

# A simple preprocessing algorithm for semidefinite programming

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## Abstract

We propose a very simple preprocessing algorithm for semidefinite programming. Our algorithm inspects the constraints of the problem, deletes redundant rows and columns in the constraints, and reduces the size of the variable matrix. It often detects infeasibility. Our algorithm does not rely on any optimization solver: the only subroutine it needs is Cholesky factorization, hence it can be implemented with a few lines of code in machine precision. We present computational results on a set of problems arising mostly from polynomial optimization.

## 1 Introduction and the preprocessing algorithm

Preprocessing is a key component in optimization solvers, in particular, in solvers of semidefinite programs (SDPs). SDPs – whether they are formulated directly by a user, or whether they are the output of an algebraic modeling language – often have redundant constraints, or they may even be infeasible. It is, of course, useful to detect these anomalies in a preprocessing stage. This paper provides a very simple preprocessing algorithm for SDPs, which can be implemented in a few line of codes in machine precision.

We consider semidefinite programming problems in the form

$$\begin{aligned} \inf \quad & C \bullet X \\ \text{s.t.} \quad & A_i \bullet X = b_i \quad (i = 1, \dots, m) \\ & X \succeq 0 \end{aligned} \tag{P}$$

where the  $A_i$  are  $n$  by  $n$  symmetric matrices, the  $b_i$  scalars,  $X \succeq 0$  means that  $X$  is symmetric, positive semidefinite (psd), and the  $\bullet$  dot product of symmetric matrices is the trace of their regular product. We also write  $X \succ 0$  to denote that  $X$  is symmetric and positive definite.

Our preprocessing algorithm is a very simple variant of facial reduction algorithms [1, 8, 6, 4], which is also able to detect infeasibility. However, instead of solving SDP subproblems, like the algorithms proposed in these papers, it reduces the size of (P) by simply inspecting the constraints. Another related work is by Permenter and Parrilo in [7], which finds reductions in SDPs by solving linear programming subproblems; our method is quite different though, since we do not rely on any optimization solver.

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To motivate our algorithm, let us consider the following example:

**Example 1.** The SDP instance (with an arbitrary objective function)

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet X &= 0 \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \bullet X &= -1 \\ X &\succeq 0, \end{aligned} \tag{1.1}$$

is infeasible. Indeed, if  $X = (x_{ij})_{i,j=1}^3$  were feasible in it, then  $x_{11} = 0$ , hence the first row and column of  $X$  are zero by positive semidefiniteness, so the second constraint implies  $x_{22} = -1$ , which is a contradiction.

Our preprocessing algorithm repeats the following basic step:

**Basic step**

- Find a constraint in (P) which, after permuting rows and columns, and possibly multiplying both sides by  $-1$ , is of the form

$$\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \bullet X = b, \tag{1.2}$$

where  $D \succ 0, b \leq 0$ .

- If  $b < 0$  STOP; (P) is infeasible.
- If  $b = 0$ , delete this constraint; also delete all rows and columns in the other constraints that correspond to rows and columns of  $D$ .

We stress that to find a constraint of the form (1.2), we are only allowed to permute rows and columns in a constraint and to multiply both sides of a constraint by  $-1$ ; we do not take linear combinations of the constraints.

In Example 1 in the first execution of the Basic step we find the first constraint, delete it, and also delete the first row and column from the second constraint matrix. In the second execution of the Basic step we find the constraint

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bullet X = -1,$$

and declare that (1.1) is infeasible.

The idea of reducing SDPs by simply inspecting constraints already appears in several papers. Gruber and Rendl [3], and Friberg [2] note that if  $A \bullet X = 0$  is a constraint in an SDP with  $A \succeq 0$ , then we can constrain  $X$  to lie in a face of the positive semidefinite cone. These papers, however, do not assume the particular form of  $A$  that we use and they do not detect infeasibility.

Also, to the best of our knowledge, no such method has been implemented.

## 2 Computational results

We implemented our algorithm in Matlab, using incomplete Cholesky factorization to check positive definiteness. Since this subroutine of Matlab works in machine precision, which is of the order  $10^{-16}$ , our entire algorithm is implemented in machine precision.

Why should we preprocess SDPs? On the one hand, as our computational results show, we cannot significantly reduce solution times on the tested instances. On the other hand, we often detect their infeasibility. Even when we do not, we bring the preprocessed instances closer to being strictly feasible, and strictly feasible problems behave better both from the theoretical, and the numerical point of view. Since we do all computations with machine precision, we believe that we should always preprocess a problem, when we can.

Also, the constraints that we detect induce a fairly simple redundancy. This redundancy should be clear even to a user not trained in optimization. Thus, returning the constraints that we find is likely to help him/her to better formulate other problems.

We compare our preprocessing algorithm with the algorithm proposed by Permenter and Parrilo in [7]. Their algorithm solves linear programming subproblems to reduce the size of an SDP. It can work either on the problem (P), which we call the *primal*; or on its dual. It can use either diagonal, or diagonally dominant reductions – for details, see [7].

We calculated the DIMACS error measures ([5]) for all instances before and after preprocessing. In general, we will use the notation

$$\text{err}_{\text{before}} \text{ and } \text{err}_{\text{after}}$$

for the worst, i.e., largest DIMACS error measure before and after preprocessing for an instance. (Which particular instance we refer to will be clear from the context.) We also solved all instances using SDPT3 before and after preprocessing.

We will say that the preprocessing (either ours, or the versions from [7]) *helped* a problem, if one of the following happens:

- (1) It detects infeasibility, or
- (2)  $\text{err}_{\text{before}} > 10^{-6}$  and

$$\frac{\text{err}_{\text{before}}}{\text{err}_{\text{after}}} < \frac{1}{10};$$

or

- (3) The optimal values before and after preprocessing differ by at least  $10^{-6}$ .

We report our computational results in Table 1 on the problem set of 49 instances from [7]. Our algorithm is denoted by FPT for the initials of the authors. The algorithms in [7] are denoted by PP, and we also indicate whether they are applied to the primal or the dual problem, and whether they use diagonal (d) or diagonally dominant (dd) approximations.

In the first row we report the number of instances on which the algorithms found *some* reduction. In the second row we report the preprocessing time. In the third row we report the number of problems on which infeasibility is found, and fourth row the number of problems which are helped by the preprocessing.

It is important to note that the algorithms of Permenter and Parrilo do not detect infeasibility by themselves; to make a fair comparison we report if SDPT3 finds a problem infeasible after it has been preprocessed by one of their algorithms.

	FPT	PP-Primal(d)	PP-Primal(dd)	PP-Dual(d)	PP-Dual(dd)
# Problems Preprocessed	31	28	35	10	13
Total Preprocessing Time	33.42	30.630	96.02	6.20	89.29
Infeasibility Detect	14	15 **	16 **	1 **	0
Number Problems Helped	24	22	25	4	5

Table 1: Computational results. "\*\*\*" means that SDPT3 detects infeasibility

The total time that SDPT3 took in solving the problems *before* preprocessing is 253.8 seconds; the time it took after preprocessing is 199.2 seconds. Thus the preprocessing only moderately helps in reducing the solution time.

In conclusion, our algorithm is competitive with the algorithms of [7] in terms of finding reductions, and detecting infeasibility. At the same time it is simpler and it does not rely on an optimization solver; thus we expect it to be at least as accurate, or more accurate.

We are currently exploring finding redundancies in (P) by detecting constraints of the form

$$A \bullet X = b,$$

where  $A \succeq 0$ ,  $b \leq 0$ .

We prepared this version of the paper to be ready for the ICCOPT 2016 conference. In the final version we will add more tables and more details about our computational results.

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