Asymptotical Analysis of a SAA Estimator for Optimal Value of a Two Stage Problem with Quadratic Recourse*

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Abstract

In this paper, we first consider the stability analysis of a convex quadratic programming problem and its restricted Wolfe dual in which all parameters in the problem are perturbed. We demonstrate the upper semi-continuity of solution mappings for the primal problem and the restricted Wolfe dual problem and establish the Hadamard directionally differentiability of the optimal value function. By expressing the optimal value function as a min-max optimization problem over two compact convex sets, we present the asymptotic distribution of a SAA estimator of the optimal value for a two stage program whose second stage problem is a convex quadratic programming problem and all parameters in the quadratic program are random variables.

Key words: quadratic programming, two stage program, stability analysis, SAA estimator, asymptotic distribution.

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1 Introduction

Consider the two-stage stochastic optimization problem of the following form:

$$
\begin{aligned}
\min \quad & g(x) + \mathbb{E}(\theta(x,\xi)) \\
\text{s.t.} \quad & x \in X, \\
& \theta(x,\xi) = \min_{y \in Y} \quad q(x,y) \\
& \qquad\qquad\quad \text{s.t.} \quad Wy + Tx = h, \\
& \qquad\qquad\qquad\quad\;\; Ay + Bx \le b,
\end{aligned}
\tag{1.1}
$$

where $x \in \Re^n$ is the first stage decision variable and $X \subset \Re^n$, $y \in \Re^m$ is the second stage decision variable and $Y \subset \Re^m$, $q : \Re^n \to \Re$ and $g : \Re^m \to \Re$ are real-valued functions.

Shapiro and Homem-de-Mello (1998) [12] considered Problem (1.1) with no inequality constraints in the second stage and $g(x) = c^T x$, $q(y) = q^T y$ and $Y = \Re^m_+$, and recourse matrix W and q are deterministic. They proposed Monte Carlo simulation based approaches to a numerical solution of this two-stage stochastic programming problem. Different from Problem (1.1), Robinson and Wets (1987) [6] studied the continuity of the optimal value function $Q(p,x,\xi) = \inf_y\{\langle h^*, y\rangle : Wy = t(p,x,\xi), y \ge 0\}$ and the upper semicontinuity of its corresponding solution set-valued mapping in which the probability distribution is regarded as a parameter and recourse matrix W is deterministic.

Most of works about the stability of two stage stochastic programming study the stability of optimal values and solution sets when the underlying probability distribution varies in some metric space of probability measures. Now we cite some of these results. Römisch and Schultz [8] considered two second stage problems. One is Problem (1.1) with no inequality constraints in the second stage and $g(x) = c^T x$, $q(x,y) = q^T y$ and $Y = \Re^m_+$ with recourse matrix W being deterministic; Another one is Problem (1.1) with no equality constraints in the second stage and $g(x) = c^T x$, $q(x,y) = \frac{1}{2} y^T H y - (a - Tx)^T y$ and $Y = \Re^m_+$ with A and H being deterministic. They derived Hölder continuity of optimal value function with respect to a Lipschitz metric $\beta(\mu,\nu)$. Furthermore, in [9] Römisch and Schultz studied the above two second stage problems in which $c^T x$ is replaced by a general convex function $g(x)$ and obtained the upper semi-continuity of solution set-valued mapping and Lipschitz continuity of optimal value function with respect to the L_p-Wasserstein metric. Dentcheva and Römisch [2] proved that optimal value function and solution set mapping of two-stage stochastic programs with random right-hand side, under certain conditions, to be directionally differentiable and semidifferentiable on appropriate functional spaces. Rachev and Römisch [5] studied quantitative stability of optimal values and solution sets to stochastic programming problems with respect to a minimal information (m.i.)

probability metric and applied the results to Problem (1.1) with no inequality constraints in the second stage and $g(x) = c^T x$, $q(x,y) = q^T y$ and $Y = \Re^m_+$ with recourse matrix W being deterministic.

In each of the above second stage problems, the recourse matrix W is deterministic. There are a few papers about the stability of two stage programming in which recourse costs, the technology matrix, the recourse matrix and the right-hand side vector are all random. For example, in Section 3 of [10], Römisch and Wets, considered a two-stage stochastic linear programs, namely Problem (1.1) with no inequality constraints in the second stage and $g(x) = c^T x$, $q(x,y) = q^T y$ and $Y = \Re^m_+$ with q, h, T and W being random variables. They obtained the Lipschitz continuity of the optimal value and the ε-approximate solution sets in Huasdorff distance with respect to Fortet-Mourier metric of probability distributions. Their analysis was based on the general perturbation results for optimization models in [7, Section 7J]. Recently Han and Chen [3] considered the same linear two-stage stochastic program as in [10]. Based on stability theory of linear programming, they derived new forms of quantitative stability results of the optimal value function and the optimal solution set with respect to the Fortet-Mourier probability metric.

In this paper, we consider a two-stage stochastic optimization problem whose second stage is a convex quadratic programming problem:

$$\begin{array}{ll} \min & g(x) + \mathbb{E}(\theta(x,\xi)) \\ \text{s.t.} & x \in X, \\ & \theta(x,\xi) = \begin{array}{ll} \min & \tilde{c}^T y + \frac{1}{2} y^T \tilde{G} y \\ \text{s.t.} & \tilde{A} y + \tilde{B} x \geq \tilde{b}, \end{array} \end{array} \tag{1.2}$$

where $x \in \Re^n$ is the decision variable, $g : \Re^n \to \Re$ is a continuous convex function, $\tilde{c} \in \Re^m$ and $\tilde{G} \in \mathbb{S}^m_+$ with $\tilde{G} = [\tilde{G}_{1\cdot}, \cdots, \tilde{G}_{m\cdot}]^T$, $\tilde{A} \in \Re^{l \times m}$ with $\tilde{A} = [\tilde{a}_1, \cdots, \tilde{a}_l]^T$, $\tilde{B} \in \Re^{l \times n}$ with $\tilde{B} = [\tilde{B}_{1\cdot}, \cdots, \tilde{B}_{l\cdot}]^T$, and $\tilde{b} \in \Re^l$.

Here we consider the case where all parameters in the second stage are random, namely $\xi = (\tilde{c}, \tilde{G}, \tilde{A}, \tilde{B}, \tilde{b})$, and discuss the asymptotical properties of a SAA estimator of the optimal value for Problem (1.2). Since $\tilde{G}$ is positively semi-definite, different from linear programming, we can not have an explicit formulation for the Lagrage dual of the quadratic programming problem in the second stage. Hence we can not use Lagrange duality theory, like the way of Chapter 2 of [11] for linear two stage problems, to study the second stage problem in (1.2). The main point in this paper is to express the optimal value $\theta(x,\xi)$ as a min-max of a convex-concave function over two compact convex nonempty sets and employ Theorem 7.24 Shapiro, Dentcheva and Ruszczynski (2009) to express the directional derivative of $\theta(x,\xi)$. After that we use Theorem 7.59 of [11], the delta theorem, to analyze the the asymptotical properties of a

SAA estimator of the optimal value for Problem (1.2). Here the so-called Wolfe dual of convex quadratic programming play an important role for the min-max expression of $\theta(x,\xi)$.

The paper is organized as follows. In Section 2, we demonstrate upper semi-continuity of the optimal solution mappings for both the quadratic programming problem and its strict Wolfe dual. In Section 3, we establish the Hadamard directional differentiability of $\theta(x,\xi)$. In Section 4, we discuss the asymptotical distribution of a SAA estimator for optimal value of the two stage problem (1.2).

2 Upper semi-continuity of optimal solution mappings

In this section we discuss the continuity properties about the solution set of the second stage problem and it's dual problem with respect to ξ and x. Let $u=(x,c,G,A,B,b)$ and $\tilde{u}=(\tilde{x},\xi)$. Consider quadratic programming problem in the second stage of problem (1.2):

$$\begin{array}{lll}(\mathrm{QP}) & \min & \tilde{c}^T y+\frac{1}{2}y^T\tilde{G}y \\ & \text{s.t.} & \tilde{A}y+\tilde{B}x\geq\tilde{b}.\end{array} \tag{2.1}$$

We denote by $\Phi(\tilde{u})$ the feasible set of problem (2.1), namely

$$\Phi(\tilde{u})=\{y\in\Re^m:\tilde{A}y+\tilde{B}\tilde{x}\geq\tilde{b}\}.$$

We denote $f(y,\tilde{u})=\tilde{c}^Ty+\frac{1}{2}y^T\tilde{G}y$ the objective function for Problem (2.1) and by $Y^*(\tilde{u})$ the set of optimal solutions when $x=\tilde{x}$. For a given parameter (c,G,A,B,b), we discuss the corresponding quadratic programming problem

$$\begin{array}{lll}(\mathrm{QP}_0) & \min & c^T y+\frac{1}{2}y^T G y \\ & \text{s.t.} & Ay+Bx\geq b,\end{array} \tag{2.2}$$

and analyze the stability of the optimal solutions when (c,G,A,B,b) is perturbed to $(\tilde{c},\tilde{G},\tilde{A},\tilde{B},\tilde{b})$. For this purpose we make the following assumptions about problem (2.2):

Assumption 2.1. *The set $X\subset\Re^n$ is a nonempty compact convex set.*

Assumption 2.2. *For each $x\in X$, the optimal value of QP problem (2.2) is finite and the solution set for problem (2.2) is compact.*

Assumption 2.3. *The following two conditions hold:*

A1 *The slater condition holds for each $x\in X$, namely for each $x\in\Re^n$, there exists y_x such that*

$$Ay_x+Bx>b.$$

A2 *The matrix* $G \in \mathbb{S}^m$ *is positively semi-definite.*

If the optimal value of QP problem (2.2) is finite and conditions of Assumption 2.3 are satisfied, then the Wolfe dual of problem (2.2)

$$\begin{aligned}\max_{\lambda} \quad & \lambda^T(b - Bx) - \frac{1}{2}y^T G y \\ \text{s.t.} \quad & c - A^T\lambda + Gy = 0, \\ & \lambda \geq 0\end{aligned} \tag{2.3}$$

has a non-empty solution set.

In the following, we use $\mathbb{B}$ to denote the unit ball and $\mathbb{B}_r(z)$ to denote the closed ball centered at z with radius $r > 0$ in a finite dimensional Hilbert space. We now recall continuity notions of set-valued mappings [7].

Definition 2.1. *[7, 5.4 Definition] A set-valued mapping* $S : \Re^n \rightrightarrows \Re^m$ *is outer semicontinuous (osc) at* $\bar{x}$ *if*

$$\limsup_{x \to \bar{x}} S(x) \subset S(\bar{x}),$$

or equivalently $\limsup\limits_{x \to \bar{x}} S(x) = S(\bar{x})$, *where* $\limsup_{x \to \bar{x}} S(x)$ *is the outer limit of* S *at* $\bar{x}$*:*

$$\limsup_{x \to \bar{x}} S(x) = \{u : \exists x^k \to \bar{x}, \exists u^k \in S(x^k), \text{ with } u^k \to u\}.$$

The mapping S *is inner semicontinuous (isc) at* $\bar{x}$ *if*

$$\liminf_{x \to \bar{x}} S(x) = S(\bar{x}),$$

where $\liminf_{x \to \bar{x}} S(x)$ *is the inner limit of* S *at* $\bar{x}$*:*

$$\liminf_{x \to \bar{x}} S(x) = \{u : \forall x^k \to \bar{x}, \exists u^k \in S(x^k) \text{ for large enough } k \text{ with } u^k \to u\}.$$

The mapping S *is continuous at* $\bar{x}$ *if*

$$\limsup_{x \to \bar{x}} S(x) = S(\bar{x}) = \liminf_{x \to \bar{x}} S(x).$$

The following proposition tells us that Slater condition is stable when (c, G, A, B, b) is perturbed to $(\tilde{c}, \tilde{G}, \tilde{A}, \tilde{B}, \tilde{b})$ with $\tilde{G} \in \mathbb{S}^m_+$.

Proposition 2.1. *For fixed* c, G, A, B *and* b, *if Assumption 2.1 and A1 of Assumption 2.3 hold, then there exists* $\delta_0 > 0$ *such that for any* $x \in X$, *Slater condition for Problem (2.1) holds when* $\|(\tilde{c}, \tilde{G}, \tilde{A}, \tilde{B}, \tilde{b}) - (c, G, A, B, b)\| \leq \delta_0$ *with* $\tilde{G} \in \mathbb{S}^m_+$.

Proof. From condition A1 of Assumption 2.3, we have that there exist $y_x \in \Re^m$ and $\varepsilon_x > 0$ such that

$$Ay_x + Bx - b \geq \varepsilon_x \mathbf{1}_l.$$

Let $M_x := \max\{\|y_x\|, \|x\|, 1\}$. When $\max_{1\leq i\leq l}\{\|\Delta\tilde{a}_i\|, \|\Delta\tilde{B}_i\|, \|\Delta\tilde{b}_i\|\} \leq \frac{\varepsilon_x}{6M_x}$ and

$$\|\tilde{x} - x\| \leq \varepsilon_x \left[6\|B\| + \frac{\varepsilon_x}{M_x}\right]^{-1},$$

we have for $i = 1, \cdots, l$,

$$\begin{aligned}
&a_i^T y_x + B_i^T x - b_i \geq \varepsilon \\
&> \|\tilde{a}_i - a_i\|\|y_x\| + \|\tilde{B}_i - B_i\|\|x\| + \|\tilde{B}_i\|\|x - \tilde{x}\| + \|\tilde{b}_i - b_i\| \\
&\geq (a_i - \tilde{a}_i)^T y_x + B_i^T x - \tilde{B}_i^T \tilde{x} - (b_i - \tilde{b}_i).
\end{aligned}$$

or equivalently,

$$\tilde{a}_i^T y_x + \tilde{B}_i^T \tilde{x} - \tilde{b}_i > 0, \; i = 1, \cdots, l.$$

Thus the Slater condition for problem (2.1) corresponding to $\tilde{x} \in X$ holds when $\|(\tilde{c}, \tilde{G}, \tilde{A}, \tilde{B}, \tilde{b}) - (c, G, A, B, b)\| \leq \delta_x$ for $\delta_x = \frac{\varepsilon_x}{6M_x}$ and $\|\tilde{x} - x\| \leq \omega_x$ with $\omega_x := \varepsilon_x \left[6\|B\| + \frac{\varepsilon_x}{M_x}\right]^{-1}$. For such $\omega_x > 0$ at $x \in X$, we have

$$X \subset \cup_{x\in X} \mathbb{B}_{\omega_x}(x).$$

From Assumption 2.1, X is compact, we have from the finite covering theorem that there are a finite number of points $x^1, \ldots, x^{n_0}$ and positive numbers $\omega_{x^1}, \ldots, \omega_{x^{n_0}}$ such that

$$X \subset \cup_{j=1}^{n_0} \mathbb{B}_{\omega_{x^j}}(x^j).$$

Let $\omega_0 = \min\{\omega_{x_j} : j = 1, \ldots, n_0\}$ and $\delta_0 = \min\{\delta_{x_j} : j = 1, \ldots, n_0\}$. Then for any $x \in X$, the Slater condition for problem (2.1) holds when $\|(\tilde{c}, \tilde{G}, \tilde{A}, \tilde{B}, \tilde{b}) - (c, G, A, B, b)\| \leq \delta_0$. □

Next we prove that the solution set $Y^*(\tilde{u})$ of Problem (2.1) is upper semicontinuous at u under Assumption 2.1, Assumption 2.2 and Assumption 2.3. For this purpose, we first establish two lemmas. For fixed c, G, A, B, b, let $\delta > 0$ be the positive number, define

$$\mathcal{U}_\delta(c, G, A, B, b) = \{(\tilde{c}, \tilde{G}, \tilde{A}, \tilde{B}, \tilde{b}, \tilde{x}) : \|(\tilde{c}, \tilde{G}, \tilde{A}, \tilde{B}, \tilde{b}) - (c, G, A, B, b)\| \leq \delta, \tilde{G} \in \mathbb{S}^m_+, \tilde{x} \in X\}.$$

Lemma 2.1. *For given A, B and b, let condition A1 in Assumption 2.3 hold. Then for any $\widehat{u} \in \mathcal{U}_{\delta_0}(c, G, A, B, b)$, where δ_0 is defined in Proposition 2.1,*

$$\lim_{\tilde{u}\to\widehat{u}} \Phi(\tilde{u}) = \Phi(\widehat{u}).$$

Proof. As the following inclusion

$$\limsup_{\tilde{u}\to\widehat{u}} \Phi(\tilde{u}) \subset \Phi(\widehat{u})$$

is obvious, we only need to verify that

$$\liminf_{\tilde{u}\to\widehat{u}} \Phi(\tilde{u}) \supset \Phi(\widehat{u}).$$

For arbitrary $\hat{y} \in \Phi(\hat{u})$, we now prove $\hat{y} \in \liminf_{\tilde{u}\to\widehat{u}} \Phi(\tilde{u})$. By Proposition 2.1, we have that

$$\exists \ \overline{y} \ \ such \ \ that \ \ \hat{A}\overline{y} + \hat{B}\hat{x} - \hat{b} \geq \hat{\epsilon}\mathbf{1}_l.$$

Let $\tilde{u}(t) = (\hat{c}+t\triangle c,\ \hat{G}+t\triangle G,\ \hat{A}+t\triangle A,\ \hat{B}+t\triangle B,\ \hat{b}+t\triangle b,\ \hat{x}+t(\tilde{x}-\hat{x}))$, $y(t) = \hat{y}+t(\overline{y}-\hat{y})$, and we obviously have $\tilde{u}(t) \to \hat{u}$ and $y(t) \to \hat{y},\ t \downarrow 0$. Then for $\triangle u = (\triangle c, \triangle G, \triangle A, \triangle B, \triangle b, \triangle x)$ we have that

$$\begin{aligned}
&\tilde{A}(t)y(t) + \tilde{B}(t)\tilde{x}(t) - \tilde{b}(t)\\
&= (\hat{A} + t\triangle A)(t\overline{y} + (1-t)\hat{y}) + (\hat{B} + t\triangle B)(\hat{x} + t\triangle x) - (\hat{b} + t\triangle b)\\
&= t(\hat{A}(\overline{y} - \hat{y}) + \hat{B}\triangle x + \triangle A\hat{y} + \triangle B\hat{x} - \triangle b) + t^2(\triangle A(\overline{y} - \hat{y}) + \triangle B\triangle x) + \hat{A}\hat{y} + \hat{B}\hat{x} - \hat{b}\\
&= t(\hat{A}\overline{y} + \hat{B}\hat{x} - \hat{b} + \hat{B}\triangle x + \triangle A\hat{y} + \triangle B\hat{x} - \triangle b)\\
&\quad +t^2(\triangle A(\overline{y} - \hat{y}) + \triangle B\triangle x) + (1-t)(\hat{A}\hat{y} + \hat{B}\hat{x} - \hat{b})\\
&\geq t\{(\hat{\epsilon}\mathbf{1}_l + \hat{B}\triangle x + \triangle A\hat{y} + \triangle B\hat{x} - \triangle b) + t(\triangle A(\overline{y} - \hat{y}) + \triangle B\triangle x)\}.
\end{aligned} \tag{2.4}$$

Therefore we have that for $\|\triangle u\|$ small enough, one has that there exits $\hat{t} > 0$ such that

$$\tilde{A}(t)y(t) + \tilde{B}(t)\tilde{x}(t) - \tilde{b}(t) \geq 0, \forall t \in [0, \hat{t}),$$

This implies $\hat{y} \in \liminf_{\tilde{u}\to\widehat{u}} \Phi(\tilde{u})$. This proof is completed. □

Define

$$\Psi(\tilde{u}, \alpha) = \Phi(\tilde{u}) \cap \mathrm{lev}_{\leq\alpha} f(\cdot, \tilde{u})$$

with

$$\mathrm{lev}_{\leq\alpha} f(\cdot, \tilde{u}) = \{y \in \Re^m : f(y, \tilde{u}) \leq \alpha\}, \alpha \in \Re.$$

Lemma 2.2. *For fixed c, G, A, B, b, let Assumptions 2.1, 2.2 and condition A2 of Assumption 2.3 hold. Then for any $\alpha \in \Re$, there exist $\delta_1 > 0$ and a bounded set $\mathcal{B} \subset \Re^m$ such that*

$$\Psi(\tilde{u}, \alpha') \subset \mathcal{B}, \forall \alpha' \leq \alpha, \forall \tilde{u} \in \mathcal{U}_{\delta_1}(c, G, A, B, b).$$

Proof. Without loss of generality, we assume that $\Psi(\tilde{u},\alpha) \neq \emptyset$. Because $\Psi(\tilde{u},\alpha') \subset \Psi(\tilde{u},\alpha), \forall \alpha' \leq \alpha$, we only need to prove $\Psi(\tilde{u},\alpha) \subset \mathcal{B}$. We prove the result by contradiction. Suppose that there exist a sequence $\tilde{u}^k = (x^k, \xi^k) \in\in \mathcal{U}_{\delta_1}(c,G,A,B,b)$ such that $x^k \in X$ and $\xi^k \to (c,G,A,B,b)$ and $y^k \in \Psi(\tilde{u}^k,\alpha)$ with $\|y^k\| \to \infty$. Let $d_y^k = y^k/\|y^k\|$, and notice X is compact, we can find a subsequence k_j such that $x^{k_j} \to x$ and $d_y^{k_j} \to d_y$ for some $x \in X$ and $d_y \in \text{bdry}\mathbf{B}$. In view of $y^{k_j} \in \Psi(\tilde{u}^{k_j},\alpha)$, one has

$$\begin{aligned} &\tilde{c}^{k_j T} y^{k_j} + \frac{1}{2} y^{k_j T} \tilde{G}^{k_j} y^{k_j} \leq \alpha \\ &\tilde{a}_i^{k_j T} y^{k_j} + \tilde{B}_i^{k_j T} x^{k_j} - \tilde{b}_i^{k_j} \geq 0, i = 1, \cdots, l. \end{aligned} \tag{2.5}$$

Dividing both sides of the above inequality by $\|y^{k_j}\|^2$ and $\|y^{k_j}\|$ respectively, we obtain

$$\begin{aligned} &\tilde{c}^{k_j T} d_y^{k_j} / \|y^{k_j}\| + \frac{1}{2} d_y^{k_j T} \tilde{G}^j d_y^{k_j} \leq \alpha / \|y^{k_j}\|^2 \\ &\tilde{a}_i^{k_j T} d_y^{k_j} + \tilde{B}_i^{k_j T} x^{k_j} / \|y^{k_j}\| - \tilde{b}_i^{k_j} / \|y^{k_j}\| \geq 0, i = 1, \cdots, l. \end{aligned}$$

From the definition of $\in \mathcal{U}_{\delta_1}(c,G,A,B,b)$, we know that $\tilde{G}^{k_j} \succeq$ so that the first inequality of (2.5) implies

$$\tilde{c}^{k_j T} d_y^{k_j} \leq \alpha / \|y^{k_j}\|.$$

Taking the limits by $j \to \infty$ in the above three inequalities, we obtain

$$\frac{1}{2} d_y^T G d_y \leq 0,\ A d_y \geq 0,\ c^T d_y \leq 0,$$

and this contradicts with the compactness of solution set in Assumption 2.2. The proof is completed. □

In the following discussions, we need to adopt Proposition 4.4 of Bonnans and Shapiro(2000) [1]. For this, we consider the parameterized optimization problem of the form

$$(\mathrm{P}_u) \qquad \min_{x \in X} f(x,u) \quad \text{s.t.} \quad G(x,u) \in K, \tag{2.6}$$

where $u \in U$, X, Y and U are Banach spaces, K is a closed convex subset of Y. $f : X \times Y \to \Re$ and $G : X \times U \to Y$ are continuous. We denote by

$$\Phi(u) := \{x \in X : G(x,u) \in K\}$$

the feasible set of problem (P_u) and the optimal value function is

$$\nu(u) := \inf_{x \in \Phi(u)} f(x,u),$$

and the associated solution set

$$S(u) := \underset{x \in \Phi(u)}{\text{argmin}}\ f(x,u).$$

Proposition 2.2. *[1, Proposition 4.4] Let u_0 be a given point in the parameter space U. Suppose that*

(i) *the function $f(x,u)$ is continuous on $X \times U$,*

(ii) *the multifunction $\Phi(\cdot)$ is closed,*

(iii) *there exist $\alpha \in \Re$ and a compact set $C \subset X$ such that every u in a neighborhood of u_0, the level set*

$$lev_{\leq \alpha} f(\cdot, u) := \{x \in \Phi(u) : f(x,u) \leq \alpha\}$$

is nonempty and contained in C,

(iv) *for any neighborhood $\mathcal{V}_X$ of the set $S(u_0)$ there exists a neighborhood $\mathcal{V}_U$ of u_0 such that $\mathcal{V}_X \cap \Phi(u)$ is nonempty for all $u \in \mathcal{V}_U$.*

Then:

(a) *the optimal value function $\nu(u)$ is continuous at $u = u_0$,*

(b) *the multifunction $S(u)$ is upper semicontinuous at u_0.*

Theorem 2.4. *For given (c, G, A, B, b), let Assumptions 2.1, 2.2 and 2.3 hold. For any $\widehat{u} \in \mathcal{U}_{\delta_1}(c, G, A, B, b)$ with δ_1 defined in Lemma 2.2, one has that $\theta(\cdot)$ is continuous at $\widehat{u}$ and the solution set mapping Y^* is upper semi-continuous at $\widehat{u}$, namely for $\epsilon > 0$ there exists a number $\delta_2 > 0$ such that*

$$Y^*(\tilde{u}) \subset Y^*(\widehat{u}) + \epsilon \boldsymbol{B}, \ \forall \tilde{u} \in \mathbb{B}_{\delta_2}(\widehat{u}) \ \textit{with} \ \tilde{G} \in \mathbb{S}^m_+.$$

Proof. Let

$$f(y, \tilde{u}) = \tilde{c}^T y + \frac{1}{2} y^T \tilde{G} y, \ \mathcal{G}(y, \tilde{u}) = \tilde{A} y + \tilde{B} \tilde{x} - \tilde{b} \text{ and } K = \Re^l_+.$$

Then the constraint set $\Phi(\tilde{u})$ is expressed as

$$\Phi(\tilde{u}) = \{y \in \Re^m : \mathcal{G}(y, \tilde{u}) \in K\}$$

and the problem is expressed in the setting of Proposition 2.2. Obviously we have that $f(y, \tilde{u})$ is continuous in $\Re^m \times \mathcal{U}_{\delta_1}(c, G, A, B, b)$, namely condition (i) of Proposition 2.2 holds. From Lemma 2.1 and noticing the equivalence between the outer semi-continuity and the closedness for set-value mappings, we have that Φ is a closed set-value mapping so that (ii) of Proposition 2.2 holds. Condition (iii) of Proposition 2.2 comes from Lemma 2.2. Since Assumption 2.3

implies Mangsarian-Fromovitz constraint qualification for $\Phi(\widehat{u})$ at any point $\widehat{y} \in Y^*(\widehat{u})$. Then it follows form Theorem 2.87 in [1] that

$$\text{dist}(\widehat{y}, \Phi(\tilde{u})) \leq \kappa(\text{dist}\,(\mathcal{G}(\widehat{y}, \tilde{u}), K)) \leq \kappa\|\mathcal{G}(\widehat{y}, \tilde{u}) - \mathcal{G}(\widehat{y}, \widehat{u})\| \tag{2.7}$$

for $\tilde{u} \in \mathcal{V}_U$, where $\mathcal{V}_U$ is some neighborhood of $\widehat{u}$ in $\Re^n \times \mathbb{S}^m_+ \times \Re^{l\times m} \times \Re^{l\times n} \times \Re^l$ and $\kappa > 0$. Since $\mathcal{G}$ is Lipschitz continuous, we have that condition (iv) of Proposition 2.2 holds.

Therefore, we have from Proposition 2.2 that the optimal value function θ is continuous at at $\widehat{u}$ and the solution set $Y^*(\tilde{u})$ is upper semicontinuous at $\widehat{u}$ in $\Re^n \times \mathbb{S}^m_+ \times \Re^{l\times m} \times \Re^{l\times n} \times \Re^l$, namely for $\epsilon > 0$ there exists a number $\delta_2 > 0$ such that

$$Y^*(\tilde{u}) \subset Y^*(\widehat{u}) + \epsilon\mathbf{B}, \forall \tilde{u} \in \mathbb{B}_{\delta_2}(\widehat{u}) \text{ with } \tilde{G} \in \mathbb{S}^m_+.$$

The proof is completed. □

Now we consider the dual of the QP problem (2.1). The Wolfe dual of problem (2.1) is

$$\begin{array}{ll} \max\limits_{\lambda} & \lambda^T(\tilde{b} - \tilde{B}x) - \frac{1}{2}y^T\tilde{G}y \\ \text{s.t.} & \tilde{c} - \tilde{A}^T\lambda + \tilde{G}y = 0, \\ & \lambda \geq 0. \end{array} \tag{2.8}$$

The restricted Wolfe dual of problem (2.1) is defined by

$$\begin{array}{ll} \max\limits_{\lambda} & \lambda^T(\tilde{b} - \tilde{B}x) - \frac{1}{2}y^T\tilde{G}y \\ \text{s.t.} & \tilde{c} - \tilde{A}^T\lambda + \tilde{G}y = 0, \\ & y \in \text{Range}\,\tilde{G},\ \ \lambda \geq 0. \end{array} \tag{2.9}$$

Remark 2.1. *The Lagrangian function of Problem (2.9) is defined by*

$$\mathcal{L}(\lambda, y, x) = \tilde{c}^Tz + z^T\tilde{G}y - (\tilde{A}z)^T\lambda + (\tilde{b} - \tilde{B}x)^T\lambda - \frac{1}{2}y^T\tilde{G}y.$$

Then the Lagrangian dual of Problem (2.9) is

$$\begin{array}{ll} & \min\limits_{z\in\Re^n} \max\limits_{\lambda\geq 0, y\in \text{Range}\,\tilde{G}} \mathcal{L}(\lambda, y, z) \\ = & \min\limits_{z\in\Re^n} \max\limits_{\lambda\geq 0, y\in \text{Range}\,\tilde{G}} \tilde{c}^Tz + (z^T\tilde{G}y - \frac{1}{2}y^T\tilde{G}y) + (\tilde{b} - \tilde{B}x - \tilde{A}z)^T\lambda \\ = & \min\limits_{z\in\Re^n}\ \tilde{c}^Tz + \frac{1}{2}z^T\tilde{G}z \\ & \text{s.t.}\ \ \tilde{A}z + \tilde{B}x \geq \tilde{b}, \end{array}$$

which is just Problem (2.1). From this observation, we have from the duality theory for convex optimization that Assumption 2.2 implies that Slater condition for Problem (2.9) holds.

We denote the feasible set for Problem (2.8) by

$$\mathcal{E}(\tilde{c}, \tilde{G}, \tilde{A}) = \{(y, \lambda) \in \Re^m \times \Re^l_+ : \tilde{c} - \tilde{A}^T \lambda + \tilde{G} y = 0\}.$$

Proposition 2.3. *Let (c, G, A) be given. If Assumption 2.2 holds, then there exists $\delta_3 > 0$ such that Slater condition for Problem (2.8) holds when $\|(\tilde{c}, \tilde{G}, \tilde{A}) - (c, G, A)\| \leq \delta_3$ with $\tilde{G} \in \mathbb{S}^m_+$, namely there exists $(\tilde{y}, \tilde{\lambda})$ such that*

$$\tilde{c} - \tilde{A}^T \tilde{\lambda} + \tilde{G}\tilde{y} = 0, \;\; \tilde{\lambda} > 0$$

when $\|(\tilde{c}, \tilde{G}, \tilde{A}) - (c, G, A)\| \leq \delta_3$ with $\tilde{G} \in \mathbb{S}^m_+$.

Proof. By Remark 2.1, we know that Slater condition of Problem (2.3) holds, namely there exists a λ depending on (c, G, A) such that

$$c - A^T \lambda + G y = 0, \;\; \lambda > 0.$$

Namely there exists $\varepsilon_1 > 0$, such that $\lambda \geq \varepsilon_1 \mathbf{1}_l$. What's more, matrix $[-A^T, G]$ is of row full rank when Assumption 2.2 holds. In fact, suppose that there exist $d_y \in \Re^m$ such that

$$\begin{pmatrix} -A \\ G \end{pmatrix} d_y = 0,$$

which implies that $Ad_y = 0$, $Gd_y = 0$ and $c^T d_y = 0$, or $d_y \in Y^*(u)^\infty$. Therefore we must have that $d_y = 0$ because otherwise $Y^*(u)$ is unbounded, a contradiction with Assumption 2.2. Thus we have that matrix $[-A^T, G]$ is of row full rank. The validity of slater condition for problem (2.8) is equivalent to the solvability of the following system in variable $(\tilde{y}, \tilde{\lambda})$:

$$\tilde{c} - \tilde{A}^T \tilde{\lambda} + \tilde{G}\tilde{y} = 0, \;\; \tilde{\lambda} > 0. \tag{2.10}$$

For $(\Delta c, \Delta A, \Delta G) = (\tilde{c}, \tilde{A}, \tilde{G}) - (c, A, G)$ with $\tilde{G} \in \mathbb{S}^m_+$ and $(\Delta y, \Delta\lambda) = (\tilde{y} - y, \tilde{\lambda} - \lambda)$, the first equality in (2.10) is equivalent to

$$\begin{aligned} 0 &= \tilde{c} - \tilde{A}^T(\lambda + \Delta\lambda) + \tilde{G}(y + \Delta y) \\ &= \tilde{c} - \tilde{A}^T\lambda - A^T\Delta\lambda + \tilde{G}y + \tilde{G}\Delta y, \\ &= \Delta c - \Delta A^T \lambda + \Delta G y - \tilde{A}^T \Delta\lambda + \tilde{G}\Delta y \end{aligned}$$

or

$$(-\tilde{A}^T, \tilde{G}) \begin{pmatrix} \Delta\lambda \\ \Delta y \end{pmatrix} = (-\Delta c + \Delta A^T \lambda - \Delta G y) \tag{2.11}$$

We define $M = (-A^T, G)$ and $\tilde{M} = (-\tilde{A}^T, \tilde{G})$, for $\Delta M = (-\Delta A^T, \Delta G)$, let $\Delta N = \Delta M M^T + M \Delta M^T + \Delta M \Delta M^T$, then when ΔM is small enough, $\tilde{M}\tilde{M}^T = MM^T + \Delta N$ is nonsingular.

We assume $\delta_3 > 0$ satisfies that $\tilde{M}\tilde{M}^T$ is nonsingular when $\|\Delta M\| \leq \delta_3$. Then we obtain from Sherman-Morrison-Woodbury formula that

$$
\begin{aligned}
\tilde{M}^+ &= \tilde{M}^T(\tilde{M}\tilde{M}^T)^{-1} \\
&= (M+\Delta M)^T(MM^T+\Delta N)^{-1} \\
&= (M+\Delta M)^T[(MM^T)^{-1} - (MM^T)^{-1}\Delta N[I_m + (MM^T)^{-1}\Delta N]^{-1}(MM^T)^{-1}] \\
&= M^+ + \Delta\Sigma,
\end{aligned}
$$

where $\Delta\Sigma$ satisfies $\|\Delta\Sigma\| = \mathrm{O}(\|\Delta M\|)$. Since $\tilde{M}\tilde{M}^T$ is nonsingular when $\|\Delta M\| \leq \delta_3$, we have that

$$
\begin{aligned}
\begin{pmatrix} \Delta\lambda^*(\Delta M) \\ \Delta y^*(\Delta M) \end{pmatrix} &:= \tilde{M}^+(-\Delta c + \Delta A^T\lambda - \Delta G y) \\
&= [M^+ + \Delta\Sigma](-\Delta c + \Delta A^T\lambda - \Delta G y)
\end{aligned} \tag{2.12}
$$

is the least square norm solution to (2.11). From the expression for $(\Delta y^*, \Delta\lambda^*)$ in (2.12), we may assume that $\delta_3 > 0$ small enough such that $\|\Delta\lambda^*(\Delta M)\| \leq \|\lambda\|/2$ when $\|(\tilde{c}, \tilde{G}, \tilde{A}) - (c, G, A)\| \leq \delta_3$ with $\tilde{G} \in \mathbb{S}^m_+$. Therefore

$$
\begin{pmatrix} \tilde{\lambda} \\ \tilde{y} \end{pmatrix} := \begin{pmatrix} \lambda \\ y \end{pmatrix} + \begin{pmatrix} \Delta\lambda^*(\Delta M) \\ \Delta y^*(\Delta M) \end{pmatrix}
$$

satisfies (2.10). The proof is completed. □

Lemma 2.3. *Let (c, G, A) be given with Assumption 2.2 being satisfied. Then, for any $(\widehat{c}, \widehat{G}, \widehat{A}) \in \mathbb{B}_{\delta_3}(c, G, A)$ with $\widehat{G} \in \mathbb{S}^m_+$, where δ_3 defined in Proposition 2.3,*

$$
\lim_{(\tilde{c},\tilde{G},\tilde{A}) \overset{\tilde{G}\in\mathbb{S}^m_+}{\longrightarrow} (\widehat{c},\widehat{G},\widehat{A})} \mathcal{E}(\tilde{c}, \tilde{G}, \tilde{A}) = \mathcal{E}(\widehat{c}, \widehat{G}, \widehat{A}).
$$

Proof. As the following inclusion

$$
\limsup_{(\tilde{c},\tilde{G},\tilde{A}) \overset{\tilde{G}\in\mathbb{S}^m_+}{\longrightarrow} (\widehat{c},\widehat{G},\widehat{A})} \mathcal{E}(\tilde{c}, \tilde{G}, \tilde{A}) \subset \mathcal{E}(\widehat{c}, \widehat{G}, \widehat{A}).
$$

is obvious, we only need to verify that

$$
\liminf_{(\tilde{c},\tilde{G},\tilde{A}) \overset{\tilde{G}\in\mathbb{S}^m_+}{\longrightarrow} (\widehat{c},\widehat{G},\widehat{A})} \mathcal{E}(\tilde{c}, \tilde{G}, \tilde{A}) \supset \mathcal{E}(\widehat{c}, \widehat{G}, \widehat{A}).
$$

For arbitrary $(\widehat{y}, \widehat{\lambda}) \in \mathcal{E}(\widehat{c}, \widehat{G}, \widehat{A})$, we now prove $(\widehat{y}, \widehat{\lambda}) \in \liminf_{(\tilde{c},\tilde{G},\tilde{A}) \overset{\tilde{G}\in\mathbb{S}^m_+}{\longrightarrow} (\widehat{c},\widehat{G},\widehat{A})} \mathcal{E}(\tilde{c}, \tilde{G}, \tilde{A})$. By Proposition 2.3, we have that there exists $(\overline{y}, \overline{\lambda})$ such that

$$
\widehat{c} - \widehat{A}^T\overline{\lambda} + \widehat{G}\overline{y} = 0, \ \overline{\lambda} > 0.
$$

For $(\Delta c, \Delta G, \Delta A)$ with $\Delta G \in \mathcal{R}_{\mathbb{S}^m_+}(\widehat{G})$, let $(\tilde{c}(t), \tilde{G}(t), \tilde{A}(t)) = (\widehat{c} + t\Delta c,\ \widehat{G} + t\Delta G,\ \widehat{A} + t\Delta A)$, we obviously have $(\tilde{c}(t), \tilde{G}(t), \tilde{A}(t)) \to (\widehat{c},\ \widehat{G},\ \widehat{A})$ as $t \downarrow 0$ and $\tilde{G}(t) \in \mathbb{S}^m_+$ for $t > 0$. Define

$$(y(t), \lambda(t)) = (\widehat{y}, \widehat{\lambda}) + t(\overline{y} - \widehat{y}, \overline{\lambda} - \widehat{\lambda}) + t(d_y(t), d_\lambda(t)). \tag{2.13}$$

We consider the system

$$\tilde{c}(t) - \tilde{A}(t)^T \lambda(t) + \tilde{G}(t) y(t) = 0. \tag{2.14}$$

Define

$$M(t) = [-\widehat{A} - t\Delta A \quad \widehat{G} + t\Delta G],\ \tilde{\lambda}_t = \widehat{\lambda} + t(\overline{\lambda} - \widehat{\lambda}),\ \tilde{y}_t = \widehat{y} + t(\overline{y} - \widehat{y}).$$

Then the equation (2.14) is equivalent to

$$M(t) \begin{pmatrix} d_\lambda(t) \\ d_y(t) \end{pmatrix} = -[\Delta c - \Delta A^T \tilde{\lambda}_t + \Delta G \tilde{y}_t]. \tag{2.15}$$

Let $(d^*_\lambda(t), d^*_y(t))$ be the following least square norm solution to (2.15):

$$\begin{pmatrix} d^*_\lambda(t) \\ d^*_y(t) \end{pmatrix} = -M(t)^\dagger [\Delta c - \Delta A^T \tilde{\lambda}_t + \Delta G \tilde{y}_t]. \tag{2.16}$$

Similar to the analysis in the proof of Proposition 2.3, we obtain $M(t)^\dagger = \widehat{M}^\dagger + \mathrm{O}(t\|\Delta M\|)$. Thus we assume that $\|M(t)^\dagger\| \leq 2\|\widehat{M}^\dagger\|$ for $(\tilde{c}, \tilde{G}, \tilde{A}) \in \mathbb{B}_{\delta_3}(\widehat{c}, \widehat{G}, \widehat{A})$. Let

$$\widehat{\epsilon} = \frac{\|\overline{\lambda}\|}{4\|\widehat{M}^\dagger\| \max\{1, \|\widehat{y}\|, \|\overline{y}\|, \|\widehat{\lambda}\|, \|\overline{\lambda}\|\}}.$$

Then for $\|(\Delta c, \Delta A, \Delta G)\| < \widehat{\epsilon}$ with $\Delta G \in \mathcal{R}_{\mathbb{S}^m_+}(\widehat{G})$, one has

$$\begin{aligned} \left\| \begin{pmatrix} d^*_\lambda(t) \\ d^*_y(t) \end{pmatrix} \right\| &\leq \|M(t)^\dagger\| \|[\Delta c - \Delta A^T \tilde{\lambda}_t + \Delta G \tilde{y}_t]\| \\ &\leq 2\|\widehat{M}^\dagger\| \max\{1, \|\widehat{y}\|, \|\overline{y}\|, \|\widehat{\lambda}\|, \|\overline{\lambda}\|\} \|(\Delta c, \Delta A, \Delta G)\| \\ &\leq \|\overline{\lambda}\|/2. \end{aligned}$$

$$(y^*(t), \lambda^*(t)) = (\widehat{y}, \widehat{\lambda}) + t(\overline{y} - \widehat{y}, \overline{\lambda} - \widehat{\lambda}) + t(d^*_y(t), d^*_\lambda(t)). \tag{2.17}$$

Then $(y^*(t), \lambda^*(t))$ satisfies equation (2.14) and for $t > 0$ small enough,

$$\lambda^*(t) = \widehat{\lambda} + t(\overline{\lambda} - \widehat{\lambda}) + t d^*_\lambda(t)) = (1 - t)\widehat{\lambda} + t(\overline{\lambda} + d^*_\lambda(t)) > 0.$$

Therefore, for small $t > 0$,

$$(y^*(t), \lambda^*(t)) \in \mathcal{E}(\widehat{c} + t\Delta c,\ \widehat{G} + t\Delta G,\ \widehat{A} + t\Delta A)$$

and $(y^*(t), \lambda^*(t)) \to (\widehat{y}, \widehat{\lambda})$. This implies

$$(\widehat{y}, \widehat{\lambda}) \in \liminf_{(\tilde{c},\tilde{G},\tilde{A}) \overset{\tilde{G}\in\mathbb{S}^m_+}{\longrightarrow} (\widehat{c},\widehat{G},\widehat{A})} \mathcal{E}(\tilde{c}, \tilde{G}, \tilde{A}),$$

and the proof is completed. □

Now we come back to Problem (2.9), the restricted Wolfe dual of problem (2.1). Denote the feasible set of Problem (2.9) by $\mathcal{R}(\tilde{u})$, namely

$$\mathcal{R}(\tilde{c}, \tilde{G}, \tilde{A}) = \mathcal{E}(\tilde{c}, \tilde{G}, \tilde{A}) \cap \operatorname{Range} \tilde{G} \times \Re^l.$$

Denote the objective function by $\phi(y, \lambda, \tilde{u}) = \lambda^T(\tilde{b} - \tilde{B}x) - \dfrac{1}{2}y^T\tilde{G}y$ and by $\Lambda^*(\tilde{u})$ the set of λ-part optimal solutions of Problem (2.9) when $x = \tilde{x}$.

The following corollary is from Lemma 2.3.

Corollary 2.1. *Let (c, G, A) be given with Assumption 2.2 being satisfied. Then, for any $(\widehat{c}, \widehat{G}, \widehat{A}) \in \mathbb{B}_{\delta_3}(c, G, A)$ with $\widehat{G} \in \mathbb{S}^m_+$, where δ_3 defined in Proposition 2.3,*

$$\lim_{(\tilde{c},\tilde{G},\tilde{A}) \overset{\tilde{G}\in\mathbb{S}^m_+}{\longrightarrow} (\widehat{c},\widehat{G},\widehat{A})} \mathcal{R}(\tilde{c}, \tilde{G}, \tilde{A}) = \mathcal{R}(\widehat{c}, \widehat{G}, \widehat{A}).$$

Define

$$\Gamma(\tilde{u}, \alpha) = \mathcal{R}(\tilde{c}, \tilde{G}, \tilde{A}) \cap \operatorname{lev}_{\geq\alpha}\phi(\cdot, \tilde{u})$$

with

$$\operatorname{lev}_{\geq\alpha}\phi(\cdot, \tilde{u}) = \{(y, \lambda) \in \Re^m \times \in \Re^l : \phi(y, \lambda, \tilde{u}) \geq \alpha\}, \alpha \in \Re.$$

Lemma 2.4. *For given (c, G, A, B, b), let Assumptions 2.1, 2.2 and 2.3 hold. Then for any $\alpha \in \Re^n$, there exists $\delta_3 > 0$ and a bounded set $\mathcal{D} \subset \Re^m \times \Re^l$ such that*

$$\Gamma(\tilde{u}, \alpha') \subset \mathcal{D}, \forall \alpha' \geq \alpha, \forall \tilde{u} \in \mathcal{U}_{\delta_3}(c, G, A, B, b).$$

Proof. Without loss of generality, we assume that $\Gamma(\tilde{u}, \alpha) \neq \emptyset$. Because $\Gamma(\tilde{u}, \alpha') \subset \Gamma(\tilde{u}, \alpha), \forall \alpha' \leq \alpha$, we only need to prove $\Gamma(\tilde{u}, \alpha) \subset \mathcal{D}$.

We first prove that, for any $(y, \lambda) \in \Gamma(\tilde{u}, \alpha)$, λ is bounded by contradiction. Suppose that there exist a sequence $\tilde{u}^k = (x^k, \xi^k)$ with $G^k \in \mathbb{S}^m_+$ such that $x^k \in X$ and $\xi^k \to (c, G, A, B, b)$ and $(y^k, \lambda^k) \in \Gamma(\tilde{u}^k, \alpha)$ with $\|\lambda^k\| \to \infty$. Let $d^k_\lambda = \lambda^k/\|\lambda^k\|$, $d^k_y = y^k/\|\lambda^k\|$, and notice X is compact, we can find a subsequence k_j such that $x^{k_j} \to x$ and $d^{k_j}_\lambda \to d_\lambda$ for some $x \in X$ with, $d_\lambda \in \operatorname{bdry}\mathbf{B}$. In view of $(y^{k_j}, \lambda^{k_j}) \in \Gamma(\tilde{u}^{k_j}, \alpha)$, one has

$$\begin{aligned}
&\lambda^{k_j T}(\tilde{b}^{k_j} - \tilde{B}^{k_j}x^{k_j}) - \frac{1}{2}y^{k_j T}\tilde{G}^{k_j}y^{k_j} \geq \alpha, \\
&\tilde{c}^{k_j} - \tilde{A}^{k_j T}\lambda^{k_j} + \tilde{G}^{k_j}y^{k_j} = 0, \\
&y^{k_j} \in \operatorname{Range} \tilde{G}^{k_j}, \\
&\lambda^{k_j} \geq 0
\end{aligned} \tag{2.18}$$

Dividing the first inequality in (2.19) by $\|\lambda^{k_j}\|^2$, we obtain from the positive semi-definiteness $\tilde{G}^{k_j}$ that

$$0 \geq -\frac{1}{2} d_y^{k_j T} \tilde{G}^{k_j} d_y^{k_j} \geq \alpha/\|\lambda^{k_j}\|^2 - \lambda^{k_j T}(\tilde{b}^{k_j} - \tilde{B}^{k_j} x^{k_j})/\|\lambda^{k_j}\|^2,$$

Taking the limits by $j \to \infty$, we have $d_y^{k_j T} \tilde{G}^{k_j} d_y^{k_j} \to 0$ or $\tilde{G}^{k_j \frac{1}{2}} d_y^{k_j} \to 0$, this implies $\tilde{G}^{k_j} d_y^{k_j} = \tilde{G}^{k_j \frac{1}{2}} \tilde{G}^{k_j \frac{1}{2}} d_y^{k_j} \to 0$ when $j \to \infty$. Combining the positive semi-definiteness of $\tilde{G}^{k_j}$ and the inequalities in (2.19), we have

$$\begin{aligned}
&\lambda^{k_j T}(\tilde{b}^{k_j} - \tilde{B}^{k_j} x^{k_j}) \geq \alpha, \\
&\tilde{c}^{k_j} - \tilde{A}^{k_j T} \lambda^{k_j} + G^{k_j} y^{k_j} = 0, \\
&y^{k_j} \in \operatorname{Range} \tilde{G}^{k_j}, \\
&\lambda^{k_j} \geq 0.
\end{aligned}$$

Dividing the above inequalities by $\|\lambda^{k_j}\|$, we get

$$\begin{aligned}
&d_\lambda^{k_j T}(\tilde{b}^{k_j} - \tilde{B}^{k_j} x^{k_j}) \geq \alpha/\|\lambda^{k_j}\|, \\
&\tilde{c}^{k_j}/\|\lambda^{k_j}\| - \tilde{A}^{k_j T} d_\lambda^{k_j} + G^{k_j} d_y^{k_j} = 0, \\
&d_y^{k_j} \in \operatorname{Range} \tilde{G}^{k_j}, \\
&d_\lambda^{k_j} \geq 0,
\end{aligned}$$

Taking the limits by $j \to \infty$, we have

$$d_\lambda^T(b - Bx) \geq 0, \;\; A^T d_\lambda = 0, \;\; d_\lambda \geq 0, \;\; \|d_\lambda\| = 1,$$

which contradicts with the compactness of the optimal solution set assumed in Assumption 2.2.

Now we prove that, for any $(y, \lambda) \in \Gamma(\tilde{u}, \alpha)$, y is bounded by contradiction. Suppose that there exist a sequence $\tilde{u}^k = (x^k, \xi^k)$ such that $x^k \in X$, $\xi^k \to (c, G, A, B, b)$ and $(y^k, \lambda^k) \in \Gamma(\tilde{u}^k, \alpha)$ with $\|y^k\| \to \infty$. From the first part of this lemma, we know that $\{\lambda^k\}$ is bounded. Let $d_\lambda^k = \lambda^k/\|\lambda^k\|$, $d_y^k = y^k/\|y^k\|$, and notice X is compact, we can find a subsequence k_j such that $x^{k_j} \to x$, $d_y^{k_j} \to d_y$ and $d_\lambda^{k_j} \to 0$ for some $x \in X$ with $d_y \in \operatorname{bdry}\mathbf{B}$. In view of $(y^{k_j}, \lambda^{k_j}) \in \Gamma(\tilde{u}^{k_j}, \alpha)$, one has

$$\begin{aligned}
&\lambda^{k_j T}(\tilde{b}^{k_j} - \tilde{B}^{k_j} x^{k_j}) - \frac{1}{2} y^{k_j T} \tilde{G}^{k_j} y^{k_j} \geq \alpha, \\
&\tilde{c}^{k_j} - \tilde{A}^{k_j T} \lambda^{k_j} + \tilde{G}^{k_j} y^{k_j} = 0, \\
&y^{k_j} \in \operatorname{Range} \tilde{G}^{k_j}, \\
&\lambda^{k_j} \geq 0
\end{aligned} \tag{2.19}$$

Dividing the first inequality in (2.19) by $\|y^{k_j}\|^2$, taking the limit for $j \to \infty$, we obtain from the positive semi-definiteness $\tilde{G}^{k_j}$ that $d_y^T G d_y = 0$ and $G d_y = 0$. From the third inclusion in

(2.19), we obtain $d_y \in \text{Range}\, G$. The relations $Gd_y = 0$ and $d_y \in \text{Range}\, G$ imply $d_y = 0$. This contradicts with $\|d_y\| = 1$. The proof is completed. □

Theorem 2.5. *For given (c, G, A, B, b), let Assumptions 2.1, 2.2 and 2.3 hold. For any $\widehat{u} \in \mathcal{U}_{\delta_3}(c, A, G, B, b)$ with δ_3 defined above, one has that the solution set mapping Λ^* is upper semi-continuous at $\widehat{u}$, namely for $\epsilon > 0$ there exists a number $\delta > 0$ such that*

$$\Lambda^*(\tilde{u}) \subset \Lambda^*(\widehat{u}) + \epsilon \boldsymbol{B}, \forall \tilde{u} \in \mathbb{B}_{\delta_3}(\widehat{u}) \text{ with } \tilde{G} \in \mathbb{S}^m_+.$$

Proof. The results in this theorem can be proved by Corollary 2.1 and Lemma 2.4. The proof is similar to that of Theorem 2.4. We omit it here. □

3 Differentiability of the optimal value function

In this section we discuss the differential properties of the optimal value of the lower level problem. The Lagrangian function of problem (2.1) is defined by

$$L(y, \lambda; \tilde{u}) = \tilde{c}^T y + \frac{1}{2} y^T \tilde{G} y + \lambda^T (\tilde{b} - \tilde{A} y - \tilde{B} \tilde{x}). \tag{3.1}$$

Define

$$\Theta(\tilde{u}, \alpha) = \{\lambda \in \Re^l : \exists y \in \Re^m \text{ such that } (y, \lambda) \in \Gamma(\tilde{u}, \alpha)\}.$$

From Lemma 2.2 and Lemma 2.4, we assume that for some $\delta_4 > 0$, $\alpha \in \Re$, and bounded sets $\mathcal{B}_p \in \Re^m$, $\mathcal{B}_d \subset \Re^l$,

$$\Psi(\tilde{u}, \alpha) \subset \mathcal{B}_p, \quad \Theta(\tilde{u}, \alpha) \subset \mathcal{B}_d$$

for any for any $\tilde{x} \in X$ and $\|\xi - (c, G, A, B, b)\| \leq \delta_4$. From Theorem 2.4 and Theorem 2.5, $\theta(\cdot)$ is continuous at $\tilde{u}$ when $\tilde{x} \in X$ and $\|\xi - (c, G, A, B, b)\| \leq \delta_1$, we may assume that $\delta_4 < \delta_1$ and $\alpha \in \Re$ such that

$$Y^*(\tilde{u}) \subset \Psi(\tilde{u}, \alpha),\ \Lambda^*(\tilde{u}) \subset \Theta(\tilde{u}, \alpha).$$

Therefore, by the Lagrange duality theory, the optimal value can be written as

$$\theta(\tilde{x}, \tilde{c}, \tilde{G}, \tilde{A}, \tilde{B}, \tilde{b}) = \max_{\lambda \in \mathcal{B}_d} \min_{y \in \mathcal{B}_p} L(y, \lambda; \tilde{u}). \tag{3.2}$$

The next proposition shows that the optimal value function $\theta(\tilde{x}, \tilde{c}, \tilde{G}, \tilde{A}, \tilde{B}, \tilde{b},)$ is locally Lipschitz continuous.

Proposition 3.1. *For given (c, G, A, B, b) and $x \in X$, let Assumptions 2.1, 2.2 and 2.3 hold. Then $\theta(\tilde{x}, \tilde{c}, \tilde{G}, \tilde{A}, \tilde{B}, \tilde{b})$ in $X \times \Re^n \times \mathbb{S}^m_+ \times \Re^{l \times m} \times \Re^{l \times n} \times \Re^l$ is locally Lipschitz continuous around (x, c, G, A, B, b), namely there exists some $\kappa \geq 0$ depending on (x, c, G, A, B, b) such that*

$$|\theta(\tilde{u}) - \theta(u')| \leq \kappa \|\tilde{u} - u'\|, \tag{3.3}$$

when $\tilde{u}, u' \in \mathbb{B}_{\delta_5}(x, c, G, A, B, b)$ *with* $\tilde{G}, G' \in \mathbb{S}^m_+$ *for some positive constant* $\delta_5 > 0$ *depending on* (x, c, G, A, B, b)*. Here* $\tilde{u} = (\tilde{x}, \tilde{c}, \tilde{G}, \tilde{A}, \tilde{B}, \tilde{b})$*,* $u' = (x', c', G', A', B', b')$ *and*

$$\|\tilde{u} - u'\| = \|c - c'\| + \|\tilde{G} - G'\| + \|\tilde{A} - A'\| + \|\tilde{B} - B'\| + \|\tilde{b} - b'\| + \|\tilde{x} - x'\|.$$

Proof. Since $L(\cdot, \cdot; \tilde{u})$ is continuous, the max-min values for parameters $\tilde{u}$ and u' can be arrived. Assume that

$$\theta(\tilde{u}) = L(\tilde{y}, \tilde{\lambda}; \tilde{u}), \quad \theta(u') = L(y', \lambda'; u')$$

for $(\tilde{y}, \tilde{\lambda}), (y', \lambda') \in \mathcal{B}_p \times \mathcal{B}_d$. Without less of generality, we assume that $\theta(\tilde{u}) \leq \theta(u')$. Then we have

$$\begin{aligned}
&|\theta(\tilde{u}) - \theta(u')| \\
&= |\sup_{\lambda \in \mathcal{B}_d} \inf_{y \in \mathcal{B}_p} L(y, \lambda, \tilde{u}) - \sup_{\lambda \in \mathcal{B}_d} \inf_{y \in \mathcal{B}_p} L(y, \lambda, u')| \\
&= |L(\tilde{y}, \tilde{\lambda}; \tilde{u}) - L(y', \lambda'; u')| \\
&= |L(\tilde{y}, \tilde{\lambda}; \tilde{u}) - L(\tilde{y}, \lambda'; \tilde{u}) + L(\tilde{y}, \lambda'; \tilde{u}) - L(y', \lambda'; u')| \\
&\leq |L(\tilde{y}, \lambda'; \tilde{u}) - L(y', \lambda'; u')| \\
&\leq |L(\tilde{y}, \lambda'; \tilde{u}) - L(\tilde{y}, \lambda'; u')| \\
&\leq \sup_{y \in \mathcal{B}_p} \sup_{\lambda \in \mathcal{B}_d} |L(y, \lambda; \tilde{u}) - L(y, \lambda; u')|
\end{aligned} \tag{3.4}$$

Choose $\delta_5 \leq \delta_4$ and define

$$\kappa = \max\{\text{Diam}(\mathcal{B}_p), \frac{1}{2}(\text{Diam}(\mathcal{B}_p))^2, \{1, \|B\| + \delta_5, \text{Diam}(X), \text{Diam}(\mathcal{B}_p)\} \times \text{Diam}(\mathcal{B}_d)\}.$$

Then, when $\|\tilde{u} - u'\| \leq \delta_5$, for $y \in \mathcal{B}_p$ and $\lambda \in \mathcal{B}_d$, we have

$$\begin{aligned}
&|L(y, \lambda; \tilde{u}) - L(y, \lambda; u')| \\
&\leq (\tilde{c} - c')^T y + \tfrac{1}{2} y^T (\tilde{G} - G') y - \lambda^T (\tilde{A} - A') y - \lambda^T (\tilde{B} - B') x' - \lambda^T \tilde{B} (\tilde{x} - x') + \lambda^T (\tilde{b} - b')| \\
&\leq \kappa \left\{ \|c - c'\| + \|\tilde{G} - G'\| + \|\tilde{A} - A'\| + \|\tilde{B} - B'\| + \|\tilde{b} - b'\| + \|\tilde{x} - x'\| \right\} \\
&= \kappa \|\tilde{u} - u'\|.
\end{aligned}$$

Combing the above inequality with (3.4), we obtain the inequality (3.3) when $\tilde{u}, u' \in \mathbb{B}_{\delta_5}(x, c, G, A, B, b)$. □

We recall the perturbation result about the minimax problem from Theorem 7.24 Shapiro, Dentcheva and Ruszczynski (2009) [11]. Consider the following min-max problem:

$$\min_{x \in X} \{\phi(x) := \sup_{y \in Y} f(x, y)\}, \tag{3.5}$$

and its dual:

$$\sup_{y \in Y} \{\iota(y) := \min_{x \in X} f(x, y)\}. \tag{3.6}$$

We assume that the set $X \subset \Re^n$ and $Y \subset \Re^m$ are convex and compact and the function $f : X \times Y \to \Re$ is continuous. Moreover, we assume that $f(x, y)$ is convex in $x \in X$ and concave in $y \in Y$. Consider the perturbation of the minimax problem (3.5):

$$\min_{x\in X} \sup_{y\in Y} \{f(x, y) + t\eta_t(x, y)\}, \tag{3.7}$$

where $\eta_t(x, y)$ is continuous in $X \times Y$, $t \geq 0$. Denoted by $\upsilon(t)$ the optimal value of the above problem (5). Clearly $\upsilon(0)$ is the optimal value of the unperturbed problem (3.5). Then the following lemma holds.

Lemma 3.1. *[11, Theorem 7.24] Suppose that:*

(i) The sets $X \subset R^n$ and $Y \subset R^m$ are convex and compact,

(ii) For all $t \geq 0$ the function $\zeta_t := f + t\eta_t$ is continuous on $X \times Y$, convex in $x \in X$ and concave in $y \in Y$,

(iii) η_t converges uniformly as $t \downarrow 0$ to a function $\gamma(x, y) \in C(X, Y)$.

Then

$$\lim_{t\to 0} \frac{\upsilon(t) - \upsilon(0)}{t} = \inf_{x\in X^*} \sup_{y\in Y^*} \gamma(x, y).$$

Theorem 3.1. *For given (c,G,A,B,b) and $x \in X$, let Assumptions 2.1, 2.2 and 2.3 hold. Then the optimal value function θ is directionally differentiable at (x, c, G, A, B, b) along any $(\Delta x, \Delta c, \Delta G, \Delta A, \Delta B, \Delta b) \in \Re^n \times \Re^n \times \mathcal{R}_{\mathbb{S}^m_+}(G) \times \Re^{l\times m} \times \Re^{l\times n} \times \Re^l$. Moreover, θ is Hadamard directionally differentiable at (x, c, G, A, B, b) in $\Re^n \times \Re^n \times \mathbb{S}^m_+ \times \Re^{l\times m} \times \Re^{l\times n} \times \Re^l$. Thus we obtain the following Taylor expansion of $\theta(\tilde{u})$ with $\tilde{G} \in \mathbb{S}^m_+$ at $u = (x, c, G, A, B, b)$:*

$$\begin{aligned} &\theta(\tilde{u}) = \theta(u) + \\ &+ \inf_{y\in Y^*(u)} \sup_{\lambda\in\Lambda^*(u)} \left\{ \Delta c^T y + \frac{1}{2} y^T \Delta G y + \lambda^T \Delta b - \lambda^T [\Delta A y + \Delta B x + B\Delta x] \right\} \\ &+ \mathrm{o}(\|\tilde{u} - u\|), \end{aligned} \tag{3.8}$$

where $\tilde{u} = (\tilde{x}, \tilde{c}, \tilde{G}, \tilde{A}, \tilde{B}, \tilde{b})$ with $\tilde{G} \in \mathbb{S}^m_+$, $u = (x, c, G, A, B, b)$ and $\Delta u = \tilde{u} - u$ satisfying $\|\Delta u\| \leq \delta_4$.

Proof. In the setting of Lemma 3.1, for the direction $\tilde{u} - u$, we define

$$\zeta_t(y, \lambda; u, \tilde{u}) = L(y, \lambda; u_t), \quad f(y, \lambda; u, \tilde{u}) = L(y, \lambda; u) \tag{3.9}$$

where $u_t = u + t(\tilde{u} - u)$, $\tilde{u} = (\tilde{x}, \tilde{c}, \tilde{G}, \tilde{A}, \tilde{B}, \tilde{b})$, $u = (x, c, G, A, B, b)$ and $\Delta u = \tilde{u} - u$. It is obvious that ζ_t is convex in y and concave in λ. Since

$$\begin{aligned}
&\eta_t(y,\lambda,u,\tilde{u}) := \frac{1}{t}[\zeta_t(y,\lambda;u,\tilde{u}) - f(y,\lambda;u,\tilde{u})] \\
&= L(y,\lambda;u_t) - L(y,\lambda;u)] \\
&= \Delta c^T y + \tfrac{1}{2} y^T \Delta G y + \lambda^T \Delta b - \lambda^T [\Delta A y + \Delta B x + B \Delta x + t \Delta B \Delta x] \\
&\longrightarrow \Delta c^T y + \tfrac{1}{2} y^T \Delta G y + \lambda^T \Delta b - \lambda^T [\Delta A y + \Delta B x + B \Delta x] =: \gamma(y,\lambda,u,\tilde{u})
\end{aligned}$$

and the convergence is uniform with respect to t, we have that condition (iii) in Lemma 3.1 is satsified. Therefore all conditions in Lemma 3.1 are satisfied and in turn we obtain

$$\begin{aligned}
&\lim_{t \downarrow 0} \frac{\theta(u + t(\tilde{u} - u)) - \theta(u)}{t} = \inf_{y \in Y^*(u)} \sup_{\lambda \in \Lambda^*(u)} \gamma(y,\lambda;u,\tilde{u}) \\
&= \inf_{y \in Y^*(u)} \sup_{\lambda \in \Lambda^*(u)} \left\{ \Delta c^T y + \frac{1}{2} y^T \Delta G y + \lambda^T \Delta b - \lambda^T [\Delta A y + \Delta B x + B \Delta x] \right\},
\end{aligned} \tag{3.10}$$

which means that θ is differentiable at u, and the directional derivative of θ at u along $\tilde{u} - u$ is given by

$$\theta'(u; \tilde{u} - u) = \inf_{y \in Y^*(u)} \sup_{\lambda \in \Lambda^*(u)} \left\{ \Delta c^T y + \frac{1}{2} y^T \Delta G y + \lambda^T \Delta b - \lambda^T [\Delta A y + \Delta B x + B \Delta x] \right\}.$$

From Proposition 3.1, we know that θ is locally Lipschitz continuous, thus it follows from Proposition 2.49 of [1] that θ is Hadmard directionally differentiable at (x, c, G, A, B, b) and the Taylor expansion of $\theta(\tilde{u})$ at $u = (x, c, G, A, B, b)$ can be expressed as in formula (3.8). □

Remark 3.1. *In the book Lee, Tam and Yen (2005) [4], a similar expression for $\theta'(u; \Delta u)$ is given, but the assumptions it adopts are strict when G is positively semi-definite. In Theorem 14.2 of this book, the similar result is obtained under three conditions: (1) the system $Ay \geq b$ is regular, namely Slater condition holds; (2) $\min_z z^T G z$ s.t. $Az \geq 0$ has only zero solution; (3) condition (G) (see Page 246 of [4]). If G is positively semi-definite, Condition (G) holds automatically, but in this case the following example shows that our assumptions for the expression of $\theta'(u; \Delta u)$ are weaker than those used in Theorem 14.2 of [4]. Let us consider the following quadratic program problem:*

$$\begin{aligned}
\min \quad & \tfrac{1}{2} y^T G y + c^T y \\
\text{s.t.} \quad & Ay \geq b,
\end{aligned}$$

where

$$G = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, c = [0, 1, 2]^T, b = [3, 0]^T.$$

The unique optimal solution is $(\frac{1}{2}, \frac{5}{2}, 0)^T$ *with optimal value* $\frac{11}{4}$*. The quadratic programming problem*

$$\begin{array}{ll} \min & \frac{1}{2} z^T G z \\ \text{s.t.} & Az \geq 0. \end{array}$$

has not only zero solution, but also nonzero solutions. For example, any point $(0, t, 0)^T$ *with* $t > 0$ *is a solution to this problem. This means that the second condition in Theorem 14.2 of Lee et. al (2005) fails and we can not derive the expression for* $\theta'(u; \Delta u)$*. However, conditions 2.2 and 2.3 are satisfied for this example and we can still obtain the expression for* $\theta'(u; \Delta u)$ *from Theorem 3.1.*

4 Asymptotical distribution of an SAA estimator for optimal value

In this section, we consider the asymptotic properties of the optimal value of the two stage problem (1.2). For $\theta(x, \xi)$ with $\xi = (\tilde{c}, \tilde{G}, \tilde{A}, \tilde{B}, \tilde{b})$ being a random variable, define

$$f(x, \xi) = g(x) + \theta(x, \xi),$$

then the two stage stochastic optimization problem is expressed as

$$\begin{array}{ll} \min & \mathbb{E}[f(x, \xi)] \\ \text{s.t.} & x \in X. \end{array} \tag{4.1}$$

Let $\xi^1, \ldots, \xi^N$ be an i.i.d. sample, then the sample average approximation problem is defined by

$$\begin{array}{ll} \min & \widehat{f}_N(x) \\ \text{s.t.} & x \in X, \end{array} \tag{4.2}$$

where

$$\widehat{f}_N(x) = f\left(x, \frac{1}{N} \sum_{i=1}^{N} \xi^i\right) = g(x) + \theta\left(x, \frac{1}{N} \sum_{i=1}^{N} \xi^i\right).$$

We denote the optimal value and the solution set of the two stage stochastic optimization problem, namely the optimal value of problem (4.1) by ν^* and S^*, respectively, and the optimal value of problem (4.2) by $\widehat{\nu}_N$.

Assumption 4.1. *Assume any two random elements in* $\{\tilde{c}, \tilde{G}, \tilde{A}, \tilde{B}, \tilde{b}\}$ *with* $\tilde{G} \in \mathbb{S}^m_+$ *are independent to each other. The expectation of* $\xi = (\tilde{c}, \tilde{G}, \tilde{A}, \tilde{B}, \tilde{b})$ *is* $\overline{p} = (c, G, A, B, b)$ *with* $G \in \mathbb{S}^m_+$*,* *i.e.,* $\mathbb{E}(\xi) = \overline{p}$. *Let* $\xi^1, \ldots, \xi^N$ *be an i.i.d. sample. For* $\xi^i = (\tilde{c}^i, \tilde{G}^i, \tilde{A}^i, \tilde{B}^i, \tilde{b}^i)$ *with* $\tilde{G}^i \in \mathbb{S}^m_+$, $i = 1, \ldots, N$ *and*

$$\widehat{\xi}_N = (\widehat{c}_N, \widehat{G}_N, \widehat{A}_N, \widehat{B}_N, \widehat{b}_N) = \frac{1}{N}\sum_{i=1}^{N}(\tilde{c}^i, \tilde{G}^i, \tilde{A}^i, \tilde{B}^i, \tilde{b}^i).$$

Assume that

$$\begin{aligned}
\sqrt{N}[\widehat{c}_N - c] &\xrightarrow{d} \mathcal{N}(0, \Sigma^c), \\
\sqrt{N}[\widehat{[G_{i\cdot}]}_N - [G_{i\cdot}]] &\xrightarrow{d} \mathcal{N}(0, \Sigma^G_i),\ \ i = 1, \ldots, m, \\
\sqrt{N}[\widehat{[B_{i\cdot}]}_N - [B_{i\cdot}]] &\xrightarrow{d} \mathcal{N}(0, \Sigma^B_i),\ \ i = 1, \ldots, l, \\
\sqrt{N}[\widehat{a}_{iN} - a_i] &\xrightarrow{d} \mathcal{N}(0, \Sigma^A_i),\ \ i = 1, \ldots, l, \\
\sqrt{N}[\widehat{b}_N - b] &\xrightarrow{d} \mathcal{N}(0, \Sigma^b),
\end{aligned}$$

where $\xrightarrow{d}$ *denotes convergence in distribution.*

The following lemma is Theorem 7.59 of [1], the delta theorem, which will be used to analyze the first order asymptotical property of the SAA optimal value ν_N.

Lemma 4.1. *[1, Theorem 7.59] Let* B_1 *and* B_2 *be Banach spaces, equipped with their Borel* σ*-algebras,* Z_N *be a sequence of random elements of* B_1, $G : B_1 \to B_2$, *be a mapping, Suppose that:*

(i) the space B_1 *is separable,*

(ii) the mapping G *is Hadamard directionally differentiable at a point* $\mu \in B_1$,

(iii) for some sequence τ_N *of positive numbers tending to infinity, as* $N \to \infty$*,the sequence* $X_N := \tau_N(Z_N - \mu)$ *converges in distribution to a random element* Z *of* B_1.

Then

$$\tau_N[G(Z_N) - G(\mu))] \xrightarrow{d} G'(\mu; Z).$$

Now we are in a position to present the main theorem about the asymptotical property of the SAA optimal value ν_N.

Theorem 4.2. *Let Assumption 4.1 hold. Then*

$$N^{1/2}(\widehat{\nu}_N - \nu^*) \xrightarrow{d} \inf_{x\in S^*} \inf_{y\in Y^*(\overline{p})} \sup_{\lambda\in\Lambda^*(\overline{p})} \{V(x, y, \lambda)\}, \tag{4.3}$$

where $V(x, y, \lambda)$ *is the random variable depending on* (x, y, λ):

$$V(x, y, \lambda) \sim \mathcal{N}\left(0, y^T\Sigma^c y + \frac{1}{4}\sum_{i=1}^{m} y_i^2 y^T \Sigma^G_i y + \lambda^T\Sigma^b\lambda + \sum_{i=1}^{l}\lambda_i^2[y^T\Sigma^A_i y + x^T\Sigma^B_i x]\right). \tag{4.4}$$

Moreover, if $S^* = \{\overline{x}\}, Y^*(\overline{p}) = \{\overline{y}\}$, $\Lambda^*(\overline{p}) = \{\bar{\lambda}\}$, *we have*

$$N^{1/2}(\widehat{\nu}_N - \nu^*) \xrightarrow{d} \mathcal{N}\left(0, \overline{y}^T\Sigma^c\overline{y} + \frac{1}{4}\sum_{i=1}^{m}\overline{y}_i^2\overline{y}^T\Sigma_i^G\overline{y} + \overline{\lambda}^T\Sigma^b\overline{\lambda} + \sum_{i=1}^{l}\overline{\lambda}_i^2[\overline{y}^T\Sigma_i^A\overline{y} + \overline{x}^T\Sigma_i^B\overline{x}]\right). \quad (4.5)$$

Proof. First, we use Lemma 4.1 to analyze the asymptotical property of $N^{1/2}(\widehat{f}_N(x) - \mathbb{E}(f(x,\xi))$. Let $B_1 = \Re^m \times \Re^{m\times m} \times \Re^{l\times m} \times \Re^{l\times n} \times \Re^l$, $B_2 = C(X)$, and $G : B_1 \to B_2$:

$$G(\xi) = g(x) + \theta(x,\xi).$$

Let $\tau_N = N^{1/2}$,$Z_N = \widehat{\xi}_N$, $\mu = \overline{p}$. Then we have $\tau_N(Z_N - \mu) \xrightarrow{d} Z$ from Assumption 4.1 with

$$\begin{aligned} Z^c &\sim \mathcal{N}(0,\Sigma^c), \\ Z^{[G_{i.}]} &\sim \mathcal{N}(0,\Sigma_i^G),\ i = 1,\dots,m, \\ Z^{[B_{i.}]} &\sim \mathcal{N}(0,\Sigma_i^B),\ i = 1,\dots,l, \\ Z^{a_i} &\sim \mathcal{N}(0,\Sigma_i^A),\ i = 1,\dots,l, \\ Z^b &\sim \mathcal{N}(0,\Sigma^b). \end{aligned} \quad (4.6)$$

Then (i) in Lemma 4.1 is obvious and (iii) in Lemma 4.1 is guaranteed by Assumption 4.1. It follows from Theorem 3.1 that G is Hadamard directionally differentiable at a point μ, namely (ii) in Lemma 4.1 is satisfied. Therefore, we have from Lemma 4.1 that

$$N^{1/2}[f(x,\widehat{\xi}_N) - f(x,\mathbb{E}\xi)] \xrightarrow{d} G'(\mu;Z).$$

Noting that $G(\mu;Z) = \theta'(\mu;Z)$, one has from Theorem 3.1 that

$$G'(\mu;Z) = \inf_{y\in Y^*(\overline{p})}\sup_{\lambda\in\Lambda^*(\overline{p})}\left\{Z^{cT}y + \frac{1}{2}\sum_{i=1}^{m}y_iZ^{G_{i.}T}y + \lambda^TZ^b - \sum_{i=1}^{l}\lambda_i[Z^{a_iT}y + Z^{B_{i.}T}x]\right\}$$

Let $V(x,y,\lambda) = \left\{Z^{cT}y + \frac{1}{2}\sum_{i=1}^{m}y_iZ^{G_{i.}T}y + \lambda^TZ^b - \sum_{i=1}^{l}\lambda_i[Z^{a_iT}y + Z^{B_{i.}T}x]\right\}$, then we have from (4.6) that

$$V(x,y,\lambda) \sim \mathcal{N}\left(0, y^T\Sigma^cy + \frac{1}{4}\sum_{i=1}^{m}y_i^2y^T\Sigma_i^Gy + \lambda^T\Sigma^b\lambda + \sum_{i=1}^{l}\lambda_i^2[y^T\Sigma_i^Ay + x^T\Sigma_i^Bx]\right).$$

Property (4.3) follows from Theorem 5.7 of [11] directly. Obviously, when $S^* = \{\overline{x}\}, Y^*(\overline{p}) = \{\overline{y}\}$, $\Lambda^*(\overline{p}) = \{\bar{\lambda}\}$, we obtain (4.5) from (4.3). The proof is completed. □

5 Conclusion

Based on the upper semi-continuity of solution mappings for a convex quadratic programming problem and its restricted dual under some conditions of the primal quadratic programming problem, we establish the Hadamard directionally differentiability of the optimal value function of all parameters in the quadratic programming problem. Using a delta theorem developed in [11], we derive the asymptotic distribution of a SAA estimator for the optimal value of a two stage program whose second stage problem is a convex quadratic programming problem and all parameters in the quadratic program are random variables. There are several issues should be considered in the future study. The first problem arises in Assumption 4.1, in which we assume that $\{\tilde{c}, \tilde{G}, \tilde{A}, \tilde{B}, \tilde{b}\}$ with $\tilde{G} \in \mathbb{S}^m_+$ are independent to each other, this is a very strict assumption and should be weakened. The second problem is the study about the rate of convergence for the SAA approach for solving Problem (1.2). Noting that the SAA estimator in Section 4 is different from the one studied in [12], we would like know the asymptotical distribution of the estimator in [12], this is the third problem needed to be addressed.

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