

Polynomial Time Algorithms and Extended Formulations for Unit Commitment Problems

Kai Pan, Kezhuo Zhou,* and Yongpei Guan

Department of Industrial and Systems Engineering
University of Florida, Gainesville, FL 32611
Emails: kpan@ufl.edu; zhoukezhuo@ufl.edu; guan@ise.ufl.edu

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Abstract

Recently increasing penetration of renewable energy generation brings challenges for power system operators to perform efficient power generation daily scheduling, due to the intermittent nature of the renewable generation and discrete decisions of each generation unit. Among all aspects to be considered, unit commitment polytope is fundamental and embedded in the models at different stages of power system planning and operations. In this paper, we focus on deriving polynomial time algorithms for the unit commitment problems with general convex cost function and piecewise linear cost function respectively. We refine an $\mathcal{O}(T^3)$ time, where T represents the number of time periods, algorithm for the deterministic unit commitment problem with general convex cost function and accordingly develop an extended formulation in a higher dimensional space that provides integral solutions in which the physical meanings of the decision variables are described. Furthermore, for the case in which the cost function is piecewise linear, by exploring the optimality conditions, we derive more efficient algorithms for both deterministic (i.e., $\mathcal{O}(T)$ time) and stochastic (i.e., $\mathcal{O}(N)$ time, where N represents the number of nodes in the stochastic scenario tree) unit commitment problems. We also develop the corresponding extended formulations for both deterministic and stochastic unit commitment problems that provide integral solutions. Similarly, physical meanings of the decision variables are explored to show the insights of the new modeling approach.

Key words: Unit commitment; polynomial time algorithm; integral formulation; integral solution

*co-first author

1 Introduction

Unit commitment (UC) is fundamental in power system operations. It decides the unit commitment status (online/offline) and power generation amount at each time period for each unit over a finite discrete time horizon, with the objective of minimizing the total cost while satisfying the load (energy demand). Each unit should satisfy associated physical restrictions, such as generation upper/lower limits, ramp-rate limits, and minimum-up/-down time limits.

Due to its significant importance in power system operations, UC has brought broad attention in academic and industry. In early 1960s, a dynamic programming algorithm was developed in [16] to formulate and solve the unit commitment problem, in which the generation amount is discretized and the algorithm itself is not polynomial. Later on, in [25], a more general dynamic programming approximation algorithm was developed to solve the problem with multiple units. Since these algorithms are not polynomial time, it is almost intractable. To target large size problems, other solution approaches such as Lagrangian relaxation (see, e.g., [28, 5]), genetic algorithms (see, e.g., [13, 26]), and simulated annealing (see, e.g., [29, 17]), have been developed to solve the problem. Detailed reviews of these approaches to solve the UC problem can be found in [20] and [23]. Among these approaches, the Lagrangian relaxation approach has been broadly adopted in industry, due to its advantages of decomposing the network constrained UC problem into a master problem and a group of subproblems where each subproblem solves an individual UC problem.

However, the Lagrangian relaxation approach does have limitation. For instance, it cannot guarantee to provide an optimal or even a feasible solution at the termination, in particular, when there are transmission network constraints currently faced by most wholesale markets, operated by Independent System Operators (ISOs), in US. On the other hand, advanced mixed-integer-linear programming (MILP) techniques have been improved significantly during the past decades, and meanwhile MILP in general has advantages in terms of ease of development and maintenance, ability to specify accurate solutions, and exact modeling of complex functionality (cf. [18]). Thus, recently, optimization algorithm developments for power system operations are switching from Lagrangian relaxation to MILP approaches. For instance, MILP approaches have been adopted by all ISOs in US (see, e.g., [11, 4]) and creates more than 500 million annual savings (cf. [18]). Among different approaches MILP can contribute to the power system operations, one particularly important one is to formulate and solve the transmission network-constrained unit commitment problem (cf. [3]).

The earliest MILP UC formulation was proposed in the 1960s as described in [10], and further improvements have been developed until recently. For instance, in [7], an exact and computationally

efficient MILP formulation is provided to address the single-generator self-scheduling unit commitment problem in order to maximize the total profit. In [9], security-constrained UC problems are modeled and solved through the MILP approach for large-scale power systems with multiple generators. Considering the fact of large-scale instances to be solved in practice, it is crucial to further develop efficient approaches to speed up the branch-and-cut algorithm for MILP so as to obtain an optimal mixed-integer solution in short time. As indicated in [11] and [27], a strong (tight) MILP formulation plays a significant role in speeding up the solution procedure, as strong formulations reduce the feasible region of the linear programming (LP) relaxation of the original problem and improve the LP relaxation bounds.

There has been research progress on developing strong formulations for the unit commitment problem by exploring its special structure. For instance, in [15], alternating up/down inequalities are proposed to strengthen the minimum-up/-down time polytope of the unit commitment problem. In [22], the convex hull of the minimum-up/-down time polytope considering start-up costs is provided, in which additional start-up and shut-down variables are introduced to provide the integral formulation. Recently, new families of strong valid inequalities are proposed in [19], [6], and [21] to tighten the ramping polytope of the unit commitment problem.

In many situations the assumption of known, deterministic data (such as load or price) is not necessarily realistic for the UC problem. In addition, recently, renewable generation has been increasingly penetrating into the power grid system. Due to its intermittent nature and generation amount dependency among time periods, recently, a formulation of the stochastic version of the deterministic UC which allows generation amount or price dependence were proposed in [12]. In this approach, the extension of the deterministic UC was studied in which a stochastic programming approach (see, e.g., [24]) is adopted to address uncertain problem parameters. We refer to the resulting model as the *stochastic UC problem*. The advantage of utilizing stochastic UC modeling approaches can help make decisions adaptively.

In general, both the deterministic UC and stochastic UC problems can eventually be formulated as MIPs, with general convex (typically quadratic) cost function (leading to MIQPs) or with piecewise linear cost function approximations (leading to MILPs). The perfect cases are to (1) derive polynomial time algorithms to solve the problems and/or (2) discover extended formulations in the form of linear programs that can provide integral solutions. Developing efficient polynomial time algorithms is very important because this will help speed up the algorithm to solve each subproblem in the Lagrangian relaxation approach and meanwhile it can help solve self-scheduling

unit commitment problems efficiently. In addition, the derived extended formulations can also be embedded into the network-constrained unit commitment and help solve it efficiently. For the polynomial time algorithms, a beautiful algorithm for the deterministic UC with general convex cost function was studied in [8] in which an $\mathcal{O}(T^3)$ time, where T represents the number of time periods, algorithm is developed. In this paper, we first refine the $\mathcal{O}(T^3)$ time algorithm in [8] for the deterministic UC problem with general convex cost function by reducing the computational time to solve the problem from $\mathcal{O}(T^3)$ time in [8] to $\mathcal{O}(T^2)$ time, when economic dispatch problem has been presolved. More importantly, our developed $\mathcal{O}(T^3)$ time dynamic programming algorithm can help derive an extended formulation that provides integral solutions. Then, we study the problems with piecewise linear cost functions, which is common in practice. For these cases, we discover the optimality conditions for the generation amounts in optimal solution at each time period for both the deterministic and stochastic UC problems. This key observation helps reduce the search space significantly and thus leads to a very efficient polynomial time algorithm. Accordingly, we develop efficient dynamic programming algorithms, different from the ones described in [16] and [25], that takes only $\mathcal{O}(T)$ time to solve the deterministic UC problem and $\mathcal{O}(N)$ time to solve the stochastic one, where N represents the number of nodes in the scenario tree. Towards the extended formulation of UC, a recent study was provided in [14], in which an integral formulation is provided by using the theorem described in [1, 2]. In this paper, we provide extended formulations based on the innovative dynamic programming algorithms we developed and furthermore physical meanings of the decision variables in the extended formulations are elaborated. To summarize, the main contributions of this paper can be described as follows:

- (1) We refine an $\mathcal{O}(T^3)$ time algorithm for the deterministic UC problem with general convex cost function, which solve the problem in $\mathcal{O}(T^2)$ time when the economic dispatch problem has been presolved.
- (2) When the general convex cost function is approximated by a piecewise linear function, by exploring the optimality conditions for the deterministic unit commitment with piecewise linear cost function, we derive a more efficient polynomial time dynamic programming algorithm that runs in $\mathcal{O}(T)$ time.
- (3) When uncertainty is considered, by exploring the optimality conditions for the corresponding derived multistage stochastic unit commitment with piecewise linear cost function, we derive an efficient polynomial time dynamic programming algorithm that runs in $\mathcal{O}(N)$ time.

- (4) Motivated by the dynamic programming algorithms described in (1), (2), and (3), we derive extended formulations for both the deterministic and stochastic UCs in the high dimensional space which can be solved as linear programs.

To the best of our knowledge, in this paper, we provide the most efficient polynomial time algorithms to solve the deterministic UC and the first studies on the polynomial time algorithm development and extended formulations for stochastic unit commitment problems. The remaining part of this paper is organized as follows. In Section 2, we propose an efficient dynamic programming algorithm to solve the deterministic unit commitment problem with ramping constraints and general convex cost function. We also derive an extended formulation that can provide integral solutions. In Section 3, we derive the optimality conditions for the deterministic unit commitment problem with piecewise linear cost function and accordingly develop a more efficient polynomial time dynamic programming algorithm to solve the problem. This study is extended to the case in which uncertainty is considered and accordingly a stochastic unit commitment is formulated in Section 4. In this section, we also explore the optimality conditions for the stochastic unit commitment problem and develop an efficient polynomial time algorithm to solve the problem. Extended formulations, which can also provide integral solutions, are provided for both the deterministic and stochastic UC models. Finally we conclude this paper in Section 5.

2 Deterministic Unit Commitment with General Convex Cost Function

We first introduce the notation and describe the deterministic unit commitment problem. We let T be the number of time periods for the whole operational horizon, L (ℓ) be the minimum-up (-down) time limit, \bar{C} (\underline{C}) be the generation upper (lower) bound when the machine is online, \bar{V} be the start-up/shut-down ramp rate (which is usually between \underline{C} and \bar{C} , i.e., $\underline{C} \leq \bar{V} \leq \bar{C}$), and V be the ramp-up/-down rate in the stable generation region. In addition, we let binary decision variable y represent the machine's online (i.e., $y_t = 1$) or offline (i.e., $y_t = 0$) status, binary decision variable u to represent whether the machine starts up (i.e., $u_t = 1$) or not (i.e., $u_t = 0$), and continuous decision variable x represent the generation amount. Moreover, we define two more continuous variables, i.e., SU_t and SD_t , to represent the start-up and shut-down costs respectively. In particular, we let SU_t denote the start-up cost at stage t following the start-up profile and SD_t denote the shut-down cost at stage $t + 1$. We let a general convex cost function $f(\cdot)$ denote the fuel cost minus revenue as a function of its electricity generation amount, online/offline status, and

electricity price. We assume that the machine has been offline for s_0 time periods ($s_0 \geq \ell$) before time 1. Therefore, the corresponding deterministic unit commitment problem can be described as follows:

$$\min \quad \sum_{t=1}^T (SU_t + f_t(x_t, y_t)) + \sum_{t=L}^{T-1} SD_t \quad (1a)$$

$$\text{s.t.} \quad \sum_{i=t-L+1}^t u_i \leq y_t, \quad \forall t \in [L, T]_{\mathbb{Z}}, \quad (1b)$$

$$\sum_{i=t-\ell+1}^t u_i \leq 1 - y_{t-\ell}, \quad \forall t \in [\ell, T]_{\mathbb{Z}}, \quad (1c)$$

$$y_t - y_{t-1} - u_t \leq 0, \quad \forall t \in [1, T]_{\mathbb{Z}}, \quad (1d)$$

$$-x_t + \underline{C}y_t \leq 0, \quad \forall t \in [1, T]_{\mathbb{Z}}, \quad (1e)$$

$$x_t - \bar{C}y_t \leq 0, \quad \forall t \in [1, T]_{\mathbb{Z}}, \quad (1f)$$

$$x_t - x_{t-1} \leq Vy_{t-1} + \bar{V}(1 - y_{t-1}), \quad \forall t \in [1, T]_{\mathbb{Z}}, \quad (1g)$$

$$x_{t-1} - x_t \leq Vy_t + \bar{V}(1 - y_t), \quad \forall t \in [1, T]_{\mathbb{Z}}, \quad (1h)$$

$$SU_t \geq SU(t + s_0 - 1)(u_t - \sum_{s=1}^{k-1} y_s), \quad \forall t \in [1, T]_{\mathbb{Z}}, \quad (1i)$$

$$SU_t \geq SU(t - k - 1)(u_t - \sum_{s=k+1}^{t-1} y_s), \quad \forall t \in [L + \ell + 1, T]_{\mathbb{Z}}, k \in [L, t - \ell - 1]_{\mathbb{Z}}, \quad (1j)$$

$$SD_t \geq SD(t - k + 1)(u_k - \sum_{s=k}^t (1 - y_s)), \quad \forall t \in [L, T - 1]_{\mathbb{Z}}, k \in [1, t - L + 1]_{\mathbb{Z}}, \quad (1k)$$

$$y_t, u_t \in \{0, 1\}, \quad SU_t, SD_t \geq 0, \quad x_0 = y_0 = 0; \quad (1l)$$

where constraints (1b) and (1c) describe the minimum-up and minimum-down time limits, respectively (if the machine starts up at time $t - L + 1$, i.e., the machine is online at time $t - L + 1$, it should stay online in the following L consecutive time periods until time t ; if the machine shuts down at time $t - \ell + 1$, i.e., the machine is offline at time $t - \ell + 1$ and online at time $t - \ell$, it should stay offline in the following ℓ consecutive time periods until time t), constraints (1d) describe the logical relationship between y and u , constraints (1e) and (1f) describe the generation lower and upper bounds, and constraints (1g) and (1h) describe the generation ramp-up and ramp-down rate limits. Constraints (1i) describe the start-up cost if the machine starts up for the first time, where $SU(\cdot)$ is a start-up cost function whose variable is the offline time length before starting

up. Constraints (1j) describe the start-up cost when the machine starts up some time later after a shut-down. Constraints (1k) describe the shut-down cost, where $SD(\cdot)$ is a shut-down cost function whose variable is the online time length before shutting down. Typically $SD(\cdot)$ is a constant function. In the above formulation, the objective is to minimize the total cost minus the revenue. For notation convenience, we define $[a, b]_{\mathbb{Z}}$ with $a < b$ as the set of integer numbers between integers a and b , i.e., $\{a, a + 1, \dots, b\}$.

2.1 A Refined $\mathcal{O}(T^3)$ Time Dynamic Programming Algorithm

The polynomial time algorithm for the unit commitment problem with general convex cost function was first developed in [8], where an $\mathcal{O}(T^3)$ time dynamic programming algorithm is proposed. This nice algorithm keeps tracking the “on” periods for the machine and use backward dynamic programming to solve the problem. In this algorithm, each state space (h, k) for $h, k \in [1, T]_{\mathbb{Z}}, k \geq h + \tau^+ - 1$, where τ^+ denotes the minimum-up time limit, represents the machine is on during the period $[h, k]_{\mathbb{Z}}$, i.e., the machine is turned on at time h and turned off at time $k + 1$. Then the Bellman equation can be written as follows:

$$V((h, k)) = ED_{hk} + \min_{r \geq k + \tau^- + 1} \{C_{kr} + V((r, q)), 0\},$$

for all possible $(h, k), (r, q)$ in the state space. In this equation, ED_{hk} represents the optimal value of economic dispatch problem for state (h, k) , τ^- denotes the minimum-down time limit and C_{kr} corresponds to the start-up cost when the machine shuts down at time $k + 1$ and starts up at time r . We call this “part II” of the algorithm. An efficient shortest path algorithm was developed in [8] to solve this “part II” in $\mathcal{O}(T^3)$. To speed up the algorithm, the economic dispatch problem for all possible “on” intervals can be precalculated. We call this “part I” of this algorithm. In [8], an intelligent algorithm was developed to solve “part I” problem in $\mathcal{O}(T^3)$ time as well.

As compared to [8], we propose a more efficient polynomial-time dynamic programming algorithm for “part II”. We first define the optimal value function and then develop the Bellman equations accordingly. The key difference as compared to [8] is that we use different state spaces and double value functions corresponding to each time period t . For instance, we let $V_{\uparrow}(t)$ represent the cost from time t to the end when the machine starts up at time t and $V_{\downarrow}(t)$ represent the cost from time t to the end when the machine shuts down at time $t + 1$ (i.e., t is the last “on” period for the current “on” interval). Thus, we have the following dynamic programming equations:

$$V_{\uparrow}(t) = \min_{k \in [\min\{t+L-1, T-1\}, T-1]_{\mathbb{Z}}} \{SD(k - t + 1) + C(t, k) + V_{\downarrow}(k), C(t, T) + V_{\downarrow}(T)\},$$

$$\forall t \in [1, T]_{\mathbb{Z}}, \quad (2a)$$

$$V_{\downarrow}(t) = \min_{k \in [t+L+1, T]_{\mathbb{Z}}} \{SU(k-t-1) + V_{\uparrow}(k), 0\}, \quad \forall t \in [L, T-\ell-1]_{\mathbb{Z}}, \quad (2b)$$

$$V_{\downarrow}(t) = 0, \quad \forall t \in [T-\ell, T]_{\mathbb{Z}}, \quad (2c)$$

where $C(t, k)$ represents the optimal generation cost (i.e., the objective value of economic dispatch problem) if the machine starts up at time t and shuts down at time $k+1$ (i.e., online at k). Equations (2a) indicates that when the machine starts up at time t , it can keep online until time k when $k-t+1 \geq L$. Equations (2b) indicates that when the machine shuts down at time t , it can keep offline to the end or starts up again when minimum-down time limit is satisfied. Following the start-up (resp. shut-down) profile, our start-up (resp. shut-down) function can capture the length of offline (resp. online) time before starting up (resp. shutting down). Equations (2c) describe that the machine cannot start up again due to minimum-down time limit.

As we consider the unit commitment problem from time period 1 to T and assume the machine is kept offline for s_0 time periods, our goal is to find out the value of the following function:

$$z = V_{\downarrow}(-s_0) := \min_{t \in [1, T]_{\mathbb{Z}}} \{SU(s_0+t-1) + V_{\uparrow}(t), 0\}. \quad (3)$$

In order to obtain the optimal objective value and corresponding optimal solution, we calculate $V_{\uparrow}(t)$ and $V_{\downarrow}(t)$ for all t and record the optimal candidate for them. To calculate the value of each optimal value function in Bellman equations (2a) – (2c) when t is given with $t \leq T$, we search among the candidate solution for each $k \leq T$, which takes $\mathcal{O}(T)$ time. Thus, the total time to calculate $V_{\downarrow}(-s_0)$ is $\mathcal{O}(T^2)$ for “part II”. The optimal solution for UC can be obtained by tracing the optimal candidate for the optimal value function starting from $V_{\downarrow}(-s_0)$, and this takes $\mathcal{O}(T)$ time in total. In summary, our backward induction dynamic programming algorithm for the deterministic unit commitment problem takes $\mathcal{O}(T^2)$ time for “part II” (i.e., if all $C(t, k)$ are presolved). Our algorithm refines the algorithm in [8]. More importantly, our algorithm is very beautiful to derive a better reformulation in the following section.

2.2 Linear Program Reformulation for Dynamic Programming

In this section, we reformulate the dynamic program in Section 2.1 into a linear program and derive its dual formulation to approach the final extended formulation. By incorporating the dynamic equations (i.e., (2a) - (2c) and (3)) as constraints, we obtain the following equivalent linear program:

$$\max \quad z \quad (4a)$$

$$\text{s.t.} \quad z \leq SU(s_0 + t - 1) + V_{\uparrow}(t), \quad \forall t \in [1, T]_{\mathbb{Z}}, \quad (4b)$$

$$V_{\uparrow}(t) \leq SD(k - t + 1) + C(t, k) + V_{\downarrow}(k),$$

$$\forall k \in [\min\{t + L - 1, T - 1\}, T - 1]_{\mathbb{Z}}, \quad \forall t \in [1, T]_{\mathbb{Z}}, \quad (4c)$$

$$V_{\uparrow}(t) \leq C(t, T) + V_{\downarrow}(T), \quad \forall t \in [1, T]_{\mathbb{Z}}, \quad (4d)$$

$$V_{\downarrow}(t) \leq SU(k - t - 1) + V_{\uparrow}(k), \quad \forall k \in [t + \ell + 1, T]_{\mathbb{Z}}, \forall t \in [L, T - \ell - 1]_{\mathbb{Z}}, \quad (4e)$$

$$V_{\downarrow}(t) = 0, \quad \forall t \in [T - \ell, T]_{\mathbb{Z}}, \quad (4f)$$

$$z \leq 0, V_{\downarrow}(t) \leq 0, \quad \forall t \in [L, T - \ell - 1]_{\mathbb{Z}}. \quad (4g)$$

Note here that the optimal value functions in the dynamic program become decision variables in the above formulation and to obtain the value $V_{\downarrow}(-s_0)$ under the dynamic programming framework it is equivalent to maximize the variable z in the linear program.

Since the above linear program cannot be solved directly as $C(t, k)$ (the objective value of economic dispatch problem) are unknown, we first show how to obtain the value of $C(t, k)$ by discussing two cases, i.e., the cases $k \leq T - 1$ and $k = T$.

When $k \leq T - 1$, we have the following formulation to calculate $C(t, k)$ with (t, k) given:

$$C(t, k) = \min \sum_{s=t}^k \phi_s \quad (5a)$$

$$\text{s.t.} \quad -x_s \leq -\underline{C}, \quad \forall s \in [t, k]_{\mathbb{Z}}, \quad (5b)$$

$$x_s \leq \overline{C}, \quad \forall s \in [t, k]_{\mathbb{Z}}, \quad (5c)$$

$$x_t \leq \overline{V}, \quad (5d)$$

$$x_k \leq \overline{V}, \quad (5e)$$

$$x_s - x_{s-1} \leq V, \quad \forall s \in [t + 1, k]_{\mathbb{Z}}, \quad (5f)$$

$$x_{s-1} - x_s \leq V, \quad \forall s \in [t + 1, k]_{\mathbb{Z}}, \quad (5g)$$

$$\phi_s \geq a_j x_s + b_j, \quad \forall s \in [t, k]_{\mathbb{Z}}, j \in [1, N]_{\mathbb{Z}}. \quad (5h)$$

When $k = T$, we have the corresponding formulation by removing constraint (5e) as the machine is not required to shut down at time $T + 1$ if it stays online until time T . Note here that we assume the generation cost function to be piecewise linear at this moment in order to reformulate model 4, but we return it back to be the general convex function in the final extended formulation.

Next, to incorporate the economic dispatch constraints (e.g., (5b) - (5h)) into our proposed linear program (4), we take the dual of the economic dispatch model and embed its dual formulation into

model 4. For instance, for $k \leq T - 1$ we have the dual formulation as follows.

$$C(t, k) = \max \quad \sum_{s=t}^k (\lambda_s^+ \bar{C} - \lambda_s^- \underline{C}) + \bar{V}(\mu_t + \mu_k) + \sum_{s=t+1}^k V(\sigma_s^+ + \sigma_s^-) + \sum_{s=t}^k \sum_{j=1}^N b_j \delta_{sj} \quad (6a)$$

$$\text{s.t.} \quad \lambda_t^+ - \lambda_t^- + \mu_t - \sigma_{t+1}^- + \sigma_{t+1}^+ - \sum_{j=1}^N a_j \delta_{tj} = 0, \quad (6b)$$

$$\lambda_k^+ - \lambda_k^- + \mu_k + \sigma_k^- - \sigma_k^+ - \sum_{j=1}^N a_j \delta_{kj} = 0, \quad (6c)$$

$$\lambda_s^+ - \lambda_s^- + \sigma_s^- - \sigma_{s+1}^- - \sigma_s^+ + \sigma_{s+1}^+ - \sum_{j=1}^N a_j \delta_{sj} = 0, \quad (6d)$$

$$\forall s \in [t+1, k-1]_{\mathbb{Z}}, \quad (6d)$$

$$\sum_{j=1}^N \delta_{sj} = 1, \quad \forall s \in [t, k]_{\mathbb{Z}}, \quad (6e)$$

$$\lambda_s^{\pm} \leq 0, \quad \forall s \in [t, k]_{\mathbb{Z}}, \quad \mu_t \leq 0, \quad \sigma_s^{\pm} \leq 0, \quad \forall s \in [t+1, k]_{\mathbb{Z}},$$

$$\delta_{sj} \geq 0, \quad \forall j \in [1, N]_{\mathbb{Z}}, \quad s \in [t, k]_{\mathbb{Z}}, \quad (6f)$$

where λ_s^+ and λ_s^- are dual variables corresponding to constraints (5b) and (5c), μ_t and μ_k are the dual variables corresponding to constraint (5d) and (5e), σ_s^+ and σ_s^- are dual variables corresponding to constraints (5f) and (5g), and δ_{sj} are dual variables corresponding to constraints (5h). For $k = T$, we obtain the corresponding dual formulation by removing the dual variable μ_k from model (6). Now we obtain an integrated linear program, as shown in the following, by plugging the dual formulation of economic dispatch problem and redefine $C(t, k)$ to be a decision variable in the following model.

$$\max \quad z \quad (7a)$$

$$\text{s.t.} \quad (4b) - (4g), \quad (7b)$$

$$C(t, k) \leq \sum_{s=t}^k (\lambda_s^+ \bar{C} - \lambda_s^- \underline{C}) + \bar{V}(\mu_t + \mu_k) + \sum_{s=t+1}^k V(\sigma_s^+ + \sigma_s^-) + \sum_{s=t}^k \sum_{j=1}^N b_j \delta_{sj},$$

$$\forall k \in [\min\{t+L-1, T-1\}, T-1]_{\mathbb{Z}}, \quad \forall t \in [1, T]_{\mathbb{Z}}, \quad (7c)$$

$$C(t, T) \leq \sum_{s=t}^T (\lambda_s^+ \bar{C} - \lambda_s^- \underline{C}) + \bar{V}\mu_t + \sum_{s=t+1}^T V(\sigma_s^+ + \sigma_s^-) + \sum_{s=t}^T \sum_{j=1}^N b_j \delta_{sj},$$

$$\forall t \in [1, T]_{\mathbb{Z}}, \quad (7d)$$

$$(6b) - (6f), \quad \forall k \in [\min\{t+L-1, T\}, T]_{\mathbb{Z}}, \quad \forall t \in [1, T]_{\mathbb{Z}}. \quad (7e)$$

Note here that the right-hand-side of constraints (7c) and (7d) correspond to the objective function in (6a).

In the following, we try to obtain the extended formulation of the original MIP model (1). Before that, we take the dual of the above linear program (7) and obtain the following dual linear program:

$$\begin{aligned} \min \quad & \sum_{t=1}^T SU(s_0 + t - 1)\alpha_t + \sum_{t=1}^T \sum_{k=t+L-1}^{T-1} SD(k - t + 1)\beta_{tk} + \\ & \sum_{t=L}^{T-\ell-1} \sum_{k=t+\ell+1}^T SU(k - t - 1)\gamma_{tk} + \sum_{tk \in \mathcal{TK}} \sum_{s=t}^k w_{tk}^s \end{aligned} \quad (8a)$$

$$\text{s.t.} \quad \sum_{t=1}^T \alpha_t \leq 1, \quad (8b)$$

$$-\alpha_t + \sum_{k=t+L-1}^T \beta_{tk} = 0, \quad \forall t \in [1, L + \ell]_{\mathbb{Z}}, \quad (8c)$$

$$-\alpha_t + \sum_{k=t+L-1}^T \beta_{tk} - \sum_{k=L}^{t-\ell-1} \gamma_{kt} = 0, \quad \forall t \in [L + \ell + 1, T]_{\mathbb{Z}}, \quad (8d)$$

$$-\sum_{k=1}^{t-L+1} \beta_{kt} + \sum_{k=t+\ell+1}^T \gamma_{tk} \leq 0, \quad \forall t \in [L, T - \ell - 1]_{\mathbb{Z}}, \quad (8e)$$

$$\theta_t - \sum_{k=1}^{t-L+1} \beta_{kt} = 0, \quad \forall t \in [T - \ell, T]_{\mathbb{Z}}, \quad (8f)$$

$$p_{tk} - \beta_{tk} = 0, \quad \forall tk \in \mathcal{TK}, \quad (8g)$$

$$q_{tk}^s \leq \bar{C}p_{tk}, \quad \forall s \in [t, k]_{\mathbb{Z}}, \forall tk \in \mathcal{TK}, \quad (8h)$$

$$-q_{tk}^s \leq -\underline{C}p_{tk}, \quad \forall s \in [t, k]_{\mathbb{Z}}, \forall tk \in \mathcal{TK}, \quad (8i)$$

$$q_{tk}^t \leq \bar{V}p_{tk}, \quad \forall tk \in \mathcal{TK}, \quad (8j)$$

$$q_{tk}^k \leq \bar{V}p_{tk}, \quad \forall tk \in \mathcal{TK}, k \leq T - 1 \quad (8k)$$

$$q_{tk}^{s-1} - q_{tk}^s \leq Vp_{tk}, \quad \forall s \in [t + 1, k]_{\mathbb{Z}}, \forall tk \in \mathcal{TK}, \quad (8l)$$

$$q_{tk}^s - q_{tk}^{s-1} \leq Vp_{tk}, \quad \forall s \in [t + 1, k]_{\mathbb{Z}}, \forall tk \in \mathcal{TK}, \quad (8m)$$

$$w_{tk}^s - a_j q_{tk}^s \geq b_j p_{tk}, \quad \forall s \in [t, k]_{\mathbb{Z}}, j \in [1, N]_{\mathbb{Z}}, \forall tk \in \mathcal{TK}, \quad (8n)$$

$$\alpha, \beta, \gamma, p \geq 0, \quad (8o)$$

where \mathcal{TK} represents the set of all possible combination of $t \in [1, T]_{\mathbb{Z}}, k \in [\min\{t + L - 1, T\}, T]_{\mathbb{Z}}$. In the above dual formulation, dual variables $\alpha, \beta, \gamma, \theta$ correspond to constraints (4b) – (4f) respectively, and dual variables p, q, w correspond to constraints (7c) – (7e) for each $tk \in \mathcal{TK}$ respectively.

Note here that we can remove constraints (8n) by letting $N \rightarrow \infty$ and consider w_{tk}^s as a general convex cost function of q_{tk}^s and β_{tk} . In the following, we will remove constraints (8n) and consider

w_{tk}^s as a general convex cost function.

After replacing p with β (due to (8g)) in the dual formulation (8), we obtain the following simplified model:

$$\begin{aligned} \min \quad & \sum_{t=1}^T SU(s_0 + t - 1)\alpha_t + \sum_{t=1}^T \sum_{k=t+L-1}^{T-1} SD(k - t + 1)\beta_{tk} + \\ & \sum_{t=L}^{T-\ell-1} \sum_{k=t+\ell+1}^T SU(k - t - 1)\gamma_{tk} + \sum_{tk \in \mathcal{TK}} \sum_{s=t}^k w_{tk}^s \end{aligned} \quad (9a)$$

$$\text{s.t.} \quad \sum_{t=1}^T \alpha_t \leq 1, \quad (9b)$$

$$-\alpha_t + \sum_{k=t+L-1}^T \beta_{tk} - \sum_{k=L}^{t-\ell-1} \gamma_{kt} = 0, \quad \forall t \in [1, T]_{\mathbb{Z}}, \quad (9c)$$

$$-\sum_{k=1}^{t-L+1} \beta_{kt} + \sum_{k=t+\ell+1}^T \gamma_{tk} \leq 0, \quad \forall t \in [L, T - \ell - 1]_{\mathbb{Z}}, \quad (9d)$$

$$\theta_t - \sum_{k=1}^{t-L+1} \beta_{kt} = 0, \quad \forall t \in [T - \ell, T]_{\mathbb{Z}}, \quad (9e)$$

$$\underline{C}\beta_{tk} \leq q_{tk}^s \leq \overline{C}\beta_{tk}, \quad \forall s \in [t, k]_{\mathbb{Z}}, \forall tk \in \mathcal{TK}, \quad (9f)$$

$$q_{tk}^t \leq \overline{V}\beta_{tk}, \quad \forall tk \in \mathcal{TK}, \quad (9g)$$

$$q_{tk}^k \leq \overline{V}\beta_{tk}, \quad \forall tk \in \mathcal{TK}, k \leq T - 1 \quad (9h)$$

$$q_{tk}^{s-1} - q_{tk}^s \leq V\beta_{tk}, \quad \forall s \in [t + 1, k]_{\mathbb{Z}}, \forall tk \in \mathcal{TK}, \quad (9i)$$

$$q_{tk}^s - q_{tk}^{s-1} \leq V\beta_{tk}, \quad \forall s \in [t + 1, k]_{\mathbb{Z}}, \forall tk \in \mathcal{TK}, \quad (9j)$$

$$\alpha, \beta, \gamma \geq 0. \quad (9k)$$

In the next section, we will prove that an extended formulation for MIP model (1) can be derived based on linear program reformulation (9).

2.3 Extended Formulation for Deterministic Unit Commitment with General Convex Cost Function

We first show that the polytope (9b) – (9k) is an integral polytope in the following lemma.

Lemma 1 *The extreme points of the polytope (9b) – (9k) are binary with respect to decision variables $\alpha, \beta, \gamma, \theta$.*

Proof: To prove Lemma 1, we assume a linear objective function so that the optimal solution lies in the extreme point set. Thus, we make the following assumption without modifying the polytope (9b) – (9k).

- $w_{tk}^s = a_{tk}^s q_{tk}^s + b_{tk}^s \beta_{tk}$, where a_{tk}^s and b_{tk}^s are pre-specified parameters;
- $V_{\downarrow}(t) = E_t, \forall t \in [T - \ell, T]_{\mathbb{Z}}$, where E_t is a pre-specified cost.

Based on the above assumptions, here we consider the objective function as

$$\begin{aligned}
\min \quad & \sum_{t=1}^T SU(s_0 + t - 1)\alpha_t + \sum_{t=1}^T \sum_{k=t+L-1}^{T-1} SD(k - t + 1)\beta_{tk} + \sum_{t=L}^{T-\ell-1} \sum_{k=t+\ell+1}^T SU(k - t - 1)\gamma_{tk} \\
& + \sum_{t=T-\ell}^T E_t \theta_t + \sum_{tk \in \mathcal{TK}} \sum_{s=t}^k a_{tk}^s q_{tk}^s + b_{tk}^s \beta_{tk} \tag{10}
\end{aligned}$$

For notation brevity, we denote (10) as $\min c^T(\alpha, \beta, \gamma, \theta, q)$ where c is the column vector including all coefficients in (10). Now we prove that for any value of c , we can provide an optimal solution that is integral with respect to $\alpha, \beta, \gamma, \theta$ to the linear program with (10) as the objective function and (9b) – (9k) as constraints, which means that Lemma 1 holds.

Considering the assumptions we make, we can obtain an optimal solution with the dynamic programming algorithm (2) – (3). Under this optimal solution, we let $\alpha_t^* = 1$ if the machine starts up for the first time at time t ; otherwise, we let $\alpha_t^* = 0$. $\beta_{tk}^* = 1$ if the machine starts up at time t and shuts down at time $k + 1$; otherwise, we let $\beta_{tk}^* = 0$. $\gamma_{tk}^* = 1$ if the machine shuts down at time $t + 1$ and starts up at time k ; otherwise, we let $\gamma_{tk}^* = 0$. $\theta_t^* = 1$ if the machine shuts down at time $t + 1$ and stays offline to the end; otherwise, we let $\theta_t^* = 0$. We also let q_{tk}^{s*} for each $s \in [t, k]_{\mathbb{Z}}$ take the value of optimal generation output if the machine starts up at time t and shuts down at time $k + 1$; otherwise, $q_{tk}^{s*} = 0$. Now we claim our constructed $(\alpha^*, \beta^*, \gamma^*, \theta^*, q^*)$ is an optimal solution to the linear program with (10) as the objective function and (9b) – (9k) as constraints.

We first verify the feasibility. Since at most one $\alpha_t^* = 1$, constraint (9b) is satisfied. For each t in constraints (9c), if all $\beta_{tk}^* = 0$, by definition, $\alpha_t^* = 0$ and all $\gamma_{kt}^* = 0$. When one $\beta_{tk}^* = 1$, then if the machine starts up at time t for the first time, $\alpha_t^* = 1$ and all $\gamma_{kt}^* = 0$; otherwise, $\alpha_t^* = 0$ and there exists exactly one $\gamma_{kt}^* = 1$. For all these cases, constraints (9c) are satisfied. For each t in constraints (9d), if one $\beta_{kt}^* = 1$, then it may start up again after shutting down at time $t + 1$ and the minimum-down time limit should be satisfied, which indicates $\sum_{k=t+\ell+1}^T \gamma_{tk} = 1$; moreover, the machine may stay offline after shutting down at $t + 1$, which indicates $\sum_{k=t+\ell+1}^T \gamma_{tk} = 0$. If all $\beta_{kt}^* = 0$, then all γ_{tk} should be 0. Thus, constraints (9d) are satisfied. For each t in constraints (9e), if one $\beta_{kt}^* = 1$, due to the minimum-down time limit, the machine cannot start up again after then, so θ_t should be 1. If all $\beta_{kt}^* = 0$, θ_t should be 0. So constraints (9e) are satisfied. For constraints (9f) – (9j), they are immediately satisfied by the construction of our solution and the definition of the economic dispatch problem. Also, (9k) is satisfied obviously.

We then verify the optimality. We claim that the objective function (10) under the constructed solution equals to the objective value of the dynamic programming algorithm (2) – (3) as follows.

$$\begin{aligned}
& \sum_{t=1}^T SU(s_0 + t - 1)\alpha_t^* + \sum_{t=1}^T \sum_{k=t+L-1}^{T-1} SD(k - t + 1)\beta_{tk}^* + \sum_{t=L}^{T-\ell-1} \sum_{k=t+\ell+1}^T SU(k - t - 1)\gamma_{tk}^* \\
& + \sum_{t=T-\ell}^T E_t\theta_t^* + \sum_{tk \in \mathcal{TK}} \sum_{s=t}^k (a_{tk}^s q_{tk}^{s*} + b_{tk}^s \beta_{tk}^*) \\
= & SU(s_0 + t_1 - 1) + \sum_{tk: \beta_{tk}^*=1, k \leq T-1} SD(k - t + 1) + \sum_{tk: \gamma_{tk}^*=1} SU(k - t - 1) + E_{t_2} \\
& + \sum_{tk: \beta_{tk}^*=1} \sum_{s=t}^k a_{tk}^s q_{tk}^{s*} + b_{tk}^s \beta_{tk}^* \tag{11a}
\end{aligned}$$

$$\begin{aligned}
= & SU(s_0 + t_1 - 1) + \sum_{tk: \beta_{tk}^*=1, k \leq T-1} SD(k - t + 1) + \sum_{tk: \gamma_{tk}^*=1} SU(k - t - 1) + E_{t_2} \\
& + \sum_{tk: \beta_{tk}^*=1} C(t, k) \tag{11b}
\end{aligned}$$

$$= V_{\downarrow}(-s_0), \tag{11c}$$

where t_1 and t_2 in (11a) represent $\alpha_{t_1}^* = 0$ and $\theta_{t_2}^* = 0$ respectively, (11b) is based on our assumption at the beginning of the proof, and (11c) is based on the construction of our solution. By the Strong Duality Theorem, the constructed solution $(\alpha^*, \beta^*, \gamma^*, \theta^*, q^*)$ is an optimal solution for model (9).

From the above analysis, we notice that $(\alpha^*, \beta^*, \gamma^*, \theta^*, q^*)$ is binary with respect to $\alpha, \beta, \gamma, \theta$ and optimal for the dual program for all possible cost coefficient c . Thus, we have proved our claim.

■

From Lemma 1, we can further observe that this formulation itself has specific physical meanings. In particular, we notice that the optimal solution α_t^* in the dual program represents whether in stage t the machine starts up for the first time or not. If yes, then $\alpha_t^* = 1$; otherwise, $\alpha_t^* = 0$. Similarly, if the machine starts up from stage t and shuts down at stage $k + 1$, $\beta_{tk}^* = 1$; otherwise, $\beta_{tk}^* = 0$. If the machine shuts down at stage $t + 1$ and starts up again at stage k , $\gamma_{tk}^* = 1$; otherwise, $\gamma_{tk}^* = 0$. If the machine shuts down at stage $t + 1$ and stays offline to the end, $\theta_t^* = 1$; otherwise, $\theta_t^* = 0$. That is, the lemma provides insight to formulate the problem in a different way. In the following, we present the detailed extended formulation for the problem in this way and justify the correctness of the model. Before that, we have the following Corollary holds, which shows the property of the optimal solution of model (9).

Corollary 1 *The optimal solution of model (9) are binary with respect to decision variables $\alpha, \beta, \gamma, \theta$.*

Proof: The detailed proof is omitted here since we can follow the same proof in Lemma 1 to construct a solution for the dual program (9) and show its feasibility and optimality. ■

Proposition 1 *If $(\alpha^*, \beta^*, \gamma^*, \theta^*, q^*)$ is an optimal solution to dual program (9), then*

$$x_s^* = \sum_{tk \in \mathcal{TK}, t \leq s \leq k} q_{tk}^{s*}, \quad y_s^* = \sum_{tk \in \mathcal{TK}, t \leq s \leq k} \beta_{tk}^*, \quad u_s^* = \sum_{tk \in \mathcal{TK}, k=s} \gamma_{tk}^*, \quad \forall s \in [1, T]_{\mathbb{Z}} \quad (12)$$

is an optimal solution to the deterministic UC problem (1).

Proof: From Corollary 1 and constraints (9f) – (9j), we can easily conclude that y^* and u^* are binary and x^*, y^*, u^* satisfy constraints (1b) – (1h). That is, x^*, y^*, u^* are feasible to the unit commitment problem. Meanwhile, through plugging x^*, y^*, u^* into (1a), we can observe that

$$\begin{aligned} & \sum_{t=1}^T (SU_t + f_t(x_t^*, y_t^*)) + \sum_{t=L}^{T-1} SD_t \\ = & \sum_{t=1}^T SU(s_0 + t - 1)\alpha_t^* + \sum_{tk: \beta_{tk}^* = 1, k \leq T-1} SD(k - t + 1) \\ & + \sum_{t=L}^{T-\ell-1} \sum_{k=t+\ell+1}^T SU(k - t - 1)\gamma_{tk}^* + \sum_{tk \in \mathcal{TK}} \sum_{s=t}^k w_{tk}^{s*} \end{aligned} \quad (13)$$

Thus, x^*, y^*, u^* are optimal to the unit commitment problem. ■

Now we are ready to establish the extended formulations for deterministic unit commitment problem. We replace the constraints (1b) – (1l) with constraints (9b) – (9k) and add equations (12) to represent the relation between original decisions and the dual decision variables.

Theorem 1 *The extended formulation of the deterministic unit commitment problem can be written as follows:*

$$\min \quad \sum_{t=1}^T (SU_t + f_t(x_t, y_t)) + \sum_{t=L}^{T-1} SD_t \quad (14a)$$

$$s.t. \quad x_s = \sum_{tk \in \mathcal{TK}, t \leq s \leq k} q_{tk}^s, \quad y_s = \sum_{tk \in \mathcal{TK}, t \leq s \leq k} \beta_{tk},$$

$$u_s = \sum_{tk \in \mathcal{TK}, k=s} \gamma_{tk}, \quad \forall s \in [1, T]_{\mathbb{Z}}, \quad (14b)$$

$$(1i) - (1k), \quad (14c)$$

$$(9b) - (9k), \quad (14d)$$

and if $(x^, y^*, u^*, \alpha^*, \beta^*, \gamma^*, \theta^*, q^*)$ is an optimal solution to the extended formulation, then (x^*, y^*, u^*) is an optimal solution to the deterministic UC problem (1).*

Proof: The conclusion holds immediately by replacing (x^*, y^*, u^*) in the objective function and constraints (14c) with the expressions in (12) and following the proof described in Proposition 1 and the conclusion described in Corollary 1. ■

Remark 1 *From the reformulation described above, we can observe that our extended formulation not only provides integral optimal solution to UC problem (1), but also is an integral formulation with respect to variables $y, u, \alpha, \beta, \gamma, \theta$ due to Lemma 1.*

3 Deterministic Unit Commitment with Piecewise Linear Cost Function

In practice, the general convex cost function is usually approximated by a piecewise linear function. In this section, we propose a more efficient dynamic programming algorithm for this type of problems. To explore the property more conveniently, we consider simplified start-up/shut-down cost first. That is, we consider objective function in the following way.

$$\min \quad \sum_{t=1}^T \bar{U}u_t + \underline{U}(y_{t-1} - y_t + u_t) + f_t(x_t, y_t). \quad (15)$$

Thus, here we have constraints (1b) - (1h) plus (11) and decision variables x, y, u . We first explore the optimality condition of this problem and develop a new algorithm to solve the deterministic UC problem. In addition, extended formulation in a higher dimensional space is derived from our proposed algorithm.

3.1 An Optimality Condition

We denote $\mathcal{D} = \{(x, y, u) \in \mathbb{R}^T \times \mathbb{B}^{2T} : (1b) - (1h)\}$, $\alpha_1 = \max\{n \in [1, T]_{\mathbb{Z}} : \underline{C} + nV \leq \bar{C}\}$, $\alpha_2 = \max\{n \in [1, T]_{\mathbb{Z}} : \bar{V} + nV \leq \bar{C}\}$, and $\mathcal{Q} = \{0, (\underline{C} + nV)_{n=0}^{\alpha_1}, (\bar{V} + nV)_{n=0}^{\alpha_2}, (\bar{C} - nV)_{n=0}^{\alpha_1}\}$. Note here that $\alpha_2 \leq \alpha_1 \leq T$ because $\bar{V} \geq \underline{C}$. We let $\text{conv}(\mathcal{D})$ represent the convex hull description of \mathcal{D} .

Proposition 2 *For any extreme point $(\bar{x}, \bar{y}, \bar{u})$ of $\text{conv}(\mathcal{D})$, $\bar{x}_t \in \mathcal{Q}$ for all $t \in [1, T]_{\mathbb{Z}}$.*

Proof: By contradiction. Suppose that there exists some $t \in [1, T]_{\mathbb{Z}}$ such that $\bar{x}_t \notin \mathcal{Q}$ for an extreme point $(\bar{x}, \bar{y}, \bar{u})$ of $\text{conv}(\mathcal{D})$, i.e., $\bar{x}_t \in (\underline{C}, \bar{C}) \setminus \mathcal{Q}$. In the following we construct two feasible points of $\text{conv}(\mathcal{D})$ to represent $(\bar{x}, \bar{y}, \bar{u})$ in a convex combination of these two points, leading to the contradiction. If $t \geq 2$ and $|\bar{x}_t - \bar{x}_{t-1}| = V$, we let $s_1 \leq t - 1$ be the smallest index such that $|\bar{x}_{s_1+1} - \bar{x}_{s_1}| = |\bar{x}_{s_1+2} - \bar{x}_{s_1+1}| = \dots = |\bar{x}_t - \bar{x}_{t-1}| = V$; otherwise, we let $s_1 = t$. If $t \leq T - 1$ and $|\bar{x}_{t+1} - \bar{x}_t| = V$, we let $s_2 \geq t + 1$ be the largest index such that $|\bar{x}_{t+1} - \bar{x}_t| = |\bar{x}_{t+2} - \bar{x}_{t+1}| = \dots =$

$|\bar{x}_{s_2} - \bar{x}_{s_2-1}| = V$; otherwise, we let $s_2 = t$. We construct two points $(\bar{x}^1, \bar{y}, \bar{u})$ and $(\bar{x}^2, \bar{y}, \bar{u})$ such that $\bar{x}_r^1 = \bar{x}_r + \epsilon$ for $r \in [s_1, s_2]_{\mathbb{Z}}$, $\bar{x}_r^2 = \bar{x}_r - \epsilon$ for $r \in [s_1, s_2]_{\mathbb{Z}}$, and $\bar{x}_r^1 = \bar{x}_r^2 = \bar{x}_r$ for $r \notin [s_1, s_2]_{\mathbb{Z}}$, where ϵ is an arbitrarily small positive number.

Now we show these two points constructed are feasible for $\text{conv}(\mathcal{D})$ by considering the following three possible cases.

- (1) If $\bar{x}_{s_1-1} = 0$, then we have $\underline{C} < \bar{x}_{s_1} < \bar{V}$. Otherwise, 1) If $\bar{x}_{s_1} = \underline{C}$, it follows that $\bar{x}_t = \underline{C} + kV$ for some $k \in [0, t - s_1]_{\mathbb{Z}}$ due to $|\bar{x}_{s_1+1} - \bar{x}_{s_1}| = |\bar{x}_{s_1+2} - \bar{x}_{s_1+1}| = \dots = |\bar{x}_t - \bar{x}_{t-1}| = V$. Since $\bar{x}_t \notin \mathcal{Q}$, it further follows that $k \geq \alpha_1 + 1 = \min\{T, \lfloor (\bar{C} - \underline{C})/V \rfloor\} + 1$, which contradicts to the fact that $k \leq T$ and $\bar{x}_t < \bar{C}$. 2) If $\bar{x}_{s_1} = \bar{V}$, it follows that $\bar{x}_t = \bar{V} + kV$ for some $k \in [0, t - s_1]_{\mathbb{Z}}$ due to $|\bar{x}_{s_1+1} - \bar{x}_{s_1}| = |\bar{x}_{s_1+2} - \bar{x}_{s_1+1}| = \dots = |\bar{x}_t - \bar{x}_{t-1}| = V$. Since $\bar{x}_t \notin \mathcal{Q}$, it further follows that $k \geq \alpha_2 + 1 = \min\{T, \lfloor (\bar{C} - \bar{V})/V \rfloor\} + 1$, which contradicts to the fact that $k \leq T$ and $\bar{x}_t < \bar{C}$. Similarly, if $\bar{x}_{s_2+1} = 0$, then we have $\underline{C} < \bar{x}_{s_2} < \bar{V}$. Therefore, in either case, it is feasible to increase or decrease \bar{x}_{s_1} and \bar{x}_{s_2} by ϵ .
- (2) If $\bar{x}_{s_1-1} > 0$, then we have $\underline{C} < \bar{x}_{s_1} < \bar{C}$. Otherwise, 1) If $\bar{x}_{s_1} = \underline{C}$, we can similarly show the contradiction as above. 2) If $\bar{x}_{s_1} = \bar{C}$, it follows that $\bar{x}_t = \bar{C} - kV$ for some $k \in [0, t - s_1]_{\mathbb{Z}}$ due to $|\bar{x}_{s_1+1} - \bar{x}_{s_1}| = |\bar{x}_{s_1+2} - \bar{x}_{s_1+1}| = \dots = |\bar{x}_t - \bar{x}_{t-1}| = V$. Since $\bar{x}_t \notin \mathcal{Q}$, it further follows that $k \geq \alpha_1 + 1 = \min\{T, \lfloor (\bar{C} - \underline{C})/V \rfloor\} + 1$, which contradicts to the fact that $k \leq T$ and $\bar{x}_t > \underline{C}$. Similarly, if $\bar{x}_{s_2+1} > 0$, then we have $\underline{C} < \bar{x}_{s_2} < \bar{C}$. Therefore, in either case, it is feasible to increase or decrease \bar{x}_{s_1} and \bar{x}_{s_2} by ϵ since $|\bar{x}_{s_1} - \bar{x}_{s_1-1}| < V$ and $|\bar{x}_{s_2+1} - \bar{x}_{s_2}| < V$ by definition.
- (3) If $s_1 - 1, s_2 + 1 \notin [1, T]_{\mathbb{Z}}$, i.e., $s_1 = 1$ or $s_2 = T$, then similarly we can follow the arguments above to show that $\underline{C} < \bar{x}_{s_1} < \bar{C}$ and $\underline{C} < \bar{x}_{s_2} < \bar{C}$. It follows that it is feasible to increase or decrease \bar{x}_{s_1} and \bar{x}_{s_2} by ϵ .

In summary, we show that in all cases it is feasible to increase or decrease \bar{x}_{s_1} and \bar{x}_{s_2} by ϵ and thus feasible to increase or decrease \bar{x}_r by ϵ for all $r \in [s_1, s_2]_{\mathbb{Z}}$. It follows that both $(\bar{x}^1, \bar{y}, \bar{u})$ and $(\bar{x}^2, \bar{y}, \bar{u})$ are feasible points of $\text{conv}(\mathcal{D})$, and $(\bar{x}, \bar{y}, \bar{u}) = \frac{1}{2}(\bar{x}^1, \bar{y}, \bar{u}) + \frac{1}{2}(\bar{x}^2, \bar{y}, \bar{u})$. Therefore, $(\bar{x}, \bar{y}, \bar{u})$ is not an extreme point of $\text{conv}(\mathcal{D})$, which is a contradiction. ■

Now, we begin to characterize the optimality condition for Problem (15). Generally $f_t(x_t, y_t) = ax_t^2 + bx_t + cy_t - q_t x_t$, where (a, b, c) is determined by the generator physics, and it is often approximated by a K -piece piecewise linear function $\varphi_t = f_t(x_t, y_t) \geq \mu_k^t x_t + \nu_k y_t, \forall 1 \leq k \leq K$ so that

the unit commitment problem can be formulated as a mixed-integer linear programming model, where $\mu_k^t = 2a\tilde{x}_k + b - q_t$ and $\nu_k = c - a\tilde{x}_k$ with \tilde{x}_k being the x -value corresponding to the k -th supporting node on the curve of $f_t(x_t, y_t)$ at each time period t and $\tilde{x}_1 = \underline{C}$, $\tilde{x}_K = \overline{C}$. Therefore Model (15) can be reformulated as

$$\begin{aligned} \min \quad & \sum_{t=1}^T \overline{U}u_t + \underline{U}(y_{t-1} - y_t + u_t) + \varphi_t \\ \text{s.t.} \quad & (x, y, u) \in \mathcal{D}. \\ & \varphi_t \geq \mu_k^t x_t + \nu_k y_t, \forall k \in [1, K]_{\mathbb{Z}}, \forall t \in [1, T]_{\mathbb{Z}}. \end{aligned} \quad (16)$$

Note here that the cost function φ_t is a linear function if there is only one piece, i.e., $K = 1$. It is easy to observe that any two adjacent pieces at each time period t , e.g., $\varphi_t \geq \mu_k^t x_t + \nu_k y_t$ and $\varphi_t \geq \mu_{k+1}^t x_t + \nu_{k+1} y_t$, intersect at $A_k = (\tilde{x}_k + \tilde{x}_{k+1})/2$. Therefore, we can obtain $K-1$ turning points with x -value $A_k, \forall k \in [1, K-1]_{\mathbb{Z}}$ on the K -piece piecewise linear function for each time period. We denote $\chi_k = \{n \in [1, T]_{\mathbb{Z}} : \underline{C} \leq A_k + nV \leq \overline{C}\}$ and $\mathcal{Q}_d = \mathcal{Q} \cup \{(A_k + nV)_{n \in \chi_k}, \forall k \in [1, K-1]_{\mathbb{Z}}\}$.

Proposition 3 *Problem (15) has at least one optimal solution $(\bar{x}, \bar{y}, \bar{u})$ with $\bar{x}_t \in \mathcal{Q}_d$ for all $t \in [1, T]_{\mathbb{Z}}$.*

Proof: By contradiction. Suppose that there exists some $t \in [1, T]_{\mathbb{Z}}$ such that $\bar{x}_t \notin \mathcal{Q}_d$ for the optimal solution $(\bar{x}, \bar{y}, \bar{u})$ of Problem (15), i.e., $\bar{x}_t \in (\underline{C}, \overline{C}) \setminus \mathcal{Q}_d$, with the optimal value $\bar{z} = \sum_{t=1}^T \overline{U}\bar{u}_t + \underline{U}(\bar{y}_{t-1} - \bar{y}_t + \bar{u}_t) + \bar{\varphi}_t$. In the following we construct a feasible solution to obtain a better objective value. If $t \geq 2$ and $|\bar{x}_t - \bar{x}_{t-1}| = V$, we let $s_1 \leq t-1$ be the smallest index such that $|\bar{x}_{s_1+1} - \bar{x}_{s_1}| = |\bar{x}_{s_1+2} - \bar{x}_{s_1+1}| = \dots = |\bar{x}_t - \bar{x}_{t-1}| = V$; otherwise, we let $s_1 = t$. If $t \leq T-1$ and $|\bar{x}_{t+1} - \bar{x}_t| = V$, we let $s_2 \geq t+1$ be the largest index such that $|\bar{x}_{t+1} - \bar{x}_t| = |\bar{x}_{t+2} - \bar{x}_{t+1}| = \dots = |\bar{x}_{s_2} - \bar{x}_{s_2-1}| = V$; otherwise, we let $s_2 = t$. Following the proof in Proposition 2, we can always construct two feasible points of \mathcal{D} , $(\bar{x}^1, \bar{y}, \bar{u})$ and $(\bar{x}^2, \bar{y}, \bar{u})$, such that $\bar{x}_r^1 = \bar{x}_r + \epsilon$ for $r \in [s_1, s_2]_{\mathbb{Z}}$, $\bar{x}_r^2 = \bar{x}_r - \epsilon$ for $r \in [s_1, s_2]_{\mathbb{Z}}$, and $\bar{x}_r^1 = \bar{x}_r^2 = \bar{x}_r$ for $r \notin [s_1, s_2]_{\mathbb{Z}}$, where ϵ is an arbitrarily small positive number.

Now we show one of $(\bar{x}^1, \bar{y}, \bar{u})$ and $(\bar{x}^2, \bar{y}, \bar{u})$ produces a better objective value and is also feasible for constraints (16). It is easy to observe that for each time period s , if $\bar{x}_s > 0$, then at most two adjacent pieces of linear functions of constraints (16) are tight; otherwise, it leads to $\bar{x}_s = 0$. Since $\bar{x}_t \notin \mathcal{Q}_d$, there is at most one piece of linear function of constraints (16), e.g., piece k_r , is tight for each time period $r \in [s_1, s_2]_{\mathbb{Z}}$, as two adjacent pieces (e.g., pieces k and $k+1$) intersect at A_k , which belongs to \mathcal{Q}_d . Note that if there exist two adjacent pieces (e.g., pieces k and $k+1$)

intersecting at A_k for some time period $r \in [s_1, s_2]_{\mathbb{Z}}$, then $\bar{x}_r = A_k$ and it follows that $\bar{x}_t = A_k + sV$ for some $s \in [-|s-r|, |s-r|]_{\mathbb{Z}}$ by definition, which contradicts the fact that $\bar{x}_t \notin \mathcal{Q}_d$ and $s \leq T$. Therefore, we can increase or decrease \bar{x}_r by ϵ for $r \in [s_1, s_2]_{\mathbb{Z}}$ to decrease the optimal value by at least $|\sum_{r=s_1}^{s_2} \mu_{k_r}^r| \epsilon$ and the resulting solution, $(\bar{x}^1, \bar{y}, \bar{u})$ or $(\bar{x}^2, \bar{y}, \bar{u})$, is feasible for both \mathcal{D} and constraints (16), which is a contradiction. ■

In other words, in order to find an optimal solution to the UC problem (1), we only need to consider the feasible solutions (x, y, u) where $x_t \in \mathcal{Q}_d$ for all $t \in [1, T]_{\mathbb{Z}}$.

3.2 An $\mathcal{O}(T)$ Time Dynamic Programming Algorithm

As \mathcal{Q}_d is a finite set and the cardinality of \mathcal{Q}_d , denoted as \aleph_d , does not depend on the total time period T , rather than solve the original MILP model (15), we can explore a backward induction dynamic programming framework by defining the corresponding states and decisions for each stage t . We first define the state space for the dynamic programming algorithm and then describe the state-decision relation through a directed graph, as shown in Figure 1, associated with predetermined parameters of the UC problem. The Bellman equations can be derived accordingly based on the state space and the directed graph.

To begin with, we define the state space as $\mathcal{S} = \mathcal{S}_* \cup \mathcal{S}_0 \cup \mathcal{S}_1$. In detail, \mathcal{S}_* consists of a dummy state that we consider it as a “source” here, which is used as initial state for the decision maker at stage 1. $\mathcal{S}_0 \cup \mathcal{S}_1$ represent all the other states that have the structure of (x, y, u, d) where x, y, u here have the same meaning as the notation defined in the beginning of Section 2 except that here they are variables for any single time period and the variable d represents the duration for current online/offline status. $\mathcal{S}_0 = \{(x, y, u, d) \in \mathcal{Q}_d \times \mathbb{B} \times \mathbb{B} \times [1, \ell]_{\mathbb{Z}} : x = 0, y = 0, u = 0\}$, representing the set of all states when the generator is offline. Note here that the candidate decisions for the decision maker remain the same whenever the duration for offline status $d \geq \ell$, so it is enough to set the upper bound for duration variable d as minimum-down time limit ℓ . $\mathcal{S}_1 = \{(x, y, u, d) \in \mathcal{Q}_d \times \mathbb{B} \times \mathbb{B} \times [1, L]_{\mathbb{Z}} : x > 0, y = 1, u = 1 \text{ when } d = 1; x > 0; y = 1, u = 0 \text{ when } d > 1\}$ represents the set of all states when the generator stays online. To be specific, we let $d = 1$, if the generator just starts up at its current state so we have $x > 0, y = 1, u = 1$; we let $d > 1$ if the generator has been online for at least two time periods so we have $x > 0, y = 1, u = 0$. Note here that it is also enough to set the upper bound for the duration variable d as minimum-up time limit L when it is online since the candidate decisions remain the same whenever $d \geq L$.

Based on our construction of state space, we can observe that \mathcal{S} is a finite set and does not

depend on the number of time periods T . To formulate the Bellman equations, we construct a directed graph where the nodes consist of all the elements in the state space and the arcs represent possible decisions from one state to another (i.e., correspondingly from one time period to the next). More specifically, we add directed arcs from the “source” node to $(0, 0, 0, \ell)$ and all possible $(x, 1, 0, L)$. These nodes form the set of decision candidates at stage 1. We also add possible directed arcs between any two nodes in $\mathcal{S}_0 \cup \mathcal{S}_1$. For example, there are arcs from $(0, 0, 0, 1)$ to $(0, 0, 0, 2)$, from $(0, 0, 0, 2)$ to $(0, 0, 0, 3)$, \dots , from $(0, 0, 0, \ell - 1)$ to $(0, 0, 0, \ell)$, from $(0, 0, 0, \ell)$ to itself and all possible $(x, 1, 1, 1) \in \mathcal{S}_1$ that satisfies start-up ramp constraint $x \leq \bar{V}$. Note here that self-loop arc is allowed since, as mentioned above, it is enough to take d as ℓ if the offline status d lasts longer than the minimum-down time limit. For node $(x, y, u, d) \in \mathcal{S}_1$ with $d < L$, we add an arc from it to all possible $(x', y', u', d') \in \mathcal{S}_1$ satisfying ramp-up/down constraints $|x - x'| \leq V$ and logical constraint $d' = d + 1$. For node $(x, y, u, L) \in \mathcal{S}_1$, we add an arc from it to all possible $(x', y', u', L) \in \mathcal{S}_1$ satisfying ramp-up/down constraints $|x - x'| \leq V$. Self-loop arc is also allowed here. For node $(x, y, u, L) \in \mathcal{S}_1$ with $x \leq \bar{V}$, i.e., shut-down ramp constraint is satisfied, we add an arc from it to $(0, 0, 0, 1)$. Furthermore, we label the nodes with positive integers starting from the “source” node with index 1 and denote all the direct successors of node i as $S(i)$ and all the immediate predecessors of node i as $P(i)$. We denote the values of x, y, u in node i as i_x, i_y, i_u .

Now we are ready to establish the dynamic programming framework. Let $F_t(i)$ represent the optimal value function for stage t considering node i as the state of the previous stage. Based on Proposition 3, an optimal decision for current stage lies in $S(i)$. The Bellman equations can be formulated as follows:

$$F_t(i) = \min_{j \in S(i)} \bar{U}j_u + \underline{U}(i_y - j_y + j_u) + f_t(j_x, j_y) + F_{t+1}(j), \quad \forall i \in \mathcal{S}, \forall t \in [1, T-1]_{\mathbb{Z}}, \quad (17a)$$

$$F_T(i) = \min_{j \in S(i)} \bar{U}j_u + \underline{U}(i_y - j_y + j_u) + f_T(j_x, j_y), \quad \forall i \in \mathcal{S}, \quad (17b)$$

where $f_t(j_x, j_y)$ describes the generation cost minus revenue, $\bar{U}j_u$ represents the start-up cost, and $\underline{U}(i_y - j_y + j_u)$ represents the shut-down cost. For notation brevity, we let $E_{tij} = \bar{U}j_u + \underline{U}(i_y - j_y + j_u) + f_t(j_x, j_y)$ for $t \in [1, T]_{\mathbb{Z}}$. Accordingly, the objective of our backward induction dynamic programming is to find out the value of $F_1(1)$.

In order to obtain the optimal objective value and optimal solution, we need to calculate $F_t(i)$ for all possible t and i and record the optimal candidate for them. To calculate the value of each optimal value function in the Bellman equations (17), we search among the candidate solution

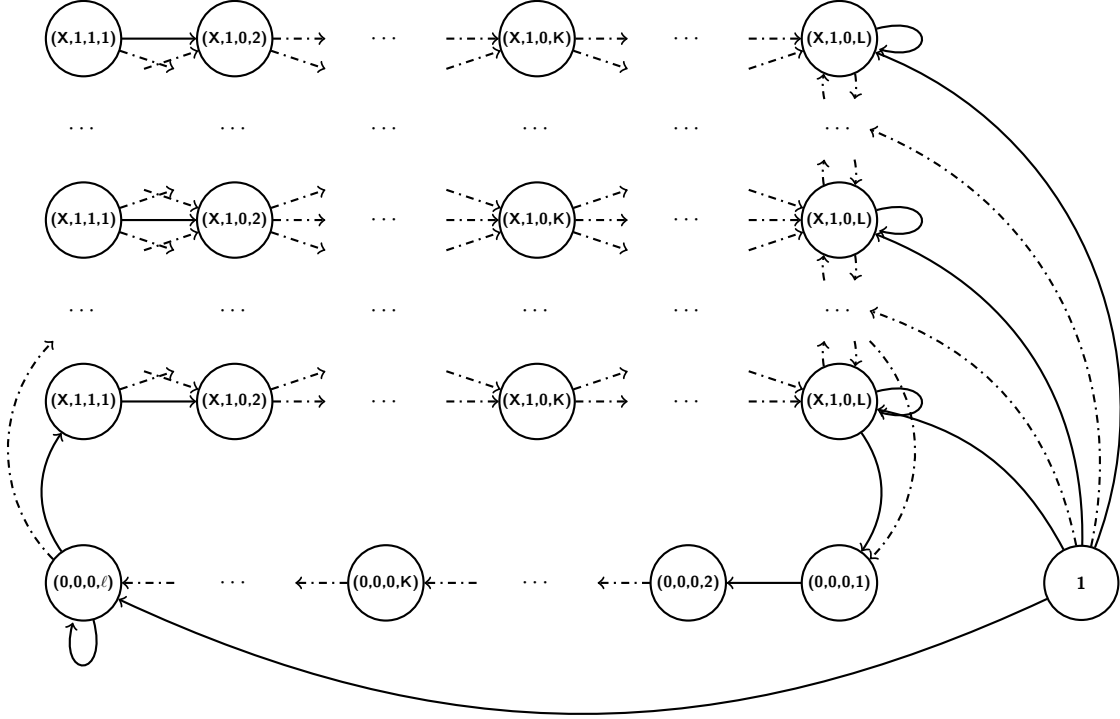


Figure 1: Directed graph

$j \in S(i)$ for each i and this step takes $\mathcal{O}(\aleph_d)$ time. Since there are in total $\aleph_d L + \ell$ number of nodes in the state graph, the computational time at each time period is $\mathcal{O}(\aleph_d(\aleph_d L + \ell))$. Thus, the total time to calculate the value of objective $F_1(1)$ is $\mathcal{O}(\aleph_d(\aleph_d L + \ell)T)$. The optimal solution for UC can be obtained by tracing the optimal candidate for the optimal value function starting from $F_1(1)$, and this takes $\mathcal{O}(T)$ time in total. Because \aleph_d, L, ℓ are constant numbers with respect to the physical parameters of UC problem, we conclude that our backward induction dynamic programming algorithm for the deterministic unit commitment problem is an $\mathcal{O}(T)$ time algorithm. When start-up profile is considered, we have the following observations.

Remark 2 *If the start-up profile is taken into account in the UC model (15), then we need to extend the upper bound of the offline duration variable d from ℓ to T , which as a result will increase the computational complexity from $\mathcal{O}(T)$ time to $\mathcal{O}(T^2)$ time.*

3.3 Extended Formulation for Deterministic Unit Commitment with Piecewise Linear Cost Function

Based on the Bellman equations (17) for the deterministic UC model, now in this section, we reformulate the problem as a linear program incorporating dynamic programming Bellman equations

as constraints. The incentive for the reformulation is to develop a linear program that can provide integral solutions to the unit commitment problem in a higher dimensional space. The primal form of our linear program can be formulated as follows:

$$\max \quad F_1(1) \tag{18a}$$

$$\text{s.t.} \quad F_t(i) \leq E_{tij} + F_{t+1}(j), \forall i \in \mathcal{S}, j \in S(i), \forall t \in [1, T-1]_{\mathbb{Z}}, \tag{18b}$$

$$F_T(i) \leq E_{Tij}, \forall i \in \mathcal{S}, j \in S(i), \tag{18c}$$

where the parameters $E_{tij} = \bar{U}j_u + \underline{U}(i_y - j_y + j_u) + f_t(j_x, j_y)$ as defined under equations (17).

But the primal linear program cannot provide solutions to the UC problem directly. Although we can solve the primal linear program and search for tight constraints to find solutions for unit commitment, this takes us back to the dynamic program framework. Thus, we resort to the dual formulation and then provide an extended linear formulation, which we show can provide solution to the deterministic UC problem directly.

By taking Lagrangian duality, we can obtain the dual formulation for the linear program (18) as follows:

$$\min \quad \sum_{t \in [1, T]_{\mathbb{Z}}, i \in \mathcal{S}, j \in S(i)} E_{tij} w_{tij} \tag{19a}$$

$$\text{s.t.} \quad \sum_{j \in S(1)} w_{11j} = 1, \tag{19b}$$

$$\sum_{j \in S(i)} w_{1ij} = 0, \forall i \in \mathcal{S} \setminus \mathcal{S}_*, \tag{19c}$$

$$\sum_{j \in S(i)} w_{tij} - \sum_{k \in P(i)} w_{t-1, ki} = 0, \forall i \in \mathcal{S}, \forall t \in [2, T]_{\mathbb{Z}}, \tag{19d}$$

$$w_{tij} \geq 0, \forall i \in \mathcal{S}, j \in S(i), \forall t \in [1, T]_{\mathbb{Z}}, \tag{19e}$$

where w_{tij} are dual variables corresponding to each constraint in the primal linear program.

In the following lemma, we demonstrate that the above dual linear program can automatically generate integral solutions for w . Furthermore, we explore physical meaning of these dual variables.

Lemma 2 *The extreme points of the polytope (19b) – (19e) are binary.*

Proof: To prove the lemma, it is equivalent to prove that for $\forall E_{tij} \in (-\infty, +\infty)$, there exists at least one optimal solution to the dual program (19) that is binary. First of all, by solving the UC problem with dynamic program approach with respect to a given E , we can obtain an optimal decision of (x, y, u) . We then construct \hat{w} , which is binary, to represent the optimal decision. For

a given optimal decision $(x, y, u) \in \mathcal{Q}_d^T \times \mathbb{B}^{2T}$, we can correspondingly draw a path in the graph starting from the node 1. If in step t of the path the state of the generator goes from node i to node j , we let $\hat{w}_{tij} = 1$. Otherwise, $\hat{w}_{tij} = 0$. In the following, we prove that this generated \hat{w} is an optimal solution to the dual program (19), which is sufficient to prove our claim.

To verify the feasibility, we plug \hat{w} into constraints (19b) – (19e). For constraint (19b), since in step 1 the state of the generator goes from node 1 to one node in $S(1)$, exactly one $\hat{w}_{11j} = 1$ and all the other $\hat{w}_{11j} = 0$. Constraint (19c) is satisfied since all $\hat{w}_{1ij} = 0$, $\forall i \in \mathcal{S} \setminus \mathcal{S}_*$. In step $t \geq 2$, only when it goes from node i to one node in $S(i)$, exactly one $\hat{w}_{tij} = 1$ and other $\hat{w}_{tij} = 0$. As a result, in step $t - 1$, it should go from one node in $P(i)$ to node i so exactly one $\hat{w}_{t-1,ki} = 1$ and other $\hat{w}_{t-1,ki} = 0$. It follows that $\sum_{j \in S(i)} \hat{w}_{tij} - \sum_{k \in P(i)} \hat{w}_{t-1,ki} = 1 - 1 = 0$. For all other cases, $\hat{w}_{tij} = 0$, and (19d) is also satisfied. Constraint (19e) is satisfied by definition. Thus, \hat{w} is feasible for the dual program.

To verify the optimality, we denote the optimal objective value obtained by dynamic program as F^* and the objective value of dual program (19) with respect to \hat{w} as $H(\hat{w})$. We want to prove $H(\hat{w}) = F^*$. Recalling the definition of \hat{w} , we use a path, denoted as $(i_0, i_1, i_2, \dots, i_T)$, to determine \hat{w} . Thus, we have

$$H(\hat{w}) = \sum_{t \in [1, T]_{\mathbb{Z}}, i \in \mathcal{S}, j \in S(i)} E_{tij} \hat{w}_{tij} \quad (20a)$$

$$= \sum_{t=1}^T E_{ti_{t-1}i_t} \hat{w}_{ti_{t-1}i_t} \quad (20b)$$

$$= \sum_{t=1}^T E_{ti_{t-1}i_t}, \quad (20c)$$

$$F^* = F_1(1) = \min_{j \in S(1)} \{E_{11j} + F_2(j)\} \quad (21a)$$

$$= E_{1i_0i_1} + F_2(i_1) \quad (21b)$$

$$= E_{1i_0i_1} + \min_{j \in S(i_1)} \{E_{2i_1j} + F_3(j)\} \quad (21c)$$

$$= E_{1i_0i_1} + E_{2i_1i_2} + F_3(i_2) \quad (21d)$$

$$= \dots \quad (21e)$$

$$= \sum_{t=1}^T E_{ti_{t-1}i_t} \quad (21f)$$

$$= H(\hat{w}). \quad (21g)$$

Therefore, we claim that \hat{w} is the optimal solution to the dual program.

From the above analysis, we notice that \hat{w} is binary and optimal for the dual program for any possible cost coefficient E . Thus, we have proved our claim. ■

From Lemma 2, we can further observe that this formulation itself has specific physical meanings. In particular, we notice that the optimal solution w_{tij}^* in the dual program represents whether in stage t the UC optimal decision corresponds to a state change from node i (in stage $t-1$) to node j (in stage t) or not. If yes, then $w_{tij}^* = 1$; otherwise, $w_{tij}^* = 0$. That is, the lemma provides insight to formulate the problem in a different way. In the following, we present the detailed extended formulation for the problem in this way and justify the correctness of the model.

Proposition 4 *If w^* is an optimal solution to dual program (19), then*

$$x_t^* = \sum_{i \in \mathcal{S}, j \in \mathcal{S}(i)} j_x w_{tij}^*, y_t^* = \sum_{i \in \mathcal{S}, j \in \mathcal{S}(i)} j_y w_{tij}^*, u_t^* = \sum_{i \in \mathcal{S}, j \in \mathcal{S}(i)} j_u w_{tij}^*, \forall t \in [1, T]_{\mathbb{Z}}, \quad (22)$$

is an optimal solution to the deterministic UC problem (15).

Proof: From the proof of Lemma 2, we can observe that for a given t , exactly one $w_{tij}^* = 1$ and all other $w_{tij}^* = 0$. Thus x_t^*, y_t^*, u_t^* are the values included in a certain node along a path which starts from node 1 in the directed graph. If we denote the entire path as $(i_0, i_1, i_2, \dots, i_T)$, then we have that $x_t^* = (i_t)_x, y_t^* = (i_t)_y$, and $u_t^* = (i_t)_u$ for each $t \in [1, T]_{\mathbb{Z}}$. Recalling the definition of the graph we can conclude that any ‘‘T-step’’ path (i.e., a path for the whole time horizon) starting from node 1 results in a feasible solution to the UC problem. Thus, the solution (x^*, y^*, u^*) is feasible.

If we plug (x^*, y^*, u^*) with expressions in (22) into the objective function of the deterministic UC problem (15), we have

$$\sum_{t=1}^T \bar{U} u_t^* + \underline{U}(y_{t-1}^* - y_t^* + u_t^*) + f(x_t^*, y_t^*) = \sum_{t \in [1, T]_{\mathbb{Z}}, i \in \mathcal{S}, j \in \mathcal{S}(i)} E_{tij} w_{tij}^*.$$

Hence, (x^*, y^*, u^*) is the optimal solution to the deterministic UC problem. ■

Now we are ready to establish the extended formulation for deterministic unit commitment problem. We replace constraints (1b) - (1h) plus (11) with constraints (19b) - (19e) and add equations (22) to represent the relation between original decisions and the dual decision variables.

Theorem 2 *The extended formulation of the deterministic unit commitment problem (15) can be written as follows:*

$$\min \quad \sum_{t=1}^T \bar{U} u_t + \underline{U}(y_{t-1} - y_t + u_t) + f_t(x_t, y_t) \quad (23a)$$

$$\begin{aligned}
s.t. \quad x_t &= \sum_{i \in \mathcal{S}, j \in \mathcal{S}(i)} j_x w_{tij}, y_t = \sum_{i \in \mathcal{S}, j \in \mathcal{S}(i)} j_y w_{tij}, \\
u_t &= \sum_{i \in \mathcal{S}, j \in \mathcal{S}(i)} j_u w_{tij}, \quad \forall t \in [1, T]_{\mathbb{Z}}, \\
& (19b) - (19e),
\end{aligned} \tag{23b}$$

$$\tag{23c}$$

and if (x^*, y^*, u^*, w^*) is an optimal solution to the extended formulation, then (x^*, y^*, u^*) is an optimal solution to the deterministic UC problem.

Proof: The conclusion holds immediately by replacing (x^*, y^*, u^*) in the objective function with the expressions in (22) and following the proof described in Proposition 4 and the conclusion described in Lemma 2. ■

4 Stochastic Unit Commitment with Piecewise Linear Cost Function

In this section, we extend our study to the multistage stochastic UC setting to incorporate uncertainty. With the consideration of renewable generation and/or price uncertainties, as well as dependency among different time periods, scenario-tree based stochastic UC is introduced in [12]. Under this setting, the uncertain problem parameters are assumed to follow a discrete-time stochastic process with finite probability space and a scenario tree $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ is utilized to describe the resulting information structure, as shown in Figure 2. Each node $i \in \mathcal{V}$ at time t of the tree provides the state of the system that can be distinguished by information available up to time t (corresponding to a scenario realization from time 1 to time t). Accordingly, corresponding to each node $i \in \mathcal{V}$, we let $t(i)$ be its time period, $\mathcal{P}(i)$ be the set of nodes along the path from the root node (denoted as node 1) to node i , and p_i be the probability associated with the state represented by node i . We also denote \mathcal{V}_* as the set of root node, i.e., $\mathcal{V}_* = \{1\}$. In addition, each node i in the scenario tree has a unique parent i^- and could have multiple children, denoted as set $\mathcal{C}_*(i)$. Meanwhile, we define $\mathcal{C}(i) = \mathcal{C}_*(i) \cup \{i\}$. Moreover, we let $i_0^- = i$, $i_1^- = i^-$, and i_k^- be the unique parent node of i_{k-1}^- , for $k \geq 2$. In other words, we define i_k^- be the k -fold parent of node i . We let $\mathcal{V}(i)$ represent the set of all descendants of node i , including itself. Finally, we let $\mathcal{H}_r(i) = \{k \in \mathcal{V}(i) : 0 \leq t(k) - t(i) \leq r - 1\}$ be the set of nodes used to describe minimum-up and minimum-down time constraints (e.g., in Figure 2, $r = t(j) - t(i)$). The decisions corresponding to each node i are assumed to be made after observing the realizations of the problem parameters along the path from the root node to this node i , but are nonanticipative with respect to future realizations.

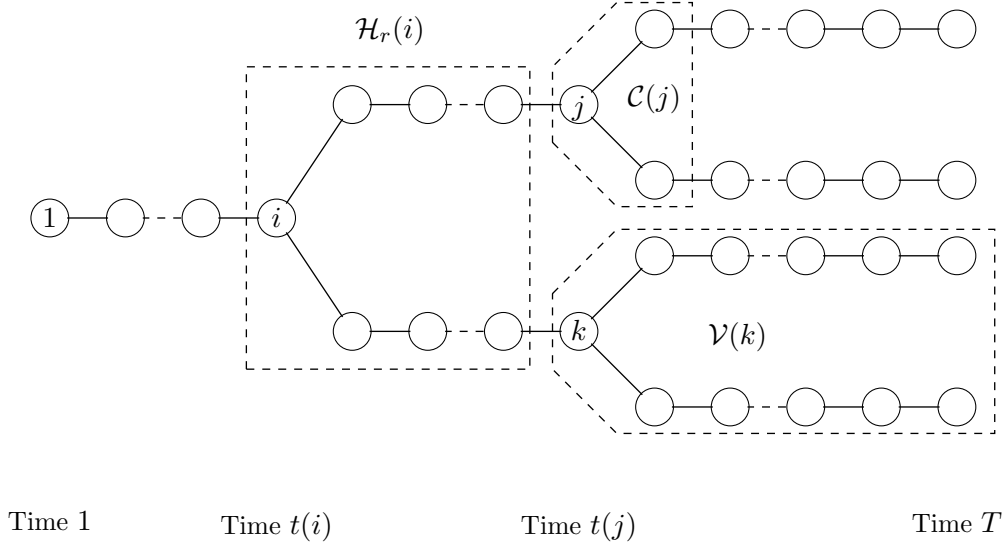


Figure 2: Multistage stochastic scenario tree

For the multistage stochastic unit commitment problem, following the notation described above, we let \bar{U} (\underline{U}) denote its start-up (shut-down) cost and a nondecreasing convex function $f(\cdot)$ denote the generation cost minus the revenue as a function of its electricity generation amount, online/offline status, and electricity price. The decision variables include the online/offline status, start-up decision, and the generation amount at each node in the scenario tree. Accordingly, for each node i , we let binary variables (y_i, u_i) denote the unit commitment decisions: (1) y_i represents if the generator is online or offline at node i (i.e., $y_i = 1$ if yes; $y_i = 0$ otherwise) and (2) u_i represents if the generator starts up or not at node i (i.e., $u_i = 1$ if yes; $u_i = 0$ otherwise). We also let continuous variable x_i denote the electricity generation amount at node i . We also assume the generator has been offline for s_0 time periods ($s_0 \geq \ell$).

Based on the notation described above, the formulation for this problem can be described as follows:

$$\min \sum_{i \in \mathcal{V}} \bar{U} u_i + \underline{U}(y_{i-} - y_i + u_i) + f_i(x_i, y_i) \quad (24a)$$

$$\text{s.t. } y_i - y_{i-} \leq y_k, \quad \forall i \in \mathcal{V}, \forall k \in \mathcal{H}_L(i), \quad (24b)$$

$$y_{i-} - y_i \leq 1 - y_k, \quad \forall i \in \mathcal{V}, \forall k \in \mathcal{H}_\ell(i), \quad (24c)$$

$$y_i - y_{i-} \leq u_i, \quad \forall i \in \mathcal{V}, \quad (24d)$$

$$u_i \leq \min\{y_i, 1 - y_{i-}\}, \quad \forall i \in \mathcal{V}, \quad (24e)$$

$$\underline{C}y_i \leq x_i \leq \bar{C}y_i, \quad \forall i \in \mathcal{V}, \quad (24f)$$

$$x_i - x_{i^-} \leq V y_{i^-} + \bar{V}(1 - y_{i^-}), \quad \forall i \in \mathcal{V}, \quad (24g)$$

$$x_{i^-} - x_i \leq V y_i + \bar{V}(1 - y_i), \quad \forall i \in \mathcal{V}, \quad (24h)$$

$$y_i, u_i \in \{0, 1\}, \quad \forall i \in \mathcal{V}, \quad x_{1^-} = y_{1^-} = 0. \quad (24i)$$

In the above formulation, the objective is to minimize the expected total cost, which is equal to the total generation cost (i.e., start-up, shut-down, and fuel costs) minus the revenue, where $f_i(x_i, y_i)$ indicates the fuel cost minus the revenue. Constraints (24b) represent the minimum-up time limits for the generator. That is, if the generator starts up at node i , then it should stay online for all the nodes in $\mathcal{H}_L(i)$. Similarly, constraints (24c) represent the minimum-down time limits. If the generator shuts down at node i , then it should be kept offline for all the nodes in $\mathcal{H}_\ell(i)$. Constraints (24d) and (24e) describe the relationship between u and y . Constraints (24f) describe the upper and lower bounds of electricity generation amount if the generator is online at node i . Constraints (24g) and (24h) describe the ramp-up rate and ramp-down rate limits, respectively. Typically the function $f(\cdot)$ is quadratic and can be approximated by a piecewise linear function. With this approximation, the deterministic equivalent formulation above can be reformulated as an MILP formulation.

4.1 An Optimality Condition

We denote $\mathcal{W} = \{(x, y, u) \in \mathbb{R}^{|\mathcal{V}|} \times \mathbb{B}^{2|\mathcal{V}|} : (24b) - (24h)\}$, $\beta_1 = \max\{n \in [1, 2T]_{\mathbb{Z}} : \underline{C} + nV \leq \bar{C}\}$, $\beta_2 = \max\{n \in [1, 2T]_{\mathbb{Z}} : \bar{V} + nV \leq \bar{C}\}$, and $\hat{\mathcal{Q}} = \{0, (\underline{C} + nV)_{n=0}^{\beta_1}, (\bar{V} + nV)_{n=0}^{\beta_2}, (\bar{C} - nV)_{n=0}^{\beta_1}\}$. Note that $\beta_2 \leq \beta_1 \leq 2T$ because $\bar{V} \geq \underline{C}$.

Proposition 5 *For any extreme point $(\bar{x}, \bar{y}, \bar{u})$ of $\text{conv}(\mathcal{W})$, $\bar{x}_i \in \hat{\mathcal{Q}}$ for all $i \in \mathcal{V}$.*

Proof: By contradiction. Suppose that there exists some $i \in \mathcal{V}$ such that $\bar{x}_i \notin \hat{\mathcal{Q}}$ for an extreme point $(\bar{x}, \bar{y}, \bar{u})$ of $\text{conv}(\mathcal{W})$, i.e., $\bar{x}_i \in (\underline{C}, \bar{C}) \setminus \hat{\mathcal{Q}}$. In the following we construct two feasible points of $\text{conv}(\mathcal{W})$ to represent $(\bar{x}, \bar{y}, \bar{u})$ in a convex combination of these two points, leading to the contradiction. Before that, we first construct a subtree of \mathcal{V} . If one of or all of the following conditions hold,

- $t(i) \geq 2$ and $|x_i - x_{i^-}| = V$,
- $t(i) \leq T - 1$ and $|x_i - x_j| = V$ for some j such that $i = j^-$,

then we construct a subtree of \mathcal{V} that consists of nodes around node i (e.g., the subtree that consists of blue nodes in Figure 3), denoted as $\mathcal{K}(i)$, such that for $\forall n_1, n_2 \in \mathcal{K}(i)$ with $n_1^- = n_2$ or $n_2 = n_1^-$,

$|x_{n_1} - x_{n_2}| = V$, and for some node $n \in \mathcal{K}(i)$ (denoted as boundary node of $\mathcal{K}(i)$), $t(n) \in \{1, T\}$ or $\exists m \in \mathcal{V} \setminus \mathcal{K}(i)$ with $n = m^-$ or $m = n^-$ such that $|x_m - x_n| \neq V$. Otherwise, we let $\mathcal{K}(i) = \{i\}$. It is easy to observe that for any node $s \in \mathcal{K}(i)$ with $s \neq i$, there exists a unique shortest path to connect nodes s and i , and we define the distance between nodes s and i , denoted as $dist(s, i)$, as the number of edges on this unique path, i.e., the number of nodes on this unique path minus one. For example, in Figure 3, $dist(s, i) = 4$. We consider two points $(\bar{x}^1, \bar{y}, \bar{u})$ and $(\bar{x}^2, \bar{y}, \bar{u})$ such that $\bar{x}_r^1 = \bar{x}_r + \epsilon$ for $r \in \mathcal{K}(i)$, $\bar{x}_r^2 = \bar{x}_r - \epsilon$ for $r \in \mathcal{K}(i)$, and $\bar{x}_r^1 = \bar{x}_r^2 = \bar{x}_r$ for $r \notin \mathcal{K}(i)$, where ϵ is an arbitrarily small positive number.

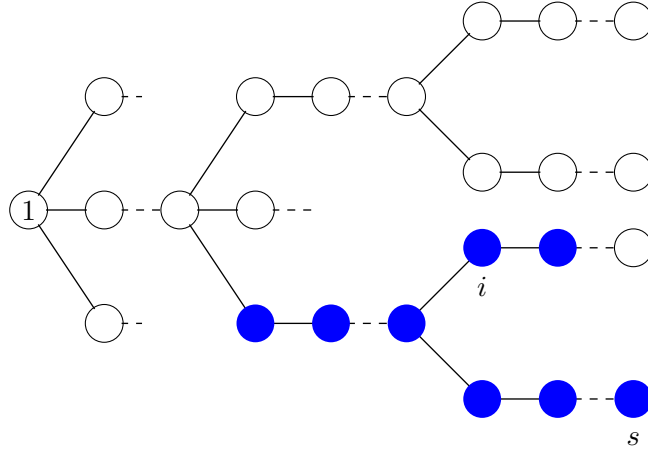


Figure 3: Subtree representation

Now we show these two points constructed are feasible for $\text{conv}(\mathcal{W})$ by considering the following three possible cases.

- (1) If there exists some boundary node n of $\mathcal{K}(i)$ such that $\bar{x}_n = 0$ with $m = n^-$ or $n = m^-$, then we have $\underline{C} < \bar{x}_n < \bar{V}$. Otherwise, 1) If $\bar{x}_n = \underline{C}$, it follows that $\bar{x}_i = \underline{C} + kV$ for some $k \in [0, dist(n, i)]_{\mathbb{Z}}$ by definition. Since $\bar{x}_i \notin \hat{\mathcal{Q}}$, it further follows that $k \geq \alpha_1 + 1 = \min\{2T, \lfloor (\bar{C} - \underline{C})/V \rfloor\} + 1$, which contradicts to the fact that $k \leq 2T$ and $\bar{x}_i < \bar{C}$. 2) If $\bar{x}_n = \bar{V}$, it follows that $\bar{x}_i = \bar{V} + kV$ for some $k \in [0, dist(n, i)]_{\mathbb{Z}}$ by definition. Since $\bar{x}_i \notin \hat{\mathcal{Q}}$, it further follows that $k \geq \alpha_2 + 1 = \min\{2T, \lfloor (\bar{C} - \bar{V})/V \rfloor\} + 1$, which contradicts to the fact that $k \leq 2T$ and $\bar{x}_i < \bar{C}$. Therefore, it is feasible to increase or decrease \bar{x}_n by ϵ .
- (2) If there exists some boundary node n of $\mathcal{K}(i)$ such that $\bar{x}_n > 0$ with $m = n^-$ or $n = m^-$, then we have $\underline{C} < \bar{x}_n < \bar{C}$. Otherwise, 1) If $\bar{x}_n = \underline{C}$, we can similarly show the contradiction as above. 2) If $\bar{x}_n = \bar{C}$, it follows that $\bar{x}_i = \bar{C} - kV$ for some $k \in [0, dist(n, i)]_{\mathbb{Z}}$ by definition.

Since $\bar{x}_i \notin \hat{\mathcal{Q}}$, it further follows that $k \geq \alpha_1 + 1 = \min\{2T, \lfloor (\bar{C} - \underline{C})/V \rfloor\} + 1$, which contradicts to the fact that $k \leq 2T$ and $\bar{x}_t > \underline{C}$. Therefore, it is feasible to increase or decrease \bar{x}_n by ϵ since $|\bar{x}_n - \bar{x}_m| < V$ by definition.

- (3) If there does not exist node $m \in \mathcal{V}$ with $m = n^-$ or $n = m^-$ for some boundary node n of $\mathcal{K}(i)$, i.e., $t(n) \in \{1, T\}$, then similarly we can follow the arguments above to show that $\underline{C} < \bar{x}_n < \bar{C}$. It follows that it is feasible to increase or decrease \bar{x}_n by ϵ .

In summary, we show that in all cases it is feasible to increase or decrease \bar{x}_n by ϵ for each boundary node n of $\mathcal{K}(i)$ and thus feasible to increase or decrease \bar{x}_r by ϵ for all $r \in \mathcal{K}(i)$. It follows that both $(\bar{x}^1, \bar{y}, \bar{u})$ and $(\bar{x}^2, \bar{y}, \bar{u})$ are feasible points of $\text{conv}(\mathcal{W})$, and $(\bar{x}, \bar{y}, \bar{u}) = \frac{1}{2}(\bar{x}^1, \bar{y}, \bar{u}) + \frac{1}{2}(\bar{x}^2, \bar{y}, \bar{u})$. Therefore, $(\bar{x}, \bar{y}, \bar{u})$ is not an extreme point of $\text{conv}(\mathcal{W})$, which is a contradiction. ■

Now, we begin to characterize the optimality condition for Problem (24). Similar to the deterministic unit commitment problem, $f_i(x_i, y_i) = ax_i^2 + bx_i + cy_i - q_i x_i$ is often approximated by a K -piece piecewise linear function $\varphi_i = f_i(x_i, y_i) \geq \mu_k^i x_i + \nu_k y_i, \forall 1 \leq k \leq K$, where $\mu_k^i = 2a\tilde{x}_i + b - q_i$ and $\nu_k = c - a\tilde{x}_i$ with \tilde{x}_i is the x -value corresponding to the k -th supporting node on the curve of $f_i(x_i, y_i)$ at node i and $\tilde{x}_1 = \underline{C}$, $\tilde{x}_K = \bar{C}$. Therefore Problem (24) can be reformulated as

$$\begin{aligned} \min \quad & \sum_{i \in \mathcal{V}} \bar{U} u_i + \underline{U}(y_{i^-} - y_i + u_i) + \varphi_i \\ \text{s.t.} \quad & (x, y, u) \in \mathcal{W}. \\ & \varphi_i \geq \mu_k^i x_i + \nu_k y_i, \forall k \in [1, K]_{\mathbb{Z}}, \forall i \in \mathcal{V}. \end{aligned} \quad (25)$$

Note that the cost function φ_i is a linear function if there is only one piece, i.e., $K = 1$. It is easy to observe that any two adjacent pieces at node i , e.g., $\varphi_i \geq \mu_k^i x_i + \nu_k y_i$ and $\varphi_i \geq \mu_{k+1}^i x_i + \nu_{k+1} y_i$, intersect at $A_k = (\tilde{x}_k + \tilde{x}_{k+1})/2$. Therefore, we can obtain $K - 1$ turning points with x -value $A_k, \forall k \in [1, K - 1]_{\mathbb{Z}}$ on the K -piece piecewise linear function for each node. We denote $\hat{\chi}_k = \{n \in [1, 2T]_{\mathbb{Z}} : \underline{C} \leq A_k + nV \leq \bar{C}\}$ and $\mathcal{Q}_s = \hat{\mathcal{Q}} \cup \{(A_k + nV)_{n \in \hat{\chi}_k}, \forall k \in [1, K - 1]_{\mathbb{Z}}\}$.

Proposition 6 *Problem (24) has at least one optimal solution $(\bar{x}, \bar{y}, \bar{u})$ with $\bar{x}_i \in \mathcal{Q}_s$ for all $i \in \mathcal{V}$.*

Proof: By contradiction. Suppose that there exists some $i \in \mathcal{V}$ such that $\bar{x}_i \notin \mathcal{Q}_s$ for the optimal solution $(\bar{x}, \bar{y}, \bar{u})$ of Problem (24), i.e., $\bar{x}_i \in (\underline{C}, \bar{C}) \setminus \mathcal{Q}_s$, with the optimal value $\bar{z} = \sum_{i \in \mathcal{V}} \bar{U} \bar{u}_i + \underline{U}(\bar{y}_{i^-} - \bar{y}_i + \bar{u}_i) + \bar{\varphi}_i$. In the following we construct a feasible solution to obtain a better objective value. Following the proof in Proposition 5, we construct a subtree $\mathcal{K}(i)$ as shown

in Figure 3 and we can always construct two feasible points of \mathcal{W} , $(\bar{x}^1, \bar{y}, \bar{u})$ and $(\bar{x}^2, \bar{y}, \bar{u})$, such that $\bar{x}_r^1 = \bar{x}_r + \epsilon$ for $r \in \mathcal{K}(i)$, $\bar{x}_r^2 = \bar{x}_r - \epsilon$ for $r \in \mathcal{K}(i)$, and $\bar{x}_r^1 = \bar{x}_r^2 = \bar{x}_r$ for $r \notin \mathcal{K}(i)$, where ϵ is an arbitrarily small positive number. Similar to the proof in Proposition 3, we can increase or decrease \bar{x}_r by ϵ for $r \in \mathcal{K}(i)$ to decrease the optimal value by at least $|\sum_{r \in \mathcal{K}(i)} \mu_{k_r}^r| \epsilon$ and the resulting solution, $(\bar{x}^1, \bar{y}, \bar{u})$ or $(\bar{x}^2, \bar{y}, \bar{u})$, is feasible for both \mathcal{W} and constraints (25), where there is at most one piece of linear function of constraints (25), e.g., piece k_r , is tight for each node $r \in \mathcal{K}(i)$. Therefore we obtain the contradiction. ■

In other words, in order to find an optimal solution to the multistage stochastic UC problem, we only need to consider the feasible solutions (x, y, u) where $x_i \in \mathcal{Q}_s$ for all $i \in \mathcal{V}$.

4.2 An $\mathcal{O}(N)$ Time Dynamic Programming Algorithm

As \mathcal{Q}_s is a finite set and the cardinality of \mathcal{Q}_s , denoted as \aleph_s , does not depend on the total number of nodes in the scenario tree, rather than solve the MILP model (24), we can again explore the backward induction dynamic programming framework by reusing the directed graph we defined in Section 3.2. More specifically, we use a similar state space \mathcal{S} except that the optimal generation candidate set \mathcal{Q}_d there is replaced by \mathcal{Q}_s here, and the same state-decision relationship as shown in Figure 1.

Now we are ready to establish the dynamic programming framework. Let $F_m(i)$ represent the optimal value function for node $m \in \mathcal{V}$ in the scenario tree considering state i as the state of m^- in the scenario tree. Based on Proposition 6, an optimal decision for current scenario lies in $S(i)$. The Bellman equations can be formulated as follows:

$$F_m(i) = \min_{j \in S(i)} E_{mij} + \sum_{n \in C_*(m)} F_n(j), \quad \forall i \in \mathcal{S}, \forall m \in \mathcal{V}, \quad (26)$$

where node m , replacing the stage t in (18), represents a scenario tree node. $E_{mij} = p_m(\overline{U}j_u + \underline{U}(i_y - j_y + j_u) + f_m(j_x, j_y))$ describes the total generation cost minus the revenue at node $m \in \mathcal{V}$, including start-up cost $\overline{U}j_u$ and shut-down cost $\underline{U}(i_y - j_y + j_u)$. Moreover, probability p_m is incorporated in each parameter E_{mij} . Note here that when m is a leaf node, $C_*(m) = \emptyset$ and correspondingly $\sum_{n \in C_*(m)} F_n(j) = 0$. The objective is to find out the value of $F_1(1)$ where the 1 in the subscript represents the root node in the scenario tree while the 1 in the bracket represents the ‘‘source’’ node in the state space.

In order to obtain the optimal objective value and optimal solution for the scenario-tree based multistage stochastic unit commitment problem, we need to calculate $F_m(i)$ for all possible m and

i and record the optimal candidate for them. To calculate the value of each optimal value function $F_m(i)$ in the Bellman equations (26), we search among the candidate solution $j \in S(i)$ and this step takes $\mathcal{O}(\aleph_s |\mathcal{C}_*(m)|)$ time. Since there are in total $\aleph_s L + \ell$ number of nodes in the state graph, the computational time at each node m is $\mathcal{O}(\aleph_s (\aleph_s L + \ell) |\mathcal{C}_*(m)|)$. Thus, the total time to calculate the value of objective $F_1(1)$ is $\mathcal{O}(\aleph_s (\aleph_s L + \ell) N)$, where N represents the total number of nodes of the scenario tree. The optimal solution for UC can be obtained by tracing the optimal candidate for the optimal value function from $F_1(1)$, and this takes $\mathcal{O}(N)$ time in total. Because \aleph_s, L, ℓ are constant numbers with respect to the physical parameters of UC problem, we conclude that our backward induction dynamic programming algorithm for the stochastic unit commitment problem is an $\mathcal{O}(N)$ time algorithm. Similarly, when start-up profile is considered, we have the following observations.

Remark 3 *If the start-up profile is taken into account in the UC model (24), then we need to extend the upper bound of the offline duration variable d from ℓ to T , which as a result will increase the computational complexity from $\mathcal{O}(N)$ time to $\mathcal{O}(N^2)$ time.*

4.3 Extended Formulation for Stochastic Unit Commitment with Piecewise Linear Cost Function

Following the same approach as that in Section 3.3, we develop an extended formulation in linear program form for multistage stochastic unit commitment problem, which is proved to provide integral solutions. By incorporating the Bellman equations (26) as constraints, we can derive the following primal linear program similarly:

$$\max \quad F_1(1) \tag{27a}$$

$$\text{s.t.} \quad F_m(i) \leq E_{mij} + \sum_{n \in \mathcal{C}_*(m)} F_n(j), \quad \forall i \in \mathcal{S}, j \in S(i), \forall m \in \mathcal{V}, \tag{27b}$$

where E_{mij} are parameters defined under equations (26) and $F_m(i)$ are decision variables.

Similar to the deterministic case in Section 18, here the primal linear program cannot provide solutions to the UC problem directly either. Thus, we also resort to the dual formulation and then provide an extended linear program, which we show can provide solution to the stochastic UC problem directly.

The dual formulation can be derived accordingly as follows:

$$\min \quad \sum_{m \in \mathcal{V}, i \in \mathcal{S}, j \in S(i)} E_{mij} w_{mij} \tag{28a}$$

$$\text{s.t.} \quad \sum_{j \in S(1)} w_{11j} = 1, \quad (28b)$$

$$\sum_{j \in S(i)} w_{1ij} = 0, \quad \forall i \in \mathcal{S} \setminus \mathcal{S}_*, \quad (28c)$$

$$\sum_{j \in S(i)} w_{mij} - \sum_{k \in P(i)} w_{m-,ki} = 0, \quad \forall i \in \mathcal{S}, \forall m \in \mathcal{V} \setminus \mathcal{V}_*, \quad (28d)$$

$$w_{mij} \geq 0, \quad \forall i \in \mathcal{S}, j \in S(i), \forall m \in \mathcal{V}, \quad (28e)$$

where w_{mij} are dual variables corresponding to each constraint in the primal linear program (27).

In the following, we first prove in a similar way to show that an extended formulation in linear form can be developed that provides integral solutions to the stochastic UC problem (24).

Lemma 3 *The extreme points of the polytope (28b) – (28e) are binary.*

Proof: The proof is similar to that in Lemma 2. For any $E_{mij} \in (-\infty, +\infty)$, we prove there exists at least one optimal solution to the dual program (28) that is binary. By solving the dynamic program for the stochastic UC model, we can obtain an optimal decision of (x, y, u) . We then construct \hat{w} , which is binary, to represent the optimal solution. We let $\hat{w}_{mij} = 1$ if on the scenario tree node m the optimal dynamic programming solution shows the decision from state i to state j in the state space. Otherwise, we let $\hat{w}_{mij} = 0$.

We can prove that the constructed \hat{w} is feasible to the dual program by verifying that it satisfies constraints (28b) – (28e). The optimality can also be proved following the same approach in Lemma 2. As \hat{w} is binary and optimal to the dual program any possible cost coefficient E , we have proved our claim. ■

Next, we show that an optimal solution to the stochastic UC problem (24) can be obtained in terms of the dual variables.

Proposition 7 *If w^* is an optimal solution to dual program (28), then*

$$x_m^* = \sum_{i \in \mathcal{S}, j \in S(i)} j_x w_{mij}^*, y_m^* = \sum_{i \in \mathcal{S}, j \in S(i)} j_y w_{mij}^*, u_t^* = \sum_{i \in \mathcal{S}, j \in S(i)} j_u w_{mij}^*, \quad m \in \mathcal{V}, \quad (29)$$

is an optimal solution to the stochastic UC problem (24).

Proof: The proof is similar to that in Proposition 4. The feasibility is satisfied because, from the proof of Proposition 4, for each scenario we can consider the multistage decision as a path in the directed graph and thus it satisfies all the physical and logical constraints. The optimality can be

proved by replacing the decision x^*, y^*, u^* with the corresponding expressions in equations (29) in the objective function and verify that

$$\sum_{m \in \mathcal{V}} \bar{U}u_m^* + \underline{U}(y_{m-}^* - y_m^* + u_m^*) + f_m(x_m^*, y_m^*) = \sum_{m \in \mathcal{V}, i \in \mathcal{S}, j \in \mathcal{S}(i)} E_{mij}w_{mij}^*. \quad (30)$$

Hence, (x^*, y^*, u^*) is the optimal solution to the stochastic UC problem. ■

Now we are ready to establish the extended formulations for stochastic unit commitment problem. We replace constraints (24b) – (24i) with constraints (28b) – (28e) and add equations (29) to represent the relation between original decisions and the dual decision variables.

Theorem 3 *The extended formulation of the stochastic UC problem (24) can be written as follows:*

$$\max \quad \sum_{i \in \mathcal{V}} \bar{U}u_i + \underline{U}(y_{i-} - y_i + u_i) + f_i(x_i, y_i) \quad (31a)$$

$$s.t. \quad x_m = \sum_{i \in \mathcal{S}, j \in \mathcal{S}(i)} j_x w_{mij}, y_m = \sum_{i \in \mathcal{S}, j \in \mathcal{S}(i)} j_y w_{mij},$$

$$u_m = \sum_{i \in \mathcal{S}, j \in \mathcal{S}(i)} j_u w_{mij}, \quad \forall m \in \mathcal{V}, \quad (31b)$$

$$(28b) - (28e), \quad (31c)$$

and if (x^*, y^*, u^*, w^*) is an optimal solution to the extended formulation, then (x^*, y^*, u^*) is an optimal solution to the stochastic UC problem.

Proof: The proof is similar to that in Theorem 2. The conclusion holds immediately by replacing (x^*, y^*, u^*) in the objective function with the expressions in (29) and following the proof described in Proposition 7 and the conclusion described in Lemma 3. ■

5 Conclusion

In this paper, efficient dynamic programming algorithms and linear program reformulations were proposed to solve the deterministic and stochastic unit commitment problems. We started with deriving a more efficient dynamic programming algorithm to solve the deterministic unit commitment problem with general convex cost function. Our proposed algorithm refines a previous work by enhancing the computational time for unit commitment from $\mathcal{O}(T^3)$ time to $\mathcal{O}(T^2)$ time when economic dispatch problems are solved in advance. Motivated by this, we obtained an efficient extended reformulation in a higher dimensional space that can provide integral solutions. In addition, for the most common cases in which piecewise linear cost function is taken, by exploiting

the optimality condition for the deterministic UC problem, we proposed a more efficient dynamic programming algorithm that runs in $\mathcal{O}(T)$ time and furthermore, our study was adapted to solve the stochastic unit commitment problem with a scenario tree in $\mathcal{O}(N)$ time by also deriving the corresponding optimality condition. Extended formulations were further derived for both deterministic and stochastic UC problems and integral solutions for both of them were provided in a similar way.

Our studies provide efficient polynomial time algorithms for a class of unit commitment problems, especially linear time for certain cases. Furthermore, we provide one of the first studies on deriving extended formulations for various unit commitment problems based on efficient dynamic programming algorithms. Besides solving single generator bidding problems, the proposed polynomial time algorithms and/or extended formulations could have impact on speeding up the MILP and Lagrangian Relaxation approaches.

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