On the Existence of Ideal Solutions in Multi-objective 0-1 Integer Programs

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Abstract

We study conditions under which the objective functions of a multi-objective 0-1 integer linear program guarantee the existence of an ideal point, meaning the existence of a feasible solution that simultaneously minimizes all objectives. In addition, we study the complexity of recognizing whether a set of objective functions satisfies these conditions: we show that it is NP-hard, but can be solved in pseudo-polynomial time. Few multi-objective 0-1 integer programs have objectives satisfying the conditions for the existence of an ideal point, but the conditions may be satisfied for a subset of the objective functions and/or be satisfied when the objective functions are restricted to a subset of the variables. We illustrate how such occurrences can be exploited to reduce the number of objective functions and/or to derive cuts in the space of the decision variables.

Keywords. Multi-objective 0-1 integer linear program, ideal point, nondominated frontier, computational complexity

1 Introduction

Multi-objective 0-1 integer programs or multi-objective binary programs (MBPs), in which all decision variables are binary and there is a set of linear objective functions, arise naturally in many contexts, e.g., transportation, machine scheduling, and network design. The goal in multi-objective optimization is to identify the set of Pareto optimal solutions, which are solutions for which it is impossible to improve the value of one objective function without deteriorating the value of at least one other objective function. For a general introduction to multi-objective optimization, see Ehrgott [4] and Ehrgott and Gandibleux [6]. MBPs are sometimes treated as a case of multi-objective combinatorial optimization; for an overview of this area, see Ehrgott and Gandibleux [5, 7]. The image of the set of Pareto-optimal solutions in criterion space is known as the set

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of nondominated points. Not surprisingly, the number of nondominated points has a significant impact on the effort required to solve a multi-objective optimization problem.

A pragmatic approach to deal with the, potentially, huge number of nondominated points is to generate only a subset of nondominated points rather than all of them. Examples of such approaches are presented by Özpeynirci and Köksalan [19] and Przybylski et al. [21], who develop methods to generated all extreme supported nondominated points, and by Sayın and Kouvelis [22] who propose an algorithm that generates nondominated points until a prespecified quality guarantee is met.

Recent studies have shown that properties of the objective functions can be used to deduce information about the set of nondominated points, which may prove to be of value. For example, from properties of objective functions it may be possible to deduce that the number of nondominated points grows exponentially with the input size, or that there will only be supported nondominated points. Examples of results that have been obtained recently for MBPs, in part based on studying or characterizing objective functions (where N denotes the number of objective functions and n denotes the number of 0-1 variables) are:

- When the coefficients of $N-1$ of the objective functions are binary, then the number of nondominated points is bounded by $(n+1)^{N-1}$, and when the coefficients of $N-1$ of the objective functions are integer and the largest coefficient in any of these $N-1$ objective functions is polynomially bounded in n, then the number of nondominated points is also polynomially bounded (Figueira et al. [9]).
- When the set of feasible solutions is the set of bases of a matroid, $N=2$, and the coefficients of one of the objective functions are binary, then all nondominated points are supported and can be computed in polynomial time (Gabow and Tarjan [10], Gorski [12]).
- All nondominated points of a biobjective minimum spanning tree problem with objective coefficients in $\{0,1,2\}$ are supported, and the number of nondominated points is bounded by $2n-1$ (Seipp [23]).
- MBPs with binary coefficients in $N-1$ of the objective functions can be solved in $\mathcal{O}(n \log n)$ for $N=2$ and in $\mathcal{O}(n^2)$ for $N=3$ (Gorski et al. [13]).

We refer the interested reader to Figueira et al. [9] for more information on these recent developments.

Our primary reason for studying objective functions of MBPs is to develop concepts and theory with the potential to enhance the performance of algorithms for solving general MBPs. For an introduction to algorithms for solving general MBPs as well as examples of recent developments in this area, see Belotti et al. [1], Boland et al. [2, 3], Lokman and Köksalan [16], Parragh and Tricoire [20] and Stidsen et al. [24].

The starting point for our investigation is studying conditions under which the objective functions of a MBP guarantee the existence of an ideal point, i.e., conditions under which the nondominated frontier consists of a single point. We say that a set of objective functions is *universally co-ideal* if any MBP with at least one feasible solution and this set of objective functions has an ideal point.

The following core results concerning universally co-idealness are obtained.

- Whether a set of objective functions is universally co-ideal can be determined in pseudo-polynomial time. This is the best possible (unless NP=P), because the problem of determining whether a set of objective functions is universally co-ideal is (weakly) NP-hard.

- When a subset of objectives of a MBP is universally co-ideal, this subset of objective functions can be replaced by a single objective function without changing the set of Pareto optimal solutions. (In fact, an even stronger statement can be made, see Section 5.1).
- When there exists a subset of variables with the property that the set of objective functions, when restricted to the variables in the subset, is universally co-ideal, then, given an efficient solution $\bar{x}$, a cut can be generated that is valid for all efficient solutions, x, that do not yield the same nondominated point as $\bar{x}$.

In the cut referred to above, the nondominated point corresponding to the efficient solution $\bar{x}$ is an ideal point for the objective functions restricted to variables in the, so-called, *objective-aligning* set. This property can be generalized: *from* a given efficient solution, a subset of variables that is *not* necessarily objective-aligning, *per se*, but that nevertheless induces the corresponding nondominated point to be ideal, can be sought. Again, a cut can be derived, but in this case under weaker conditions.

In summary, our search for conditions on a set of objective functions of a MBP that guarantee the existence of an ideal point, i.e., our investigation of universally co-idealness, has not only resulted in further insights into the relationship between the characteristics of the objective functions of a MBP and the characteristics of its nondominated frontier, but has also lead to the discovery of a number of techniques that have the potential to enhance the performance of algorithms for solving general MBPs.

The remainder of paper is organized as follows. In Section 2, we introduce notation and a few basic results. In Section 3, we formally introduce the universally co-ideal concept and some basic theoretical results. In Section 4, we discuss the computational complexity of recognizing universally co-idealness. In Section 5, we introduce different ways to exploit the concept of universally co-idealness, so as to reduce the objective space dimension and/or deduce cuts. In Section 6, we indicate how the concept of universally co-idealness can be extended to make it even more powerful. Finally, in Section 7, we give some concluding remarks.

2 Preliminaries

A *Multi-objective Binary Problem* can be formulated as follows,

$$\min_{x \in F} \{c^1 x, c^2 x, \ldots, c^N x\} \tag{1}$$

where $c^1, c^2, \ldots, c^N \in \mathbb{R}^n$ are row vectors that represent the objective functions and $F \subseteq \{0,1\}^n$ represents the *feasible set in the decision space*, assumed to be a subset of the binary column vectors of dimension n. We denote the $N \times n$ matrix with ith row given by c^i, for each $i = 1, \ldots, N$, by C, which is called the *objective function matrix.* For a given *solution*, $x \in \{0,1\}^n$, in decision space, its image in criterion space may be compactly written as Cx. Note that it is convenient to treat row and column vectors in criterion space as indistinguishable, and so $(c^1 x, \ldots, c^N x)$ and Cx will be treated as identical. We use $Z := \{Cx : x \in \{0,1\}^n\}$ to denote the set of all *points* in the criterion space that are the image of a solution in decision space and use $O \subseteq Z$, given by $O := \{Cx : x \in F\}$, to denote the *feasible set in the criterion space.* To aid in distinguishing the two spaces, we usually refer to elements of the decision space as solutions and elements of the criterion space as points. A MBP is defined by the pair (C, F).

Definition 1. A solution $x^I \in F$ is *ideal* if $c^i x^I = \min_{x \in F} c^i x$ for all $i \in \{1, \ldots, N\}$. We refer to $z^I := Cx^I$ as the *ideal point*.

Definition 2. A feasible solution $x \in F$ is called *efficient* or *Pareto optimal*, if there is no $x' \in F$ such that $Cx' \leq Cx$ and $Cx' \neq Cx$. If x is efficient, then Cx is called a *nondominated point*. The set of all efficient solutions is denoted by F_E. The set of all nondominated points, $Cx \in O$ for some $x \in F_E$, is denoted by O_N, and referred to as the *nondominated frontier* or the *efficient frontier*.

Note that we use the usual vector inequality notation, $p \leq q$, for a pair of vectors, $p, q \in \mathbb{R}^m$, for some $m \in \mathbb{Z}_+$, to indicate that $p_i \leq q_i$ for all $i = 1, \ldots, m$; we do not assume, as is sometimes the case in multi-objective optimization, that $p \leq q$ implies that $p \neq q$.

Given a solution $x \in \{0,1\}^n$, we denote the *support* of x by $supp(x)$, i.e., $supp(x) := \{j \in \{1, \ldots, n\} : x_j = 1\}$. Moreover, given a set $S \subseteq \{1, \ldots, n\}$, and a row vector $c \in \mathbb{R}^n$, we write $c(S)$ to denote $\sum_{j \in S} c_j$, and have that $c(supp(x)) = \sum_{j \in supp(x)} c_j = \sum_{j=1}^n c_j x_j = cx$.

A key definition in understanding ideal point properties of a MBP is the following. There are several different, equivalent, ways of stating this property, each of which is convenient, depending on the circumstance in which it is used.

Definition 3. A set of objectives, $c^1, c^2, \ldots, c^N \in \mathbb{R}^n$, is *sortable-by-subsets* when, for every pair of sets $S, S' \subseteq \{1, \ldots, n\}$, either

(a) $c^i(S) \leq c^i(S')$ for all $i = 1, \ldots, N$, or

(b) $c^i(S) \geq c^i(S')$ for all $i = 1, \ldots, N$.

Stating that $c^1, c^2, \ldots, c^N$ is sortable-by-subsets is equivalent to stating that the vector inequality relation, $\leq$, totally orders Z. To see this, assume that $c^1, c^2, \ldots, c^N$ is sortable-by-subsets. Then, if $p, p' \in Z$ are two arbitrarily chosen points in Z, say $p = Cx$ and $p' = Cx'$, for $x, x' \in \{0,1\}^n$, either $p = p'$ or, for some i, $p_i < p'_i$ (without loss of generality). In the latter case, $p_i = c^i(supp(x)) < c^i(supp(x')) = p'_i$, so, by Definition 3 (a), $c^h(supp(x)) \leq c^h(supp(x'))$, for all $h = 1, \ldots, N$. But $p_h = c^h(supp(x)) \leq c^h(supp(x')) = p'_h$, for all $h = 1, \ldots, N$, so $p \leq p'$. This shows that $\leq$ totally orders Z. The converse is similar.

Another, equivalent, way of stating the sortable-by-subsets property, is obtained from the definition below.

Definition 4. Two sets $S, S' \subseteq \{1, \ldots, n\}$ are *mutually nondominated* (MND) for a set of objectives, $c^1, c^2, \ldots, c^N \in \mathbb{R}^n$, if there exist $i, j \in \{1, \ldots, N\}$ such that $c^i(S) < c^i(S')$ and $c^j(S) > c^j(S')$.

Stating that the set of objectives, $c^1, c^2, \ldots, c^N \in \mathbb{R}^n$, is sortable-by-subsets is equivalent to stating that there does **not** exist a pair of subsets of $\{1, \ldots, n\}$ that are MND for this set of objectives.

The following lemma and its corollary are also helpful. As the proof of each is straightforward, these proofs are omitted.

Lemma 5. *Let $S, S' \subseteq \{1, \ldots, n\}$. Then S and S' are MND if and only if $S \setminus S'$ and $S' \setminus S$ are MND.*

Corollary 6. *For a set of objectives, $c^1, c^2, \ldots, c^N$, to be sortable-by-subsets, it suffices that either $c^i(S) \leq c^i(S')$ for all $i = 1, \ldots, N$, or $c^i(S) \geq c^i(S')$ for all $i = 1, \ldots, N$ holds for every pair of* disjoint *sets $S, S' \subseteq \{1, \ldots, n\}$.*

3 Universally co-ideal

Next, we introduce the concept that forms the foundation of this study. It defines a property of objectives that ensures that any (feasible) MBP with this set of objectives, irrespective of its feasible set, must have an ideal point.

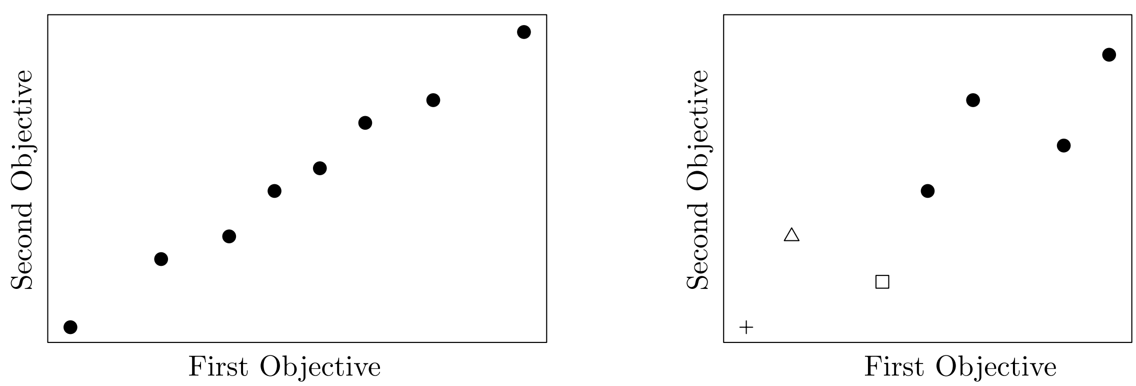
Definition 7. A set of objectives, $c^1, c^2, \ldots, c^N \in \mathbb{R}^n$, is *universally co-ideal* if, for **any** nonempty feasible set, $F \subseteq \{0,1\}^n$, the MBP

$$\min_{x \in F} \{c^1 x, c^2 x, \ldots, c^N x\}$$

has an ideal point, i.e.,

$$\bigcap_{i=1}^{N} \arg\min_{x \in F} c^i x \neq \emptyset.$$

The following proposition shows that the sortable-by-subsets property (or, equivalently, the property that $\leq$ totally orders Z) characterizes when a set of objectives is universally co-ideal. The correspondence between $\leq$ totally ordering Z and the set of objectives being universally co-ideal is illustrated in Figure 1, which plots Z for two sets of (two) objectives. In Figure 1a, it is clear that Z is totally ordered by $\leq$, and that, irrespective of which subset of Z is feasible for a MBP, (which points constitute O), there must be a "bottom left" feasible point that dominates the others, and hence the MBP has an ideal point. By contrast, in Figure 1b, the pair of points plotted as an unfilled triangle and an unfilled square are not ordered by $\leq$, and clearly any feasible set that contains both those points and does not contain the bottom left point, plotted as a plus symbol, cannot have an ideal point.



(a) Objectives that are universally co-ideal: $c^1 = (4, 7, 9)$ and $c^2 = (3, 4, 6)$.

(b) Objectives that are not universally co-ideal: $c^1 = (1, 3, 4)$ and $c^2 = (2, 1, 3)$.

Figure 1: Illustration of sets of objectives that are, and are not, universally co-ideal. The figures show the set Z, for each set of objectives.

Proposition 8. *A set of objectives, $c^1, c^2, \ldots, c^N \in \mathbb{R}^n$, is universally co-ideal if and only if it is sortable-by-subsets.*

Proof. Suppose the set of objectives is not universally co-ideal, so there exists $F \subseteq \{0,1\}^n$, $F \neq \emptyset$, with

$$\bigcap_{i=1}^{N} \arg\min_{x \in F} c^i x = \emptyset.$$

Then there must exist $k \in \{2, \ldots, N\}$ with

$$G^{k-1} := \bigcap_{i=1}^{k-1} \arg\min_{x \in F} c^i x \neq \emptyset \qquad \text{and} \qquad \bigcap_{i=1}^{k} \arg\min_{x \in F} c^i x = \emptyset, \qquad \text{i.e.,} \qquad G^{k-1} \cap \arg\min_{x \in F} c^k x = \emptyset.$$

Now let $y \in G^{k-1}$ and $y' \in \arg\min_{x \in F} c^k x$. Then $y \notin \arg\min_{x \in F} c^k x$, so $c^k y > c^k y'$. Also, there must exist $i \in \{1, \ldots, k-1\}$ with $y' \notin \arg\min_{x \in F} c^i x$. But $y \in G^{k-1} \subseteq \arg\min_{x \in F} c^i x$, so $c^i y < c^i y'$. Let $S = supp(y)$, and $S' = supp(y')$. Then

$$c^i(S) = c^i y < c^i y' = c^i(S')$$

while

$$c^k(S) = c^k y > c^k y' = c^k(S'),$$

which shows the objectives are not sortable-by-subsets.

Now suppose that the set of objectives is not sortable-by-subsets, so there exists $S, S' \subseteq \{0,1\}^n$ and $i, k \in \{1, \ldots, N\}$, $i \neq k$, with $c^i(S) < c^i(S')$ and $c^k(S) > c^k(S')$ (so obviously $S \neq S'$). Let $y \in \{0,1\}^n$ be the indicator vector of S and $y' \in \{0,1\}^n$ be the indicator vector of S', (so $y \neq y'$), and set $F = \{y, y'\}$. Then $\arg\min_{x \in F} c^i x = \{y\}$ and $\arg\min_{x \in F} c^k x = \{y'\}$, and the result follows. □

To decide that a set of objectives is sortable-by-subsets, direct use of the sortable-by-subsets definition would appear to require enumeration of $\mathcal{O}(2^n)$ sets. Here we give a necessary condition that is easy to check.

Definition 9. A set of objectives, $c^1, c^2, \ldots, c^N \in \mathbb{R}^n$, is *elementwise sortable* if, for every $j, k \in \{1, \ldots, n\}$, either $c^i_j \geq c^i_k$ for all $i = 1, \ldots, N$, or $c^i_j \leq c^i_k$ for all $i = 1, \ldots, N$. In this case, there exists a permutation, $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ of the cost vector indices such that

$$c^i_{\sigma(1)} \leq c^i_{\sigma(2)} \leq \cdots \leq c^i_{\sigma(n)}, \qquad \forall i = 1, \ldots, N.$$

Corollary 10 (to Proposition 8)**.** *If a set of objectives $c^1, c^2, \ldots, c^N \in \mathbb{R}^n$ is not elementwise sortable then it is not universally co-ideal.*

Proof. Elementwise sortable is necessary for a set of objectives to be sortable-by-subsets, since it is precisely the same condition restricted to pairs of sets of cardinality 1. □

Of course, if a set of objectives is sortable-by-subsets then it is sortable by subsets of cardinality not greater than a specified value, k, and checking this property can be done efficiently, in practice, for any small value of k; elementwise sortable is simply the special case of $k = 1$.

We can also give a sufficient condition that is easy to check, in the case that all entries in all objectives have the same sign, and that each objective "increases" at a sufficiently fast rate. For example, it is not difficult to check that $c^1 = (1, 3, 5, 11), c^2 = (1, 5, 7, 13)$ are universally ideal, since $3 \geq 1$, $5 \geq 1 + 3$ and $11 \geq 1 + 3 + 5$ and since $5 \geq 1$, $7 \geq 1 + 5$ and $13 \geq 1 + 5 + 7$. (Similarly, $c^1 = (-1, -3, -5, -11), c^2 = (-1, -5, -7, -13)$ are universally co-ideal.) The sufficient condition may be stated is as follows.

Corollary 11 (to Proposition 8)**.** *If a set of non-negative objectives,* $c^1, c^2, \ldots, c^N \in \mathbb{R}^n_+$*, or non-positive objectives,* $c^1, c^2, \ldots, c^N \in \mathbb{R}^n_-$*, satisfies*

$$|c^i_j| \geq \sum_{k=1}^{j-1} |c^i_k|, \qquad \forall j = 2, \ldots, n, \ \forall i = 1, \ldots, N, \tag{2}$$

where $|.|$ *denotes the absolute value, then they are universally co-ideal.*

Proof. We only prove the case of non-negative objectives; the case of non-positive objectives can be proved similarly.

Suppose that (2) holds, but the set of objectives is not sortable-by-subsets. Then there exists $i, i' \in \{1, \ldots, N\}$ and disjoint sets $S, S' \subseteq \{1, \ldots, n\}$ with $c^i(S) > c^i(S')$ but $c^{i'}(S) < c^{i'}(S')$. Let j^* be the highest index of any element of S, i.e, $j^* = \arg\max\{j \ : \ j \in S\}$ and k^* be the highest index of any element of S', i.e, $k^* = \arg\max\{k \ : \ k \in S'\}$. Since S and S' are disjoint, $j^* \neq k^*$. Since $c^i(S) > c^i(S')$, it must be that $j^* > k^*$, as, otherwise, $S \subseteq \{1, \ldots, j^*\} \subseteq \{1, \ldots, k^* - 1\}$ and $c^i(S') \geq c^i_{k^*} \geq \sum_{j=1}^{k^*-1} c^i_j \geq c^i(S)$, which is a contradiction. Similarly, it must be that $k^* < j^*$ since $c^{i'}(S) < c^{i'}(S')$. Since $j^* > k^*$ and $j^* < k^*$ cannot both hold, we obtain a contradiction, and the objectives must be sortable-by-subsets. The result follows by Proposition 8. □

This sufficient condition is not necessary, as the example given in Figure 1a shows: the objectives $c^1 = (4, 7, 9), c^2 = (3, 4, 6)$ are universally co-ideal, but $9 \not\geq 4 + 7$. In fact, as we will show in the next section, there is no easily checked condition that is both necessary and sufficient, however the universally co-ideal property can be checked in time that is only polynomially dependent on n (and N); it can be checked in pseudo-polynomial time.

4 Computational Complexity

We now consider the computational complexity of the decision problem `UNIVERSALLY CO-IDEAL`: given inputs $c^1, \ldots, c^N \in \mathbb{Z}^n$, are these universally co-ideal? Note that for this analysis, we assume integer-valued objective vectors.

We first observe that `UNIVERSALLY CO-IDEAL` can be reduced to `2-UNIVERSALLY CO-IDEAL`: given instance (f, g), where $f, g \in \mathbb{Z}^n$, is it true that for all $S, S' \subseteq \{1, \ldots, n\}$, if $f(S) < f(S')$ then $g(S) \leq g(S')$ and if $f(S) > f(S')$ then $g(S) \geq g(S')$? By Proposition 8, this is equivalent to asking that (f, g) is universally co-ideal. We claim that an instance, $(c^1, \ldots, c^N)$, of `UNIVERSALLY CO-IDEAL`, is a YES instance if and only if for all $i, k \in \{1, \ldots, N\}$, the instance (c^i, c^k) of `2-UNIVERSALLY CO-IDEAL` is a YES instance. To see the "if" direction, we use Proposition 8, as follows. Let $S, S' \subseteq \{1, \ldots, n\}$, and suppose, without loss of generality, that $c^1(S) < c^1(S')$. Then for all $i = \{2, \ldots, N\}$, since (c^1, c^i) is a YES instance of `2-UNIVERSALLY CO-IDEAL`, it must be that $c^i(S) \leq c^i(S')$. The "only if" direction is obvious. Thus an instance of `UNIVERSALLY CO-IDEAL` can be solved by solving $\mathcal{O}(N^2)$ instances of `2-UNIVERSALLY CO-IDEAL`, and hence the latter must be at least as hard.

We now show that `2-UNIVERSALLY CO-IDEAL` is pseudo-polynomially solvable, and hence so is `UNIVERSALLY CO-IDEAL`. Given an instance (f, g), of `2-UNIVERSALLY CO-IDEAL`, where $f, g \in \mathbb{Z}^n$, we construct the following 0-1 knapsack optimization problem:

$$\begin{aligned}\psi(f,g) = \max \quad & fx - fy \\ \text{s.t.} \quad & gx - gy \leq -1, \\ & x, y \in \{0,1\}^n.\end{aligned}$$

Then the instance (f, g) is a YES instance if and only if $\psi(f, g) \leq 0$. Since 0-1 knapsack optimization is pseudo-polynomial, the result follows.

Note that the 0-1 knapsack optimization problem above solves both the `2-UNIVERSALLY CO-IDEAL` instance and its complement: does there exist $S, S' \subseteq \{1, \ldots, n\}$ with either $f(S) < f(S')$ and $g(S) > g(S')$ or $f(S) > f(S')$ and $g(S) < g(S')$? The answer is YES if and only if $\psi(f, g) > 0$, or, equivalently, $\psi(f, g) \geq 1$.

We now show that the complement of `2-UNIVERSALLY CO-IDEAL` is NP-complete.

Theorem 12. *The complement of `2-UNIVERSALLY CO-IDEAL` is NP-complete.*

Proof. Note first that the complement of `2-UNIVERSALLY CO-IDEAL` is obviously in NP.

We now show that `EQUAL-SUBSET-SUM` [26] is polynomially reducible to the complement of `2-UNIVERSALLY CO-IDEAL`, where `EQUAL-SUBSET-SUM` is defined as follows: given a set $\{a_1, \ldots, a_n\}$ of positive integers, does there exist non-empty disjoint subsets, $S, S' \subset \{1, \ldots, n\}$, with $\sum_{j \in S} a_j = \sum_{j \in S'} a_j$?

Given such an instance of `EQUAL-SUBSET-SUM`, we construct an instance of `2-UNIVERSALLY CO-IDEAL`, (f, g), by

$$f_j = a_j - 2^{-j},$$

and

$$g_j = a_j + 2^{-j},$$

for all $j = 1, \ldots, n$. Observe that for any $S \subseteq \{1, \ldots, n\}$,

$$\lceil f(S) \rceil = \lfloor g(S) \rfloor = \sum_{j \in S} a_j, \tag{3}$$

since $0 \leq \sum_{j \in S} 2^{-j} < 1$ and hence

$$\lceil f(S) \rceil = \lceil \sum_{j \in S} a_j - \sum_{j \in S} 2^{-j} \rceil = \sum_{j \in S} a_j = \lfloor \sum_{j \in S} a_j + \sum_{j \in S} 2^{-j} \rfloor = \lfloor g(S) \rfloor.$$

Now suppose that this instance, (f, g), of `2-UNIVERSALLY CO-IDEAL`, is a `NO` instance, so there exist, (by Proposition 8 and Corollary 6), non-empty disjoint sets, $S, S' \subset \{1, \ldots, n\}$, with $f(S) < f(S')$ and $g(S) > g(S')$. Then $f(S) < f(S')$ implies that

$$\lceil f(S) \rceil \leq \lceil f(S') \rceil = \lfloor g(S') \rfloor \leq \lfloor g(S) \rfloor,$$

by (3) and since $g(S') < g(S)$. But, also by (3), we have $\lceil f(S) \rceil = \lfloor g(S) \rfloor$, and hence, again by (3), it must be that

$$\sum_{j \in S} a_j = \lceil f(S) \rceil = \lfloor g(S) \rfloor = \lceil f(S') \rceil = \lfloor g(S') \rfloor = \sum_{j \in S'} a_j,$$

so `EQUAL-SUBSET-SUM` is a `YES` instance.

Finally, suppose that `EQUAL-SUBSET-SUM` is a `YES` instance, with non-empty, disjoint sets, $S, S' \subset \{1, \dots, n\}$, with $\sum_{j \in S} a_j = \sum_{j \in S'} a_j$. Let $m := \min\{j \ : \ j \in S \cup S'\}$ and, without loss of generality, assume that $m \in S$. Observe that

$$\sum_{j \in S} 2^{-j} \geq 2^{-m} > \sum_{j \in S'} 2^{-j}.$$

Hence

$$g(S) = \sum_{j \in S} a_j + \sum_{j \in S} 2^{-j} \geq \sum_{j \in S} a_j + 2^{-m} = \sum_{j \in S'} a_j + 2^{-m} > \sum_{j \in S'} a_j + \sum_{j \in S'} 2^{-j} = g(S').$$

Similarly,

$$f(S) = \sum_{j \in S} a_j - \sum_{j \in S} 2^{-j} \leq \sum_{j \in S} a_j - 2^{-m} = \sum_{j \in S'} a_j - 2^{-m} < \sum_{j \in S'} a_j - \sum_{j \in S'} 2^{-j} = f(S').$$

Thus the `2-UNIVERSALLY CO-IDEAL` instance, (f, g), is a `NO` instance. The result follows. □

Corollary 13. *`2-UNIVERSALLY CO-IDEAL`, and hence `UNIVERSALLY CO-IDEAL`, is NP-hard.*

5 Exploiting universally co-idealness

5.1 Reducing the number of objectives

In this section, we show how universally co-idealness can be exploited to reduce the number of objectives in a MBP. Reducing the number of objectives in a multi-objective optimization problem has, so far, been studied mostly in the context of multi-objective linear programming; interested readers are referred to Engau and Wiecek [8], Lindroth et al. [15], Malinowska and Torres [17], Malinowska [18], and Thoai [25].

We begin by defining a notion of equivalence for MBPs. Note that if F_E is the efficient set of a MBP, $\min_{x \in F}\{c^1 x, \dots, c^N x\}$, with objective function matrix C, and O_N is the set of its nondominated points, so $O_N = \{Cx : x \in F_E\}$, then O_N induces a partition of F_E, as follows. For each $z \in O_N$, let $F_E(z)$ denote the set of efficient points that yield criterion space image, z, so $F_E(z) = \{x \in F_E : Cx = z\}$. Then the collection of sets $\{F_E(z) : z \in O_N\}$ partitions F_E.

Definition 14. Let $P^1 := \min_{x \in F}\{c^1 x, c^2 x, \dots, c^{k-1}, c^k, \dots, c^N x\}$ with $c^i \in \mathbb{R}^n$ for $i = 1, \dots, N$ be a MBP and let $P^2 := \min_{x \in F}\{c^1 x, c^2 x, \dots, c^{k-1}, \bar{c}^k x\}$ with $\bar{c}^k \in \mathbb{R}^n$ be another MBP. Furthermore, let F_E^1, O_N^1 and F_E^2, O_N^2 be the sets of efficient solutions and nondominated points of P^1 and P^2, respectively. We say P^2 is *equivalent* to P^1 if $F_E^1 = F_E^2 =: F_E$ and the partition of F_E induced by O_N^2 is identical to that induced by O_N^1.

If P^2 is equivalent to P^1, then solving P^2 solves P^1. To see this, observe that solving P^2 means finding O_N^2 and, for each $z \in O_N^2$, identifying at least one $x(z) \in F_E^2$ with $(c^1 x, c^2 x, \dots, c^{k-1}, \bar{c}^k x) = z$, i.e., finding at least one member of each element of the partition of F_E^2 induced by O_N^2. But $F_E^1 = F_E^2$ and the partition of F_E^1 induced by O_N^1 is identical to the partition induced by O_N^2, thus, by solving P^2, we have, in fact, identified at least one member of each element in the partition of F_E^1 induced by O_N^1. By calculating the objective function, $(c^1 x, c^2 x, \dots, c^N x)$, for each such member, x, we must obtain O_N^1, as required.

We now show that if a MBP has a subset of the objectives that is universally co-ideal, then replacing these objectives by any positive combination of them yields an equivalent MBP, having fewer objectives.

Proposition 15. *Let* $P^1 := \min_{x\in F}\{c^1x, c^2x, \ldots, c^Nx\}$ *be a MBP and let* $c^k, \ldots, c^N$ *for* $k \geq 2$ *be universally co-ideal. Then* $P^2 := \min_{x\in F}\{c^1x, c^2x, \ldots, c^{k-1}x, \sum_{i=k}^N \lambda_i c^i x\}$*, with* $\lambda_i > 0$ *for* $i = k, \ldots, N$*, is an equivalent MBP.*

Proof. Let F_E^1, O_N^1 and F_E^2, O_N^2 denote the sets of efficient solutions and nondominated points of P^1 and P^2, respectively. Let C denote the $N \times n$ objective function matrix of P^1, with ith row given by (row vector) c^i, for $i = 1, \ldots, N$. Let $\overline{C}$ denote the $k \times n$ objective function matrix of P^2, with ith row given by c^i for $i = 1, \ldots, k-1$ and kth row given by $\sum_{i=k}^N \lambda_i c^i$.

It is straightforward to show that $F_E^2 \subseteq F_E^1$. Let $x \in F_E^2$. Now suppose that $x \notin F_E^1$, so it must be that $Cx' \leq Cx$ and $Cx' \neq Cx$, for some $x' \in F$. Then $c^ix' \leq c^ix$ for all $i = 1, \ldots, k-1$, and also $c^ix' \leq c^ix$ for all $i = k, \ldots, N$, so, since $\lambda_i > 0$ for all $i = k, \ldots, N$, it follows that $\sum_{i=k}^N \lambda_i c^i x' \leq \sum_{i=k}^N \lambda_i c^i x$. Thus $\overline{C}x' \leq \overline{C}x$ and, since $x \in F_E^2$, it must be that $\overline{C}x' = \overline{C}x$. Now since $Cx' \neq Cx$, it must be that $c^ix' < c^ix$ for some $i \in \{k, \ldots, N\}$. But then $\sum_{i=k}^N \lambda_i c^i x' < \sum_{i=k}^N \lambda_i c^i x$, which contradicts $\overline{C}x' = \overline{C}x$. So it must be that $x \in F_E^1$.

To show that $F_E^1 \subseteq F_E^2$, we proceed as follows. Let $x \in F_E^1$. Now suppose that $x \notin F_E^2$, so there exists $x' \in F$ with $\overline{C}x' \leq \overline{C}x$ and $\overline{C}x' \neq \overline{C}x$. Hence $c^ix' \leq c^ix$ for all $i = 1, \ldots, k-1$ and it must be that $c^ix' > c^ix$, for some $i \in \{k, \ldots, N\}$, as, otherwise, $x \in F_E^1$ is contradicted. Also, $\overline{C}x' \leq \overline{C}x$ implies $\sum_{i=k}^N \lambda_i c^i x' \leq \sum_{i=k}^N \lambda_i c^i x$, so there must exist $h \in \{k, \ldots, N\}$ with $c^hx' < c^hx$. Thus $S = supp(x)$ and $S' = supp(x')$ are mutually nondominated (MND) for $c^k, \ldots, C^N$, so $c^k, \ldots, c^N$ are not sortable-by-subsets, and hence, by Proposition 8, cannot be universally co-ideal, yielding a contradiction.

Thus $F_E^1 = F_E^2$ and we may define $F_E := F_E^1 = F_E^2$.

Now consider distinct $x, x' \in F_E$ in the same element of the partition induced by O_N^1, i.e., with $Cx = Cx'$ and $x \neq x'$. Then obviously $\overline{C}x = \overline{C}x'$, so x and x' are also in the same element of the partition induced by O_N^2.

It remains to consider distinct $x, x' \in F_E$ in the same element of the partition induced by O_N^2, i.e., with $\overline{C}x = \overline{C}x'$ and $x \neq x'$. Suppose they are in different elements of the partition induced by O_N^1, i.e., suppose $Cx \neq Cx'$. Then, since $c^ix = c^ix'$ for all $i = 1, \ldots, k-1$, (as $\overline{C}x = \overline{C}x'$), and since both x and x' are efficient for P^1, it must be that $c^ix' > c^ix$ and $c^hx' < c^hx$ for some $i, h \in \{k, \ldots, N\}$. Again, this shows that $S = supp(x)$ and $S' = supp(x')$ are mutually nondominated (MND) for $c^k, \ldots, C^N$, so $c^k, \ldots, c^N$ are not sortable-by-subsets, and hence, by Proposition 8, cannot be universally co-ideal, yielding a contradiction. We conclude that x and x' must also be in the same element of the partition of F_E induced by O_N^1.

Thus the partitions of F_E induced by O_N^1 and O_N^2 are identical, as required. □

The following example shows that solving the equivalent MBP obtained by aggregating universally co-ideal objectives may require less computational effort than solving the original MBP.

Example 16. Let P^1 be

$$\begin{aligned} \min\ & \{x_1 + 4x_2 + 2x_3,\ 4x_1 + x_2 + 2x_3,\ 6x_1 + 2x_2 + 3x_3\} \\ \text{s.t. } & x_1 + x_2 + x_3 \geq 2, \\ & x_1, x_2, x_3 \in \{0, 1\}. \end{aligned}$$

It is easy to verify that $F_E^1 = \{(1,1,0),\ (1,0,1),\ (0,1,1)\}$ and that the last two objectives are universally co-ideal. Let P^2 be the MPB obtained from P^1 by combining the last two objectives (with unit weights) into a single objective:

$$\begin{aligned} \min\ & \{x_1 + 4x_2 + 2x_3,\ 10x_1 + 3x_2 + 5x_3\} \\ \text{s.t. } & x_1 + x_2 + x_3 \geq 2, \\ & x_1, x_2, x_3 \in \{0,1\}. \end{aligned}$$

Suppose we use the method proposed by Kirlik-Sayın [14] to compute the nondominated frontier. First consider P^1. In each iteration, the methods first solves

$$x^i \in \arg\min\{6x_1 + 2x_2 + 3x_3 :\ x \in F,\ x_1 + 4x_2 + 2x_3 \leq u_1,\ 4x_1 + x_2 + 2x_3 \leq u_2\},$$

for some upper bounds (right-hand sides) u_1 and u_2, and then, if a solution exists, solves

$$x^n \in \arg\min\{5x_1 + 5x_2 + 4x_3 :\ x \in F,\ 6x_1 + 2x_2 + 3x_3 = 6x_1^i + 2x_2^i + 3x_3^i\}.$$

Next, consider P^2. In each iteration, the methods first solves

$$x^i \in \arg\min\{10x_1 + 3x_2 + 5x_3 :\ x \in F,\ x_1 + 4x_2 + 2x_3 \leq u\},$$

for some upper bound u, and then, if a solution exists, solves

$$x^n \in \arg\min\{x_1 + 4x_2 + 2x_3 :\ x \in F,\ 10x_1 + 3x_2 + 5x_3 = 10x_1^i + 3x_2^i + 5x_3^i\}.$$

More specifically, to solve P^1, the method explores the following bounds (u_1, u_2): $(+\infty, +\infty)$, $(2, +\infty)$, $(+\infty, 5)$, $(4, 5)$, $(+\infty, 4)$, $(5, 4)$, and $(+\infty, 2)$. Consequently, 10 optimization problems have to be solved, 4 of which are infeasible. To solve P^2, the method explores the following bounds u: $+\infty$, 14, 12, and 7. Consequently, 7 optimization problems have to be solved, only one of which is infeasible.

Since equivalence of MBPs, as in Definition 14, is transitive, aggregation of objectives that are universally co-ideal can be repeated for multiple subsets of universally co-ideal objectives. Specifically, we have the following corollary to Proposition 15.

Corollary 17 (to Proposition 15)**.** *Let $P^1 := \min_{x \in F}\{c^1x, c^2x, \ldots, c^Nx\}$ be a MBP and also let $Q_1, \ldots, Q_D \subseteq \{1, \ldots, N\}$ be disjoint sets such that the objectives c^i for $i \in Q_d$ are universally co-ideal, for each $d = 1, \ldots, D$. Without loss of generality, say $\bigcup_{d=1}^D Q_d = \{k, \ldots, N\}$ for some $k \in \{1, \ldots, N\}$. Then*

$$P^2 := \min_{x \in F}\{c^1x, c^2x, \ldots, c^{k-1}x, \sum_{i \in Q_1} \lambda_i^1 c^i x, \sum_{i \in Q_2} \lambda_i^2 c^i x, \ldots, \sum_{i \in Q_D} \lambda_i^D c^i x\},$$

with $\lambda_i^d > 0$ for all $i \in Q_d$ and $d = 1, \ldots, D$, is an equivalent MBP.

This suggests the possibility of partitioning the objectives of a MBP into subsets, each of which is a universally co-ideal set of objectives, and seeking a partition of minimum cardinality. Aggregating the objectives in each universally co-ideal element of such a partition would yield an equivalent MBP, having minimum criterion space dimension. Recall that a set of objectives is universally co-ideal if and only if every pair in the set is universally co-ideal. So such a partition can be found by finding a minimum cardinality partition of the nodes of a graph into cliques, where the graph has a node for each objective and an edge for each universally co-ideal pair of objectives. By results in Section 4, the graph can be constructed in pseudo-polynomial time. Although this partitioning problem is equivalent to `PARTITION INTO CLIQUES`, and so is NP-hard [11], for small values of N it may be practically tractable to solve to optimality; otherwise heuristic algorithms may suffice.

5.2 Deriving cuts

In this section, we introduce the concept of *objective-aligning* sets of variables and show how this concept can be used to derive a class of cuts for MBPs. These cuts are not inequalities that are valid for all feasible, or even all efficient, solutions of a MBP, but, rather, they are cuts that must be satisfied by any efficient solution that yields a nondominated point distinct from one already discovered. In this way, they may aid in the search for discovery of new nondominated points, during the solution of a MBP.

Definition 18. Given a set of objectives $c^1, c^2, \dots, c^N \in \mathbb{R}^n$, a set of variables $A \subseteq \{1, \dots, n\}$ is *objective-aligning* if the set of objectives $(c^1_j)_{j\in A}, (c^2_j)_{j\in A}, \dots, (c^N_j)_{j\in A}$ is universally co-ideal.

Before we show how objective-aligning sets of variables may be exploited, we first illustrate the concept with an example.

Example 19. Consider the two objective function vectors

$$c^1 = (1, 3, 2, 4) \qquad \text{and} \qquad c^2 = (2, 1, 5, 3).$$

It is obvious from Figure 2a, which is a plot of the points in Z, that c^1, c^2 are not universally co-ideal. For a given A, we can visualize $((c^1_j)_{j\in A}, (c^2_j)_{j\in A})$ by plotting $(\sum_{j\in A} c^1_j x_j, \sum_{j\in A} c^2_j x_j)$ for all $x \in \{0, 1\}^4$. Figure 2b shows the result for three alternative variable subsets, $A = \{1, 3\}$, $A = \{2, 4\}$ and $A = \{2, 3\}$. It is clear that the former two sets align the objective functions, since the points are simultaneously sorted in both objectives; $\{1, 3\}$ and $\{2, 4\}$ are both objective-aligning sets of variables for c^1, c^2. The set $\{2, 3\}$ does not align the objectives, since, for example, the points $(2, 5)$ and $(3, 1)$ cannot be simultaneously sorted; $\{2, 3\}$ is not an objective-aligning set for c^1, c^2.

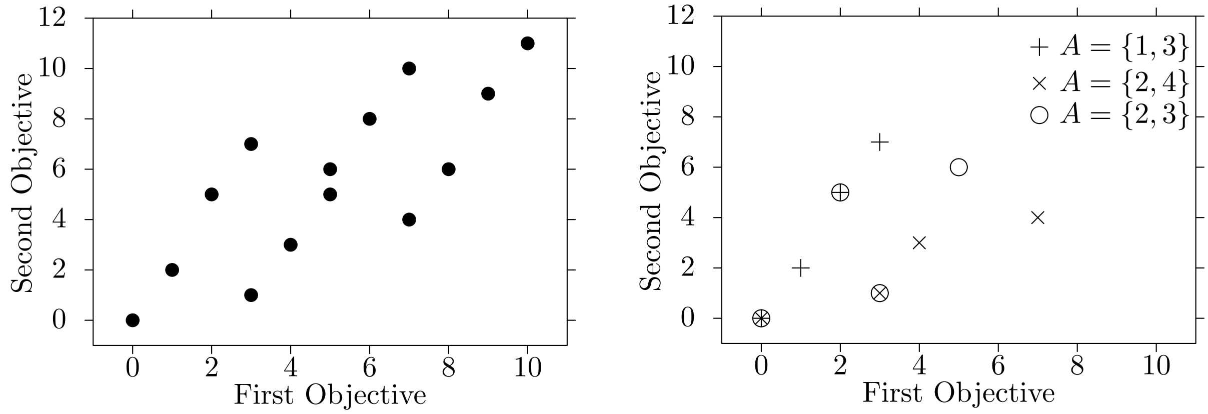
Proposition 20. *Let $\bar{x}$ be an efficient solution for a MBP with objectives $c^1, c^2, \dots, c^N \subseteq \mathbb{R}^n$. If $A \subseteq \{1, \dots, n\}$ is an objective-aligning set of variables, then*

$$\sum_{j\notin A,\ \bar{x}_j=0} x_j + \sum_{j\notin A,\ \bar{x}_j=1} (1 - x_j) \geq 1 \tag{4}$$

is valid for all efficient solutions, x, that do not yield the same nondominated point as $\bar{x}$, i.e., it is satisfied by all efficient solutions, x, with $Cx \neq C\bar{x}$, where C is the objective function matrix.

Proof. Suppose not, i.e., suppose there is an efficient solution, x, with $Cx \neq C\bar{x}$ and

$$\sum_{j\notin A,\ \bar{x}_j=0} x_j + \sum_{j\notin A,\ \bar{x}_j=1} (1 - x_j) = 0.$$



(a) The points in Z for $c^1 = (1, 3, 2, 4), c^2 = (2, 1, 5, 3)$. (b) Points $(\sum_{j\in A} c^1_j x_j, \sum_{j\in A} c^2_j x_j)$ for all $x \in \{0, 1\}^4$, for three different variable sets A.

Figure 2: Illustration of objective-aligning sets of variables, from Example 19.

This implies that $x_j = \bar{x}_j$ for all $j \notin A$. Define $S := supp(x)$, $S' := supp(\bar{x})$, $S^1 := S \backslash S'$ and $S^2 := S' \backslash S$. Since $Cx \neq C\bar{x}$ and both x and $\bar{x}$ are efficient solutions, S and S' must be MND. So, by Lemma 5, S^1 and S^2 are also MND. This, together with the fact that $S^1, S^2 \subseteq A$, (because $x_j = \bar{x}_j$ for all $j \notin A$), shows that $(c^1_j)_{j\in A}, (c^2_j)_{j\in A}, \ldots, (c^N_j)_{j\in A}$ is not sortable-by-subsets. Then, by Proposition 8, the set of objectives $(c^1_j)_{j\in A}, (c^2_j)_{j\in A}, \ldots, (c^N_j)_{j\in A}$ is not universally co-ideal, and hence is not objective-aligning. □

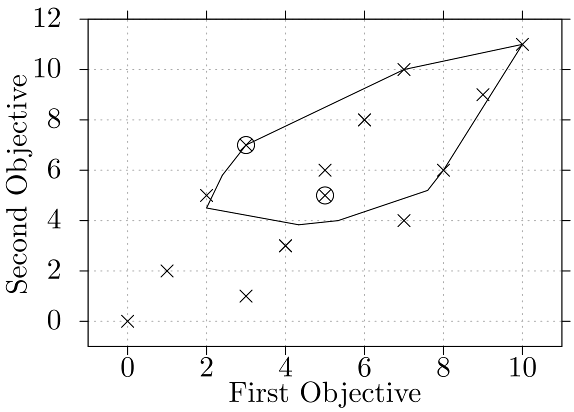
The example below shows the effect of adding cuts of type (4) to the LP relaxation of a MBP. In this example, the projection of the LP relaxation becomes smaller after adding each of the cuts. More importantly, the nondominated frontier of the LP relaxation improves, in each case. This demonstrates the potential utility of these inequalities in decision-space search methods, which (usually) fathom a node based on the nondominated frontier of the LP-relaxation at the node.

Example 21. Suppose that we want to compute all distinct nondominated points of

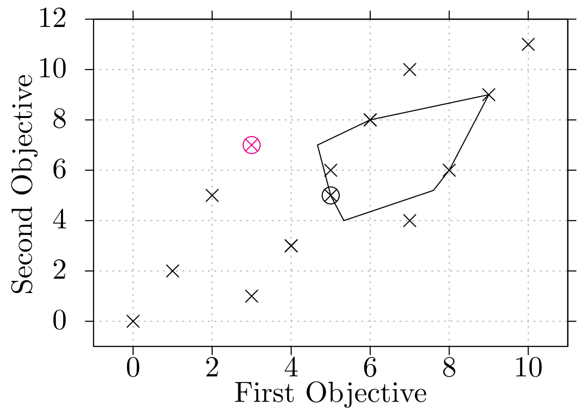
$$\begin{aligned} \min\ & \{x_1 + 3x_2 + 2x_3 + 4x_4,\ 2x_1 + x_2 + 5x_3 + 3x_4\} \\ \text{s.t. } & 5x_1 + 2x_2 + 6x_3 + 3x_4 \geq 8, \\ & x_1, x_2, x_3, x_4 \in \{0, 1\}. \end{aligned}$$

Note that this MBP has the same objective functions as are given in Example 19. It has two nondominated points, $(3, 7)$ and $(5, 5)$, corresponding to solutions $(1, 0, 1, 0)$ and $(1, 0, 0, 1)$, respectively. Projections of all $x \in \{0, 1\}^4$ and the LP-relaxation of this problem into the criterion space are shown in Figure 3a. To distinguish the nondominated points, we have drawn a circle around each in the figure.

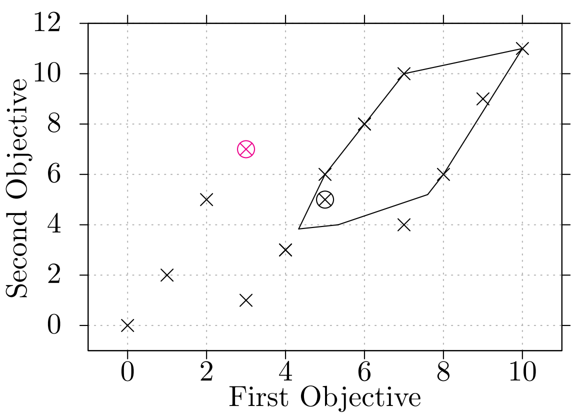
It is easy to verify that $\{2, 4\}$, $\{1, 3\}$, and $\{1, 4\}$ are objective-aligning sets (indeed, the first two of these were observed so from Figure 2b). Suppose that we have found one of the nondominated points, say $(3, 7)$. When searching for another nondominated point, we can use Proposition 20 to generate and add a cut of the form (4) for any objective-aligning set. The cuts corresponding to



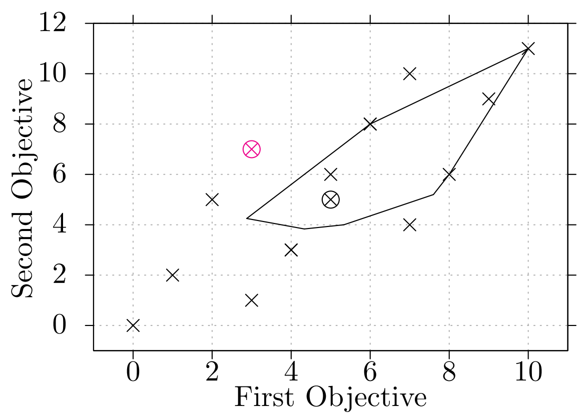
(a) No additional cut.



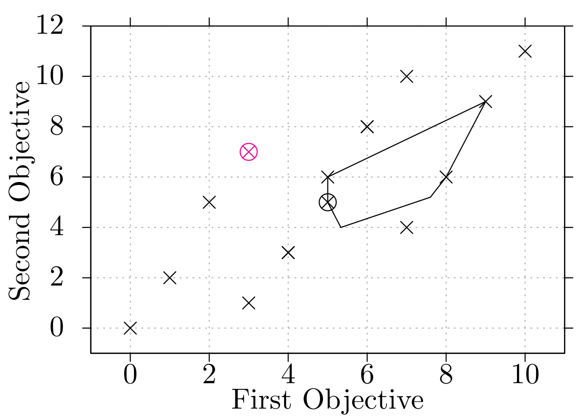
(b) After adding $(1 - x_1) + (1 - x_3) \geq 1$.



(c) After adding $x_2 + x_4 \geq 1$.



(d) After adding $x_2 + (1 - x_3) \geq 1$.



(e) After adding $(1 - x_1) + (1 - x_3) \geq 1$, $x_2 + x_4 \geq 1$, and $x_2 + (1 - x_3) \geq 1$.

Figure 3: Projections of $\{0, 1\}^4$ and the LP-relaxation of Example 21 into the criterion space.

$\{2,4\}$, $\{1,3\}$ and $\{1,4\}$ are

$$(1 - x_1) + (1 - x_3) \geq 1, \qquad x_2 + x_4 \geq 1, \qquad \text{and} \qquad x_2 + (1 - x_3) \geq 1,$$

respectively. The projection of the LP-relaxation after adding each of these three inequalities can be found in Figures 3b, 3c, and 3d, respectively. Figure 3e shows the projection of the LP-relaxation of the problem after adding all three. For convenience, we show the point $(3, 7)$, which was used to construct the cuts, in red.

Above, we derived a cut from an objective-aligning set of variables and a known, efficient, solution. In this case, the corresponding nondominated point is an ideal point for the objective functions restricted to variables in the objective-aligning set. This property can be generalized: *from* a given efficient solution, a subset of variables that is *not* necessarily objective-aligning, *per se*, but that nevertheless induces the corresponding nondominated point to be ideal, can be sought. Again, a cut is derived, but in this case under weaker conditions than those required for those discussed above. Thus a cut of the type we derive in this section may be available even when none of the type above can be found.

Definition 22. The point $p \in \mathbb{R}^N$ is a *nexus point* for $c^1, c^2, \ldots, c^N \in \mathbb{R}^n$, a set of objective functions, if, for any $x \in \{0,1\}^n$, either $p_i \leqslant c^i x$ for all $i = 1, \ldots, N$ or $p_i \geqslant c^i x$ for all $i = 1, \ldots, N$. Equivalently, p is a nexus point if $Z \subseteq (\{p\} + \mathbb{R}^N_+) \cup (\{p\} - \mathbb{R}^N_+)$.

Figure 4 illustrates a nexus point: it shows Z for two objectives that are not universally co-ideal, but that have a nexus point. Recall Figure 1a: here we see Z in the case of two objectives that are universally co-ideal, in which case *every* point in Z is a nexus point.

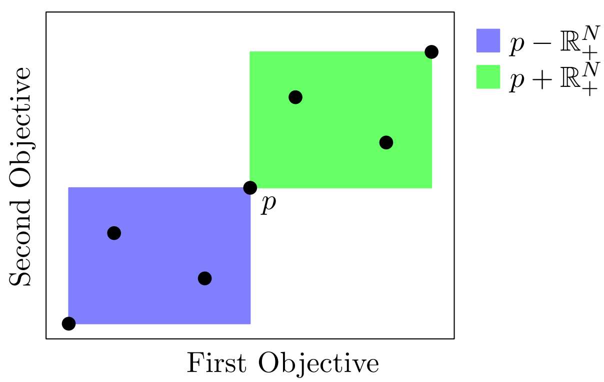


Figure 4: The points in Z for the objectives $c^1 = (1, 3, 4)$ and $c^2 = (2, 1, 3)$. These objectives are not universally co-ideal, but have $p = (4, 3)$ a nexus point.

It is not difficult to see (for example, by considering Figure 4), that if p is a nexus point for $c^1, c^2, \ldots, c^N$, then for any MBP with these objectives, we have that if p is a nondominated point, i.e., $p \in O_N$, then p is an ideal point. The following defines a special case of nexus point, for which the only condition needed to ensure that p is ideal is that it is the image of a feasible solution.

Definition 23. The point $p \in \mathbb{R}^N$ is a *lower bound point* for $c^1, c^2, \ldots, c^N \in \mathbb{R}^n$, a set of objective functions, if, for any $x \in \{0,1\}^n$, $p_i \leqslant c^i x$ for all $i = 1, \ldots, N$. Equivalently, p is a lower bound point if $Z \subseteq \{p\} + \mathbb{R}^N_+$.

We generalize these definitions to consider optimization over only a subset of the variables, with variables outside this subset fixed so as to coincide with a given vector. Given $A \subseteq \{1, \ldots, n\}$, $y \in \{0,1\}^n$, we define $Z_{A,y} = \{Cx : x \in \{0,1\}^n,\ x_j = y_j, \forall j \notin A\}$, where C is a given objective function matrix, to indicate the image, in criterion space, of every binary vector of dimension n that coincides with y for variables outside A. We call the pair (A, y), with A nonempty, a *subspace-fixing*. Note that the values of y_j for $j \in A$ are irrelevant, and can safely be ignored; we take y to be in $\{0,1\}^n$ purely for notational convenience.

Definition 24. Given a point $p \in \mathbb{R}^N$ and a set of objectives $c^1, c^2, \ldots, c^N \in \mathbb{R}^n$, a subspace-fixing, (A, y), is *objective-aligning for* p if p is a nexus point with respect to solutions fixed to y for variables not indexed by A. Equivalently, the subspace-fixing (A, y) is objective-aligning for p if $Z_{A,y} \subseteq (\{p\} + \mathbb{R}^N_+) \cup (\{p\} - \mathbb{R}^N_+)$. In the special case that $Z_{A,y} \subseteq \{p\} + \mathbb{R}^N_+$, we say that p is a lower bound point for the subspace-fixing (A, y).

We illustrate the concept using the same objectives as in Examples 19 and 21. Recall that for these objectives, $c^1 = (1, 3, 2, 4)$ and $c^2 = (2, 1, 5, 3)$, the variables sets $\{1, 3\}$, $\{2, 4\}$ and $\{1, 4\}$ are objective-aligning. In fact, these are the *only* (non-trivial) objective-aligning sets for these objectives. So, for example, $\{2, 3\}$ and $\{1, 2, 4\}$ are not objective-aligning. In Figure 5a, we show $Z_{A,y}$ for $A = \{1, 2, 4\}$ and $y = (-, -, 0, -)$, (we do not specify y_j for $j \in A$, since these values are irrelevant), indicated with the plus symbol in the figure, and observe that $p = (4, 3)$ is a nexus point: the subspace-fixing $(\{1, 2, 4\}, (-, -, 0, -))$ is objective-aligning for $(4, 3)$. This Figure 5a also shows $Z_{A,y}$ for $A = \{1, 2, 4\}$ and $y = (-, -, 1, -)$, indicated with the unfilled circle in the figure, and observe that $p = (6, 8)$ is a nexus point: the subspace-fixing $(\{1, 2, 4\}, (-, -, 1, -))$ is objective-aligning for $(6, 8)$. In Figure 5b we show $Z_{A,y}$ for $A = \{2, 3\}$ and $y = (1, -, -, 1)$, indicated with the triangle symbol, and observe that $p = (5, 5)$ is a lower bound point for the subspace-fixing $(\{2, 3\}, (1, -, -, 1))$.

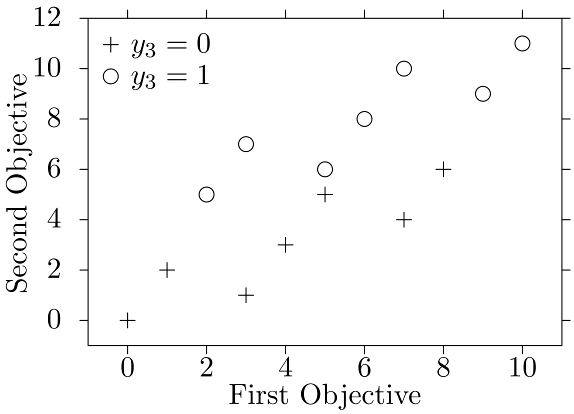
To explain the relationship between sets of variables that are objective-aligning, *per se*, (i.e., satisfy Definition 18), and subspace-fixings that are objective-aligning for some point, we show, in Figures 5c and 5d, $Z_{A,y}$ for *every* possible y, for the objective-aligning sets $A = \{1, 3\}$ and $A = \{2, 4\}$, respectively. We see that for A an objective-aligning set of variables, the set $Z_{A,y}$, for any y, is totally ordered, and hence for every $p \in Z_{A,y}$, the subspace-fixing (A, y) is objective-aligning for p. In fact, this property follows from the observation below, which can be seen to hold in all subfigures of Figure 5.

Observation 25. For any y, $Z_{A,y}$ is translation of $Z_{A,\mathbf{0}}$, where $\mathbf{0}$ is the zero vector. Specifically, $Z_{A,y} = \{p + CI_{\bar{A}}y : p \in Z_{A,\mathbf{0}}\}$, where $I_{\bar{A}}$ denotes the matrix with diagonal entries given by the indicator vector of $\bar{A}$ and all other entries zero, and $\bar{A} = \{1, \ldots, n\} \setminus A$ denotes the complement of A. Note also that $Z_{A,\mathbf{0}}$ is the image of the objectives restricted to A, $((c^1_j)_{j \in A}, (c^2_j)_{j \in A}, \ldots, (c^N_j)_{j \in A})$, i.e.,

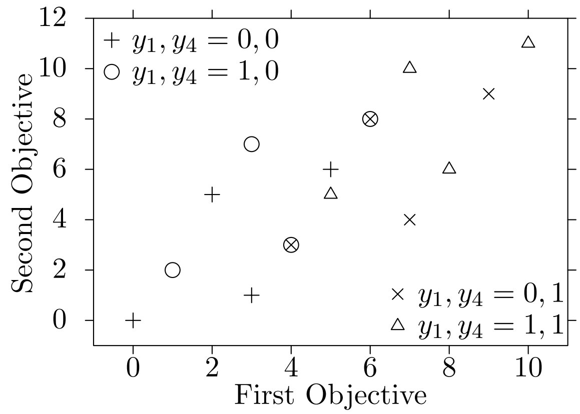
$$Z_{A,\mathbf{0}} = \{((c^1_j)_{j \in A}\hat{x}, (c^2_j)_{j \in A}\hat{x}, \ldots, (c^N_j)_{j \in A}\hat{x}) : \hat{x} \in \{0,1\}^{|A|}\}.$$

The latter part of this observation can be illustrated by comparing Figures 4 and 5a: the former consists precisely of the points in the latter marked with plus symbol, i.e., those for the case $y_3 = 0$. (Note that Figure 4 is scaled so that the set of points shown fills the region.)

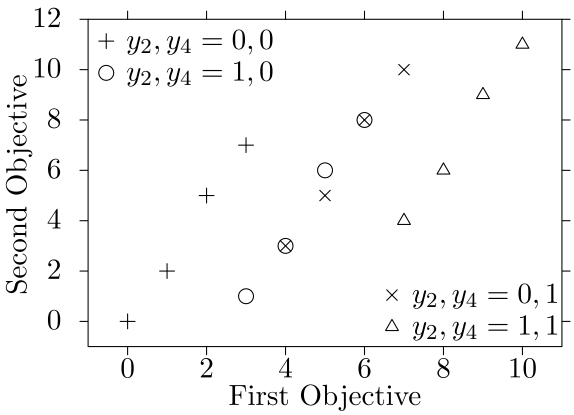
If A is a set of variables that is objective-aligning, then the subspace-fixing $(A, \mathbf{0})$ is objective-aligning for every $p \in Z_{A,\mathbf{0}}$, since this set is totally ordered. Thus for every $y \in \{0,1\}^n$ the subspace-fixing (A, y) is objective-aligning for every $p \in Z_{A,y}$, since this set is simply a translation



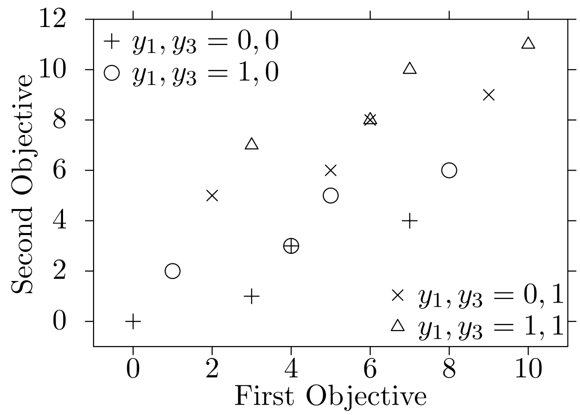
(a) All (both) subspace-fixings (A, y) with $A = \{1, 2, 4\}$ ($y_3 = 0$ and $y_3 = 1$).



(b) All (four) subspace-fixings (A, y) with $A = \{2, 3\}$ ($(y_1, y_4) = (0, 0), (1, 0), (0, 1)$ and $(1, 1)$).



(c) All (four) subspace-fixings (A, y) with $A = \{1, 3\}$ ($(y_2, y_4) = (0, 0), (1, 0), (0, 1)$ and $(1, 1)$).



(d) All (four) subspace-fixings (A, y) with $A = \{2, 4\}$ ($(y_1, y_3) = (0, 0), (1, 0), (0, 1)$ and $(1, 1)$).

Figure 5: Illustration of subspace-fixings, showing $Z_{A,y}$ for $c^1 = (1, 3, 2, 4)$ and $c^2 = (2, 1, 5, 3)$, for different values of A and y.

of $Z_{A,\mathbf{0}}$. However, (A, y) objective-aligning for some p does *not* imply that A is objective-aligning, *per se* (as we saw in Figures 5a and 5b). Thus it may be that, for a given p, subspace-fixings that are objective-aligning for p can be found, and hence cuts are available, having the form given in the following proposition, that could not be deduced from objective-aligning sets.

Proposition 26. *If $\bar{x}$ is an efficient solution of the MBP (C, F) and the subspace-fixing (A, y) is objective-aligning for $p = C\bar{x}$, then*

$$\sum_{j \notin A, y_j = 1} (1 - x_j) + \sum_{j \notin A, y_j = 0} x_j \geq 1 \tag{5}$$

is valid for all efficient solutions, x, that do not yield the same nondominated point as $\bar{x}$, i.e., it is satisfied by all $x \in F_E$ with $Cx \neq C\bar{x}$. If $p = C\bar{x}$ is a lower bound point for the subspace-fixing (A, y), then, to ensure validity of (5), it suffices to require that $\bar{x}$ is a feasible solution of the MBP, not necessarily efficient.

Proof. Let $\bar{x} \in F_E$ and (A, y) be a subspace-fixing that is objective-aligning for $p = C\bar{x}$. $\bar{x}$ an efficient solution implies that $O_N \cap (\{p\} + \mathbb{R}^N_+) \cup (\{p\} - \mathbb{R}^N_+) = \{p\}$ and (A, y) objective-aligning for p is equivalent to $Z_{A,y} \subseteq (\{p\} + \mathbb{R}^N_+) \cup (\{p\} - \mathbb{R}^N_+)$. So $O_N \cap Z_{A,y} \subseteq \{p\}$.

Now suppose $x \in F_E$ with $Cx \neq C\bar{x}$ and $\sum_{j \notin A, y_j = 1}(1 - x_j) + \sum_{j \notin A, y_j = 0} x_j < 1$, i.e., since x is binary, $x_j = 1$ for all $j \notin A$ with $y_j = 1$ and $x_j = 0$ for all $j \notin A$ with $y_j = 0$. Then $Cx \in O_N$, $Cx \neq p$, $x_j = y_j$ for all $j \notin A$, and hence $Cx \in Z_{A,y}$. But then $Cx \in O_N \cap Z_A \subseteq \{p\}$ and $Cx \neq p$, which is a contradiction.

We conclude that if x is an efficient solution yielding a nondominated point distinct from p, then x must satisfy (5).

To complete the proof, let $\bar{x} \in F$, not necessarily efficient, and let (A, y) be a subspace-fixing for which $p = C\bar{x}$ is a lower bound point, so $Z_{A,y} \subseteq \{p\} + \mathbb{R}^N_+$. Again, suppose $x \in F_E$ with $Cx \neq C\bar{x}$ and $\sum_{j \notin A, y_j = 1}(1 - x_j) + \sum_{j \notin A, y_j = 0} x_j < 1$, i.e., $x_j = y_j$ for all $j \notin A$. Then $Cx \in O_N$, $Cx \neq p$, and $Cx \in Z_{A,y}$. But then $Cx \in O_N \cap (\{p\} + \mathbb{R}^N_+)$ and $Cx \neq p$, which is a contradiction, since for $p \in O$, the only nondominated point in $\{p\} + \mathbb{R}^N_+$ must be p itself. □

In the case that A is a set of variables that is objective-aligning and $p = C\bar{x}$ for $\bar{x} \in F_E$, cuts of *both* types (4) and (5) may be applied. Taking $y = \bar{x}$ ensures that $p \in Z_{A,y}$, so in this case the two inequalities coincide. Otherwise neither dominates the other. For example, if $(A, \mathbf{0})$, (where $\mathbf{0}$ is the zero vector), is objective-aligning for p, then (5) is given by

$$\sum_{j \notin A} x_j \geq 1,$$

and if $\bar{x}_j \neq 0$ for some $j \notin A$ then this can be combined with (4) to form

$$1 - \sum_{j \notin A, \bar{x}_j = 0} x_j \leq \sum_{j \notin A, \bar{x}_j = 1} x_j \leq |\{j \notin A, \bar{x}_j = 1\}| - 1 + \sum_{j \notin A, \bar{x}_j = 0} x_j.$$

The following example illustrates the effect on the LP relaxation of a cut of type (5).

Example 27. Consider the MBP

$$\begin{aligned} \min\ & \{x_1 + 3x_2 + 2x_3 + 4x_4,\ 2x_1 + x_2 + 5x_3 + 3x_4\} \\ \text{s.t. } & 5x_1 + 2x_2 + 6x_3 + 3x_4 \geq 7, \\ & x_1, x_2, x_3, x_4 \in \{0,1\}. \end{aligned}$$

Note that this has the same objectives as Example 21, but has a slightly different (enlarged) feasible set. The points in Z, and the image of the LP relaxation in criterion space, are shown in Figure 6a.

This problem has two nondominated points, $(4,3)$ and $(3,7)$, corresponding to solutions $(1,1,0,0)$ and $(1,0,1,0)$, respectively. Suppose we have discovered the former, so $\bar{x} = (1,1,0,0) \in F_E$, and use the three, maximal, objective-aligning sets of variables $\{1,3\}$, $\{1,4\}$, $\{2,4\}$ to derive cuts of type (4). We obtain the cuts $x_4 \geq x_2$, $x_3 \geq x_2$, and $x_3 \geq x_1$, respectively. The effect of adding these cuts on the LP relaxation is shown in Figure 6b.

Furthermore, $p = C\bar{x} = (4,3)$ is a nexus point for the subspace-fixing $(\{1,2,4\}, (-,-,0,-))$: the corresponding cut of type (5) is $x_3 \geq 1$. Figure 6c shows the effect of this cut on the LP relaxation, while Figure 6d shows the effect of all four cuts, in combination.

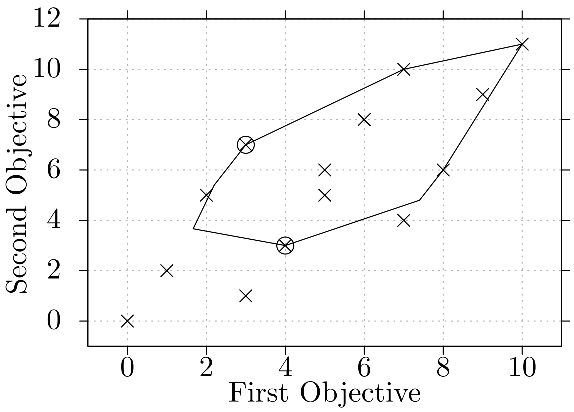
5.2.1 Complexity

Naturally, the questions of how difficult it is to (i) recognize that a set of variables, A, is objective-aligning, (ii) decide if a given point, p, is a nexus point for a subspace-fixing (A,μ), and (iii) decide if a given point, p, is a lower bound point for a subspace-fixing (A,μ), arise. The answer to (i) follows immediately from the results in Section 4: recognizing that the objectives restricted to A are universally co-ideal is NP-hard, but can be solved in pseudo-polynomial time. The answer to (ii) is the same. It suffices to check that p is a nexus point for every pair of objectives, say $c^i, c^{i'}$ with $i \neq i'$, for the subspace fixing (A,μ), and this can be done using the 0-1 knapsack optimization problem, $\psi(c^i, c^{i'})$, given in Section 4, simply by replacing the term $c^i y$ by p_i and the term $c^{i'} y$ by $p_{i'}$, removing the y variables, and setting $x_j = \mu_j$ for all $j \notin A$. The remaining problem is still a 0-1 knapsack problem, in $|A|$ variables, and hence can be solved in pseudo-polynomial time. The problem can be proved to be NP-hard using ideas similar to those in Section 4; proof is given in Appendix A. The answer to (iii) is that the problem is trivial: p is a lower bound point for the subspace-fixing (A,μ) if and only if, for every objective, c^i, $i = 1,\ldots,N$, $\min\{c^i x : x \in \{0,1\}^n, x_j = \mu_j, \forall j \notin A\} = \sum_{j \notin A} c^i_j \mu_j + \sum_{j \in A} \min\{c^i_j, 0\} \geq p_i$, which can be checked in $\mathcal{O}(Nn)$ time.

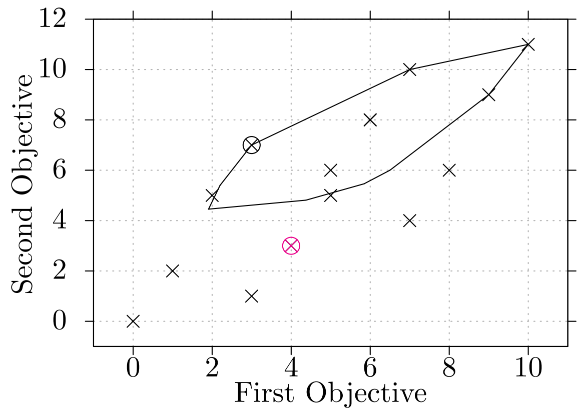
5.2.2 Variable fixing

Subspace-fixings that induce nexus points may be used in preprocessing, to fix variables. Given an MPB (C,F) and given $x \in F_E$, if $p = Cx$ is a nexus point for the subspace-fixing $(\{1,\ldots,n\} \backslash \{j\}, y)$, for some $j \in \{1,\ldots,n\}$, where $y_j = 0$, then (5) has the effect of fixing $x_j = 1$. Indeed, the cut derived from the subspace-fixing $(\{1,2,4\}, (-,-,0,-))$ in Example 27, $x_3 \geq 1$, has exactly this effect. Alternatively, if $p = Cx$ is a nexus point for the subspace-fixing $(\{1,\ldots,n\} \setminus \{j\}, y)$, for some $j \in \{1,\ldots,n\}$, where $y_j = 1$, then (5) has the effect of fixing $x_j = 0$. To illustrate this, note that, in Example 27, the nondominated point $(4,3)$ is a nexus point for the subspace-fixing $(\{1,3,4\}, (-,1,-,-))$, giving cut $x_2 \leq 0$.

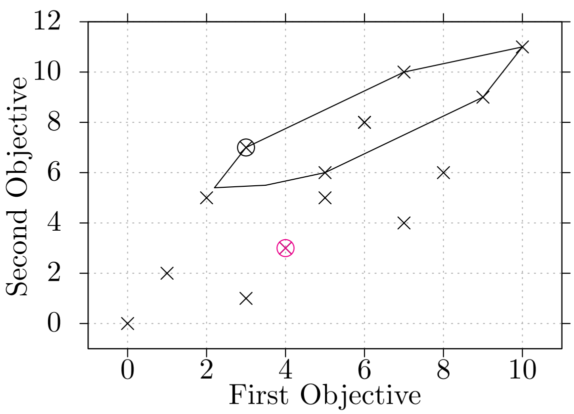
Although checking that p is a nexus point for such a subspace-fixing requires pseudo-polynomial time in the worst case, in practice it may still be tractable. Furthermore, the 0-1 knapsack problems



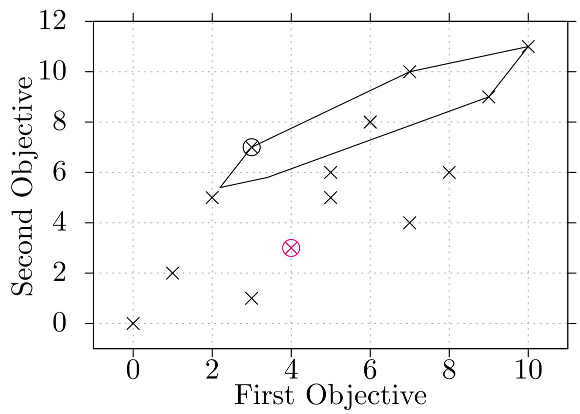
(a) No additional cut.



(b) After adding $x_4 \geq x_2$, $x_3 \geq x_2$, and $x_3 \geq x_1$.



(c) After adding $x_3 \geq 1$.



(d) After adding $x_4 \geq x_2$, $x_3 \geq x_2$, $x_3 \geq x_1$, and $x_3 \geq 1$.

Figure 6: The effect of cuts derived from objective-aligning sets and a subspace-fixing, in conjunction with the knowledge of the nondominated point, $(4, 3)$, on the LP relaxation, for Example 27.

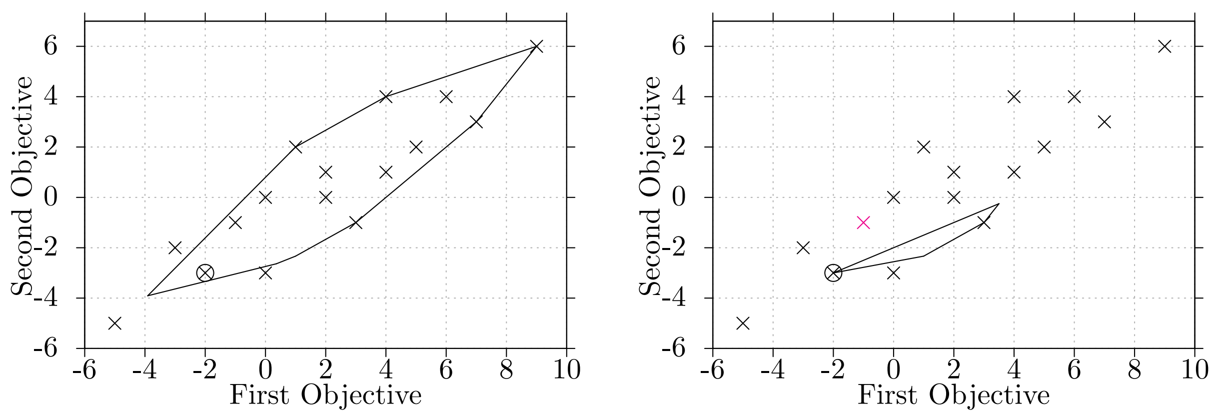
to be solved do not necessarily need to be solved to optimality; they can be halted as soon as their dual (upper) bound does not exceed zero, or whenever their primal (lower) bound does exceed zero.

As discussed in the previous section, subspace-fixings that induce lower bounds can be recognized efficiently, so these may be a particularly attractive special case of nexus point to check. Furthermore, variable fixing can be deduced from $p = Cx$ for x a feasible solution, ($x \in F$), not necessarily efficient. An illustration of this is given in Example 28.

Example 28. Consider the MBP

$$\begin{aligned} \min\ & \{-3x_1 + 4x_2 + 5x_3 - 2x_4,\ -2x_1 + 4x_2 + 2x_3 - 3x_4\} \\ \text{s.t.}\ & -3x_1 + 11x_2 + 2x_3 + 8x_4 \geq 8, \\ & x_1, x_2, x_3, x_4 \in \{0,1\}. \end{aligned}$$

This problem has an ideal point of $(-2,-3)$ corresponding to solution $(0,0,0,1)$. Projections of all $x \in \{0,1\}^4$ and the LP-relaxation of this problem into the criterion space are shown in Figure 7a. Suppose that we know that $(-1,-1)$ is a point in criterion space (corresponding to feasible solution $(1,1,0,1)$). Then $(-1,-1)$ is a lower bound point for the subspace-fixing $(\{1,3,4\},(-,1,-,-))$, and hence $x_2 = 0$ must be satisfied by any efficient solution yielding nondominated point distinct from $(-1,-1)$. Figure 7b shows the projection of the LP-relaxation of the problem after fixing $x_2 = 0$.



(a) No additional cut.

(b) After fixing $x_2 = 0$.

Figure 7: Projections of all $x \in \{0,1\}^4$ and the LP relaxation, into the criterion space, for Example 28.

6 Relaxations

A useful extension of the concept of universally co-ideal is given in Definition 29.

Definition 29. A set of objectives, $c^1, c^2, \dots, c^N \in \mathbb{R}^n$, is *universally co-ideal with respect to* $R \subseteq \{0,1\}^n$ if, for **any** nonempty feasible set contained in R, $F \subseteq R$, the MBP

$$\min_{x \in F}\{c^1x, c^2x, \dots, c^Nx\}$$

has an ideal point, i.e.,

$$\bigcap_{i=1}^{N} \arg\min_{x \in F} c^i x \neq \emptyset.$$

We observe that the above results concerning universally co-idealness can be extended easily to universally co-idealness with respect to R. For example, when we consider feasible sets known to have bounded cardinality.

Definition 30. A set of objectives, $c^1, c^2, \ldots, c^N \in \mathbb{R}^n$, is *sortable-by-$K$-$L$-sets* if for every pair of sets $S, S' \subseteq \{1, \ldots, n\}$, with $K \leq |S|, |S'| \leq L$ either
(a) $c^i(S) \leq c^i(S')$ for all $i = 1, \ldots, N$, or
(b) $c^i(S) \geq c^i(S')$ for all $i = 1, \ldots, N$.

The following result is easy to prove.

Proposition 31. *A set of objectives, $c^1, c^2, \ldots, c^N \in \mathbb{R}^n$, is universally co-ideal with respect to $R = \{x \in \{0,1\}^n \ : \ K \leq \sum_{j=1}^n x_j \leq L\}$ if and only if it is sortable-by-K-L-sets.*

Note that in the case of feasible sets $F \subseteq \{x \in \{0,1\}^n \ : \ K \leq \sum_{j=1}^n x_j \leq L\}$, the sortable-by-$K$-$L$-sets property can also be tested in pseudo-polynomial time: the 0-1 cardinality-constrained knapsack problem

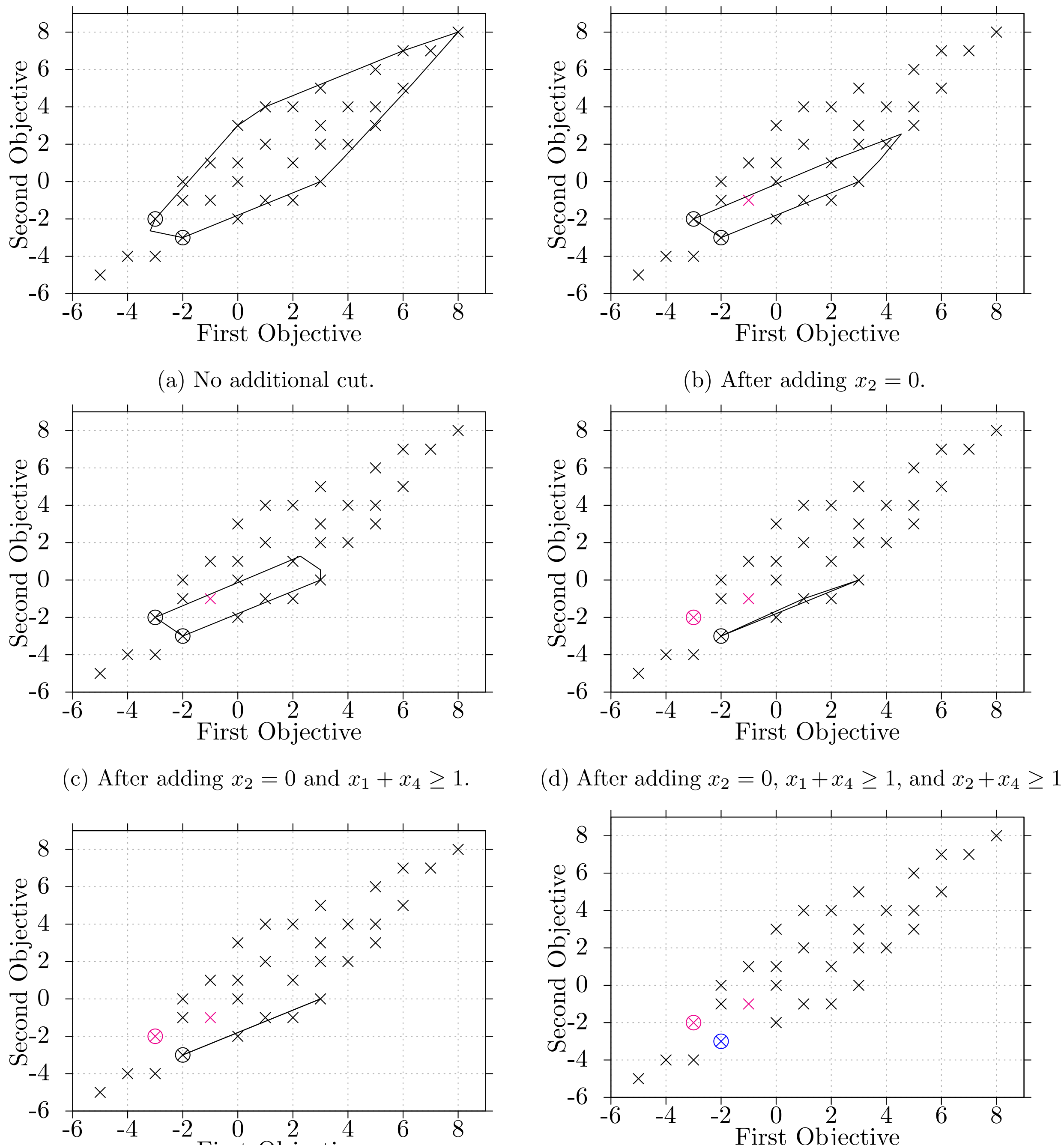
$$\begin{aligned} \psi_{K,L}(f,g) = \max \quad & fx - fy \\ \text{s.t.} \quad & gx - gy \leq -1, \\ & K \leq \sum_{j=1}^n x_j \leq L \\ & K \leq \sum_{j=1}^n y_j \leq L \\ & x, y \in \{0,1\}^n \end{aligned}$$

is also pseudo-polynomially solvable. This observation extends to any set of cardinality constraints on disjoint sets of variables.

All the concepts discussed so far, including objective aligning sets of variables, subspace-fixings, nexus points, and lower bound points, as well as the cuts derived from them, can readily be extended to make use of a relaxation of the feasible set, $R \subseteq \{0,1\}^n$, simply by replacing Z by $\{Cx : x \in R\}$ and $Z_{A,y}$ by $\{Cx : x \in R, x_j = y_j, \forall j \notin A\}$. The following example illustrates the value of such relaxations.

Example 32. Consider the MBP

$$\begin{aligned} \min \ & \{-2x_1 + 3x_2 + 5x_3 - 2x_4 - x_5, \ -x_1 + 5x_2 + 3x_3 - 3x_4 - x_5\} \\ \text{s.t.} \ & -3x_1 + 11x_2 + 3x_3 + 8x_4 + 11x_5 \geq 8, \\ & x_4 + x_5 \leq 1, \\ & x_1, x_2, x_3, x_4, x_5 \in \{0,1\}. \end{aligned}$$



(a) No additional cut.

(b) After adding $x_2 = 0$.

(c) After adding $x_2 = 0$ and $x_1 + x_4 \geq 1$.

(d) After adding $x_2 = 0$, $x_1 + x_4 \geq 1$, and $x_2 + x_4 \geq 1$.

(e) After adding $x_2 = 0$, $x_1 + x_4 \geq 1$, $x_2 + x_4 \geq 1$, and $(1 - x_1) + x_2 \geq 1$.

(f) After adding $x_2 = 0$, $x_1 + x_4 \geq 1$, $x_2 + x_4 \geq 1$, $(1 - x_1) + x_2 \geq 1$, and $x_3 = 0$. (The point $(-2, -3)$ is the projection of the LP-relaxation.)

Figure 8: Projections of all $x \in \{0, 1\}^5$ and the LP-relaxation of the final example into the criterion space.

The points in Z and the projection of the LP-relaxation into the criterion space, for this MBP, are shown in Figure 8a. The problem has two nondominated points, $(-3,-2)$ and $(-2,-3)$, corresponding to solutions $(1,0,0,0,1)$ and $(0,0,0,1,0)$, respectively.

Suppose that during the search, the point $(-1,-1)$, corresponding to feasible solution $(0,0,0,0,1)$, is found. Taking R to be

$$R := \{x \in \{0,1\}^5 : x_4 + x_5 \leq 1\}$$

we find that $(-1,-1)$ is a lower bound point for the subspace-fixing $(\{1,3,4,5\},(-,1,-,-,-))$, and so we obtain the cut $x_2 \leq 0$, and so may fix the variable $x_2 = 0$. Figure 8b shows the projection of the LP relaxation of the MBP after adding $x_2 = 0$.

Also, $(-1,-1)$ is readily observed to be a lower bound for the subspace-fixing $(\{2,3,5\},(0,-,-,0,-))$. Thus, since it is the image of a feasible solution, we can derive the cut $x_1+x_4 \geq 1$. Figure 8c shows the projection of the LP-relaxation of the problem after adding $x_2 = 0$ and $x_1 + x_4 \geq 1$.

Now suppose that the search continues after adding $x_2 = 0$ and $x_1 + x_4 \geq 1$, finding the nondominated point $(-3,-2)$, corresponding to efficient solution $(1,0,0,0,1)$. It is easy to verify that $\{1,3,5\}$ and $\{3,4,5\}$ are both objective-aligning sets with respect to the relaxation

$$R = \{x \in \{0,1\}^5 : x_4 + x_5 \leq 1\}.$$

Thus we may derive the cuts $x_2 + x_4 \geq 1$ and $(1-x_1) + x_2 \geq 1$. Figure 8d shows the projection of the LP relaxation of the problem after adding $x_2 = 0$, $x_1 + x_4 \geq 1$, and $x_2 + x_4 \geq 1$. Also, Figure 8e shows the projection of the LP relaxation of the problem after adding $x_2 = 0$, $x_1+x_4 \geq 1$, $x_2 + x_4 \geq 1$, and $(1-x_1) + x_2 \geq 1$.

Finally, we observe that the nondominated point $(-3,-2)$ is a nexus point (in fact, it is a lower bound point) for the subspace-fixing $(\{1,2,4,5\},(-,-,1,-,-))$, and hence we can derive the variable fixing $x_3 = 0$. Figure 8f shows the projection of the LP relaxation of the problem after adding all cuts (and variable fixings) derived so far. The LP relaxation is now a single point, the nondominated point $(-2,-3)$. We have shown the point $(-2,-3)$ in blue in the figure.

7 Final remarks

As well as enabling reduction of the objective space dimension and providing cuts that can be useful in solving MBPs, the properties discussed in this paper can be used to deduce bounds on the number of efficient solutions, and nondominated points, of a MBP. We give two examples.

If $A \subseteq \{1,\ldots,n\}$ is objective-aligning for a set of objectives, $c^1, c^2, \ldots, c^N \in \mathbb{R}^n$, then for any binary vector $y \in \{0,1\}^n$, the subspace-fixing (A,y) yields $Z_{A,y}$ totally ordered, and so either $Z_{A,y} \cap O$ is empty ($Z_{A,y}$ does not contain the image of any feasible solution), or $Z_{A,y} \cap O_N$ contains a single point. Since entries, y_j, in y, for $j \in A$, are irrelevant to the description of $Z_{A,y}$, there can be at most $2^{n-|A|}$ distinct sets $Z_{A,y}$, over all binary vectors y. Since each contains at most one nondominated point, it must be that $|O_N| \leq 2^{n-|A|}$.

If a set of objectives, $c^1, c^2, \ldots, c^N \in \mathbb{R}^n$, is sortable-by-$K$-$L$-sets, then, for any MBP with this set of objectives, we have, by Proposition 31, that

$$|F_E|, |O_N| \leq \sum_{i=0}^{K-1} \binom{n}{i} + \sum_{i=L+1}^{n} \binom{n}{i} + 1.$$

This suggests that if K is close to zero and L is close to n, we may be able to use an enumerative approach to compute all efficient solutions or nondominated points of the MBP.

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Appendix A: Complexity of recognizing a nexus point

We show that recognizing that a given point, $p \in \mathbb{R}^N$, is a nexus point for the subspace-fixing (A, y) is NP-hard. It suffices to show that the special case with $N = 2$ and $A = \{1, \ldots, n\}$ (in which case y is irrelevant), is NP-hard. We thus define the decision problem `2-OBJ-NEXUS-POINT`, as follows: given inputs consisting of the pair of objectives $f, g \in \mathbb{Z}^n$ and a point $p \in \mathbb{Z}^2$, is p a nexus point for f, g? Note that for this analysis, we assume integer-valued inputs. We first show that the complement of `2-OBJ-NEXUS-POINT` is NP-complete.

Theorem 33. *The complement of* `2-OBJ-NEXUS-POINT` *is NP-complete.*

Proof. Our proof is similar to the proof of Theorem 12 but we use `PARTITION PROBLEM` instead of `EQUAL-SUBSET-SUM`. Note first that the complement of `2-OBJ-NEXUS-POINT` is obviously in NP.

We will now show that `PARTITION PROBLEM` is polynomially reducible to the complement of `2-OBJ-NEXUS-POINT`, where `PARTITION PROBLEM` is defined as follows: given a set $\{a_1, \dots, a_n\}$ of positive integers, does there exist non-empty subset, $S \subset \{1, \dots, n\}$, with $\sum_{j \in \{1,\dots,n\} \setminus S} a_j = \sum_{j \in S} a_j$? In this proof, to remove trivial cases, we assume that $\frac{\sum_{j \in \{1,\dots,n\}} a_j}{2} \in \mathbb{Z}$.

Given such an instance of `PARTITION PROBLEM`, we construct an instance of `2-OBJ-NEXUS-POINT`, $\big((f, g), p\big)$, by

$$p = \Big(\frac{\sum_{j \in \{1,\dots,n\}} a_j}{2}, \frac{\sum_{j \in \{1,\dots,n\}} a_j}{2}\Big),$$
$$f_j = a_j - 2^{-j},$$

and

$$g_j = a_j + 2^{-j},$$

for all $j = 1, \dots, n$. Observe that for any $S \subseteq \{1, \dots, n\}$,

$$\lceil f(S) \rceil = \lfloor g(S) \rfloor = \sum_{j \in S} a_j, \tag{6}$$

since $0 \le \sum_{j \in S} 2^{-j} < 1$.

First suppose that this instance, $\big((f, g), p\big)$, of `2-OBJ-NEXUS-POINT`, is a `NO` instance, so there exist $S \subset \{1, \dots, n\}$, with $p_1 < f(S)$ and $p_2 > g(S)$. Then

$$\lceil p_1 \rceil \le \lceil f(S) \rceil = \lfloor g(S) \rfloor \le \lfloor p_2 \rfloor,$$

by (6). But, since $\frac{\sum_{j \in \{1,\dots,n\}} a_j}{2} \in \mathbb{Z}$, we have $\lceil p_1 \rceil = \lfloor p_2 \rfloor$, and hence, it must be that

$$\frac{\sum_{j \in \{1,\dots,n\}} a_j}{2} = \lceil p_1 \rceil = \lfloor p_2 \rfloor = \lceil f(S) \rceil = \lfloor g(S) \rfloor = \sum_{j \in S} a_j,$$

so `PARTITION PROBLEM` is a `YES` instance.

Finally, suppose that `PARTITION PROBLEM` is a `YES` instance, with non-empty $S \subset \{1, \dots, n\}$, with $\sum_{j \in \{1,\dots,n\} \setminus S} a_j = \sum_{j \in S} a_j = \frac{\sum_{j \in \{1,\dots,n\}} a_j}{2}$. Hence,

$$p_2 = \frac{\sum_{j \in \{1,\dots,n\}} a_j}{2} < \frac{\sum_{j \in \{1,\dots,n\}} a_j}{2} + \sum_{j \in S} 2^{-j} = \sum_{j \in S} a_j + \sum_{j \in S} 2^{-j} = g(S).$$

Similarly,

$$p_1 = \frac{\sum_{j \in \{1,\dots,n\}} a_j}{2} > \frac{\sum_{j \in \{1,\dots,n\}} a_j}{2} - \sum_{j \in S} 2^{-j} = \sum_{j \in S} a_j - \sum_{j \in S} 2^{-j} = f(S).$$

Thus the `2-OBJ-NEXUS-POINT` instance, $\big((f, g), S, p\big)$, is a `NO` instance. The result follows. □

Corollary 34. *`2-OBJ-NEXUS-POINT` is NP-hard.*