On the polyhedrality of closures of multi-branch split sets and other polyhedra with bounded max-facet-width

Sanjeeb Dash
IBM Research
sanjeebd@us.ibm.com

Oktay Günlük
IBM Research
gunluk@us.ibm.com

Diego A. Morán R.
School of Business, Universidad Adolfo Ibañez
diego.moran@uai.cl

August 2, 2016

Abstract

For a fixed integer $t > 0$, we say that a t-branch split set (the union of t split sets) is dominated by another one on a polyhedron P if all cuts for P obtained from the first t-branch split set are implied by cuts obtained from the second one. We prove that given a rational polyhedron P, any arbitrary family of t-branch split sets has a finite subfamily such that each element of the family is dominated on P by an element from the subfamily. The result for $t = 1$ (i.e., for split sets) was proved by Averkov (2012) extending results in Andersen, Cornuéjols and Li (2005). Our result implies that the closure of P with respect to any family of t-branch split sets is a polyhedron. We extend this result by replacing split sets with polyhedral sets with bounded max-facet-width as building blocks and show that any family of such sets also has a finite dominating subfamily. This result generalizes a result of Averkov (2012) on bounded max-facet-width polyhedra.

1 Introduction

Cutting planes (or *cuts*, for short) are essential tools for solving general mixed-integer programs (MIPs). *Split cuts* are an important family of cuts for general MIPs, and special cases of split cuts are effective in practice. Cook, Kannan and Schrijver [6] proved that when all split cuts are applied to a rational polyhedron, only finitely many split cuts imply the rest. Therefore, the *split closure* of a rational polyhedron – that is, the set of points in the polyhedron that satisfy all split cuts – is again a polyhedron. Andersen, Cornuéjols and Li [1] showed that the above results hold even when only a subset of all possible splits is used to derive split cuts. Following these papers, the question of whether other more general families of cuts have a finite subset of cuts implying the rest has been studied by various authors.

Andersen, Louveaux and Weismantel [2] generalize the result in [1] and show that, as in the case of split sets, families of polyhedral lattice-free sets with bounded max-facet-width yield polyhedral closures. Split sets are maximal convex lattice-free sets with max-facet-width one. Averkov [3] gave a short proof of a stronger version of this result using the Gordan-Dickson Lemma. He showed that for a given rational polyhedron P, any arbitrary family $\mathcal{L}$ of polyhedra with bounded max-facet-width has a finite subfamily $\mathcal{L}'$ such that for any $L \in \mathcal{L}$, there exists $L' \in \mathcal{L}'$ satisfying $\text{conv}(P \setminus L') \subseteq \text{conv}(P \setminus L)$ (for any set S, $\text{conv}(S)$ stands for the convex hull of S). When the elements of $\mathcal{L}$ are lattice-free sets, this result implies that all cuts for a rational polyhedron P obtained from a set $L \in \mathcal{L}$ are dominated by cuts for P obtained from some $L' \in \mathcal{L}'$.

Basu, Hildebrand and Köeppe [5] showed that the triangle closure (points satisfying cuts obtained from maximal lattice-free triangles) of the two-row continuous group relaxation is a polyhedron. Note that, in the context of the two-row continuous group relaxation, the triangle closure is contained in the split closure. We showed [9] that the quadrilateral closure of the two-row continuous group relaxation is a polyhedron. This result follows from the study of an alternative generalization of split cuts, called multi-branch split cuts. Li and Richard [13] defined *2-branch split cuts*, i.e., cuts that are obtained by considering the union of two split sets simultaneously. These cuts are called *cross cuts* in Dash, Dey and Günlük [7] (a *cross set* is the union of two split sets). In [9] we generalized Averkov's result and showed that given an arbitrary family $\mathcal{C}$ of 2-branch split sets, $\mathcal{C}$ has a finite subfamily $\mathcal{C}'$ such that cuts defined by the 2-branch split sets in $\mathcal{C}'$ imply all cuts defined by 2-branch split sets in $\mathcal{C}$.

In this paper, we first show that given a rational polyhedron P and a family $\mathcal{T}$ of t-branch split sets (i.e., the unions of up to t split sets), there exists a finite subfamily $\mathcal{T}'$ of $\mathcal{T}$ such that all cuts derived from any t-branch split set in $\mathcal{T}$ are dominated by cuts from a single t-branch split set in $\mathcal{T}'$. This result implies that the t-branch split closure of any rational polyhedron is also a polyhedron, generalizing our earlier result [9] for the special case when $t = 2$. Note that even in the case $t = 2$, the new result is stronger than our earlier result in [9] as it shows that all cuts derived from any 2-branch split set are dominated by cuts from a single 2-branch split set that comes from a finite family.

We also generalize Averkov's result [3] to the case where cuts are derived from the union of multiple sets with bounded max-facet-width instead of a single one. This result implies all results mentioned above on split and other bounded max-facet-width sets.

1.1 Preliminaries

In this paper we mostly work with pure-integer sets of the form $P^{IP} = P \cap \mathbb{Z}^n$ where

$$P = \{x \in \mathbb{R}^n \;:\; Ax \leq b\}, \tag{1}$$

and A, and b have m rows. At the end of the paper we extend our results to mixed-integer sets. Furthermore, we assume that all data defining P is rational. We call P the continuous relaxation of P^{IP}. For a given set $X \subseteq \mathbb{R}^n$, we denote its convex hull by $\text{conv}(X)$. A linear inequality $a^T x \leq b$ is called a valid inequality for P^{IP} if it is satisfied by all points in P^{IP} (and therefore by all points in $\text{conv}(P^{IP})$ but not necessarily by all points in P. Consequently, adding valid inequalities for

P^{IP} to the set of inequalities defining P leads to tighter relaxations of P^{IP}. We refer to valid inequalities for P^{IP} as 'cuts' for P.

One possible way to generate valid inequalities for P^{IP} employs *strictly lattice-free sets.* A set $L \subset \mathbb{R}^n$ is called strictly lattice-free if it does not contain any points in $\mathbb{Z}^n$. Given a strictly lattice-free set L, clearly $P^{IP} \cap L = \emptyset$ and consequently, any valid inequality for $P \setminus L$ is also valid for P^{IP}.

The *split set* associated with $(\pi, \beta) \in \mathbb{Z}^n \times \mathbb{Z}$ is defined to be

$$S(\pi, \beta) = \{x \in \mathbb{R}^n : \beta < \pi^T x < \beta + 1\}.$$

Clearly, $S(\pi, \beta)$ is a strictly lattice-free set. Any valid inequality for $\operatorname{conv}(P \setminus S(\pi, \beta))$ is called a *split cut* for P derived from the split set $S(\pi, \beta)$. Many split cuts can be derived from the same split set. Some authors define split cuts in terms of *split disjunctions* instead of split sets. A split disjunction derived from $S(\pi, \beta)$ corresponds to the set

$$\mathbb{R}^n \setminus S(\pi, \beta) = \big\{x \in \mathbb{R}^n : \pi^T x \leq \beta\big\} \cup \big\{\pi^T x \geq \beta + 1\big\}.$$

Let

$$\mathcal{S}^* = \Big\{S(\pi, \beta) : (\pi, \beta) \in \mathbb{Z}^n \times \mathbb{Z}\Big\}$$

denote the family of all split sets.

For a polyhedron P and a family $\mathcal{L}$ of sets in $\mathbb{R}^n$, we define the $\mathcal{L}$-closure of P by

$$\operatorname{Cl}(P, \mathcal{L}) = \bigcap_{L \in \mathcal{L}} \operatorname{conv}(P \setminus L).$$

When $\mathcal{L}$ is a family of strictly lattice-free sets in $\mathbb{R}^n$, the $\mathcal{L}$-closure of P contains P^{IP}. The $\mathcal{S}^*$-closure of P is called the *split closure* of P and is a polyhedron [6]. Furthermore, for any $\mathcal{S} \subseteq \mathcal{S}^*$, it is known that $\operatorname{Cl}(P, \mathcal{S})$ is also polyhedral [1].

A natural generalization of split cuts is obtained by constructing strictly lattice-free sets using multiple split sets simultaneously. A *multi-branch split set* T is defined by $T = \cup_{i=1}^t S_i$ where $t > 0$ is an integer and all $S_i \in \mathcal{S}^*$ for $i = 1, \dots t$. We emphasize that not all split sets defining T have to be distinct. Any inequality valid for $\operatorname{conv}(P \setminus T)$ is called a *multi-branch split cut* for P derived from T. It is known that $\operatorname{conv}(P \setminus T)$ is a polyhedron, for any multi-branch split set T, [9, Lemma 7]. We sometimes call T a *t-branch split set* to emphasize the maximum number of split sets defining it. Let $\mathcal{T}_t^*$ denote the set of all t-branch split sets (for a fixed $t > 0$),

$$\mathcal{T}_t^* = \Big\{ \cup_{i=1}^t S_i : S_i \in \mathcal{S}^* \text{ for } i = 1, \dots, t\Big\}.$$

Multi-branch split cuts were first introduced by Li and Richard [13]. 2-branch split cuts were studied in [7] where they were called *cross cuts*. The set of points in P that satisfy all 2-branch split cuts was shown to be a polyhedron in [9]. In [8] the authors show that every facet-defining inequality for $\operatorname{conv}(P^{IP})$ is a t-branch split cut where $t \leq h(n)$, and $h(n)$ is a constant that depends on the number of integer variables n, but not on the complexity of the data defining P. This result also extends to mixed-integer sets and leads to a finite cutting-plane algorithm to solve mixed-integer programs [8].

It is also possible to generate cuts using lattice-free sets other than split sets and also by taking unions of such sets. In particular, one can use the union of convex lattice-free sets with bounded max-facet-width [2]. We discuss this approach in detail later in Section 4.

1.2 Summary of results

Throughout the paper we work with pure-integer sets of the form P^{IP} and at the end of the paper we describe how our results extend to mixed-integer sets. Our main result in the first part of this paper is that for any t, adding all t-branch split cuts that can be obtained from an arbitrary family of t-branch split sets to the set of inequalities defining P yields a polyhedron. Given an arbitrary $\mathcal{T} \subseteq \mathcal{T}_t^*$, we show that there is a finite $\mathcal{T}' \subseteq \mathcal{T}$ such that for each $T \in \mathcal{T}$, there exists $T' \in \mathcal{T}'$ satisfying

$$\mathrm{conv}(P \setminus T') \subseteq \mathrm{conv}(P \setminus T).$$

A corollary of this result is the fact that $\mathrm{Cl}(P, \mathcal{T})$ is polyhedral as

$$\mathrm{Cl}(P, \mathcal{T}) = \mathrm{Cl}(P, \mathcal{T}'),$$

where $\mathcal{T}'$ is a finite set and $\mathrm{conv}(P \setminus T')$ is a polyhedron for any $T' \in \mathcal{T}_t^*$ [9].

In the second part of the paper we show that the result above also holds if one uses more general sets, namely interiors of polyhedral sets with bounded max-facet-width, instead of split sets. In [2], Andersen, Louveaux, and Weismantel study cuts generated from a family of rational, polyhedral lattice-free sets that subsume split sets. The sets they study have the property that there exists a fixed k such that if $ax \leq b$ is a facet defining inequality for the set with a integral, then $ax \geq b - k$ is valid for the set. It is possible to show that a rational polyhedron satisfies this property provided that its recession cone is the same as its lineality space. As split sets also have bounded max-facet-width (with $k = 1$) our result in the second part of the paper subsumes the one in the first part. We, however, present them separately as the proof of the result in the first part is easier to follow than the one in the second part.

1.3 Our proof technique

We next give an overview of the proof technique for our main result on multi-branch split sets. We start off by expressing any given polyhedron P as the Minkowski sum of a pointed polyhedron Q and a linear space L where Q is contained in the orthogonal complement of L. In an earlier paper ([9, Lemma 7]) we showed that the effect of a multi-branch split set T on P is the same as the effect of a multi-branch set split $T' \subseteq T$ on P where T' is the union of only those split sets in T that are "compatible" with L. Furthermore, we also showed that the effect of T' on P can be computed from the effect of T' on Q. We use this earlier result to argue that it suffices to show the main result for pointed polyhedra only. Our proof for pointed polyhedra uses a double induction, one on the dimension of Q and the other on the number of split sets in the multi-branch split set. The induction on dimension, similar to the split closure proof of Cook, Kannan and Schrijver [6], builds up a finite collection of multi-branch split sets by taking a union of the "important" collections for each face; we then argue that any multi-branch split set which is not dominated by one of these important multi-branch split sets has the property that one of its split sets belongs to a finite family. The induction on the number of split sets uses the fact that the dominance relationship – on Q – between elements of a family of t-branch split sets that all contain a common split set S can be inferred from the dominance relationship of a family of $(t-1)$-branch split sets on $Q \setminus S$, where each element of this new family is obtained from an original t-branch split set after removing S.

Even though we use a number of ideas from the proof of Cook, Kannan and Schrijver, an important technical difference is that we work with arbitrary families of multi-branch split sets, and not all possible ones, and this fact prevents us from assuming that Q is full-dimensional. We prove the extension of our result on multi-branch split sets to multi-branch sets defined by bounded max-facet-width polyhedra using similar but more involved steps.

2 Well-ordered qosets

An important component of our proof technique involves establishing a dominance relationship between the members of an infinite set, namely a family of multi-branch split or bounded max-facet-width sets. Some of the results we use to this end are not specific to cutting plane theory but instead apply to more general sets and ordering relationships among their members. Earlier, Averkov [3] gave a short proof of the fact that the split (and bounded max-facet-width) closure of a rational polyhedron is polyhedral using the Gordan-Dickson Lemma. In this section we study a generalization of this lemma and relate it to cutting planes obtained from lattice free sets.

Consider a polyhedral set P and let $\mathcal{L}$ be a family of strictly lattice-free sets that are not necessarily convex. For example any subset of $\mathcal{T}_t^*$ is such a family. We say that the set $L' \in \mathcal{L}$ *dominates* $L \in \mathcal{L}$ *on* P if

$$\operatorname{conv}(P \setminus L') \subseteq \operatorname{conv}(P \setminus L). \tag{2}$$

In other words, L' dominates L on P when all valid inequalities for P^{IP} that can be derived using L can also be derived using L'. Similarly, we say that $\hat{\mathcal{L}} \subseteq \mathcal{L}$ dominates $\mathcal{L}$ on P (or $\hat{\mathcal{L}}$ is a *dominating subset of* $\mathcal{L}$ *for* P), if each element of $\mathcal{L}$ is dominated by some element of $\hat{\mathcal{L}}$. Note that if $\hat{\mathcal{L}} \subseteq \mathcal{L}$ is a dominating subset of $\mathcal{L}$ for P, then

$$\operatorname{Cl}(P, \mathcal{L}) = \operatorname{Cl}(P, \hat{\mathcal{L}}).$$

Furthermore, if $\hat{\mathcal{L}}$ is finite, then $\operatorname{Cl}(P, \hat{\mathcal{L}})$ is a polyhedral set provided that $\operatorname{conv}(P \setminus L)$ is a polyhedron for each $L \in \hat{\mathcal{L}}$. For instance, $\operatorname{conv}(P \setminus L)$ is polyhedral for all $L \in \mathcal{T}_t^*$ [9].

The subsequent results in this section do not use the fact that P is a polyhedral set or that the elements of $\mathcal{L}$ are lattice-free. Therefore, let Q be a set in $\mathbb{R}^n$ and let $\mathcal{V}$ be a family of sets in $\mathbb{R}^n$. We say that a set $V' \in \mathcal{V}$ dominates $V \in \mathcal{V}$ on Q if

$$\operatorname{conv}(Q \setminus V') \subseteq \operatorname{conv}(Q \setminus V),$$

and $\mathcal{V}' \subseteq \mathcal{V}$ is a dominating subset of $\mathcal{V}$ for Q if each element of $\mathcal{V}$ is dominated (on Q) by some element of $\mathcal{V}'$. We use this concept of domination to define a quasi-order on $\mathcal{V}$. In particular, we define the quasi-ordered set (qoset) $(\mathcal{V}, \preceq_Q)$ with the binary relation $\preceq_Q$ on any $V, V' \in \mathcal{V}$ defined as follows:

$$V' \preceq_Q V \qquad \text{if and only if} \qquad \text{conv}(Q \setminus V') \subseteq \text{conv}(Q \setminus V). \tag{3}$$

Note that the relation $\preceq_Q$ is indeed a quasi-order (also called preorder) as it is (i) reflexive (i.e., $V \preceq_Q V$ for all $V \in \mathcal{V}$), and (ii) transitive (i.e., if $V \preceq_Q V'$ and $V' \preceq_Q V''$, then $V \preceq_Q V''$ for all $V, V', V'' \in \mathcal{V}$). This relation however is not antisymmetric (i.e., $V \preceq_Q V'$ and $V' \preceq_Q V$, does not necessarily imply $V = V'$ for all $V, V' \in \mathcal{V}$) and therefore it is not a partial order. Consequently, $(\mathcal{V}, \preceq_Q)$ is not a partially ordered set (poset) in general.

Definition 1. *Given a qoset $(X, \preceq)$, we say that Y is a* dominating subset *of X if $Y \subseteq X$ and for all $x \in X$, there exists $y \in Y$ such that $y \preceq x$. Furthermore, $(X, \preceq)$ is called* fairly well-ordered *if X' has a finite dominating subset for each $X' \subseteq X$.*

In a recent paper Averkov [3] proved that given a rational polyhedron P and a family of split sets $\mathcal{S} \subseteq \mathcal{S}^*$, there exists a finite subset $\mathcal{S}_f \subseteq \mathcal{S}$ such that each split set in $\mathcal{S}$ is dominated by a split set in $\mathcal{S}_f$. Using the binary relation $\preceq_P$ on pairs of split sets defined in (3), Averkov's result can be rephrased as showing that $(\mathcal{S}^*, \preceq_P)$ is a fairly well ordered qoset.

Let $(X, \preceq)$ be a qoset. The following properties follow directly from the definition of fairly well-ordered sets: If X' is a finite subset of X (including $\emptyset$) then $(X', \preceq)$ is fairly well-ordered. Furthermore, if $(X, \preceq)$ is fairly well-ordered, then for any $X' \subseteq X$, the qoset $(X', \preceq)$ is also fairly well-ordered. Therefore, in the context of t-branch split cuts, if we can show that $(\mathcal{T}_t^*, \preceq_P)$ is fairly well-ordered for a given polyhedron P, it would imply that the t-branch split closure $\text{Cl}(P, \mathcal{T})$ is polyhedral for any $\mathcal{T} \subseteq \mathcal{T}_t^*$.

Given a family of qosets $(X_1, \preceq_1), \ldots, (X_m, \preceq_m)$, the *cardinal product* of these qosets is the qoset $(X_1 \times \ldots \times X_m, \preceq_*)$ defined on the Cartesian product of the ground sets with the usual product order $\preceq_*$ where for $x_i, y_i \in X_i$ for all $i = 1, \ldots, m$, the relation $(x_1, \ldots, x_m) \preceq_* (y_1, \ldots, y_m)$ holds if and only if $x_i \preceq_i y_i$ for all $i = 1, \ldots, m$. See the paper by Higman [11] for more on qosets. He proved the following result on cardinal products of qosets; we give a proof here for completeness sake, as our notion of fairly well-ordered is equivalent to, but stated differently from, his notion of "having the finite basis property".

Lemma 2. *[11, Theorem 2.3] If $(X, \preceq_1)$ and $(Y, \preceq_2)$ are two fairly well-ordered qosets, then so is $(X \times Y, \preceq_*)$, where $\preceq_*$ is the product order of $\preceq_1$ and $\preceq_2$.*

Proof. In the proof we will abbreviate the term "finite dominating subset" by f.d.s. for convenience. Let $(X, \preceq_1)$ and $(Y, \preceq_2)$ be fairly well-ordered qosets. If the result is not true, then $(X \times Y, \preceq_*)$ is not fairly well-ordered which implies that there exists $L^1 \subseteq X \times Y$ such that L^1 does not have an f.d.s. with respect to the order $\preceq_*$. Clearly L^1 is infinite.

Consider the set of X components of the tuples in L^1. By definition, this set has an f.d.s, say $\{u_1, \ldots, u_t\}$ for some $t > 0$, with respect to the order $\preceq_1$. Let $v_1, \ldots, v_t$ be elements in Y such that $(u_i, v_i) \in L^1$. Let $U^0 = \{(u_i, v_i) : i = 1, \ldots, t\}$. Let $U^i = \{(x, y) \in L^1 : u_i \preceq_1 x, v_i \not\preceq_2 y\}$ for $i = 1, \ldots, t$. Let $(\bar{x}, \bar{y})$ be an arbitrary element in L^1. Then, by definition, $u_j \preceq_1 \bar{x}$ for some $j \in \{1, \ldots t\}$. Also, either $v_j \preceq_2 \bar{y}$ – in which case $(u_j, v_j) \preceq_* (\bar{x}, \bar{y})$ – or $(\bar{x}, \bar{y}) \in U^j$. In other words, each element in L^1 is either dominated by an element of U^0 or by an element of U^j (i.e., by the element itself). Therefore, if each U^j has an f.d.s for $j = 1, \ldots, t$, then the union of these dominating subsets and U^0 gives an f.d.s for L^1, a contradiction. Therefore, some U^j for $j = 1, \ldots, t$ does not have an f.d.s. Call this set L^2. We have thus showed that L^1 contains an element $(x_1, y_1) = (u_j, v_j)$ such that

$$L^2 = \{(x, y) \in L^1 : x_1 \preceq_1 x, y_1 \not\preceq_2 y\} \tag{4}$$

has no f.d.s. Using (4) iteratively, we can create two infinite sequences $((x_i, y_i))_{i=1}^{\infty}$ and $(L^i)_{i=1}^{\infty}$ such that for all $i \geq 1$, L^i contains (x_i, y_i) and $L^{i+1} = \{(x, y) \in L^i : x_i \preceq_1 x, y_i \not\preceq_2 y\} \subseteq L^i$. Therefore for all $j > i$, $y_i \not\preceq_2 y_j$. But as the set $\{y_i : i \geq 1\} \subseteq Y$, it has an f.d.s. $Y' = \{y_i : i \in I\}$ for some finite I. Then for any $j > \max(I)$ there is an $i \in I$ such that $y_i \preceq_2 y_j$, a contradiction. ∎

Applying Lemma 2 iteratively, we also conclude that if $(X_i, \preceq_i)$ for $i = 1, \ldots, m$ are fairly well-ordered qosets, then so is $(X_1 \times \ldots \times X_m, \preceq_*)$. We next specialize Lemma 2 to qosets that have a common ground set.

Lemma 3. *If $(X, \preceq_1), \ldots, (X, \preceq_m)$ are fairly well-ordered qosets, then there is a finite set $Y \subseteq X$ such that for all $x \in X$ there exists $y \in Y$ such that $y \preceq_i x$ for all $i = 1, \ldots, m$.*

Proof. Taking $X_i = X$ for $i = 1, \ldots, m$ in Lemma 2, we conclude that $(X \times \ldots \times X, \preceq_*)$ is a fairly well-ordered qoset with the product order $\preceq_*$. Therefore

$$X^= = \{(x, \ldots, x) : x \in X\},$$

which is a subset of $X \times \cdots \times X$, has a finite dominating subset $Y^= \subseteq X^=$. Therefore, there exists a finite set $Y \subseteq X$ such that $Y^= = \{(y, \ldots, y) : y \in Y\}$. Consequently, for all $x \in X$ there exists $y \in Y$ such that

$$y \preceq_i x.$$

for all $i = 1, \ldots, m$. In other words, Y is a finite dominating subset of X for all the quasi-orders $\preceq_i$ simultaneously. ∎

Using Lemma 3 on fairly well-ordered qosets, we next prove a result on dominance relationships in $\mathbb{R}^n$, which we will later relate to multi-branch closures.

Lemma 4. *Let $\mathcal{V}$ be a given family of subsets of $\mathbb{R}^n$ and let $Q_1, \ldots, Q_m \subseteq \mathbb{R}^n$. If $(\mathcal{V}, \preceq_{Q_i})$ is a fairly well-ordered qoset for $i = 1, \ldots, m$, then $(\mathcal{V}, \preceq_{\cup_{i=1}^m Q_i})$ is a fairly well-ordered qoset.*

Proof. Let $\mathcal{V}'$ be an arbitrary subset of $\mathcal{V}$. Then the qoset $(\mathcal{V}', \preceq_{Q_i})$ is fairly well-ordered for all $i = 1, \ldots, m$. Applying Lemma 3 with these qosets, we see that $\mathcal{V}'$ has a finite subset $\mathcal{V}_f$ such that for each V in $\mathcal{V}'$, there is a $\bar{V} \in \mathcal{V}_f$ such that $\bar{V} \preceq_{Q_i} V$ for all $i = 1, \ldots, m$. In other words,

$\text{conv}(Q_i \setminus \bar{V}) \subseteq \text{conv}(Q_i \setminus V)$ for $i = 1, \ldots, m$. This, combined with the fact that for all $U \in \mathcal{V}$ we have $\text{conv}\big((\cup_{i=1}^m Q_i) \setminus U\big) = \text{conv}\big(\cup_{i=1}^m \text{conv}(Q_i \setminus U)\big)$, implies that

$$\begin{aligned}
\text{conv}\big((\cup_{i=1}^m Q_i) \setminus \bar{V}\big) &= \text{conv}\big(\cup_{i=1}^m \text{conv}(Q_i \setminus \bar{V})\big) \\
&\subseteq \text{conv}\big(\cup_{i=1}^m \text{conv}(Q_i \setminus V)\big) = \text{conv}\big((\cup_{i=1}^m Q_i) \setminus V\big).
\end{aligned}$$

Therefore $\mathcal{V}_f$ is a finite dominating subset of $\mathcal{V}'$ for $\cup_{i=1}^m Q_i$. ∎

Given polyhedra $Q_1, \ldots, Q_m$ and a family of multi-branch split sets $\mathcal{T} \subseteq \mathcal{T}_t^*$, we define the closure of $P = \cup_{i=1}^m Q_i$ with respect to $\mathcal{T}$ as follows:

$$\text{Cl}(P, \mathcal{T}) = \bigcap_{T \in \mathcal{T}} \text{conv}\big((\cup_{i=1}^m Q_i) \setminus T\big) = \bigcap_{T \in \mathcal{T}} \text{conv}\big(\cup_{i=1}^m \text{conv}(Q_i \setminus T)\big),$$

Therefore an immediate corollary of Lemma 4 is the following.

Corollary 5. *Given a family of polyhedra, if every $\mathcal{T} \subseteq \mathcal{T}_t^*$ contains a finite dominating set for each polyhedron separately, then each subset of $\mathcal{T}_t^*$ has a finite dominating set for the union of the polyhedra as well.*

In an earlier paper [9, Theorem 6], we showed that given a finite union of rational polyhedra $P = \bigcup_{i \in K} P_i$ and a family of split sets $\mathcal{S} \subseteq \mathcal{S}^*$, the set $\mathcal{S}$ has a finite dominating set for P. Using the fact that $(\mathcal{S}^*, \preceq_Q)$ is a fairly well ordered qoset for any rational polyhedron Q, our earlier result becomes a corollary of Lemma 4.

3 t-branch closure of polyhedral sets

Our main result in this section is that $\text{Cl}(P, \mathcal{T})$ is polyhedral for any rational polyhedron $P \subseteq \mathbb{R}^n$ and any family of multi-branch split sets $\mathcal{T}$. We first prove the result for pointed polyhedra P, and then extend the result to the general case in Theorem 1. Furthermore, the inductive step necessary to show the result for pointed polyhedra is proved in Lemma 7.

The *lineality space* of a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is the linear space $\{x \in \mathbb{R}^n : Ax = 0\}$, denoted by $\text{ls}(P)$. A nonempty polyhedron P is pointed (i.e., it has an extreme point) if and only if $\text{ls}(P) = \{0\}$. Let $L^\perp$ denote the orthogonal complement of a linear space L. Any polyhedron P can be written as $P = Q + \text{ls}(P)$ where $Q = P \cap \text{ls}(P)^\perp$ is a pointed polyhedron. The operator "+" denotes the usual Minkowski sum of two sets in $\mathbb{R}^n$.

We start off with a result that generalizes a result in Cook, Kannan and Schrijver [6] on the effect of split sets on full-dimensional polyhedra to the effect of t-branch split sets on polyhedra which may not be full-dimensional. For a given t-branch split set T, we use $\text{atoms}(T)$ to denote the family of distinct splits defining it; clearly $|\text{atoms}(T)| \leq t$ and $T = \cup_{S \in \text{atoms}(T)} S$.

Lemma 6. *Let $P \subseteq \mathbb{R}^n$ be a rational pointed polyhedron and let $\mathcal{T}' \subset \mathcal{T}_t^*$ be finite. Let $\mathcal{T} \subseteq \mathcal{T}_t^*$ be such that for each $T \in \mathcal{T}$ there exists a $T' \in \mathcal{T}'$ such that T' dominates T on all facets of P, but not on P. Then there exists a finite set $\mathcal{S} \subseteq \mathcal{S}^*$ with the following property: for each $T \in \mathcal{T}$ there is an $S \in \text{atoms}(T)$ such that $S \cap \text{aff}(P) = S' \cap \text{aff}(P)$ for some $S' \in \mathcal{S}$.*

Proof. First assume that $|\mathcal{T}'| = 1$. In an earlier paper [9, Lemma 14] we presented this result for $t = 2$ and extending it to general $t > 0$ requires very limited changes to the proof of [9, Lemma 14]. Furthermore, Lemma 12 in Section 4 of this paper is a generalization of this result when $|\mathcal{T}'| = 1$.

To extend this result to $|\mathcal{T}'| > 1$ it suffices to consider each $T' \in \mathcal{T}'$ one at a time and apply the first part of the proof with the elements of $\mathcal{T}$ that are dominated by T' on all facets of P but not on P. This gives a finite set $\mathcal{S}_{T'} \subseteq \mathcal{S}^*$ for each $T' \in \mathcal{T}'$. Taking the union of the sets $\mathcal{S}_{T'}$ gives the desired result. ∎

Note that if two sets have the same intersection with the affine hull of P, then they also have the same intersection with P itself. Therefore, in Lemma 6 the condition $S \cap \text{aff}(P) = S' \cap \text{aff}(P)$ implies $S \cap P = S' \cap P$.

We next prove the inductive step necessary for our main result. More precisely, we show that if for any pointed polyhedron, every family of $(t-1)$-branch split sets has a finite dominating subset then the same holds for families of t-branch split sets. For convenience, we define $\mathcal{T}_0^* = \emptyset$ and say that all subsets of $\mathcal{T}_0^*$ have a finite dominating subset for all polyhedra.

Lemma 7. *Let $t \geq 1$ be a given integer. Suppose each $\mathcal{T} \subseteq \mathcal{T}_{t-1}^*$ has a finite dominating subset for any pointed polyhedron. Then each $\mathcal{T} \subseteq \mathcal{T}_t^*$ has a finite dominating subset for any pointed polyhedron.*

Proof. The claim can equivalently be stated as follows: If $(\mathcal{T}_{t-1}^*, \preceq_Q)$ is fairly well-ordered for each pointed polyhedron $Q \subseteq \mathbb{R}^n$, then $(\mathcal{T}_t^*, \preceq_P)$ is fairly well-ordered for each pointed polyhedron $P \subseteq \mathbb{R}^n$. In other words, we will show that for any given polyhedron P and $\mathcal{T} \subseteq \mathcal{T}_t^*$, there is a finite set $\mathcal{T}^f \subseteq \mathcal{T}$, such that for each $T \in \mathcal{T}$, $\text{conv}(P \setminus T') \subseteq \text{conv}(P \setminus T)$ for some $T' \in \mathcal{T}^f$.

If $P \subseteq T$ for some $T \in \mathcal{T}$ the the result trivially follows by setting $\mathcal{T}^f = \{T\}$. We therefore assume that $P \setminus T \neq \emptyset$ for all $T \in \mathcal{T}$. We will next prove that $(\mathcal{T}_t^*, \preceq_P)$ is fairly well-ordered by induction on the dimension of P, which we denote by $\dim(P)$. By definition $\dim(P) \leq n$.

If $\dim(P) = 0$, then P consists of a single point and as $P \setminus T = P$ and we can set $\mathcal{T}^f = \{T\}$, where T is an arbitrary element of $\mathcal{T}$.

Let $\dim(P) > 0$, and assume that for all pointed polyhedron $K \subseteq \mathbb{R}^n$ with $\dim(K) < \dim(P)$, the qoset $(\mathcal{T}_t^*, \preceq_K)$ is fairly well-ordered. Let $F_1, \ldots, F_N$ be the facets of P. Then the qoset $(\mathcal{T}_t^*, \preceq_{F_i})$ is fairly well-ordered for $i = 1, \ldots, N$ and so are the qosets $(\mathcal{T}, \preceq_{F_1}), \ldots, (\mathcal{T}, \preceq_{F_N})$. Lemma 3 implies that there exists a finite set $\hat{\mathcal{T}} \subseteq \mathcal{T}$ with the following property: for all $T \in \mathcal{T}$ there exists $T' \in \hat{\mathcal{T}}$ such that

$$T' \preceq_{F_i} T$$

for all $i = 1, \ldots, N$. In other words, the elements of $\hat{\mathcal{T}}$ are the dominating t-branch split sets for all facets of P simultaneously. Applying Lemma 6 with the finite set $\hat{\mathcal{T}}$ we obtain a finite set $\hat{\mathcal{S}} \subseteq \mathcal{S}^*$ such that any $T \in \mathcal{T}$ is either dominated by some $T' \in \hat{\mathcal{T}}$ on P or one of its atoms has the same intersection with P as an element of $\hat{\mathcal{S}}$.

Consider a fixed $S \in \hat{\mathcal{S}}$ and let

$$\mathcal{T}_S = \{T \in \mathcal{T} : S \cap P = S' \cap P \text{ for some } S' \in \text{atoms}(T)\}.$$

For an arbitrary $T \in \mathcal{T}_S$, let T^- to be the union of the following split sets in atoms(T):

$$\{S' \in \text{atoms}(T) : S' \cap P \neq S \cap P\}.$$

Note that T^- is the union of at most $t-1$ split sets and therefore T^- is a $(t-1)$-branch split set. Then

$$T \cap P = (T^- \cap P) \cup (S \cap P),$$

and therefore $P \setminus T = P \setminus (T^- \cup S)$. Therefore, letting $Q = P \setminus S$, we have

$$P \setminus T = Q \setminus T^-.$$

Note that Q is the union of two polyhedra, say Q_1 and Q_2. By the assumptions of the Lemma, any family of $(t-1)$-branch splits has a finite dominating subset for Q_1 or for Q_2, and therefore Lemma 4 implies that the set $\{T^- : T \in \mathcal{T}_S\}$ has a finite dominating subset for $Q = Q_1 \cup Q_2$. For any member of this dominating subset, there exists a $T \in \mathcal{T}_S$ such that T^- equals this member. Taking one such T for each member of this dominating subset, we construct a finite dominating subset $\mathcal{T}_S^f$ of $\mathcal{T}_S$ for P.

As $T \in \mathcal{T}$ is either dominated by some $T' \in \hat{\mathcal{T}}$ on P, or $T \in \mathcal{T}_S$ for some $S \in \hat{\mathcal{S}}$, we have shown that

$$\hat{\mathcal{T}} \cup \Big(\bigcup_{S \in \hat{\mathcal{S}}} \mathcal{T}_S^f \Big)$$

is a finite dominating subset of $\mathcal{T}$ for P. ∎

We now present the main result of this section.

Theorem 1. *For any rational polyhedron P and $\mathcal{T} \subseteq \mathcal{T}_t^*$ where t is a nonnegative integer, the set $\mathcal{T}$ has a finite dominating subset for P. Consequently,* $\text{Cl}(P, \mathcal{T})$ *is a polyhedron.*

Proof. If P is a pointed polyhedron the result trivially holds for $t = 0$ as $\mathcal{T}_0^* = \emptyset$. Further, using Lemma 7 inductively, the result holds for all integer $t > 0$.

If P is nonpointed, then $\text{ls}(P) \neq \{0\}$. In this case $P = Q + L$, where Q is a pointed polyhedron, $L = \text{ls}(P)$ and $Q \subseteq L^\perp$. Consider some $T \in \mathcal{T}$ such that $T = \cup_{i \in I} S_i$ and let $T^L = \cup_{i \in J} S_i$ where $J = \{i \in I : L \subseteq \text{ls}(S_i)\}$. By [9, Proposition 1] (also see Proposition 9 in this paper) we have

$$\text{conv}(P \setminus T) = \text{conv}(Q \setminus T^L) + L. \tag{5}$$

Let $\mathcal{T}^L = \{T^L : T \in \mathcal{T}\}$ be the family of multi-branch split sets obtained from $\mathcal{T}$ by deleting split sets whose lineality space does not contain L. Clearly every multi-branch split in $\mathcal{T}^L$ is a t-branch split set as it is a union of up to t split sets. As Q is a pointed polyhedron, the first part of the proof implies that $\mathcal{T}^L$ has a finite dominating subset $\mathcal{T}_f^L$ for Q. Furthermore, for any $T_1, T_2 \in \mathcal{T}$, it holds that

$$\text{conv}(Q \setminus T_1^L) \subseteq \text{conv}(Q \setminus T_2^L) \Rightarrow \text{conv}(P \setminus T_1) \subseteq \text{conv}(P \setminus T_2).$$

Consequently, $\mathcal{T}$ has a finite dominating subset $\mathcal{T}_f$ for P. In fact, one can construct $\mathcal{T}_f$ by choosing, for each $\hat{T} \in \mathcal{T}_f^L$, one t-branch split $T \in \mathcal{T}$ such that $T^L = \hat{T}$.

This also implies that

$$\mathrm{Cl}(P,\mathcal{T}) = \mathrm{Cl}(P,\mathcal{T}_f).$$

■

For a given a linear subspace L of $\mathbb{R}^n$, and $A, B \subset L^\perp$ it is well-known that

$$(A+L)\cap(B+L) = (A\cap B) + L,$$

and consequently, (5) implies that

$$\mathrm{Cl}(P,\mathcal{T}) = \mathrm{Cl}(Q,\mathcal{T}_f^L) + \mathrm{ls}(P)$$

where $Q = P \cap \mathrm{ls}(P)^\perp$ and $\mathcal{T}_f^L$ is a finite subset of t-branch sets such that for each $T \in \mathcal{T}_f^L$ $\mathrm{atoms}(T) \subseteq \mathrm{atoms}(T')$ for some $T' \in \mathcal{T}$

4 Extension to bounded max-facet-width polyhedra

In this section, we extend Theorem 1 by showing that it holds in a more general setting where we replace split sets with polyhedral sets with bounded max-facet-width. As a corollary we observe that polyhedral lattice-free sets with bounded max-facet-width yield polyhedral closures. Our result also generalizes results in [6],[1],[2],[3],[9].

A set $L \subset \mathbb{R}^n$ is called a lattice-free set if its interior, denoted by L°, does not contain integer points. L might have integer points on its boundary as long as $L^\circ \cap \mathbb{Z}^n = \emptyset$. It is known that any full-dimensional maximal (with respect to inclusion) lattice-free convex set in $\mathbb{R}^n$ is a polyhedron with at most 2^n facets [14]. If $P^{IP} = P \cap \mathbb{Z}^n$ then $P^{IP} \subseteq \mathrm{conv}(P \setminus L^\circ)$ provided that L is a lattice-free set. Furthermore, any facet defining inequality of $\mathrm{conv}(P^{IP})$ can be obtained as a facet of $\mathrm{conv}(P \setminus L^\circ)$ for some polyhedral lattice-free set L, see Jörg [12]. Consequently (polyhedral) maximal lattice-free sets are of interest to generate valid inequalities for P^{IP}. However, we note that not all maximal lattice-free convex sets are rational. Furthermore, even rational ones can have arbitrarily large max-facet-width.

4.1 Bounded max-facet-width polyhedra

Let $L \subsetneq \mathbb{R}^n$ be a full-dimensional rational polyhedron with minimal facet description:

$$L = \{x \in \mathbb{R}^n : a_i^T x \le b_i,\ i \in I\},$$

where $a_i \in \mathbb{Z}^n$, $b_i \in \mathbb{Z}$ and $\gcd(a_i, b_i) = 1$ for all $i \in I$ and I is a finite set. Here we use $\gcd(a_i, b_i)$ to denote the greatest common divisor of b_i and the components of the vector a_i. Given a vector $a \in \mathbb{R}^n$ we define

$$\mathrm{width}(L,a) = \max\{a^T x : x \in L\} - \min\{a^T x : x \in L\}$$

if both the maximum and minimum in the expression above are finite, or $+\infty$ otherwise. The *max-facet-width* of L is defined as

$$\operatorname{maxfw}(L) = \max_{i\in I}\{\operatorname{width}(L, a_i)\}.$$

Observe that the max-facet-width of L is finite if and only if its recession cone equals its lineality space. We let L° stand for the interior of L. Clearly $L^\circ = \{x \in \mathbb{R}^n : a_i^T x < b_i,\ i \in I\}$.

In [3], Averkov uses a metric $\mathbf{m}(L)$ closely related to $\operatorname{maxfw}(L)$. Just as in the case of max-facet-width, if the recession cone of L is different from the lineality space of L, then $\mathbf{m}(L)$ is defined as $\mathbf{m}(L) = +\infty$. If, on the other hand, $\operatorname{rec}(L) = \operatorname{ls}(L)$, then $\mathbf{m}(L)$ is defined as the minimal value $k \in \mathbb{Z}_+$ such that L can be written as

$$L = \{x \in \mathbb{R}^n : d_i \le c_i^T x \le d_i + k,\ i \in I\}, \tag{6}$$

where $|I| \in \mathbb{Z}_+$, and $c_i \in \mathbb{Z}^n$, $d_i \in \mathbb{Z}$ for all $i \in I$. In other words, $\mathbf{m}(L)$ is the smallest *integer* k such that L can be expressed as the intersection of strips (defined by integer coefficients) of width k. It is easy to see that

$$\mathbf{m}(L) = \lceil \operatorname{maxfw}(L) \rceil.$$

In the rest of the paper, we will use Averkov's definition of max-facet-width. Now let $\mathcal{L}$ be a family of full-dimensional rational polyhedra in $\mathbb{R}^n$, then $\mathbf{m}(\mathcal{L})$ is defined as follows:

$$\mathbf{m}(\mathcal{L}) = \sup_{L\in\mathcal{L}} \mathbf{m}(L).$$

We emphasize that if $\mathbf{m}(\mathcal{L}) < \infty$, it not only means that $\operatorname{rec}(L) \ne \operatorname{ls}(L)$ for all $L \in \mathcal{L}$, but it also means that one cannot construct a sequence $\{L_1, L_2, \ldots\} \subseteq \mathcal{L}$ in such a way that $\mathbf{m}(L_{j+1}) > \mathbf{m}(L_j)$ for all $j > 1$. For example, if $\mathcal{L}$ is a finite family of so-called rational type-2 triangles [10] in $\mathbb{R}^2$, then $\mathbf{m}(\mathcal{L})$ is finite, whereas if $\mathcal{L}$ is the family of all rational type-2 triangles in $\mathbb{R}^2$, then $\mathbf{m}(\mathcal{L}) = \infty$ even though every member of $\mathcal{L}$ is a bounded set with finite max-facet-width. Note that $\mathbf{m}(\bar{S}) = 1$ for any split set $S \in \mathcal{S}^*$.

4.2 t-branch bounded max-facet-width polyhedra

In Section 3, we worked with subsets of $\mathcal{T}_t^*$, where $\mathcal{T}_t^*$ is constructed by taking unions of simple strictly lattice-free sets, namely split sets. In this section, instead of using split sets as our building blocks, we use interiors of polyhedral sets with max-facet-width bounded by a given constant. Note that if S is a split set such that $S = \{x \in \mathbb{R}^n : \beta < \pi^T x < \beta + 1\}$, where π and β are integral, then $\operatorname{maxfw}(\bar{S}) \le 1$, where $\bar{S}$ denotes the topological closure of S. In addition, if $\gcd(\pi) = 1$ then $\operatorname{maxfw}(\bar{S}) = 1$. Also, note that if $\gcd(\pi) > 1$, then S is strictly contained in another split set, namely,

$$S' = \left\{x \in \mathbb{R}^n : \left\lfloor \frac{\beta}{\gcd(\pi)} \right\rfloor < \frac{1}{\gcd(\pi)} \pi^T x < \left\lfloor \frac{\beta}{\gcd(\pi)} \right\rfloor + 1 \right\}.$$

For a given integer $l \in \mathbb{Z}_+$, let $\mathcal{M}_l$ be the family of interiors of all rational, full-dimensional, polyhedra with max-facet-width at most l:

$$\mathcal{M}_l = \left\{P^\circ \subset \mathbb{R}^n \ :\ P \text{ is a rational polyhedron with } \dim(P) = n \text{ and } \mathbf{m}(P) \le l\right\}.$$

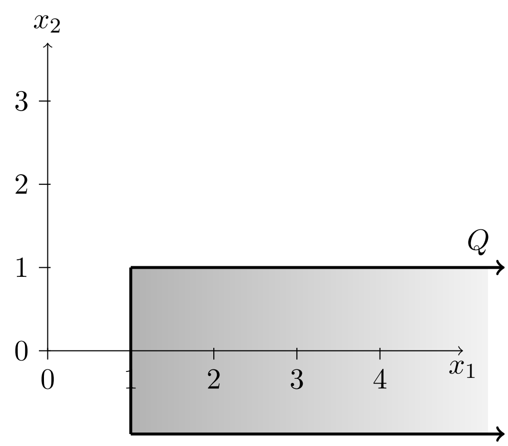
In this section, we will use $\mathcal{T}^*_{t,l}$ to denote t-branch bounded max-facet-width sets:

$$\mathcal{T}^*_{t,l} = \Big\{ T \subset \mathbb{R}^n \, : \, T = \cup_{j=1}^t M_j \text{ where } M_j \in \mathcal{M}_l \text{ for } j = 1, \ldots, t \Big\}.$$

Note that $\mathcal{T}^*_{t,l}$ contains $\mathcal{T}^*_t$ when $l = 1$.

4.3 Polyhedra for which the recession cone equals the lineality space

In this section we generalize basic results on subtracting split sets from a polyhedron in [9] to the case where we subtract the interiors of a family of polyhedral sets with bounded max-facet-width. Recall that any polyhedron with bounded max-facet-width has the property that its recession cone is the same as its lineality space. The following example (see Figure 1) shows that $\text{conv}(P \setminus Q^\circ)$ is sometimes nonpolyhedral if $P, Q \in \mathbb{R}^n$ are two polyhedra and $\text{rec}(Q) \neq \text{ls}(Q)$. Let $P = \mathbb{R}^2_+$ and $Q = \{x \in \mathbb{R}^2 \, : \, x_1 \geq 1, \; 1 \geq x_2 \geq -1\}$, then $\text{conv}(P \setminus Q^\circ) = P \setminus \{x \in \mathbb{R}^2 \, : \, x_1 > 1, \; x_2 = 0\}$, which is not a closed set. On the other hand, the next result shows that when the interiors of finitely many polyhedra for which the recession cone equals the lineality space are subtracted from a polyhedron, the convex hull of the remaining points is a polyhedron. This result is a generalization of [9, Lemma 7] which considers the case when a t-branch split set is subtracted from a polyhedron. It also generalizes a result of Andersen, Louveaux and Weismantel [2, Lemma 2.3, 2.4] who prove the result for the case $m = 1$.



(a)

Figure 1: $\text{conv}(\mathbb{R}^2_+ \setminus Q^\circ)$ is not a polyhedron

Lemma 8. *Let $P \subseteq \mathbb{R}^n$ be a polyhedron. Let m be a positive integer and let P_i be polyhedra where* $\text{rec}(P_i) = ls(P_i)$*, for $i = 1, \ldots, m$. Then,* $\text{conv}\left(P \setminus \left(\bigcup_{i=1}^m P_i^\circ\right)\right)$ *is a polyhedron, which, if nonempty, has the same recession cone as P.*

Proof. If $\mathrm{rec}(P)=\{0\}$ or $P\setminus(\bigcup_{i=1}^m P_i^\circ)=\emptyset$, the result trivially follows. Therefore, we assume that $\mathrm{rec}(P)\neq\{0\}$ and $P\setminus(\bigcup_{i=1}^m P_i^\circ)\neq\emptyset$. We first show that the recession cone of $\mathrm{conv}(P\setminus(\bigcup_{i=1}^m P_i^\circ))$ equals $\mathrm{rec}(P)$. Let $x\in\mathrm{conv}(P\setminus(\bigcup_{i=1}^m P_i^\circ))$. Then, for some $t>0$ we have

$$x=\sum_{j=1}^t \lambda_j x_j \quad \text{where} \quad x_1,\dots,x_t\in P\setminus(\bigcup_{i=1}^m P_i^\circ), \tag{7}$$

$$0\le\lambda_1,\dots,\lambda_t\le 1 \text{ and } \lambda_1+\cdots+\lambda_t=1. \tag{8}$$

Let $d\in\mathrm{rec}(P)$. By definition, for all $j=1,\dots,t$ we have $x_j+\alpha d\in P$ for all $\alpha\ge 0$. Let i be an index in $\{1,\dots,m\}$. Suppose $d\in\mathrm{rec}(P_i)$; by definition $-d\in\mathrm{rec}(P_i)$. Then $x_j+\alpha d\notin P_i^\circ$ for all $\alpha\ge 0$; otherwise, $\hat{x}=x_j+\alpha d\in P_i^\circ\Rightarrow x_j=\hat{x}+\alpha(-d)\in P_i^\circ$, a contradiction to (7). If, on the other hand, $d\notin\mathrm{rec}(P_i)$, then there must exist an $\alpha_{ij}>0$ such that $x_j+\alpha d\notin P_i^\circ$ for all $\alpha\ge\alpha_{ij}$. Consequently, letting $\alpha^*\ge 0$ be an upper bound on all α_{ij}s (which exist only for i such that $d\notin\mathrm{rec}(P_i^\circ)$), we conclude that

$$\sum_{j=1}^t \lambda_j(x_j+\alpha d)=x+\alpha d\in\mathrm{conv}(P\setminus(\bigcup_{i=1}^m P_i^\circ)) \quad \text{for all } \alpha\ge\alpha^*.$$

As x is contained in $\mathrm{conv}(P\setminus(\bigcup_{i=1}^m P_i^\circ))$, so is $x+\alpha d$ for all $\alpha\ge 0$. Therefore $\mathrm{rec}(\mathrm{conv}(P\setminus(\bigcup_{i=1}^m P_i^\circ)))=\mathrm{rec}(P)$.

It is clear that $P\setminus\bigcup_{i=1}^m P_i^\circ$ is the union of a finite family of polyhedra, say $Q_1,\dots,Q_l$ for some $l>0$. Then

$$\begin{aligned}\mathrm{conv}(P\setminus(\bigcup_{i=1}^m P_i^\circ))=\mathrm{conv}(\bigcup_{i=1}^l Q_i) &= \mathrm{conv}(\bigcup_{i=1}^l Q_l)+\mathrm{rec}(P)\\ &= \mathrm{conv}(\bigcup_{i=1}^l Q_i+\mathrm{rec}(P))=\mathrm{conv}(\bigcup_{i=1}^l(Q_i+\mathrm{rec}(P))).\end{aligned}$$

But the last convex hull is a polyhedron as each $Q_i+\mathrm{rec}(P)$ is a polyhedron with the same recession cone $\mathrm{rec}(P)$. The result follows. ∎

We next show that when subtracting multiple polyhedra from a given polyhedron and taking the convex hull of the remainder, it suffices to consider only the polyhedra that contain the lineality space of the given polyhedron. Based on this, we also show that one can simply work with the pointed polyhedron given by projecting the original polyhedron onto the orthogonal complement of its lineality space. This result generalizes [9, Proposition 1] and the proof is similar.

Proposition 9. *Let $P\subseteq\mathbb{R}^n$ be a polyhedron with $P=Q+L$ where L is a linear subspace of $\mathbb{R}^n$ and $Q\subseteq L^\perp$. For $i=1,\dots,m$, let $P_i\subseteq\mathbb{R}^n$ be full-dimensional polyhedra such that* $\mathrm{rec}(P_i)=L_i$, *where L_i is a linear subspace of $\mathbb{R}^n$. Let $I=\{1,\dots,m\}$ and let $J=\{i\in I: L\subseteq L_i\}$. Then*

$$\mathrm{conv}(P\setminus(\bigcup_{i\in I}P_i^\circ))=\mathrm{conv}(P\setminus(\bigcup_{i\in J}P_i^\circ)).$$

Furthermore, if not empty,

$$\mathrm{conv}(P \setminus (\bigcup_{i\in I} P_i^\circ)) = \mathrm{conv}(Q \setminus (\bigcup_{i\in J} P_i^\circ)) + L.$$

Proof. For the first part of the proposition, if $I = J$, there is nothing to prove. Therefore assume $I \neq J$, and consequently $L \supset \{0\}$. The inclusion $\mathrm{conv}(P \setminus (\bigcup_{i\in I} P_i^\circ)) \subseteq \mathrm{conv}(P \setminus \bigcup_{i\in J} P_i^\circ)$ is straighforward.

We next prove that $\mathrm{conv}(P \setminus (\bigcup_{i\in J} P_i^\circ)) \subseteq \mathrm{conv}(P \setminus (\bigcup_{i\in I} P_i^\circ))$. For all $i \in I \setminus J$, as $L \nsubseteq L_i$, we have $L \cap L_i$ is a linear subspace of $\mathbb{R}^n$ with dimension less than that of L. Therefore

$$L \nsubseteq \bigcup_{i\in I\setminus J} L_i.$$

If this were not the case, then

$$L \subseteq \bigcup_{i\in I\setminus J} L_i \Rightarrow L = \bigcup_{i\in I\setminus J} (L \cap L_i),$$

which would imply that L equals the finite union of some sets, each with a lower dimension than that of L, which is not possible. Therefore, there exists some $v_0 \in L \setminus \bigcup_{i\in I\setminus J} L_i$.

Let $x_0 \in P \setminus (\bigcup_{i\in J} P_i^\circ)$. Since $v_0 \in L \setminus \cup_{i\in I\setminus J} L_i$, $x_0 \in P$, $L \subseteq \mathrm{rec}(P)$ and $\mathrm{rec}(P_i) = L_i$, there exists $\alpha > 0$ such that

$$x_0 + \alpha v_0,\ x_0 - \alpha v_0 \in P \setminus P_i^\circ \text{ for all } i \in I \setminus J. \tag{9}$$

Further, for $i \in J$, we have $L \subseteq L_i$ and thus $v_0 \in L_i$. Therefore, since $x_0 \in P$, $L \subseteq \mathrm{rec}(P)$, $x_0 \notin P_i^\circ$, and $\mathrm{rec}(P_i) = L_i$, we obtain

$$x_0 + \alpha v_0,\ x_0 - \alpha v_0 \in P \setminus P_i^\circ, \text{ for all } i \in J. \tag{10}$$

Now, by using (9) and (10) we obtain

$$x_0 + \alpha v_0,\ x_0 - \alpha v_0 \in P \setminus (\bigcup_{i\in I} P_i^\circ). \tag{11}$$

As $x_0 \in \mathrm{conv}(\{x_0 + \alpha v_0, x_0 - \alpha v_0\})$, (11) implies that $x_0 \in \mathrm{conv}(P \setminus (\bigcup_{i\in I} P_i^\circ))$. Therefore,

$$P \setminus \bigcup_{i\in J} P_i^\circ \subseteq \mathrm{conv}(P \setminus \bigcup_{i\in I} P_i^\circ)$$

and we conclude that $\mathrm{conv}(P \setminus \bigcup_{i\in J} P_i^\circ) \subseteq \mathrm{conv}(P \setminus \bigcup_{i\in I} P_i^\circ)$.

For the second part of the proposition, note that since $L \subseteq L_i$ for $i \in J$, we can write $P_i = Q_i + L$

where $Q_i \subseteq L^\perp$ for all $i \in J$. Using this equality, we obtain

$$\begin{aligned}
P \setminus (\bigcup_{i\in J} P_i^\circ) &= (Q+L) \setminus [\bigcup_{i\in J} (Q_i^\circ + L)] \\
&= (Q+L) \cap [\bigcap_{i\in J} \mathbb{R}^n \setminus (Q_i^\circ + L)] \\
&= (Q+L) \cap [\bigcap_{i\in J} ((L^\perp \setminus Q_i^\circ) + L)] \\
&= [Q \cap \bigcap_{i\in J} (L^\perp \setminus Q_i^\circ)] + L \\
&= [Q \setminus (\bigcup_{i\in J} Q_i^\circ)] + L \\
&= [Q \setminus (\bigcup_{i\in J} P_i^\circ)] + L.
\end{aligned}$$

For any two convex sets $A, B \subseteq \mathbb{R}^n$, it is well-known that $\mathrm{conv}(A+B) = \mathrm{conv}(A) + \mathrm{conv}(B)$. Therefore, $\mathrm{conv}(P \setminus (\bigcup_{i\in J} P_i^\circ)) = \mathrm{conv}(Q \setminus (\bigcup_{i\in J} P_i^\circ) + L) = \mathrm{conv}(Q \setminus (\bigcup_{i\in J} P_i^\circ)) + L$, as desired. ▪

The next example illustrates the previous result.

Example 1. *Let $P = \{x \in \mathbb{R}^2 : -1 \leq x_2 \leq 1\}$ with lineality space $L = \{x \in \mathbb{R}^2 : x_2 = 0\}$; here $L^\perp = \{x \in \mathbb{R}^2 : x_1 = 0\}$ and $P = Q + L$ where $Q = \{x \in \mathbb{R}^2 : x_1 = 0, -1 \leq x_2 \leq 1\} \subseteq L^\perp$. In Figure 2(a), we depict P. In Figure 2(b), we show three polyhedra, P_1, P_2, P_3, where P_1 is a polytope, $P_2 = \{x \in \mathbb{R}^2 : 3 \leq x_1 \leq 4\}$ and $P_3 = \{x \in \mathbb{R}^2 : 0.5 \leq x_2 \leq 1.5\}$. The lineality space of P_3 contains L whereas the lineality spaces of P_1 and P_2 do not contain L. We see that $\mathrm{conv}(P \setminus P_1^\circ) = \mathrm{conv}(P \setminus P_2^\circ) = P$, but $\mathrm{conv}(P \setminus P_3^\circ) \subsetneq P$. Furthermore, $\mathrm{conv}(P \setminus (P_1^\circ \cup P_2^\circ \cup P_3^\circ)) = \mathrm{conv}(P \setminus P_3^\circ)$ and this convex hull can be computed as*

$$\mathrm{conv}(Q \setminus P_3^\circ) + L = \{x \in \mathbb{R}^2 : x_1 = 0, -1 \leq x_2 \leq 0.5\} + L.$$

4.4 Main result

In this section we present our main result. The main steps of the proof are similar to the proof of Cook, Kannan and Schrijver [6] which shows that the split closure of a polyhedral set is again polyhedral. Cook, Kannan and Schrijver start off by showing that if a full-dimensional pointed polyhedron strictly contains another such polyhedron, then any inequality that cuts off a vertex of the inner polyhedron must also cut off a part of the outer polyhedron that contains a fixed size ball. Using this technical result, they then argue that if the inequality cutting of the vertex of the inner polyhedron is derived using a split set, then the split set cannot be arbitrarily thin and therefore it must belong to a finite set. In an earlier paper we gave a generalization of the first part of their result by extending it to polyhedra that are not necessarily full-dimensional.

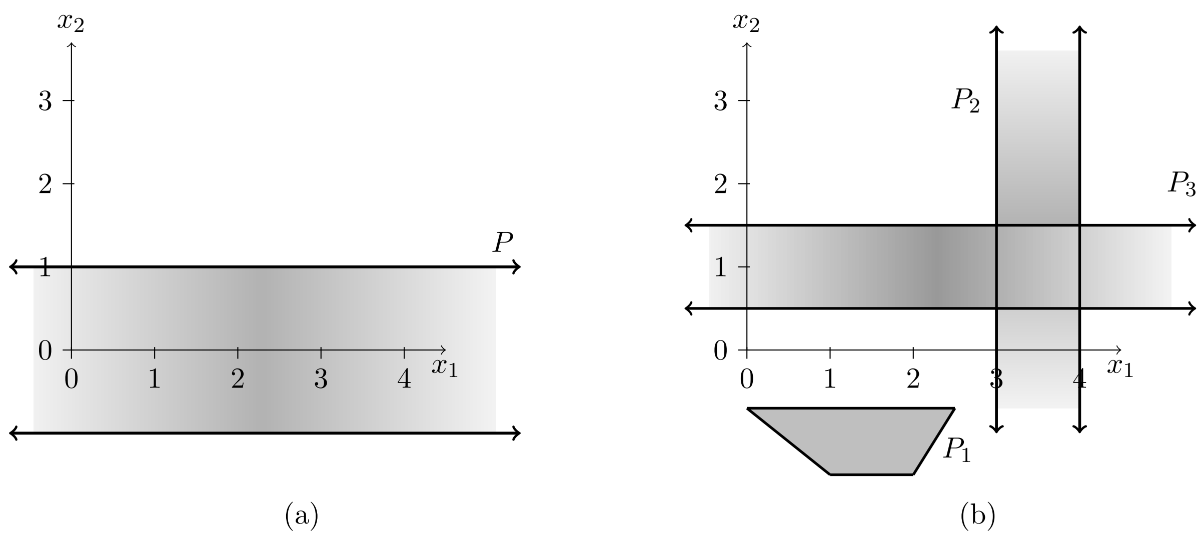


Figure 2: Subtracting the interiors of either P_1 or the split set P_2 from P and convexifying results in P. If P_3 is subtracted, P shrinks.

Lemma 10. *[9, Lemma 14] Let P and W be pointed polyhedra such that $W \subset P$. Then there exists a constant $r > 0$ such that any inequality that cuts off a vertex of W that lies in the relative interior of P excludes a $\dim(P)-$dimensional ball $B \subset P$ of radius r.*

Using this observation, we next show that any finite set of bounded max-facet-width polyhedra that contains a ball of fixed radius must contain at least one polyhedron that is, in some sense, not too thin. We will first show this claim for full-dimensional polyhedra and use it later for the general case.

A *strip* is a full-dimensional polyhedron of the form $Q = \{x \in \mathbb{R}^n : b' \leq a^T x \leq b,\}$, where $0 \neq a \in \mathbb{R}^n$, $b, b' \in \mathbb{R}$ and $b' < b$. The *geometric width* of a strip is defined as

$$\text{geow}(Q) = \frac{b - b'}{\|a\|},$$

and it denotes the Euclidian distance between the hyperplanes $a^T x = b$ and $a^T x = b'$. A split set is a strip with $a \in \mathbb{Z}^n$, and $b, b' \in \mathbb{Z}$ with $b' = b + 1$. Therefore, if Q is a split set, $\text{geow}(Q) = 1/\|a\|$.

Bang's theorem [4] states that if a union of N strips covers a ball of radius $r > 0$, then at least one of the strips S covering the ball satisfies $\text{geow}(S) \geq 2r/N$. This implies that if all the strips covering the ball are split sets, then one of the split sets comes from a finite family that only depends on the ball and on N, see [9]. We next generalize this result from a finite family of splits to a finite family of polyhedra with bounded max-facet-width.

Lemma 11. *Let $l, t \in \mathbb{Z}_+$ and $B \subseteq \mathbb{R}^n$ be a ball of radius $r > 0$. Assume that $B \subseteq \bigcup_{k=1}^{t} P_k^\circ$ where $P_k^\circ \in \mathcal{M}_l$ for all $k = 1, \ldots, t$. Then there exists a finite family of polyhedra $\mathcal{P}$, that depends only on B, t, l, such that $P_{k^*} \in \mathcal{P}$ for some $k^* \in \{1, \ldots, t\}$.*

Proof. Since $\mathbf{m}(P_k) \le l$, $k = 1, \ldots, t$, each polyhedron P_k can be described as a finite intersection of strips (using the description in the definition of the operator $\mathbf{m}(\cdot)$). For each $k = 1, \ldots, t$, let Q_k be the strip that minimizes geow$(\cdot)$ among all the strips in the description of P_k used to compute $\mathbf{m}(P_k)$. As $\bigcup_{k=1}^t P_k \supseteq \bigcup_{k=1}^t P_k^\circ \supseteq B$ and $Q_k \supseteq P_k$ for all $k = 1, \ldots, t$, we obtain that

$$\bigcup_{k=1}^{t} Q_k \supseteq B.$$

Therefore, by Bang's theorem[4], there exists $k^* \in \{1, \ldots, t\}$ such that $P_{k^*} \cap B \neq \emptyset$ and $\text{geow}(Q_{k^*}) \ge 2r/t$.

Since $\mathbf{m}(P_{k^*}) \le l$, we can write P_{k^*} in terms of strips as

$$P_{k^*} = \{x \in \mathbb{R}^n : b_i \le a_i^T x \le b_i + l, i = 1, \ldots, m\},$$

where $a_1, \ldots, a_m \in \mathbb{Z}^n$, $b_1, \ldots, b_m, \in \mathbb{Z}$.

Since $\text{geow}(Q_{k^*}) \ge 2r/t$ and Q_{k^*} is the strip that minimizes geow$(\cdot)$ among all the strips in the description of P_{k^*} used to compute $\mathbf{m}(P_{k^*})$, it follows that

$$\frac{(b_i + l) - b_i}{\|a_i\|} = \frac{l}{\|a_i\|} \ge \text{geow}(Q_{k^*}) \ge \frac{2r}{t} \qquad \text{for all } i = 1, \ldots, m.$$

Therefore, we obtain

$$\|a_i\| \le \frac{lt}{2r} \qquad \text{for all } i = 1, \ldots, m. \tag{12}$$

On the other hand, since $P_{k^*} \cap B \neq \emptyset$, then for all $i = 1, \ldots, m$ we must have

$$\min\{a_i^T x : x \in B\} - l \le b_i \le \max\{a_i^T x : x \in B\}. \tag{13}$$

Since $a_1, \ldots, a_m \in \mathbb{Z}^n$, (12) implies that the coordinates of $a_1, \ldots, a_m$ can only have finitely many values. This fact, along with $b_1, \ldots, b_m, \in \mathbb{Z}$ and (13) implies that $b_1, \ldots, b_m$ can only have finitely many values. Therefore, we conclude that there are only finitely many possibilities for the polyhedron P_{k^*}. Let $\mathcal{P}$ stand for the set of all such polyhedra. Then $\mathcal{P}$ only depends on B, t, l. Furthermore, $|\mathcal{P}| < +\infty$ and $P_{k^*} \in \mathcal{P}$. ∎

Consider an $n \times n$ unimodular matrix U (i.e., an integral matrix with determinant ± 1). Then the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ given by $f(x) = Ux$ is an invertible linear transformation, which we call a *unimodular transformation*. As f is an invertible linear transformation, it follows that for any sets $A, B \subseteq \mathbb{R}^n$, we have $f(\text{conv}(A)) = \text{conv}(f(A))$, $f(\text{aff}(A)) = \text{aff}(f(A))$, and $f(A \setminus B) = f(A) \setminus f(B)$. Furthermore, if P is a polyhedron, then $f(P)$ is a polyhedron and facets of P are mapped to facets of $f(P)$. Equation (6) implies that if $L \in \mathcal{M}_l$ then

$$L = \{x \in \mathbb{R}^n : d_i < c_i^T x < d_i + l, \ i \in I\}$$

where $c_i \in \mathbb{Z}^n$, $d_i \in \mathbb{Z}$ for all $i \in I$. As $c^T x = c^T U^{-1} U x$, it follows that

$$f(L) = \{f(x) : d_i < c^T U^{-1} U x < d_i + l, \ i \in I\} = \{x \in \mathbb{R}^n : d_i < c^T U^{-1} x < d_i + l, \ i \in I\}.$$

As U is unimodular, U^{-1} is also unimodular, and therefore $c^T U^{-1}$ is integral. It follows that $f(L) \in \mathcal{M}_l$. Consequently, for any $T \in \mathcal{T}^*_{t,l}$, $f(T) \in \mathcal{T}^*_{t,l}$. Furthermore, let $T_1, T_2 \in \mathcal{T}^*_{t,l}$ and let $P \subseteq \mathbb{R}^n$ be a polyhedron. If $\text{conv}(P \setminus T_1) \subseteq \text{conv}(P \setminus T_2)$, then $\text{conv}(f(P) \setminus f(T_1)) \subseteq \text{conv}(f(P) \setminus f(T_2))$. Therefore T_1 dominates T_2 on P if and only if $f(T_1)$ dominates $f(T_2)$ on $f(P)$. We will use this observation in the proof of Lemma 12 when P is not full-dimensional. Recall that if $T \in \mathcal{T}^*_{t,l}$, then $T = \cup_{j=1}^t M_j$ for some $M_j \in \mathcal{M}_l$ and we define $\text{atoms}(T) = \{M_1, \ldots, M_t\}$.

Lemma 12. *Let $P \subseteq \mathbb{R}^n$ be a rational pointed polyhedron. Let $T' \in \mathcal{T}^*_{t,l}$ and $\mathcal{T} \subseteq \mathcal{T}^*_{t,l}$ be such that T' dominates each $T \in \mathcal{T}$ on all facets of P, but not on P. Then there exists a finite set $\mathcal{M} \subseteq \mathcal{M}_l$ such that for any $T \in \mathcal{T}$, there exists $M' \in \mathcal{M}$ such that $M \cap \text{aff}(P) = M' \cap \text{aff}(P)$ for some $M \in \text{atoms}(T)$.*

Proof. If $\mathcal{T} = \emptyset$, the claim trivially holds, and so we assume that $\mathcal{T} \neq \emptyset$. If $P \subseteq T' \Rightarrow \text{conv}(P \setminus T') = \emptyset$, then T' dominates all $T \in \mathcal{T}^*_{t,l}$ on P and therefore $\mathcal{T} = \emptyset$, which contradicts the previous assumption. We therefore assume that $\text{conv}(P \setminus T') \neq \emptyset$. Furthermore, for any $T \in \mathcal{T}$, as T is not dominated by T' on P, it follows that $P \cap T \neq \emptyset$.

Suppose P is not full-dimensional and let $k = \dim(P) < n$. Then there exists a unimodular transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ that maps $\text{aff}(P)$ to the set $\{x \in \mathbb{R}^n : x_{k+1} = \alpha_1, \ldots, x_n = \alpha_{n-k}\}$, where $\alpha \in \mathbb{R}^{n-k}$ is a rational vector (see Schrijver [15, Corollary 4.3b]). If T, T' are two elements of $\mathcal{T}^*_{t,l}$, then as discussed prior to this lemma, T' dominates T on P if and only if $f(T')$ dominates $f(T)$ on $f(P)$. Furthermore, the same holds for facets of P. In addition, if $M_1, M_2 \in \mathcal{M}_l$, then $M_1 \cap \text{aff}(P) = M_2 \cap \text{aff}(P)$ if and only if $f(M_1) \cap \text{aff}(f(P)) = f(M_2) \cap \text{aff}(f(P))$. Therefore, without loss of generality, we can assume that $\text{aff}(P) = \{x \in \mathbb{R}^n : x_{k+1} = \alpha_1, \ldots, x_n = \alpha_{n-k}\}$ for some rational $\alpha \in \mathbb{R}^{n-k}$ when $\dim(P) = k < n$.

We will next show the existence of a $\dim(P)$-dimensional ball $B \subset P$ with a radius r that is independent of $\mathcal{T}$ such that $B \subset T$ for all $T \in \mathcal{T}$. Let T be an arbitrary element in $\mathcal{T}$. Suppose $P \subseteq T$. Let B be a ball of some radius $r > 0$ contained in the relative interior of P and having the same dimension as P. Then $B \subseteq T$.

On the other hand, assume $P \not\subseteq T$. As T' does not dominate T on P, there exists a valid inequality $c^T x \leq \mu$ for $\text{conv}(P \setminus T)$ that is not valid for $\text{conv}(P \setminus T')$. As both $\text{conv}(P \setminus T')$ and $\text{conv}(P \setminus T)$ are nonempty, they have the same recession cone as P, see Lemma 8. Therefore the finiteness of μ implies that $c^T d \leq 0$ for all $d \in \text{rec}(\text{conv}(P \setminus T)) = \text{rec}(\text{conv}(P \setminus T'))$. Therefore, $\max\{c^T x : x \in \text{conv}(P \setminus T')\}$ is bounded and has an extreme point solution x^* which violates the inequality $c^T x \leq \mu$.

For any facet F of P it is well-known that $\text{conv}(P \setminus B) \cap F = \text{conv}(F \setminus B)$ for any set B. As T' dominates T on any facet F of P, we have

$$\text{conv}(P \setminus T') \cap F = \text{conv}(F \setminus T') \subseteq \text{conv}(F \setminus T) = \text{conv}(P \setminus T) \cap F.$$

Therefore, $c^T x \leq \mu$ is valid for $\text{conv}(P \setminus T') \cap F$ for any facet F of P. Consequently, x^* cannot be contained in any facet of P, but must be in the relative interior of P. Applying Lemma 10 with $W = \text{conv}(P \setminus T')$, we conclude that there exists a ball B (of radius r for some fixed $r > 0$) in the relative interior of P such that

$$B \subseteq \{x \in P : c^T x > \mu\}.$$

As $c^T x \leq \mu$ is valid for $\operatorname{conv}(P \setminus T)$, the above statement implies that $B \subseteq T$.

Let $T = \cup_{j=1}^t M_j$ for some $M_j \in \mathcal{M}_l$. We have two cases.

Case 1: P is full-dimensional. In this case $\operatorname{aff}(P) = \mathbb{R}^n$, so for any $M \in \mathcal{M}_l$, $M \cap \operatorname{aff}(P) = M$. As $B \subseteq \cup_{i=j}^t M_j$, by Lemma 11 there exists a finite set $\mathcal{M} \subseteq \mathcal{M}_l$ (each element of $\mathcal{M}$ is the interior of an element of $\mathcal{P}$ in Lemma 11, and therefore $\mathcal{M}$ depends only on B, t, l) such that $M_j \in \mathcal{M}$ for some $j \in \{1, \ldots, t\}$.

Case 2: Let $k = \dim(P) < n$. Then $\operatorname{aff}(P) = \{x \in \mathbb{R}^n : x_{k+1} = \alpha_1, \ldots, x_n = \alpha_{n-k}\}$, where $\alpha \in \mathbb{R}^{n-k}$ is a rational vector. Let $\alpha_i = p_i/\Delta$ where p_i is an integer, for $i = 1, \ldots, n-k$ and Δ is a positive integer. Note that as $B \subseteq P$ and B has the same dimension as P, we have $B = \tilde{B} \times \{\alpha\}$, where $\tilde{B} \subseteq \mathbb{R}^k$ is a full-dimensional ball of radius r.

For $\pi \in \mathbb{Z}^n$ we define $\tilde{\pi} \in \mathbb{Z}^k$ to denote the first k components of π and $\hat{\pi} \in \mathbb{Z}^{n-k}$ to denote the remaining components. Then, for any $M_j = \{x \in \mathbb{R}^n : b_{ji} < a_{ji}^T x < b_{ji} + l, i = 1, \ldots, m_j\}$, we have

$$M_j \cap \operatorname{aff}(P) = \tilde{M}_j \times \{\alpha\}, \text{ where } \tilde{M}_j = \{x \in \mathbb{R}^k : b_{ji} - \hat{a}_{ji}^T \alpha < \tilde{a}_{ji}^T x < b_{ji} + l - \hat{a}_{ji}^T \alpha, i = 1, \ldots, m_j\}.$$

As $P \cap T \neq \emptyset$, $P \cap M_j \neq \emptyset$ for some $j \in \{1, \ldots, t\}$. For all such indices j, $\tilde{M}_j \subseteq \mathbb{R}^k$ has dimension k as M_j is a full-dimensional open set. Moreover, by definition of α, for all $j = 1, \ldots, t$, $i = 1, \ldots, m_j$ we have that $\hat{a}_{ji}^T \alpha$ is an integral multiple of $1/\Delta$. Therefore, since we can write

$$\tilde{M}_j = \{x \in \mathbb{R}^k : \Delta b_{ji} - \Delta \hat{a}_{ji}^T \alpha < \Delta \tilde{a}_{ji}^T x < \Delta b_{ji} + l\Delta - \Delta \hat{a}_{ji}^T \alpha, i = 1, \ldots, m_j\},$$

then by definition of the operator $\mathbf{m}(\cdot)$, we conclude that the topological closure of $\tilde{M}_j$ is a polyhedron with max-facet-width at most $l\Delta$.

Since $B \subseteq \cup_{i=j}^t M_j$, we obtain that $\tilde{B} \subseteq \cup_{i=j}^t \tilde{M}_j \subseteq \mathbb{R}^k$. By applying Lemma 11 to the full-dimensional sets $\tilde{M}_j \in \mathbb{R}^k$ for those j such that $P \cap M_j \neq \emptyset$ (as in **Case 1**), it follows that there exists a finite set $\tilde{\mathcal{M}}$ of interiors of k-dimensional polyhedra with max-facet-width at most $l\Delta$ such that $\tilde{M}_j \in \tilde{\mathcal{M}}$ for some $j \in \{1, \ldots, t\}$. We have just shown that for all $T \in \mathcal{T}$ we have that $M \cap \operatorname{aff}(P) \in \tilde{\mathcal{M}}$ for some $M \in \operatorname{atoms}(T)$.

We will next use the set $\tilde{\mathcal{M}}$ to construct a finite family $\mathcal{M} \subseteq \mathcal{M}_l$ of sets in $\mathbb{R}^n$. For each $\tilde{M} \in \tilde{\mathcal{M}}$ if there is no $\bar{T} \in \mathcal{T}$ and $\bar{M} \in \operatorname{atoms}(\bar{T})$ such that $\tilde{M} = \bar{M} \cap \operatorname{aff}(P)$, we do not do anything. Otherwise we chose one $\bar{T} \in \mathcal{T}$ and $\bar{M} \in \operatorname{atoms}(\bar{T})$ such that $\tilde{M} = \bar{M} \cap \operatorname{aff}(P)$ and include $\bar{M}$ in $\mathcal{M}$. Clearly $\mathcal{M}$ has the properties claimed in the lemma. ■

Notice that if instead of a single set $T' \in \mathcal{T}_{t,l}^*$, we have a finite family $\mathcal{T}' \subset \mathcal{T}_{t,l}^*$ that collectively dominates the elements of $\mathcal{T}$ on all facets of P, but not on P, then we can consider each $T' \in \mathcal{T}'$ one at a time and apply Lemma 12 to generalize it to the case $|T'| > 1$. This observation leads to the following result which is analogous to Lemma 6.

Corollary 13. *Let $P \subseteq \mathbb{R}^n$ be a rational pointed polyhedron and let $\mathcal{T}' \subset \mathcal{T}_{t,l}^*$ be finite. Let $\mathcal{T} \subseteq \mathcal{T}_{t,l}^*$ be such that for each $T \in \mathcal{T}$ there exists a $T' \in \mathcal{T}'$ such that T' dominates T on all facets of P,* *but not on P. Then there exists a finite set $\mathcal{M} \subseteq \mathcal{M}_l$ with the following property: for each $T \in \mathcal{T}$ there is an $M \in \text{atoms}(T)$ such that $M \cap \text{aff}(P) = M' \cap \text{aff}(P)$ for some $M' \in \mathcal{M}$.*

Note that the condition $M \cap \text{aff}(P) = M' \cap \text{aff}(P)$ in Lemma 12 and Corollary 13 implies that $M \cap P = M' \cap P$. The next lemma proves the inductive step necessary for the main result. For convenience, we define $\mathcal{T}^*_{0,l} = \emptyset$ for all l and say that all subsets of $\mathcal{T}^*_{0,l}$ have a finite dominating subset for all polyhedra.

Lemma 14. *Let $l, n, t \geq 1$ be fixed integers. Suppose each $\mathcal{T} \subseteq \mathcal{T}^*_{t-1,l}$ has a finite dominating subset for any rational pointed polyhedron. Then each $\mathcal{T} \subseteq \mathcal{T}^*_{t,l}$ has a finite dominating subset for any rational pointed polyhedron.*

Proof. In other words, we will show that if $(\mathcal{T}^*_{t-1,l}, \preceq_Q)$ is fairly well-ordered for each rational pointed polyhedron $Q \subseteq \mathbb{R}^n$, then $(\mathcal{T}^*_{t,l}, \preceq_P)$ is fairly well-ordered for each rational pointed polyhedron $P \subseteq \mathbb{R}^n$. The proof follows the same steps as the proof of Lemma 7 with a few modifications. We replace each occurrence of "split set" in the previous proof with "interior of a polyhedron with max-facet-width l". In addition, instead of Lemma 6 which gives a finite family $\hat{\mathcal{S}}$ of split sets, we now use Corollary 13 which gives a finite subset of $\mathcal{M}_l$. As we are dealing with a finite subset of $\mathcal{M}_l$, let k be the maximum number of inequalities necessary to describe any of these polyhedral sets.

Finally, when S is not a split set but is the interior of a polyhedron with max-facet-width l defined by at most k linear inequalities and P is a polyhedron, then it is easy to see that $P \setminus S$ equals the union of at most k polyhedra. As

$$S = \{x \in \mathbb{R}^n : a_i^T x < b_i, i = 1, \ldots, k\},$$

for some a_i, b_i of appropriate dimension for $i = 1, \ldots, k$, we obtain $P \setminus S = \cup_{i=1}^k Q_i$ where

$$Q_i = P \cap \{x \in \mathbb{R}^n : a_i^T x \geq b_i\}.$$

Therefore, as in the case of split sets, $P \setminus S$ is a finite union of polyhedra. As in the proof of Lemma 7, we can now argue that $(\mathcal{T}^*_{t,l}, \preceq_P)$ is fairly well-ordered as $(\mathcal{T}^*_{t-1,l}, \preceq_{Q_i})$ is fairly well ordered for all $i = 1, \ldots, k$. ∎

The proof of our main result can now be obtained by simply modifying the proof of Theorem 1. In particular, we can replace all occurrences of $\mathcal{T}^*_t$ by $\mathcal{T}^*_{t,l}$ and "split set" by "interior of a polyhedral set with max-facet-width l" and use Proposition 9 instead of [9, Proposition 1]. Consequently, we obtain a generalization of Averkov's result [3, Thm 1.1] by extending it to t-branch bounded max-facet-width polyhedra for any finite $t > 1$.

Theorem 2. *For any rational polyhedron P and $\mathcal{T} \subseteq \mathcal{T}^*_{t,l}$ where t and l are positive integers, the set $\mathcal{T}$ has a finite dominating subset for P. Consequently, $\text{Cl}(P, \mathcal{T})$ is a polyhedron.*

5 Concluding remarks

In this section we first observe that our dominance and closure results can easily be extended to mixed-integer sets. In [1] Andersen, Cornuéjols and Li show that the effect of subtracting a split set from a rational polyhedron P and convexifying the remaining points can be computed purely from the effect of the split set on the edges of P. We generalize this to t-branch bounded max-facet-width polyhedra.

5.1 Mixed-integer Sets

Consider a mixed-integer set defined by a polyhedron $P^{LP} \subseteq \mathbb{R}^n$ and the mixed-integer lattice $\mathbb{Z}^{n_1} \times \mathbb{R}^{n-n_1}$ where $0 < n_1 \leq n$:

$$P^I = P^{LP} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n-n_1}). \tag{14}$$

An inequality is called a lattice-free cut for P^{LP} with respect to the mixed-integer lattice $\mathbb{Z}^{n_1} \times \mathbb{R}^{n-n_1}$ if it is valid for $\operatorname{conv}(P^{LP} \setminus L^\circ)$ where $L = L' \times \mathbb{R}^{n-n_1}$ and $L' \subset \mathbb{R}^{n_1}$ is a lattice-free set. With slight abuse of terminology, we call the set L a lattice-free set with respect to the mixed-integer lattice $\mathbb{Z}^{n_1} \times \mathbb{R}^{n-n_1}$. Clearly, if L' has max-facet-width l, so does L.

Notice that if in Theorem 2 we choose $\mathcal{T} \subseteq \mathcal{T}^*_{t,l}$ to consist of bounded max-facet-width sets whose lineality space contains $0_{n_1} \times \mathbb{R}^{n-n_1}$ where 0_{n_1} is a vector of zeroes in $\mathbb{R}^{n_1}$, then we obtain the following straightforward extension of Theorem 2 to mixed-integer sets:

Corollary 15. *Let $P \subset \mathbb{R}^n$ be a rational polyhedron, then the bounded max-facet-width closure of $P \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n-n_1})$ is finitely generated.*

5.2 Subtracting a union of polyhedral sets from a polyhedron

The result below gives an interesting structural property of t-branch split sets and the unions of t-branch polyhedra with max-facet-width l.

Theorem 3. *Let $P \subseteq \mathbb{R}^n$ be a pointed polyhedron. Let $B_1, \ldots, B_t$ be full-dimensional polyhedra, for some $t > 0$, such that $Q = \operatorname{conv}(P \setminus \cup_{i=1}^t B_i^\circ)$ is a nonempty polyhedron. Then all vertices of Q lie on faces of P that have dimension at most t.*

Proof. Let m be the maximum number of nonredundant inequalities defining any of the polyhedra B_i. By repeating inequalities, we can assume each B_i is defined by exactly m inequalities (some may be redundant). Let $I = \{1, \ldots, t\}$ and $J = \{1, \ldots, m\}$ and let $B_i = \{x \in \mathbb{R}^n : a_i^j x \leq b_i^j, j \in J\}$ for $i \in I$. Then B_i°, the interior of B_i, is equal to $\{x \in \mathbb{R}^n : a_i^j x < b_i^j, j \in J\}$, and the complement of B_i° can be expressed as

$$\mathbb{R}^n \setminus B_i^\circ = \cup_{j=1}^m H_i^j, \text{ where } H_i^j = \{x \in \mathbb{R}^n : a_i^j x \geq b_i^j\} \text{ for } j \in J.$$

Let $\mathcal{M} = \{1, \ldots, m\}^t$, i.e., $\mathcal{M}$ consists of all ordered tuples that have t elements, where each element is from the set J. For $K \in \mathcal{M}$, let K_i stand for the ith entry of the tuple K and let $P_K = \bigcap_{i \in I} H_i^{K_i}$.

Then

$$x \notin \bigcup_{i=1}^{t} B_i^\circ \iff x \notin B_i^\circ, \text{ for all } i \in I \iff a_i^j x \geq b_i^j \text{ for some } j \in J, \text{ for all } i \in I,$$
$$\iff x \in P_K \text{ for some } K \in \mathcal{M} \iff x \in \bigcup_{K \in \mathcal{M}} P_K.$$

Therefore,

$$P \setminus \bigcup_{i \in I} B_i^\circ = \bigcup_{K \in \mathcal{M}} P \cap P_K.$$

Consequently, $Q = \text{conv}(P \setminus \bigcup_{i=1}^t B_i^\circ)$ is the convex hull of polyhedra of the form $P \cap P_K$. As Q is a polyhedron its vertices form a subset of the vertices of the polyhedra $P \cap P_K$ for $K \in \mathcal{M}$. For any $K \in \mathcal{M}$, the vertices of $P \cap P_K$ are contained in t-dimensional faces of P. This is because every vertex of $P \cap P_K$ is defined by some $t' \leq t$ inequalities of the form $a_i^j x \geq b_i$ and $n - t' \geq n - t$ inequalities defining P; the latter collection of inequalities define a face of P with dimension at most $t' \leq t$. ■

Theorem 3 generalizes a result of Andersen, Cornuéjols and Li [1], who proved the result when $t = 1$ and B_1° is a split set and a result of Averkov [3, Lemma 2.2, part 2] who proved it for $t = 1$ and a bounded max-facet-width polyhedron B_1.

Let $P \subseteq \mathbb{R}^n$ be a pointed polyhedron with dimension at least t and let F_P denote the union of its t-dimensional faces. Let $B_1, \ldots, B_t$ satisfy the conditions of the previous theorem and, in addition, assume that for each B_i the recession cone equals the lineality space. Also let $T = \cup_{i=1}^t B_i^\circ$ and $Q = \text{conv}(P \setminus T)$. By Lemma 8, $Q = \text{conv}(P \setminus T)$, if non-empty, is a pointed polyhedron and therefore

$$Q = \text{conv}(V_Q) + \text{rec}(P)$$

where $\text{rec}(P)$ denotes the recession cone of P and V_Q is the union of the extreme points of Q. As Q is a (closed) polyhedral set, $V_Q \subseteq P \setminus T$. Furthermore, by Theorem 3, we have $V_Q \subseteq F_P$, and consequently

$$V_Q \subseteq F_P \setminus T.$$

In addition, as $F_P \setminus T \subseteq Q$, we can write

$$Q = \text{conv}(F_P \setminus T) + \text{rec}(P)$$

In other words, the effect of $B_1, \ldots, B_t$ on P (i.e., the effect of subtracting $\cup_{i=1}^t B_i$ from P and convexifying the remaining points) can be computed purely from the effect of these sets on each of the t-dimensional faces F_j.

In conclusion, we have shown that t-branch split sets and t-branch sets with bounded max-facet-width behave similarly to split sets in a number of ways; in particular, they yield polyhedral closures.

References

[1] K. Andersen, G. Cornuéjols, and Y. Li, Split closure and intersection cuts, *Mathematical Programming* **102** (2005), 457–493.

[2] K. Andersen, Q. Louveaux, and R. Weismantel, An analysis of mixed integer linear sets based on lattice point free convex sets, *Math. Oper. Res.* **35** (2010), 233–256.

[3] G. Averkov, On finitely generated closures in the theory of cutting planes, *Discrete Optimization* **9** (2012), 209–215.

[4] T. Bang, A solution of the plank problem, *Proc. American Mathematical Society* **2** (1951), 990–993.

[5] Amitabh Basu, Robert Hildebrand, and Matthias Köppe, The triangle closure is a polyhedron, *Mathematical Programming* **145** (2014), 19–58.

[6] W. J. Cook, R. Kannan, and A. Schrijver, Chvátal closures for mixed integer programming problems, *Mathematical Programming* **47** (1990), 155–174.

[7] S. Dash, S. Dey, and O. Günlük, Two dimensional lattice-free cuts and asymmetric disjunctions for mixed-integer polyhedra, *Mathematical Programming* **135** (2012), 221–254.

[8] S. Dash, N. Dobbs, O. Gunluk, T. Nowicki, and G. Swirszcz, Lattice-free sets, branching disjunctions, and mixed-integer programming, *Mathematical Programming* **145** (2014), 483–508.

[9] S. Dash, O. Günlük, D. A. Moran R., On the polyhedrality of cross and quadrilateral closures, *Mathematical Programming*, DOI 10.1007/s10107-016-0982-x, 2016 .

[10] S. Dey, L. A. Wolsey, Two row mixed integer cuts via lifting, *Mathematical Programming* **124** (2010), 143–174.

[11] F. Higman, Ordering by divisibility in abstract algebras, *Proc. London Math. Soc.* **3** (1952), 326–336.

[12] M. Jörg, k-disjunctive cuts and cutting plane algorithms for general mixed integer linear programs, PhD Thesis, Technische Universität München, 2008.

[13] Yanyun Li and J. P. P. Richard, Cook, Kannan and Schrijver's example revisited, *Discrete Optimization* **5** (2008), 724–734.

[14] L. Lovász, Geometry of numbers and integer programming, *Mathematical Programming: Recent Developments and Applications*, M. Iri and K. Tanabe eds., Kluwer (1989) 177–201.

[15] A. Schrijver, Theory of Linear and Integer Programming, Wiley, New York, 1986.