(D_t)$

$$\sum_{k \in K} y_k \left(A_k + tE_k\right) \in (\mathbb{S}^n_+ + F_s^{\perp}) \setminus F_s^{\perp},\ b^T y = 0.$$

is infeasible. Suppose we have

$$F_{\min} = \left\{ Q \begin{pmatrix} O & O \\ O & X \end{pmatrix} Q^T \;\middle|\; X \in \mathbb{S}^r_+ \right\},$$

where $Q$ is an orthogonal matrix. Then $E_k \in F_{\min}^{\perp}$ means that

$$E_k \in \left\{ Q \begin{pmatrix} Y & O \\ O & O \end{pmatrix} Q^T \;\middle|\; Y \in \mathbb{S}^{n-r}_+ \right\},$$

Thus we have

$$\left(Q^T(A_k + tE_k)Q\right)_3 = \left(Q^T A_k Q\right)_3 + \left(tQ^T E_k Q\right)_3 = \left(Q^T A_k Q\right)_3$$

and hence the rank condition is satisfied. Therefore the $(s+1)$st discriminant system of $(D_t)$ is infeasible and $F_s$ is the minimal face of $(D_t)$. □

*Example* 4.4. Let $(y, W)$ be as in Example 3.11. Then $\hat{K} = \{k : \hat{y}^i_k = 0\ (\forall i, \dots, s)\}$. By Corollary 4.3, if $\{E_k\}$ has the following form:

$$E_1 = E_4 = (O_6, O_2),\ E_j = \left( \begin{pmatrix} * & & & & & \\ * & 0 & & & & \\ * & 0 & 0 & & & \\ * & 0 & 0 & 0 & & \\ * & 0 & 0 & 0 & 0 & \\ * & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} * & \\ * & 0 \end{pmatrix} \right)\ (j = 2, 3, 5, 6),$$

then the minimal face of (D) does not change under the perturbation with $\{E_k\}$. $*$ indicates that we can choose real numbers arbitrarily.

Next, we will use positive eigenvectors of reducing certificates to determine conditions for minimal faces to be invariant. We start with the following basic lemma.

**Lemma 4.5.** *Let $W \in \mathbb{S}^n_+$ and $q_1, \dots, q_r$ be the eigenvectors corresponding to the positive eigenvalues $\lambda_1, \dots, \lambda_r$ of $W$. Then we have*

$$\mathbb{S}^n_+ \cap \{W\}^{\perp} = \mathbb{S}^n_+ \cap \langle q_1 q_1^T, \dots, q_r q_r^T \rangle^{\perp}$$

*Proof.* We have $W = \sum_\ell \lambda_\ell q_\ell q_\ell^T$ and hence for $X \in \mathbb{S}^n_+$,

$$X \bullet W = \sum_\ell \lambda_\ell X \bullet q_\ell q_\ell^T = \sum_\ell \lambda_\ell q_\ell^T X q_\ell.$$

Thus $X \bullet W = 0$ if and only if $X \bullet q_\ell q_\ell^T = 0$ for all $\ell = 1, \dots r$. Then we have the desired equality. $\square$

**Proposition 4.6.** *Let $(\hat{y}^1, \widehat{W}^1), \dots, (\hat{y}^s, \widehat{W}^s)$ be a facial reduction sequence of (D) and $q_{j,1}, \dots, q_{j,r_j}$ be eigenvectors associated with the positive eigenvectors of $\widehat{W}^j$. Suppose that $(D_t)$ satisfies Condition 3, the rank condition and for each $i = 1, \dots, s$,*

$$\sum_{k \in K} \hat{y}^i_k E_k \in \left\langle q_{j,1} q_{j,1}^T, \dots, q_{j,r} q_{j,r_j}^T : j = 1, \dots, i \right\rangle,$$

*Then $(D_t)$ has the same minimal face as (D) for sufficiently small $t > 0$.*

*Proof.* Let the facial reduction for (D) generate

$$(D) \quad \mathbb{S}^n_+ \overset{(\hat{y}^1, \widehat{W}^1)}{\longrightarrow} F_1 \overset{(\hat{y}^2, \widehat{W}^2)}{\longrightarrow} F_2 \overset{(\hat{y}^3, \widehat{W}^3)}{\longrightarrow} \cdots \overset{(\hat{y}^s, \widehat{W}^s)}{\longrightarrow} F_s = F_{\min}.$$

and let $W^i := \sum_k \hat{y}^i_k (A_k + tE_k)$ for $i = 1, \dots, s$. Then we claim that $(\hat{y}, W^1), \dots, (\hat{y}, W^s)$ are reducing certificates for $(D_t)$ and generate the faces $F_1, \dots, F_s$ until the $s$-th loop. It is shown by induction on $i$. Suppose that we have up to $i - 1$

$$(D_t) \quad \mathbb{S}^n_+ \overset{(\hat{y}^1, W^1)}{\longrightarrow} F_1 \overset{(\hat{y}^2, W^2)}{\longrightarrow} F_2 \overset{(\hat{y}^3, W^3)}{\longrightarrow} \cdots \overset{(\hat{y}^{i-1}, W^{i-1})}{\longrightarrow} F_{i-1}.$$

Let $\lambda_{i,1}, \dots, \lambda_{i,r_i}$ be positive eigenvalues of $\widehat{W}^i$. Then there exists $\alpha_{j\ell} \in \mathbb{R}$ such that

$$\begin{aligned}
W^i &:= \sum_k \hat{y}^i_k (A_k + tE_k) = \sum_k \hat{y}^i_k A_k + t \sum_k \hat{y}^i_k E_k \\
&= \sum_{\ell=1}^{r_i} (\lambda_{i,\ell} + t\alpha_{i,\ell}) q_{i,\ell} q_{i,\ell}^T + t \sum_{j=1}^{i-1} \sum_{\ell=1}^{r_j} \alpha_{j,\ell} q_{j,\ell} q_{j,\ell}^T \\
&\in \operatorname{cl}\left( \mathbb{S}^n_+ + \left\langle \sum_{j=1}^{i} \widehat{W}^j \right\rangle \right) = \left( \mathbb{S}^n_+ \cap \left\langle \sum_{j=1}^{i} \widehat{W}^j \right\rangle^\perp \right)^* = F_i^* = (\mathbb{S}^n_+ + F_{i-1}^\perp).
\end{aligned}$$

for small $t > 0$. Here $\operatorname{cl}(T)$ stands for the closure of the set $T$ and the inclusion is implied by Lemma 4.5 and Corollary 16.4.2 in [21]. Since $\widehat{W}^i + t \sum_{\ell=1}^{r_i} \alpha_{i,\ell} q_{i,\ell} q_{i,\ell}^T \notin F_{i-1}^\perp$ and $t \sum_{j=1}^{i-1} \sum_{\ell=1}^{r_j} \alpha_{j,\ell} q_{j,\ell} q_{j,\ell}^T \in F_{i-1}^\perp$, we also have $W^i \notin F_{i-1}^\perp$ and thus $W^i \in (\mathbb{S}^n_+ + F_{i-1}^\perp) \setminus F_{i-1}^\perp$. Let $F_i^t$ be the face generated at the $i$-th loop of the facial reduction to $(D_t)$. Then we have

$$\begin{aligned}
F_i^t &= F_{i-1} \cap \left\{W^i\right\}^\perp = \mathbb{S}^n_+ \cap \left\{ W^i + \widehat{W}^1 + \cdots + \widehat{W}^{i-1} \right\}^\perp \\
&= \mathbb{S}^n_+ \cap \left\{ \sum_{\ell=1}^{r} (\lambda_\ell + t\alpha_\ell) q_{i,\ell} q_{i,\ell}^T + \sum_{j=1}^{i-1} \sum_{l=1}^{r_j} (\lambda_{j,\ell} + t\alpha_{j,\ell}) q_{j,\ell} q_{j,\ell}^T \right\}^\perp = F_{i-1} \cap \left\{ \widehat{W}^i \right\}^\perp = F_i.
\end{aligned}$$

We remark that the first equality holds due to the assumption of the induction. Thus the claim is shown. In addition, the above equality shows that $(\hat{y}^1, \widehat{W}^1), \dots, (\hat{y}^s, \widehat{W}^s)$ are also reducing certificates for $(D_t)$ and generate the faces $F_1^t, \dots, F_s^t$. The rank condition and Lemma 4.1 ensure that the $(s+1)$st discriminant system of $(D_t)$ is infeasible. Therefore $\{(\hat{y}^i, \widehat{W}^i)\}_{i=1}^s$ is a facial reduction sequence of $(D_t)$ and its minimal face is $F_s$. □

*Remark* 4.7. Let $\widehat{W}^i$, $\lambda_{i,\ell}$ and $q_{i,\ell}$ as in Proposition 4.6 and the proof. Then we have $\widehat{W}^i = \sum_{\ell=1}^{r_i} \lambda_{i,\ell} q_{i,\ell} q_{i,\ell}^T$. This implies

$$\left\langle q_{j,1}q_{j,1}^T, \dots, q_{j,r}q_{j,r_j}^T : j = 1, \dots, i \right\rangle \subseteq \langle \widehat{W}^1, \dots, \widehat{W}^i \rangle,$$

for each $i = 1, \dots, s$. Therefore in particular, the inclusion in Proposition 4.6 holds if we have

$$\sum_{k \in K} \hat{y}_k^i E_k \in \langle \widehat{W}^1, \dots, \widehat{W}^i \rangle,$$

for each $i = 1, \dots, s$.

*Example* 4.8. Consider the singular SDP (1). This example case is an $H_\infty$ state feedback control problem for control system (2) and is originally structured as follows:

$$\sup \left\{ -x_6 : -\begin{pmatrix} \mathrm{He}(AX_1 + B_2X_2) & * & * \\ C_1X_1 + D_{12}X_2 & -x_6 I & * \\ B_1^T & D_{11}^T & -x_6 I \end{pmatrix} \in \mathbb{S}_+^6,\ X_1 \in \mathbb{S}^2,\ X_2 \in \mathbb{R}^{1\times 2} \right\}, \tag{P}$$

where $\mathrm{He}(M) = M + M^T$ for any square matrix $M$ and

$$X_1 = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix},\ X_2 = \begin{pmatrix} x_4 & x_5 \end{pmatrix},\ \left(\begin{array}{c|c|c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \end{array}\right) = \left(\begin{array}{cc|cc|c} -1 & -1 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ \hline 2 & -1 & -1 & 0 & 2 \\ -1 & 2 & -1 & 0 & -1 \end{array}\right).$$

We remark that since (2) is stabilizable, *i.e.*, for any $\lambda \in \mathbb{C}_+$, $\mathrm{rank}(A - \lambda I_n, B_2) = n$, (P) is strictly feasible. See [31] for the detail. We show that matrix-wise perturbations preserve the minimal face of its dual of (P) or make it full-dimensional, *i.e.*, $\mathbb{S}_+^6 \times \mathbb{S}_+^2$.

Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},\ B_2 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},\ C_1 = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},\ D_{12} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}.$$

For simplicity, let $B_1, D_{11}$ are as above. Then the first block of (P) is

$$\begin{pmatrix} -2a_{11}x_1 - 2a_{12}x_2 - 2b_1x_4 & & & & & \\ -a_{21}x_1 - (a_{11}+a_{22})x_2 - a_{12}x_3 - b_2x_4 - b_1x_5 & -2a_{21}x_2 - 2a_{22}x_3 - 2b_2x_5 & & & & \\ -c_{11}x_1 - c_{12}x_2 - d_1x_4 & -c_{11}x_2 - c_{12}x_3 - d_1x_5 & x_6 & & & \\ -c_{21}x_1 - c_{22}x_2 - d_2x_4 & -c_{21}x_2 - c_{22}x_3 - d_2x_5 & 0 & x_6 & & \\ 1 & 1 & 1 & 1 & x_6 & \\ 1 & 0 & 0 & 0 & 0 & x_6 \end{pmatrix} \in \mathbb{S}_+^6.$$

The related part with $a_{11}$ can be extracted as

$$a_{11}\begin{pmatrix} -2x_1 & -x_2 & 0 & 0 & 0 & 0 \\ -x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = a_{11}(x_1E_{1,1} + x_2E_{2,1}),$$

where

$$E_{1,1} = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_{2,1} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then the perturbation on $a_{11}$ corresponds to perturbing matrices $E_1 = (E_{1,1}, O_2), E_2 = (E_{2,1}, O_2)$ and $E_k = (O_6, O_2)$ $(k = 3, \ldots, 6)$. Now the reducing sequence is $\{(y, W)\}$ as given Example 4.4. Let $e_i$ be the unit vector whose $i$th entry is 1 and others are zero. Then positive eigenvalues of $W$ are $2, 1$ and eigenvectors are $e_1, e_7$ respectively. Since $E_1 \in \langle e_1 e_1^T, e_7 e_7^T \rangle$, Proposition 4.6 implies that this perturbation does not change the minimal face of the dual of (P).

On the other hand, the related part with $a_{21}$ is

$$a_{21} \begin{pmatrix} 0 & -x_1 & 0 & 0 & 0 & 0 \\ -x_1 & -2x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = a_{21}(x_1 E_{1,1} + x_2 E_{2,1}),$$

where

$$E_{1,1} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_{2,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easily verified that the perturbation with respect to $a_{21}$, *i.e.*, $E_1 = (E_{1,1}, O_2)$, $E_2 = (E_{2,1}, O_2)$ and $E_k = (O_6, O_2)$ $(k = 3, \ldots, 6)$, makes the discriminant system of the first loop of the facial reduction infeasible. Thus the perturbed $(D_t)$ is strictly feasible for small $t > 0$. Similar arguments are provided in Table 3. Here we observe that if we perturb matrices $A$, $B_2$, $C_1$ and $D_{12}$ in the structured form, the minimal face does not become smaller.

Table 3: Behaviors of the minimal face under matrix-wise perturbations

| Perturbation | Face | Perturbation | Face |
|---|---|---|---|
| $a_{11}$ | Invariant | $c_{11}$ | Full-dimensional |
| $a_{12}$ | Invariant | $c_{12}$ | Invariant |
| $a_{21}$ | Full-dimensional | $c_{21}$ | Full-dimensional |
| $a_{22}$ | Invariant | $c_{22}$ | Invariant |
| $b_1$ | Invariant | $d_1$ | Full-dimensional |
| $b_2$ | Full-dimensional | $d_2$ | Full-dimensional |

We can discuss non-strict feasibility under perturbations by using a result in [31]. They show that the dual of (P) is *not* strictly feasible if and only if there exists $\lambda \in \mathbb{C}$ such that

$$\Re(\lambda) \leq 0 \text{ and } \operatorname{rank} \begin{pmatrix} A - \lambda I_2 & B_2 \\ C_1 & D_{12} \end{pmatrix} < 3, \tag{11}$$

where $\Re(\lambda)$ stands for the real part of $\lambda \in \mathbb{C}$. In fact, since $\lambda = -1$ satisfies (11), the dual of (P) is not strictly feasible despite of the fact that (P) is strictly feasible. Table 3 implies that the perturbations on $a_{11}$, $a_{12}$, $a_{22}$, $b_1$, $c_{12}$ and $c_{22}$ still satisfy the necessary and sufficient condition (11) for the non-strict feasibility of the dual of (P). For instance, we perturb only $a_{11}$ to $a_{11} + \epsilon$. Then we can see that the following linear system with $\lambda = -1 - \epsilon$ has a nonzero solution $(u_1, u_2, v)$:

$$\begin{pmatrix} -1-\epsilon-\lambda & -1 & 0 \\ 1 & -\lambda & 1 \\ 2 & -1 & 2 \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ v \end{pmatrix} = 0.$$

This means that (11) also holds in the perturbed control system, and thus the dual of (P) obtained by perturbing only $a_{11}$ is not strictly feasible.

*Remark* 4.9. In Example 4.8, the optimal value of $(D_t)$ changes continuously at $t = 0$ due to Theorem 3.9 in the case where perturbations preserve the minimal face, *i.e.*, perturbations with $a_{11}$, $a_{12}$, $a_{22}$, $b_1$, $c_{12}$ and $c_{22}$. We have numerically confirmed that the optimal value of $(D_t)$ also varies continuously in the case that the perturbations change the minimal face into $\mathbb{S}^6 \times \mathbb{S}^2$, *i.e.*, perturbations with $a_{21}$, $b_2$, $c_{11}$, $c_{21}$, $d_1$ and $d_2$. However the continuity in the latter case has not been proven yet. When $(D_t)$ becomes strictly feasible, one might think that direct analysis on $(D_t)$ could work, since both $(P_t)$ and $(D_t)$ are strictly feasible. However, in that case, main difficulty for proving the continuity of the optimal value is that the optimal set of (P) may be empty. Although (D) has an optimal solution, it does not satisfy the KKT condition. Thus our arguments do not work.

Although Example 4.8 shows that Proposition 4.2 and Proposition 4.6 are sufficient to analyze the behavior of the control system (2), we only consider the following simple conditions.

**Proposition 4.10.** *Suppose that for a facial reduction sequence* $(\hat{y}^1, \widehat{W}^1), \ldots, (\hat{y}^s, \widehat{W}^s)$ *of (D), $(D_t)$ satisfies Condition 3, the rank condition and*

$$E_k = w_k E$$

*for some $E \in \mathbb{S}$ and $w \in \langle \hat{y}^1, \ldots, \hat{y}^s \rangle^\perp$. Then $(D_t)$ has the same minimal face as (D) for sufficiently small $t > 0$.*

*Proof.* For $i = 1, \ldots, s$, we have

$$\sum_k \hat{y}^i_k (A_k + tE_k) = \sum_k \hat{y}^i_k A_k + \sum_k t\hat{y}^i_k w_k E = \sum_k \hat{y}^i A_k = \widehat{W}^i.$$

Thus $\hat{y}^i$ solves the discriminant system of $(D_t)$

$$\sum_k y_k (A_k + tE_k) \in (\mathbb{S}^n_+ + F_{i-1}^\perp) \setminus F_{i-1}^\perp, \; b^T y = 0,$$

for each $i = 1, \ldots, s$. Lemma 4.1 and the rank condition ensure that $(D_t)$ has the same minimal face as (D). □

## 5. Conclusions

We begin this study with the analysis of the numerical results in Table 2. It is known that the Slater condition on at least one of primal or dual SDP is sufficient for the optimal value to be continuous if one perturbs only data on the right hand side. However, Table 2 shows that if one perturbs the coefficient matrices on the left hand side, the optimal value can be discontinuous. Table 2 also provides a guideline for solving singular SDPs. In particular, when we use SDPA-GMP to solve singular SDPs, it is important not only to use the floating point computation with longer significant digits, but also to choose the appropriate tolerance for the stopping criteria of computation in this case.

We first provide the result on the continuity of the optimal value of SDPs perturbed in both sides for Theorem 3.4. Although it has been already proven for general convex programs [9], we provide a new proof and extend the result to the case of singular SDPs. In fact, the dual of our problem (1) of interest is not strictly feasible. We provide the result on the continuity of the optimal value of the duals $(D_t)$ of perturbed SDPs in Theorem 3.9. In particular, we use a minimal face of (D) for the characterization. We give a detailed analysis on numerical results in Table 2 based on Theorem 3.9 and Example 3.11.

We provide the results on linear perturbations for the dual (D) of SDP (P). We then determine change of the minimal face under the matrix-wise perturbations of the control system (2) as in Example 4.8. The SDPs generated from combinatorial optimization, matrix completion problems and sums of squares problems also have special structures and reducing certificates are obtained without solving SDPs for these problems. In future work, we could use these structures to obtain a sharper criteria for perturbations to make minimal faces invariant. In addition, it may be interesting to try to find combinatorial structures in elements of matrices which are used in linear perturbations that preserve minimal faces.

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## A. On the optimal value of LMI problem (1)

Let $\gamma^* = -\sqrt{5}$. Define the sequence $\{(x^*_{1n}, \ldots, x^*_{6n})\}_{n=1}^\infty$ as follows:

$$x_{1n} = n, x_{2n} = x_{3n} = 0, x_{4n} = -n, x_{5n} = \gamma^*/4, x^*_{6n} = -\gamma^* + 1/n.$$

for all $n \geq 1$.

First we prove that the sequence consists of feasible solutions of (1) with the objective value $\gamma^* + \frac{1}{n}$. We note that the objective value converges $\gamma^*$. It is not difficult to prove that

$\{(x^*_{1n}, \dots, x^*_{6n})\}_n$ is a sequence that is feasible in (1) and whose objective values converge to $-\sqrt{5}$.

We proved that the optimal value of (1) is less than or equal to $\gamma^* = -\sqrt{5}$. Next, we prove the optimality. To this end, we check that the dual of (1) has a feasible solution with the objective value $-\sqrt{5}$. The dual is formulated as follows:

$$\left\{\begin{array}{ll} \inf_{z_{ij}} & -2(z_{51} + z_{61} + z_{52} + z_{53} + z_{54}) \\ \text{subject to} & \mathrm{He}\left(\begin{pmatrix} -z_{11} + z_{21} + 2z_{31} - z_{41} & -z_{21} + z_{22} + 2z_{32} - z_{42} \\ -z_{11} - z_{31} + 2z_{41} & -z_{21} - z_{32} + 2z_{42} \end{pmatrix}\right) \in \mathbb{S}^2_+, \\ & z_{21} + 2z_{31} - z_{41} = 0, z_{22} + 2z_{32} - z_{42} = 0, \\ & \displaystyle\sum_{i=3}^{6} z_{ii} = 1, (z_{ij})_{1 \le i,j \le 6} \in \mathbb{S}^6_+. \end{array}\right. \tag{12}$$

where $\mathrm{He}(X) = X + X^T$ for $X \in \mathbb{R}^{n \times n}$.

We prove that the following solution is feasible in (12) with the objective value $-\sqrt{5}$:

$$(z_{ij})_{1 \le i,j \le 6} = \left(\begin{array}{cc|cc|cc} 0 & & & & & \\ 0 & 1.6 & & & & \\ \hline 0 & -0.4 & 0.1 & & & \\ 0 & 0.8 & -0.2 & 0.4 & & \\ \hline 0 & -4\delta^* & \delta^* & -2\delta^* & 0.5 & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) = \frac{1}{10}\begin{pmatrix} 0 \\ -4 \\ 1 \\ -2 \\ \gamma^* \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ -4 \\ 1 \\ -2 \\ \gamma^* \\ 0 \end{pmatrix}^T,$$

where $\delta^* = \sqrt{5}/10$. It is easy to see that the solution is feasible with the objective value $-\sqrt{5}$. Therefore since we construct a sequence in (1) whose objective value converge $\gamma^* = -\sqrt{5}$, the optimal value of (1) is $\gamma^* = -\sqrt{5}$.

Finally, we prove that (1) does not have any optimal solutions, *i.e.*, the optimal value $-\sqrt{5}$ is not attained. To this end, we suppose that (1) has an optimal solution $(x^*_1, \dots, x^*_6)$. It follows from the positive definiteness and the complementarity condition in [6, Theorem 2.6] that we have

$$\begin{pmatrix} x^*_1 & x^*_2 \\ x^*_2 & x^*_3 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = 0, \tag{13}$$

$$\begin{pmatrix} 2x_1 + 2x_2 & & & & & \\ -x_1 + x_2 + x_3 - x_4 & -2x_2 - 2x_5 & & & & \\ -2x_1 + x_2 - 2x_4 & -2x_2 + x_3 - 2x_5 & x_6 & & & \\ x_1 - 2x_2 + x_4 & x_2 - 2x_3 + x_5 & 0 & x_6 & & \\ 1 & 1 & 1 & 1 & x_6 & \\ 1 & 0 & 0 & 0 & 0 & x_6 \end{pmatrix}\begin{pmatrix} 0 \\ -4 \\ 1 \\ -2 \\ \gamma^* \\ 0 \end{pmatrix} = 0. \tag{14}$$

We obtain $x^*_2 = x^*_3 = 0$ and $x^*_2 - 4x^*_3 = -\sqrt{5}$ form (13) and (14), respectively. It is clear that the two equalities are contradictory, and thus (1) does not have any optimal solutions.

## B. On the optimal value of perturbed SDP (P1)

The dual of (P1) can be formulated as follows:

$$\left\{\begin{array}{ll} \inf_{z_{ij}} & -2(z_{51}+z_{61}+z_{52}+z_{53}+z_{54}) \\ \text{subject to} & \mathrm{He}\left(\begin{pmatrix} -z_{11}+z_{21}+2z_{31}-z_{41} & -z_{21}+z_{22}+2z_{32}-z_{42} \\ -z_{11}-z_{31}+2z_{41} & -z_{21}-z_{32}+2z_{42} \end{pmatrix}\right) \in \mathbb{S}^2_+, \\ & z_{21}+2z_{31}-z_{41}=0, (1+\epsilon)z_{22}+2z_{32}-z_{42}=0, \\ & \displaystyle\sum_{i=3}^{6} z_{ii}=1, (z_{ij})_{1\le i,j\le 6}\in\mathbb{S}^6_+. \end{array}\right. \tag{15}$$

From the first and second constraints, we obtain $z_{11}=0$, and thus $z_{i1}=z_{1i}=0$ for $i=1,\dots,6$ holds due to the positive semidefiniteness of the matrix $(z_{ij})_{1\le i,j\le 6}$. Substituting them into (15), we obtain

$$\inf_{z_{ij}}\left\{-2(z_{52}+z_{53}+z_{54}) : \begin{array}{l} -z_{32}+2z_{42}\ge 0, z_{22}+2z_{32}-z_{42}=0, \\ (1+\epsilon)z_{22}+2z_{32}-z_{42}=0, \\ \displaystyle\sum_{i=3}^{6} z_{ii}=1, (z_{ij})_{2\le i,j\le 6}\in\mathbb{S}^5_+. \end{array}\right\}. \tag{16}$$

We obtain $z_{22}=0$ from the second and third constraints in (16), and thus $z_{i2}=z_{2i}=0$ for all $i=2,\dots,6$. Then, (15) is equivalent to the following problem:

$$\inf_{z_{ij}}\left\{-2(z_{53}+z_{54}) : \sum_{i=3}^{6} z_{ii}=1, (z_{ij})_{3\le i,j\le 6}\in\mathbb{S}^4_+\right\}. \tag{17}$$

This problem (17) is the minimization of the eigenvalue of the following matrix

$$\begin{pmatrix} 0 & & & \\ 0 & 0 & & \\ -1 & -1 & 0 & \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since the minimum of the eigenvalue of the matrix is $-\sqrt{2}$, the optimal value of (P1) is $-\sqrt{2}$.

## C. Minimal faces of the dual (12) of SDP (1) and the dual (15) of perturbed SDP (P1)

We compare the dual (12) of SDP (1) with the dual (15) of its perturbed SDP (P1) from the viewpoint of minimal faces. In fact, we prove here that the minimal face of the dual (15) of its perturbed SDP (P1) is smaller than the dual (12) of the original SDP (1).

As we seen in Example 4.4, the minimal face of (12) is $F_1\times F_2$ in (9). Now the reduction certificate $(y,W)$ of the first iteration of facial reduction for (15) is the same as $(y,W)$ of (12).

The discriminant system (4) is equivalent to

$$y_6 \le 0, \begin{pmatrix} y_1 & \\ y_2 & y_3 \end{pmatrix} \in (F_2^1)^* \setminus (F_2^1)^\perp,$$

$$\begin{pmatrix} 2y_1 + 2y_2 & & & & & \\ -y_1 + y_2 + y_3 - y_4 & -2y_2 - 2(1+\epsilon)y_5 & & & & \\ -2y_1 + y_2 - 2y_4 & -2y_2 + y_3 - 2y_5 & y_6 & & & \\ y_1 - 2y_2 + y_4 & y_2 - 2y_3 + y_5 & 0 & y_6 & & \\ 0 & 0 & 0 & 0 & y_6 & \\ 0 & 0 & 0 & 0 & 0 & y_6 \end{pmatrix} \in (F_1^1)^* \setminus (F_1^1)^\perp.$$

Then the following $(y, W)$ is a reduction certificate obtained at the second iteration of facial reduction for (D):

$$y = (-1, 1, 0, 2, -1, 0)^T, W = (W_1, W_2), W_1 = \begin{pmatrix} 0 & & & & & \\ 0 & 2\epsilon & & & & \\ -1 & 0 & 0 & & & \\ -1 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, W_2 = O_2.$$

In fact, the obtained faces are

$$F_1^2 = F_1^1 \cap \{W_1\}^\perp = \left\{ X \in \mathbb{S}^6 : X = \begin{pmatrix} O_2 & \\ O & X_2 \end{pmatrix}, X_2 \in \mathbb{S}^4_+ \right\} \text{ and } F_2^2 = F_2^1.$$

We obtain the minimal face $F_1^2 \times F_2^2$ of (15) with two iterations. We see that the degree of singularity of (15) is two and that this face is smaller than the minimal face of (12). Therefore we see that for any $\epsilon > 0$, the minimal face of (1) is different from that of (P1). Since the optimal value of (P1) is $-\sqrt{2}$ for any $\epsilon > 0$, this implies that the optimal value of (P1) has a big jump at $t = 0$.