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On a Practical Notion of Geoffrion Proper Optimality in Multicriteria Optimization

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Geoffrion proper optimality is a widely used optimality notion in multicriteria optimization that prevents exact solutions having unbounded trade-offs. As algorithms for multicriteria optimization usually give only approximate solutions, we analyze the notion of approximate Geoffrion proper optimality. We show that in the limit, approximate Geoffrion proper optimality may converge to solutions having unbounded trade-offs. Therefore, we introduce a restricted notion of approximate Geoffrion proper optimality and prove that this restricted notion alleviates the problem of solutions having unbounded trade-offs. Furthermore, using a characterization based on infeasibility of a system of inequalities, we investigate two convergence properties of different approximate optimality notions in multicriteria optimization. These convergence properties are important for algorithmic reasons. The restricted notion of approximate Geoffrion proper optimality seems to be the only approximate optimality notion that shows favourable convergence properties. This notion bounds the trade-offs globally and can be used in multicriteria decision making algorithms as well. Due to these, it seems to be a practical optimality notion.

Keywords: Geoffrion proper optimality; multicriteria optimization; trade-offs; approximate solutions

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1. Introduction

Consider the following multicriteria optimization problem:

$$\begin{aligned} &\min f(x) := (f_1(x), \ldots, f_m(x)), \\ &\text{subject to } \ x \in X, \end{aligned}$$

where each criterion $f_i : \mathbb{R}^n \to \mathbb{R}$, the constraint set $X \subseteq \mathbb{R}^n$, and $n, m \in \mathbb{N}$. In what follows we shall also denote this multicriteria optimization problem by the tuple (f, X). Solving a multicriteria optimization problem requires a binary ordering relation on $\mathbb{R}^m$. Given an ordering relation, one can compare two m-dimensional vectors from the set $f(X) := \{f(x)|x \in X\}$ and define an optimality notion. This optimality notion is used in iterative algorithms to find one or many optimal solutions of the multicriteria optimization problem.

The theory of multicriteria optimization is one of the most rapidly growing areas of modern optimization theory, see for example Luc [1], Miettinen [2], Ehrgott

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[3], Jahn [4], Eichfelder [5] and the references therein. There are multiple solution concepts in mulicriteria optimization, and it often become a challenging issue both in theory and practice. Although Pareto optimality plays a central role in multicriteria optimization, other solution concepts like weak Pareto optimal solutions, strict Pareto solutions etc. are important as well.

The image of the set of Pareto optimal solutions as is well known lies on the boundary of the feasible set in the criteria space. This image in the criteria space is referred to as the efficient frontier. In the most of practical and large scale problems, the user may not get the exact efficient frontier and he has to be content with approximate solutions.

Pareto optimal solutions can only be improved in one criterion by simultaneously deteriorating another criteria. Thus, the notion of trade-offs or gain-to-loss ratios is inherent if one compares two Pareto optimal solutions. All points on the efficient frontier need not have equally nice trade-off properties which a decision maker may desire and one may need to filter out the bad Pareto optimal solutions and keep the good ones. Thus the need for potentially good solutions has always been one of the primary aims in multicriteria optimization. Such nice Pareto points are referred to in the literature as proper Pareto solutions. The study of proper Pareto solutions was first carried out by Kuhn and Tucker [6] and then followed by Geoffrion [7], Benson [8, 9], Henig [10], Borwein[11], and others [12–14]. Good solutions can be thought of as "Knee-points" on the efficient frontier or that are good in some sense like that of Geoffrion, Benson, Henig etc.

Unless (f, X) has a special structure, e.g. X being polyhedral and each f_i being linear or quadratic, almost all the algorithms for solving (f, X) generates an *infinite* sequence of iterates (or of a set of points, called as a population) that converges to the set of optimal points of (f, X). For computational reasons, it is usually necessary to terminate the algorithm after some finite number of iterations. This leads to a sub-optimal point, or, depending upon the distance of the obtained point to the set of optimal points of (f, X), an ϵ-optimal point.

In what follows, unless otherwise stated, we will consider $\epsilon \in \mathbb{R}^m_+$, i.e. $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_m)$ and $\epsilon_i \geq 0$ for each $i \in I := \{1, 2, \ldots, m\}$.

Definition 1.1 *A point $x^* \in X$ is said to be an ϵ-Pareto optimal solution of (f, X), if*

$$\nexists x \in X : \quad f(x) + \epsilon - f(x^*) \in -\mathbb{R}^m_+ \setminus \{0\}.$$

If, on the other hand,

$$\nexists x \in X : \quad f(x) + \epsilon - f(x^*) \in -\operatorname{int}(\mathbb{R}^m_+),$$

then x^* is said to be a weak ϵ-Pareto optimal solution of (f, X).

The set of all ϵ-Pareto points will be denoted by $\mathcal{S}^\epsilon(f, X)$ and the set of all weak ϵ Pareto points as $S^\epsilon_w(f, X)$. An (a weak) ϵ-Pareto optimal solution with $\epsilon = 0$ is commonly known as (weak) Pareto optimal. For clarity, the set of Pareto optimal solutions and the set of weak Pareto optimal solutions will be denoted by $\mathcal{S}(f, X)$ and $\mathcal{S}_w(f, X)$, respectively.

The notions of Pareto optimality is widely used in finite dimensional multicriteria optimization. However, even in the finite dimensional setting, it is some times useful to consider Pareto and ϵ-Pareto solutions and their weak counterpart in terms of an ordering relation that is induced by a closed, convex, and pointed cone C such that $C \supseteq \mathbb{R}^m_+$. Such a cone C is also called as an ordering cone. Eichfelder [5, 15] has provided many examples where in a natural way, one needs to consider a $C \neq \mathbb{R}^m_+$ as the ordering cone.

DEFINITION 1.2 *A point $x^* \in X$ is said to be an (ϵ, C) optimal solution of (f, X), if*

$$\nexists x \in X : \quad f(x) + \epsilon - f(x^*) \in -C \setminus \{0\}.$$

If, on the other hand,

$$\nexists x \in X : \quad f(x) + \epsilon - f(x^*) \in -\mathrm{int}(C),$$

then x^* is said to be a weak (ϵ, C) optimal solution of (f, X).

The set of all (ϵ, C) optimal solutions will be denoted by $\mathcal{S}^\epsilon(f, X, C)$ and the set of all weak (ϵ, C) optimal solutions as $\mathcal{S}^\epsilon_w(f, X, C)$.

The above optimality notions gives to a set of optimal solutions in multicriteria optimization. As argued at the beginning of this paper, different Pareto optimal solutions might have different properties that a decision maker may desire. The need to filter out bad Pareto optimal solutions lead to the notion of a properly efficient point. There are different notions of proper optimality; a nice survey can be found in [16]. Geoffrion proper optimality, for example, bounds the trade-off between criteria values and is one of the most cited and widely used notion of proper optimality. Combining Geoffrion proper optimality with approximate solutions leads to the following notion of Geoffrion ϵ-proper optimality [3, 17].

DEFINITION 1.3 *A point $x_0 \in X$ is called Geoffrion ϵ-proper solution if $x_0 \in \mathcal{S}^\epsilon(f, X)$ and if there exists a number $M > 0$ such that for all $(i, x) \in I \times X$ satisfying $f_i(x) < f_i(x_0) - \epsilon_i$, there exists a $j \in I$ such that $f_j(x_0) - \epsilon_j < f_j(x)$ and additionally*

$$\frac{f_i(x_0) - f_i(x) - \epsilon_i}{f_j(x) - f_j(x_0) + \epsilon_j} \leq M.$$

The set of Geoffrion ϵ-proper solutions will be denoted by $\mathcal{G}_\epsilon(f, X)$. When $\epsilon = 0$ then x_0 is the exact or usual Geoffrion-proper solution. For clarity, we will use $\mathcal{G}(f, X)$ to denote the set of Geoffrion-proper solutions.

Geoffrion ϵ-proper solution says that the trade-offs between criteria at two points are bounded. It is interesting to ask whether the trade-offs are bounded above for all the points. This would mean that the same M can work for all the Geoffrion ϵ-proper solutions. This may not always be the case unless the trade-offs are bounded above.

This trade-off bound M can be thought of as the utility or usefulness of the solution x_0. A decision maker may be interested in obtaining (all) solutions for which the trade-offs are bounded above by a given M. This motivates us to define the notion of a Geoffrion $(\hat{M}, \epsilon)$-proper solution which is in some sense reverse of the definition of Geoffrion ϵ-proper solution and appears to be more practical.

DEFINITION 1.4 *Given an $\hat{M} > 0$, a point $x_0 \in X$ is called Geoffrion $(\hat{M}, \epsilon)$-proper solution if $x_0 \in \mathcal{S}^\epsilon(f, X)$ and for all $(i, x) \in I \times X$ satisfying $f_i(x) < f_i(x_0) - \epsilon_i$,* *there exists a* $j \in I$ *such that* $f_j(x_0) - \epsilon_j < f_j(x)$ *and additionally*

$$\frac{f_i(x_0) - f_i(x) - \epsilon_i}{f_j(x) - f_j(x_0) + \epsilon_j} \leq \hat{M}.$$

We shall denote the set of all Geoffrion $(\hat{M}, \epsilon)$-proper solutions by $\mathcal{G}_{\hat{M},\epsilon}(f, X)$. To avoid any ambiguity with the set of Geoffrion ϵ-proper solutions, we will use $\mathcal{G}_{\hat{M},0}(f, X)$ to denote the set of (exact) Geoffrion $(\hat{M}, 0)$-proper solutions. Geoffrion $(\hat{M}, 0)$-proper solutions have been introduced in [18].

It is now important to see how the definition of a Geoffrion $(\hat{M}, \epsilon)$-proper solution differs from the standard notion of a Geoffrion proper Pareto solution. In some sense they seem to be reverse to each other. Note that in case of the Geoffrion ϵ-proper optimality we need to show the existence of an $M > 0$ such that a given ϵ-Pareto solution satisfies the trade-off inequality in terms of that M. On the other hand, in case of the Geoffrion $(\hat{M}, \epsilon)$-proper solution the decision maker chooses a global trade-off bound $\hat{M}$ beforehand which signals his or her intention to choose only those Geoffrion ϵ-proper solutions whose trade-off is bounded by the chosen value $\hat{M}$. By choosing $\hat{M}$, the decision maker has a direct control on the trade-offs. Therefore, the notion of Geoffrion $(\hat{M}, \epsilon)$-proper optimality looks more practical than Geoffrion $(\hat{M}, \epsilon)$-proper optimality.

We will shows in Section 3 that Geoffrion ϵ-proper optimality has algorithmic advantages as well. In particular, we will investigate the following two convergence aspects of different notions ϵ optimality:

(1) Does a sequence of approximate solutions converge to an exact solution?
(2) Does a similar result holds for a set of approximate solutions?

Algorithms for solving multicriteria optimization problems iteratively approach the efficient frontier. Therefore, the above two convergence aspects are relevant. We will prove that, as ϵ goes to zero, a sequence of ϵ-Pareto optimal solutions *does not converge* to Pareto optimal solution. If f is bounded below, then a sequence of Geoffrion ϵ-proper solutions also *does not converge* to a Geoffrion proper solution. In the limit, these sequence converge to weak Pareto optimal solution. Weak Pareto optimal solutions are, however, not desirable in practice as there migt be scenarios where it is possible to improve a criterion without deteriorating other criteria. On the other hand, we will prove that a sequence of Geoffrion $(\hat{M}, \epsilon)$-proper solutions always converges to a Geoffrion $(\hat{M}, 0)$-proper solution.

The flip-side is, however, that there may not be any Pareto optimal solution which is Geoffrion $(\hat{M}, \epsilon)$-proper for a very small value of $\hat{M}$. Nevertheless, we show in Section 2 that under very mild conditions $\mathcal{G}_{\hat{M},\epsilon}(f, X)$ is nonempty for every $\hat{M} \geq m - 1$.

It is important to note that that if x_0 is a Geoffrion $(\hat{M}, \epsilon)$-proper solution that it also a Geoffrion $(\tilde{M}, \epsilon)$-proper solution for every $\tilde{M} \geq \hat{M}$, i.e.

$$\forall\ \tilde{M} \geq \hat{M} : \quad \mathcal{G}_{\hat{M},\epsilon}(f, X) \subseteq \mathcal{G}_{\tilde{M},\epsilon}(f, X).$$

Moreover it is simple to observe that if x_0 is a Geoffrion ϵ-proper solution, then it is also a Geoffrion $(\hat{M}, \epsilon)$-proper for some $\hat{M} > 0$. Thus we have

$$\bigcup_{\hat{M}>0} \mathcal{G}_{\hat{M},\epsilon}(f, X) = \mathcal{G}_\epsilon(f, X).$$

Obviously the question remains as how would a decision maker choose an $\hat{M}$. Though some rough approaches may be suggested, for example the decision making approaches from [19, 20], we would like to keep the concept of Geoffrion (M, ϵ)-proper solutions open to discussion.

The paper has been organized in four sections of which this is the first. Section 2 characterizes Geoffrion proper optimality using infeasibility of a system of inequalities. Convergence aspects of different ϵ-optimal points are detailed in Section 3. Finally conclusions and algorithmic implications of these results are discussed in the last section.

2. Characterization and existence results

Geoffrion proper solutions can be characterized and analyzed by infeasibility of a system of inequalities. For an element $(x_0, \epsilon, i, M, \mathcal{X}) \in X \times \mathbb{R}^m_+ \times I \times \mathbb{R}_+ \times 2^X$, let $\Gamma(x_0, \epsilon, i, M, \mathcal{X})$ denote the system

$$\begin{cases} -f_i(x_0) + f_i(x) + \epsilon_i < 0, \\ -f_i(x_0) + f_i(x) + \epsilon_i < M(f_j(x_0) - f_j(x) - \epsilon_j) \ \forall j \in I \setminus \{i\}, \\ x \in \mathcal{X}. \end{cases}$$

To characterize Geoffrion proper solutions we will also make use of following fundamental properties in multicriteria optimization.

Definition 2.1 *If for every $x \in X \setminus \mathcal{S}(f, X)$ there exists an $x(s) \in \mathcal{S}(f, X)$ such that $f(x) - f(x(s)) \in \mathbb{R}^m_+ \setminus \{0\}$, then (f, X) is said to satisfy the* domination property. *If for every $x \in X \setminus \mathcal{S}^\epsilon(f, X)$ there exists an $x(s) \in \mathcal{S}^\epsilon(f, X)$ such that $f(x) - f(x(s)) + \epsilon \in \mathbb{R}^m_+ \setminus \{0\}$, then (f, X) is said to satisfy the* ϵ-domination property.

These properties relate to the existence of Pareto-optimal solutions and are quite weak. They are satisfied if, for every $x \in X$, the sections $f(X) \cap (f(x) - \mathbb{R}^m_+)$ are compact. Other sufficient conditions for the above properties to hold can be found in [21, 22].

The next two propositions characterize $\mathcal{G}_\epsilon(f, X)$ and $\mathcal{G}_{\hat{M},\epsilon}(f, X)$ using infeasibility of $\Gamma(x_0, \epsilon, i, M, \mathcal{X})$. These propositions can be considered as theorems of alternatives.

Proposition 2.1 *Consider the following statements:*

- *(i)* $x_0 \in \mathcal{G}_\epsilon(f, X)$.
- *(ii) There exists an $\hat{M} > 0$ such that, for each $i \in I$, the system $\Gamma(x_0, \epsilon, i, \hat{M}, X)$ is inconsistent.*
- *(iii) The problem (f, X) satisfies the domination property and there exists an $\hat{M} > 0$ such that, for each $i \in I$, the system $\Gamma(x_0, \epsilon, i, \hat{M}, \mathcal{S}(f, X))$ is inconsistent.*
- *(iv) There exists an $\tilde{M} > 0$ such that for every $M \geq \tilde{M}$ there exists an $i \in I$ such that the system $\Gamma(x_0, \epsilon, i, M, X)$ is consistent.*

Then, the following implications hold:

1. $(i) \iff (ii)$,
2. $(iii) \Rightarrow (i) \vee (ii)$, *and*
3. $(iv) \Rightarrow \neg(i)$.

Proof. Part 1. *(i) ⇒ (ii).* As $x_0 \in \mathcal{G}_\epsilon(f,X)$, then it is clear from the definition of $\mathcal{G}_\epsilon(f,X)$ that there exists an $\hat{M} > 0$ such that $\Gamma(x_0,\epsilon,i,M,X)$ is inconsistent for each $i \in I$.

(ii) ⇒ (i). Suppose that an $\hat{M} > 0$ exists such that $\Gamma(x_0,\epsilon,i,\hat{M},X)$ is inconsistent for each $i \in I$ and that $x_0 \notin \mathcal{G}_\epsilon(f,X)$. Then we will arrive at a contradiction.

We first show that (ii) implies that $x_0 \in \mathcal{S}^\epsilon(f,X)$. If this is not the case, then there exists $(\hat{x},\ell) \in X \times I$ such that $f_\ell(\hat{x}) < f_\ell(x_0) - \epsilon_\ell$ and $f_k(\hat{x}) \leq f_k(x_0) - \epsilon_k$ for all $k \in I \setminus \{\ell\}$. This easily shows that $\hat{x}$ is a solution of $\Gamma(x_0,\epsilon,\ell,\hat{M},)$, which is a contradiction. Hence, we have shown that $x_0 \in \mathcal{S}^\epsilon(f,X)$.

Let us assume that $x_0 \in \mathcal{S}^\epsilon(f,X) \setminus \mathcal{G}_\epsilon(f,X)$. This means that for all $M > 0$, there exists $(\hat{x},\ell) \in X \times I$ such that the following inequalities hold:

$$\begin{cases} -f_\ell(x_0) + f_\ell(\hat{x}) + \epsilon_i < 0, \\ -f_\ell(x_0) + f_\ell(\hat{x}) + \epsilon_i < M(f_j(x_0) - f_j(\hat{x}) - \epsilon_j) \; \forall j \in \{i \in I | f_i(x_0) - \epsilon_i < f_i(\hat{x})\}, \end{cases} \tag{1}$$

In particular, inequalities (1) also hold for $M = \hat{M}$. Note that, as $x_0 \in \mathcal{S}^\epsilon(f,X)$, the set $\{i \in I | f_i(x_0) - \epsilon_i < f_i(\hat{x})\}$ is non-empty. For other j's, i.e., for $j \in \{i \in I | f_i(x_0) - \epsilon_i \geq f_i(\hat{x})\}$,

$$-f_\ell(x_0) + f_\ell(\hat{x}) + \epsilon_i < 0 \leq \hat{M}(f_j(x_0) - f_j(\hat{x}) - \epsilon_j)$$

holds trivially. Therefore,

$$\begin{cases} -f_\ell(x_0) + f_\ell(\hat{x}) + \epsilon_i < 0, \\ -f_\ell(x_0) + f_\ell(\hat{x}) + \epsilon_i < \hat{M}(f_j(x_0) - f_j(\hat{x}) - \epsilon_j) \quad \forall j \in I \setminus \{i\}, \end{cases}$$

and, as $\hat{x}$ belongs to X, this means that $\hat{x}$ is a solution of $\Gamma(x_0,\epsilon,\ell,\hat{M},X)$. This is a contradiction, and consequently $x_0 \in \mathcal{G}_\epsilon(f,X)$.

Part 2. *(iii) ⇒ (ii).* Consider an $x \in X \setminus \mathcal{S}(f,X)$. Since the domination property holds, there exists an $s(x) \in \mathcal{S}(f,X)$ such that $f_i(s(x)) \leq f_i(x)$ for every $i \in I$ and $f_k(s(x)) < f_k(x)$ for some $k \in I$. As $\Gamma(x_0,\epsilon,i,\hat{M},\mathcal{S}(f,X))$ is inconsistent for each $i \in I$, it follows that

$$\begin{cases} -f_\ell(x_0) + f_\ell(s(x)) + \epsilon_i < f_\ell(s(x)) - f_\ell(x), \\ -f_\ell(x_0) + f_\ell(s(x)) + \epsilon_i < \hat{M}(f_j(x_0) - f_j(s(x)) - \epsilon_j) + f_\ell(s(x)) - f_\ell(x) \\ \qquad\qquad\qquad\qquad\qquad\quad +\hat{M}(f_j(s(x)) - f_j(x)) \quad \forall j \in I \setminus \{i\}, \\ x \in X \setminus \mathcal{S}(f,X), \end{cases} \tag{2}$$

is inconsistent for each $i \in I$ as well.

Rearranging the terms of (2) and by noting that the choice of $x \in X \setminus \mathcal{S}(f,X)$ was arbitrary, it follows that $\Gamma(x_0,\epsilon,i,\hat{M},X \setminus \mathcal{S}(f,X))$ is inconsistent for each $i \in I$. Furthermore, from *(iii)* we have that $\Gamma(x_0,\epsilon,i,\hat{M},\mathcal{S}(f,X))$ is inconsistent. Therefore, for each $i \in I$, $\Gamma(x_0,\epsilon,i,\hat{M},X)$ is inconsistent as well. Consequently, the implication *(iii) ⇒ (ii)* follows. As $(i) \iff (ii)$ this also means that *(iii) ⇒ (i)* is true.

Part 3. Simple rearrangements show that if $\hat{x}$ is a solutions of $\Gamma(x_0,\epsilon,i,\tilde{M},X)$, then $\hat{x}$ solves $\Gamma(x_0,\epsilon,i,M,X)$ for every $0 < M \leq \tilde{M}$ as well. Therefore, statement (iv) implies that for every $M > 0$ there exists an $i \in I$ such that the system $\Gamma(x_0,\epsilon,i,M,X)$ is consistent. This is the same as the negation of statement (ii), and from Part 1 of this proposition, we have that $x_0 \notin \mathcal{G}_\epsilon(f,X)$. □

PROPOSITION 2.2 *Consider the following statements:*

(i) $x_0 \in \mathcal{G}_{\hat{M},\varepsilon}(f, X)$.
(ii) *For each* $i \in I$, *the system* $\Gamma(x_0, \varepsilon, i, \hat{M}, X)$ *is inconsistent.*
(iii) *The problem* (f, X) *satisfies the domination property and, for each* $i \in I$, *the system* $\Gamma(x_0, \varepsilon, i, \hat{M}, \mathcal{S}(f, X))$ *is inconsistent.*
(iv) *The problem* (f, X) *satisfies the* ϵ*-domination property for some* $\epsilon \in [0, 1]\varepsilon$ *and, for each* $i \in I$, *the system* $\Gamma(x_0, \varepsilon, i, \hat{M}, \mathcal{S}^{\varepsilon}(f, X))$ *is inconsistent.*

Then, the following implications hold:

1. $(i) \iff (ii)$ *and*
2. $(iii) \vee (iv) \Rightarrow (i) \vee (ii)$.

Proof. The proofs of Part 1 and $(iii) \Rightarrow (i) \vee (ii)$ of Part 2 follow along the lines of the proofs of the corresponding results from Proposition 2.1. As $(i) \iff (ii)$, it is sufficient to prove that $(iv) \Rightarrow (ii)$.

Consider an $x \in X \setminus \mathcal{S}^{\varepsilon}(f, X)$ and an $\epsilon \in [0, \varepsilon]$. Since the ϵ-domination property holds, there exists an $s(x) \in \mathcal{S}^{\epsilon}(f, X)$ such that $f_i(s(x)) \leq f_i(x) - \epsilon_i$ for every $i \in I$ and $f_k(s(x)) < f_k(x) - \epsilon_k$ for some $k \in I$. Since $\epsilon \in [0, 1]\varepsilon$,

$$\mathcal{S}^{\epsilon}(f, X) \subseteq \mathcal{S}^{\varepsilon}(f, X).$$

Therefore, the point $s(x)$ belongs to $\mathcal{S}^{\varepsilon}(f, X)$ as well.

As $\Gamma(x_0, \varepsilon, i, \hat{M}, \mathcal{S}^{\epsilon}(f, X))$ is inconsistent for each $i \in I$, it follows that

$$\begin{cases} -f_\ell(x_0) + f_\ell(s(x)) + \varepsilon_i < f_\ell(s(x)) - f_\ell(x) + \epsilon_\ell, \\ -f_\ell(x_0) + f_\ell(s(x)) + \varepsilon_i < \hat{M}(f_j(x_0) - f_j(s(x)) - \varepsilon_j) + f_\ell(s(x)) - f_\ell(x) \\ \qquad\qquad\qquad\qquad +\epsilon_\ell + \hat{M}(f_j(s(x)) - f_j(x) + \epsilon_j) \quad \forall j \in I \setminus \{i\}, \\ x \in X \setminus \mathcal{S}^{\varepsilon}(f, X), \end{cases} \tag{3}$$

is inconsistent for each $i \in I$ as well. Rearranging the terms of (3) we see that it is equivalent to the inconsistency of

$$\begin{cases} -f_\ell(x_0) + f_\ell(x) + \varepsilon_\ell < \epsilon_\ell, \\ -f_\ell(x_0) + f_\ell(x) + \varepsilon_\ell < \hat{M}(f_j(x_0) - f_j(x) - \varepsilon_j) + \hat{M}\epsilon_j + \epsilon_\ell \quad \forall j \in I \setminus \{i\}, \\ x \in X \setminus \mathcal{S}^{\varepsilon}(f, X), \end{cases} \tag{4}$$

for each $i \in I$. As $\epsilon_\ell, \epsilon_j, \hat{M}$ are all non-negative, inconsistency of (4) implies that,

$$\begin{cases} -f_\ell(x_0) + f_\ell(x) + \varepsilon_\ell < 0, \\ -f_\ell(x_0) + f_\ell(x) + \varepsilon_\ell < \hat{M}(f_j(x_0) - f_j(x) - \varepsilon_j) \quad \forall j \in I \setminus \{i\}, \\ x \in X \setminus \mathcal{S}^{\varepsilon}(f, X), \end{cases} \tag{5}$$

is inconsistent for each $i \in I$. The system (5) is the same as $\Gamma(x_0, \varepsilon, i, \hat{M}, X \setminus \mathcal{S}^{\varepsilon}(f, X))$. Furthermore, from *(iv)* we have that $\Gamma(x_0, \varepsilon, i, \hat{M}, \mathcal{S}^{\varepsilon}(f, X))$ is inconsistent. Hence, for each $i \in I$, the system $\Gamma(x_0, \varepsilon, i, \hat{M}, X)$ is inconsistent as well. Consequently, the implication *(iv)* $\Rightarrow$ *(ii)* follows. $\square$

Corollary 2.3 *Let (f, X) satisfy the domination property. Then, we have*

$$\mathcal{G}_\epsilon(f, X) = \mathcal{G}_\epsilon(f, \mathcal{S}^\epsilon(f, X)) \text{ and}$$
$$\mathcal{G}_{\hat{M},\epsilon}(f, X) = \mathcal{G}_{\hat{M},\epsilon}(f, \mathcal{S}^\epsilon(f, X)).$$

Remark 1 The relations in Corollary 2.3 can be used in the design of efficient population based algorithms for approximating $\mathcal{G}_{\hat{M},\epsilon}(f, X)$ (or approximating $f\left(\mathcal{G}_{\hat{M}}(f, X)\right)$ in the criteria space). Many of the population based approximation algorithms, i.e. multisearch methods [23] and stochastic techniques [24–26], employ a set of points P in their search procedure. As a finite set is trivially compact, it also satisfies the domination property. Computing $\mathcal{G}_{\hat{M},\epsilon}(f, X)$ from P using a naïve method requires pairwise comparisons of all the points in P in all the criteria. This leads to a quadratic asymptotic complexity. An efficient computation method would be to first compute $S^\epsilon(f, X)$ from P and then compute Geoffrion ϵ-proper solutions of $S^\epsilon(f, X)$. According to Corollary 2.3, this would lead to the set of Geoffrion ϵ-proper solutions of P. The algorithm in [27] uses this technique for the special case of $\epsilon = 0$, and uses the algorithms to solve large-scale energy optimization problems in smart grid [28]. Design of algorithms that employ $\epsilon > 0$ during their search procedure and successively reduce it to compute Geoffrion ϵ-proper solutions using relations in Corollary 2.3 is a relevant open task.

The next proposition shows that under very mild conditions $\mathcal{G}_{\hat{M},\epsilon}(f, X)$ is nonempty for every $\hat{M} \geq m - 1$.

Proposition 2.4 *Let $\epsilon > 0$ and $\hat{M} \geq m - 1$. If the weighted-sum problem*

$$\min_{x \in X} \sum_{i=1}^{m} f_i(x) \tag{6}$$

is bounded from below, then $\mathcal{G}_{\hat{M},\epsilon}(f, X) \neq \emptyset$.

Proof. If the weighted-sum problem (6) is bounded from below, then it has an infimum, and, from [29, Theorem 2.4], the set of ε-minimum of (6) is non-empty for every $\varepsilon > 0$. The proof of the proposition proceeds by showing that if x_0 is a $\sum_{j=1}^{m} \epsilon_j$-minimum of the problem (6), then $x_0 \in \mathcal{G}_{\hat{M},\epsilon}(f, X)$. We will prove that $x_0 \in \mathcal{G}_{m-1,\epsilon}(f, X)$. From the definition of Geoffrion $(\hat{M}, \epsilon)$-proper optimality, this would imply that for every $\hat{M} \geq m - 1$, the point $x_0 \in \mathcal{G}_{\hat{M},\epsilon}(f, X)$ as well.

Let us assume on the contrary that $x_0 \notin \mathcal{G}_{m-1,\epsilon}(f, X)$. Therefore, from Proposition 2.2 we obtain an $i \in I$ such that $\Gamma(x_0, \epsilon, i, \hat{M}, X)$ is consistent. Without loss of generality, we assume that $i = 1$. Thus, the system $\Gamma(x_0, \epsilon, 1, \hat{M}, X)$, written as

$$\begin{cases} -f_1(x_0) + f_1(x) + \epsilon_1 < 0, \\ -f_1(x_0) + f_1(x) + \epsilon_1 < (m-1)(f_j(x_0) - f_j(x) - \epsilon_j) \quad \forall j \in I \setminus \{1\}, \\ x \in X. \end{cases} \tag{7}$$

has a solution. Summing (7) for all $j \in I \setminus \{1\}$, it follows that

$$-f_1(x_0) + f_1(x) + \epsilon_1 < \sum_{j=2}^{m} (f_j(x_0) - f_j(x) - \epsilon_j),$$

which furthermore implies that

$$\sum_{j=1}^{m} f_j(x_0) - \sum_{j=1}^{m} f_j(x) - \sum_{j=1}^{m} \epsilon_j > 0. \tag{8}$$

Inequality (8) is a contradiction to the $\sum_{j=1}^{m} \epsilon_j$-minimality of point x_0 for the problem (6). Therefore, the assertion $x_0 \in \mathcal{G}_{m-1,\epsilon}(f, X)$ follows. □

Remark 2 Following the proof of Proposition 2.4, we can easily show that, for every $\varepsilon \geq 0$ and every $\epsilon \geq 0$ such that $\sum_{i=1}^{m} \epsilon_i \leq \varepsilon$, all the ε-minima of (6) are also Geoffrion $(\hat{M}, \epsilon)$-proper solutions.

COROLLARY 2.5 *Let each f_i be strongly convex and X a closed convex set. Then for any $\hat{M} \geq m - 1$ we have $\mathcal{G}_{\hat{M},\epsilon}(f, X) \neq \emptyset$.*

Proof. The proof follows from the fact that the sum of strongly convex function is strongly convex and it achieves a unique minimizer on a closed convex set. Thus, the objective function of problem (6) is bounded below in this case and this Proposition 2.4 immediately applies. □

Remark 3 If there exists an $\hat{x} \in X$ such that the following scalarized problem (originally by Guddat et al. [30])

$$\begin{aligned} &\min_{x \in X} \sum_{i=1}^{m} f_i(x) \\ &\text{s.t.} \;\; f_j(x) \leq f_j(\hat{x}) \quad \forall j \in I \end{aligned} \tag{9}$$

is unbounded, then a recent result [31, Corollary 1] shows that $\mathcal{G}(f, X) = \emptyset$. This also implies that, for any $\tilde{M} > 0$, the set $\mathcal{G}_{\tilde{M}}(f, X) = \emptyset$. Unboundedness of (9) is a stronger condition than that of (6). We conjecture that unboundedness of (6) is a sufficient condition for $\mathcal{G}(f, X) = \emptyset$. This seems to be the case for at least bicriteria optimization problems having a connected efficient front.

The next example shows that solution for $\hat{M} < m - 1$ are also possible. Our analysis suggests that $\mathcal{G}_{\hat{M}}(f, X)$ is a singleton for $\hat{M} < 1$, irrespective of the number of criteria. In a multicriteria optimization, a singleton as an optimal set occurs if a total order is used as the underlying ordering relation. In our context, this would mean that, for $\hat{M} < 1$, Geoffrion $\hat{M}$-proper optimality is equivalent to a total order. A general proof evades us so far.

EXAMPLE 2.2 *For the parameter $\theta \in \left[0, \frac{\pi}{2}\right)$, let us consider the parametric bicriteria linear optimization problem $(f(x, \theta), X(\theta))$ defined by $X = \{x \in \mathbb{R}^2_+ \,|\, x_2 = (1 - x_1)\tan\theta\}$ and $f = \mathrm{id}_{X(\theta)}$, where $\mathrm{id}_{X(\theta)}$ the identity function on $X(\theta)$. For $\theta = 0$, we have a trivial scenario where $f(\mathcal{S}_w(f(\theta), X(\theta)))$ is a singleton. Our aim is to find the minimum value of $\hat{M}$, denoted by $M(\theta)$, such that $\mathcal{G}_{\hat{M}}(f(\theta), X(\theta)) \neq \emptyset$.*

The Pareto optimal front of $(f(x, \theta), X(\theta))$ comprises of the line segment joining $(0, \tan\theta)^\top$ and $(1, 0)^\top$ (Figure 1, left). The trade-off appearing in the Definition 1.4 at $(0, \tan\theta)^\top$ is $\tan\theta$, while the corresponding trade-off at $(1, 0)^\top$ is $\cot\theta$. For a point u in the relative interior of the line segment, the trade-off is $\max\{\tan\theta, \cot\theta\}$. Consequently, we have that

$$M(\theta) = \min\{\min\{\tan\theta, \cot\theta\},\, \max\{\tan\theta, \cot\theta\}\} = \min\{\tan\theta, \cot\theta\}.$$

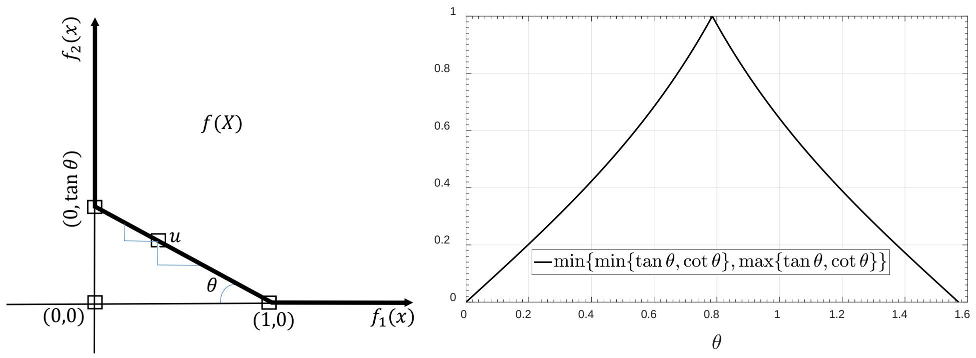

Figure 1. Illustration of Example 2.2. The set $f\left(\mathcal{S}_w(f(\theta), X(\theta))\right)$ comprises of three thick black lines (left). The minimum value of $\hat{M}$ such that $\mathcal{G}_{\hat{M}}(f(\theta), X(\theta)) \neq \emptyset$ is plotted as a function of θ (right).

We see that, for $\theta \in \left(0, \frac{\pi}{2}\right)$, $M(\theta)$ takes every possible value from $(0,1)$ (Figure 1, right). Furthermore, we can easily compute that, for every $M(\theta) < 1$, the set $\mathcal{G}_{M(\theta)}(f, X)$ is a singleton, consisting of one of the end points of the line segment.

3. Convergence properties

Let $\hat{M} > 0$; $\epsilon > 0$; and $C \supseteq \mathbb{R}^m_+$ a closed, convex, and pointed cone be arbitrary but fixed. For a given $\tau > 0$, let the hextuples $\mathbb{T}$ and $\mathbb{T}^\tau$ be defined by

$$\mathbb{T} = \Big(\mathcal{G}_{\hat{M},0}(f,X), \mathcal{G}(f,X), \mathcal{S}(f,X), \mathcal{S}_w(f,X), \mathcal{S}(f,X,C), \mathcal{S}_w(f,X,C)\Big) \text{ and}$$

$$\mathbb{T}^\tau = \Big(\mathcal{G}_{\hat{M},\tau\epsilon}(f,X), \mathcal{G}_{\tau\epsilon}(f,X), \mathcal{S}^{\tau\epsilon}(f,X), \mathcal{S}^{\tau\epsilon}_w(f,X), \mathcal{S}^{\tau\epsilon}(f,X,C), \mathcal{S}^{\tau\epsilon}_w(f,X,C)\Big).$$

For every $i \in \{1, 2, \ldots, 6\}$, the i-th element of $\mathbb{T}$ and $\mathbb{T}^\tau$ will be denoted by $\mathbb{T}_i$ and $\mathbb{T}^\tau_i$, respectively. In this section, for every $i \in \{1, 2, \ldots, 6\}$, we investigate answers to the following questions:

*Q*1. *If $u^\tau \in \mathbb{T}^\tau_i$ for $\tau > 0$ and $\lim_{\tau\to 0^+} u^\tau = u$, does $u \in \mathbb{T}_i$?*
*Q*2. *What is the relation between $\mathbb{T}_i$ and $\bigcap_{\tau>0} \mathbb{T}^\tau_i$?*

For $\mathbb{T}_3$, $\mathbb{T}_4$, and their approximate counterparts $\mathbb{T}^\tau_3$, $\mathbb{T}^\tau_4$, these questions have been discussed in [29]. The answers to these questions are important for algorithmic reasons. For example, an *affirmative* answer to $Q1$ would imply that a point-by-point, iterative algorithm which computes points in $\mathbb{T}^\tau_i$ with smaller τ value in successive iterations would indeed converge to a point in $\mathbb{T}_i$. An *equality* between $\mathbb{T}_i$ and $\bigcap_{\tau>0} \mathbb{T}^\tau_i$ as an answer to $Q2$ would lead to a similar convergence result for population based approximation algorithms. On the other hand, if $\bigcap_{\tau>0} \mathbb{T}^\tau_i$ is a superset of $\mathbb{T}_i$, then there are more points in the intersection. This situation is not desirable as this implies that, for some sequences in $\mathbb{T}^\tau_i$, the answer of $Q1$ cannot be affirmative. Furthermore, if $\bigcap_{\tau>0} \mathbb{T}^\tau_i$ is a proper subset of $\mathbb{T}_i$, then $Q1$ might have an affirmative answer, yet there will be points in $\mathbb{T}_i$ that would not attainable in the sense of $Q1$.

It is easy to see that the inclusion $\mathbb{T}_i \subseteq \mathbb{T}_{i+1}$ for every $i \in \{1, 2, 3\}$ holds. Consequently, for each $i \in \{1, 2, 3\}$, if we additionally assume that every $u^\tau \in \mathbb{T}_i$, an affirmative answer to $Q1$ for $\mathbb{T}_i$ would yield an affirmative answer to $Q1$ for $\mathbb{T}_{i+1}$ as well.

To start with we will use the weighted sum method (6) to investigate $Q1$ for

$\mathbb{T}_1$. To this end, let $\mathbb{W}$ and $\mathbb{W}^\tau$ denote the minimum and $\tau\epsilon$-minimum of (6), respectively.

PROPOSITION 3.1 *Let X be closed and f be continuous. If $u^\tau \in \mathbb{W}^\tau \subseteq \mathcal{G}_{\hat{M},\tau\epsilon}(f,X)$ for every $\tau > 0$ and $\lim_{\tau\to 0^+} u^\tau = u$, then $u \in \mathbb{W} \subseteq \mathcal{G}_{\hat{M},0}(f,X)$.*

Proof. From Remark 2, it follows that $u^\tau \in \mathcal{G}_{\hat{M},\tau\epsilon}(f,X)$ and $u \in \mathcal{G}_{\hat{M},0}(f,X)$. As $u^\tau \in \mathbb{W}^\tau$, we have

$$\sum_{i=1}^m f_i(u^\tau) - \tau \sum_{i=1}^m \epsilon_i \le f(x) \qquad \forall x \in X.$$

Taking the limit $\tau \to 0^+$, and by observing that X is closed and f is continuous, we obtain that $\sum_{i=1}^m f_i(u) \le f(x)$ for all $x \in X$. Hence, the limit point $u \in \mathbb{W}$. □

The next two theorems answer $Q1$ without the additional assumption that $u^\tau \in \mathbb{W}^\tau$ for every $\tau > 0$.

THEOREM 3.2 *Let X be closed and f be continuous. If $u^\tau \in \mathcal{S}_w^{\tau\epsilon}(f,X,C)$ for every $\tau > 0$ and $\lim_{\tau\to 0^+} u^\tau = u$, then $u \in \mathcal{S}_w(f,X,C)$.*

Proof. Under the assumptions of the theorem, from Definition 1.2 we have

$$\forall (x,\tau) \in X \times \mathbb{R}_+ : \quad f(x) + \tau\epsilon - f(u^\tau) \in \tilde{C} := C \setminus (-\mathrm{int}(C)).$$

As C is assumed to be a closed, convex, and pointed cone, the set $\tilde{C}$ is a closed cone. Taking the limit $\tau \to 0^+$, and by observing that X is closed, $\tilde{C}$ is closed, and f is continuous, we obtain that

$$\forall x \in X : \quad f(x) - f(u) \in \tilde{C}.$$

Therefore, $u \in \mathcal{S}_w(f,X,C)$ follows. □

Remark 4 In particular, Theorem 3.2 also holds for the choice of $C = \mathbb{R}^m_+$. Thus, a similar result also holds for $\mathbb{T}_4$.

THEOREM 3.3 *Let X be closed and f be continuous. If $u^\tau \in \mathcal{G}_{\hat{M},\tau\epsilon}(f,X)$ for every $\tau > 0$ and $\lim_{\tau\to 0^+} u^\tau = u$, then $u \in \mathcal{G}_{\hat{M},0}(f,X)$.*

Proof. As $u^\tau \in \mathcal{G}_{\hat{M},\tau\epsilon}(f,X)$. Applying Part 1 of Proposition 2.2 it follows that, for each $i \in I$, the system $\Gamma(x_0,\tau\epsilon,i,\hat{M},X)$, written as

$$\begin{cases} -f_i(u^\tau) + f_i(x) + \tau\epsilon_i < 0, \\ -f_i(u^\tau) + f_i(x) + \tau\epsilon_i < \hat{M}(f_j(u^\tau) - f_j(x) - \tau\epsilon_j) \ \forall j \in I \setminus \{i\}, \\ x \in X, \end{cases}$$

is inconsistent.

Let $W = \mathbb{R}^m \setminus (-\mathrm{int}(\mathbb{R}^m_+))$ and, for every $(i,x,\tau) \in I \times X \times \mathbb{R}_+$, consider the vectors $F^i(\tau,x,\hat{M})$ defined by

$$F^i_j(\tau,x,\hat{M}) = \begin{cases} -f_i(u^\tau) + f_i(x) + \tau\epsilon & \text{if } j = 1; \\ -f_i(u^\tau) + f_i(x) + \tau\epsilon_i - \hat{M}(f_j(u^\tau) - f_j(x) - \tau\epsilon_j) & \text{if } j \in I \setminus \{1\}. \end{cases}$$

Since W is a closed cone and f is continuous, for every $(i,x) \in I \times X$,

$$\lim_{\tau \to 0^+} F^i(\tau, x, \hat{M}) = F^i(0, x, \hat{M}) \in W. \tag{10}$$

From (10) we obtain that, for each $i \in I$, the system

$$\begin{cases} -f_i(u) + f_i(x) < 0 \\ -f_i(u) + f_i(x) - \hat{M}(f_j(u) - f_j(x))) < 0, \\ x \in X, \end{cases} \tag{11}$$

is inconsistent. As (11) is the same as $\Gamma(u, 0, i, \hat{M}, X)$, applying Part 1 of Proposition 2.2, we have $u \in \mathcal{G}_{\hat{M},0}(f, X)$. □

Theorem 3.3 shows a favorable convergence property for Geoffrion $(\hat{M}, \epsilon)$-proper solutions. We will show later in Theorems 3.4 and 3.5 that this convergence property is not satisfied in general for Geoffrion ϵ-proper and ϵ-Pareto optimal solutions.

To answer $Q2$, we start with the case of $\mathbb{T}_5 = \mathcal{S}(f, X, C)$ and $\mathbb{T}_6 = \mathcal{S}_w(f, X, C)\}$.

Theorem 3.4 *The following set relations hold:*

$$\begin{aligned} \bigcap_{\tau>0} \mathcal{S}_w^{\tau\epsilon}(f, X, C) = \bigcap_{\tau>0} \mathcal{S}^{\tau\epsilon}(f, X, C) = \mathcal{S}_w(f, X, C) \\ \supseteq \mathcal{S}(f, X, C) \supseteq \mathcal{G}(f, X) \supseteq \mathcal{G}_{\hat{M},0}(f, X). \end{aligned}$$

Proof. It is a well known result that

$$\mathcal{S}_w(f, X, C) \supseteq \mathcal{S}(f, X, C) \supseteq \mathcal{G}(f, X).$$

For a proof, see [3], for example. The inclusion $\mathcal{G}(f, X) \supseteq \mathcal{G}_{\hat{M},0}(f, X)$ follows from definitions of these sets. Therefore, it is only remains to prove that

$$\bigcap_{\tau>0} \mathcal{S}_w^{\tau\epsilon}(f, X, C) = \bigcap_{\tau>0} \mathcal{S}^{\tau\epsilon}(f, X, C) = \mathcal{S}_w(f, X, C). \tag{12}$$

To prove (12), we will show that the relative complements of the sets $\bigcap_{\tau>0} \mathcal{S}_w^{\tau\epsilon}(f, X, C)$, $\bigcap_{\tau>0} \mathcal{S}^{\tau\epsilon}(f, X, C)$, and $\mathcal{S}_w(f, X, C)$ with respect to X are identical.

Let $x_0 \in X$ be such that $x_0 \notin \bigcap_{\tau>0} \mathcal{S}_w^{\tau\epsilon}(f, X, C)$. Consequently, using Definition 1.2, we obtain

$$\exists \hat{\tau} > 0, \hat{x} \in X, c^+ \in \mathrm{int}(C) : f(x_0) - f(\hat{x}) - \hat{\tau}\epsilon = c^+. \tag{13}$$

As c^+ is also an element of $C \setminus \{0\}$, statement (13) shows that x_0 cannot be an element of $\bigcap_{\tau>0} \mathcal{S}^{\tau\epsilon}(f, X, C)$. Hence,

$$X \setminus \left(\bigcap_{\tau>0} \mathcal{S}_w^{\tau\epsilon}(f, X, C) \right) \subseteq X \setminus \left(\bigcap_{\tau>0} \mathcal{S}^{\tau\epsilon}(f, X, C) \right). \tag{14}$$

On the other hand, if $x_0 \in X$ is such that $x_0 \notin \bigcap_{\tau>0} \mathcal{S}^{\tau\epsilon}(f, X, C)$, then

$$\exists \hat{\tau} > 0, \hat{x} \in X, c \in C \setminus \{0\} : f(x_0) - f(\hat{x}) - \hat{\tau}\epsilon = c.$$

As $\hat{\tau}\epsilon \in \operatorname{int}(C)$, we easily obtain,

$$f(x_0) - f(\hat{x}) = c + \hat{\tau}\epsilon \in \operatorname{int}(C).$$

Consequently, the point x_0 cannot belong to $S_w(f, X, C)$. Hence,

$$X \setminus \left(\bigcap_{\tau>0} \mathcal{S}^{\tau\epsilon}(f, X, C) \right) \subseteq X \setminus S_w(f, X, C). \tag{15}$$

Finally, if $x_0 \in X$ is such that $x_0 \notin S_w(f, X, C)$, then

$$\exists \hat{x} \in X, c^+ \in \operatorname{int}(C) : f(x_0) - f(\hat{x}) = c^+.$$

As $c^+ \in \operatorname{int}(C)$, for every $\epsilon > 0$, there exists a $\hat{\tau} > 0$ such that

$$f(x_0) - f(\hat{x}) - \hat{\tau}\epsilon = c^+ - \hat{\tau}\epsilon \in \operatorname{int}(C).$$

This shows that $x_0 \notin \bigcap_{\tau>0} \mathcal{S}_w^{\tau\epsilon}(f, X, C)$. Therefore,

$$X \setminus S_w(f, X, C) \subseteq X \setminus \left(\bigcap_{\tau>0} \mathcal{S}_w^{\tau\epsilon}(f, X, C) \right) \tag{16}$$

and the proof follows form the inclusions (14), (15), and (16). □

Remark 5 Parts of the results in Theorem 3.4 has been stated in different works. For example, Dutta et al. [32] showed $\bigcap_{\tau>0} \mathcal{S}_w^{\tau\epsilon}(f, X, C) = \mathcal{S}_w(f, X, C)$ with the additional assumptions that $C = \mathbb{R}^m_+$ and that $f(x)$ is convex. The relation $\bigcap_{\tau>0} \mathcal{S}_w^{\tau\epsilon}(f, X, C) \subseteq \mathcal{S}_w(f, X, C)$ has been shown in [33, Theorem 2.1].

It is interesting that Theorem 3.4 makes no assumptions about f and X. For example, the result holds even if $f(X)$ is a bounded open set. In this scenario, one can easily show that for every $\tau > 0$ the set $\mathcal{S}_w^{\tau\epsilon}(f, X, C)$ is nonempty, while $\mathcal{S}_w(f, X, C) \subseteq \text{bd(f(X))}$ is empty (as $\text{bd(f(X))} = \emptyset$).

Theorem 3.4 shows a favorable convergence property for weak (ϵ, C) optimal points–an equality as an answer for $Q2$. In particular, Theorem 3.4 also holds for the choice of $C = \mathbb{R}^m_+$. Thus, similar results also hold for $\mathbb{T}_3$ and $\mathbb{T}_4$.

Weak optimality is the weakest optimality concept. Nevertheless, as weak optimal points can be made strictly better in some criteria without worsening others, this is not a practical optimality notion. The most widely-used optimality notion is that Pareto optimality. For Pareto optimal points, Theorem 3.4 shows that an affirmative answer to $Q1$ is not possible in general. Given that the intersection of $\tau\epsilon$-Pareto points (for $\tau > 0$) is the set of weakly Pareto optimal points, equality as an answer to $Q2$ is not possible. Due to the fact that the set of weakly Pareto optimal points can be substantially larger than that of Pareto-optimal solutions, this intersection result shows that Pareto-optimality is also not a practical optimality notion.

We next investigate $\mathbb{T}_2$, i.e. the set of Geoffrion proper solutions. Geoffrion proper optimality is a stronger notion of optimality than Pareto optimality.

THEOREM 3.5 *The following set relations hold:*

$$\mathcal{G}(f,X) \subseteq \bigcap_{\tau>0} \mathcal{G}_{\tau\epsilon}(f,X) \subseteq \mathcal{S}_w(f,X). \tag{17}$$

If f is bounded below, then we have that

$$\bigcap_{\tau>0} \mathcal{G}_{\tau\epsilon}(f,X) = \mathcal{S}_w(f,X). \tag{18}$$

Proof. We will prove by contraposition. Suppose that $x_0 \in X \setminus \mathcal{S}_w(f,X)$. Thus, there exists an $\hat{x} \in X$ such that $f_i(\hat{x}) < f_i(x_0)$ for each $i \in I$. This implies that

$$\exists \hat{\tau} > 0 : -f_i(x_0) + f_i(\hat{x}) + \hat{\tau}\epsilon_i < 0 \quad \forall i \in I.$$

Hence, the system $\Gamma(x_0, \hat{\tau}\epsilon, 1, M, X)$ is consistent for every $M > 0$. Applying Part 1 of Proposition 2.1, it follows that $x_0 \notin \bigcap_{\tau>0} \mathcal{G}_{\tau\epsilon}(f,X)$. Hence,

$$\bigcap_{\tau>0} \mathcal{G}_{\tau\epsilon}(f,X) \subseteq \mathcal{S}_w(f,X). \tag{19}$$

We next prove by contraposition that

$$\mathcal{G}(f,X) \subseteq \bigcap_{\tau>0} \mathcal{G}_{\tau\epsilon}(f,X). \tag{20}$$

Suppose that $x_0 \in X \setminus \left(\bigcap_{\tau>0} \mathcal{G}_{\tau\epsilon}(f,X)\right)$. Using Part 1 of Proposition 2.1, it follows that there exists a $\hat{\tau} > 0$ such that

$$\forall M > 0, \exists i \in I : \Gamma(x_0, \hat{\tau}\epsilon, i, M, X) \text{ is consistent.} \tag{21}$$

Equivalently, for every $M > 0$, there exists $(i, \hat{x}) \in I \times X$ such that

$$\begin{cases} -f_i(x_0) + f_\ell(\hat{x}) + \hat{\tau}\epsilon_i < 0, \\ -f_i(x_0) + f_i(\hat{x}) + \hat{\tau}\epsilon_i < M(f_j(x_0) - f_j(\hat{x}) - \hat{\tau}\epsilon_j) \quad \forall j \in I \setminus \{i\}. \end{cases} \tag{22}$$

As $\hat{\tau} > 0$ and $\epsilon \geq 0$, the system (22) implies that

$$\begin{cases} -f_i(x_0) + f_\ell(\hat{x}) < 0, \\ -f_i(x_0) + f_i(\hat{x}) < M(f_j(x_0) - f_j(\hat{x})) \quad \forall j \in I \setminus \{i\}. \end{cases} \tag{23}$$

As $\hat{x} \in X$, the system (23) shows that $\Gamma(x_0, 0, i, M, X)$ is consistent. Therefore, applying Part 1 of Proposition 2.1, we have $x_0 \notin \mathcal{G}(f,X)$. This proves (20), and together with (19), the first part of the proposition.

Finally, we prove by contraposition that if f is bounded below, then

$$\bigcap_{\tau>0} \mathcal{G}_{\tau\epsilon}(f,X) \supseteq \mathcal{S}_w(f,X). \tag{24}$$

To this end, let, for each $i \in I$, the function f_i be bounded below by β. Suppose that $x_0 \in X \setminus \left(\bigcap_{\tau>0} \mathcal{G}_{\tau\epsilon}(f,X)\right)$. Proceeding as in the proof of (20), we obtain (21). Let us consider a sequence $(M_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} M_n = \infty$. As I is a finite set, we can assume without loss of generality that $i = 1$, i.e., for every M_n, the system $\Gamma(x_0, \hat{\tau}\epsilon, 1, M_n, X)$ is consistent. Therefore, for every M_n, there exist a $x^n \in X$ such that

$$\begin{cases} -f_1(x_0) + f_1(x^n) + \hat{\tau}\epsilon_1 < 0, \\ -f_1(x_0) + f_1(x^n) + \hat{\tau}\epsilon_1 < M_n(f_j(x_0) - f_j(x^n) - \hat{\tau}\epsilon_j) \quad \forall j \in \{2,3,\dots,m\} \end{cases} \tag{25}$$

holds.

As f_1 be bounded below by β and as $\lim_{n\to\infty} M_n = \infty$, from (25) it follows that

$$0 \geq \lim_{n\to\infty} \frac{-f_1(x_0) + f_1(x^n) + \hat{\tau}\epsilon_1}{M_n} \geq \lim_{n\to\infty} \frac{-f_1(x_0) + \beta + \hat{\tau}\epsilon_1}{M_n} = 0.$$

Consequently,

$$\lim_{n\to\infty} \frac{-f_1(x_0) + f_1(x^n) + \hat{\tau}\epsilon_1}{M_n} = 0. \tag{26}$$

Using (26) in (25), it follows that

$$\exists n_0 \in \mathbb{N} : f_j(x_0) - f_j(x^{n_0}) - \hat{\tau}\epsilon_j \geq -\frac{\hat{\tau}\epsilon_1}{2} \quad \forall j \in \{2,3,\dots,m\} \tag{27}$$

From the first inequality of (25) together with (27) show that $f_i(x^{n_0}) < f_i(x_0)$ for each $i \in I$. Hence, $x_0 \notin \mathcal{S}_w(f,X)$ follows and the proof is complete. □

The boundedness condition in Theorem 3.5 seems very mild. For example, if one of the criteria is unbounded below, then $\mathcal{S}(f,X)$ would be unbounded as well. Furthermore, from Remark 3, if each of f_i's are unbounded such that (9) is unbounded, then Proposition 2.4 shows that $\mathcal{G}(f,X)$ is empty. For the bicriteria case, the next proposition shows that, if there is a weakly Pareto optimal point that is not even Pareto optimal and is not in the intersection of Geoffrion ϵ-proper solutions, then the set of Geoffrion proper solutions would be empty. This means that if (18) does not hold, then we have a pathological situation where the set of Geoffrion proper solutions is empty.

We will need the notion of $\mathbb{R}^m_+$-closedness [3] in the next proposition. A set $S \subset \mathbb{R}^m$ is called as $\mathbb{R}^m_+$-closed if, for every $s \in S$, the sections $(\{s\} - \mathbb{R}^m_+) \cap S$ are closed.

Proposition 3.6 *Let $m = 2$ and let $f(X)$ be $\mathbb{R}^m_+$-closed. Then, the following implication holds*

$$\left(\mathcal{S}_w(f,X) \setminus \left(\mathcal{S}(f,X) \bigcup \left(\bigcap_{\tau>0} \mathcal{G}_{\tau\epsilon}(f,X)\right)\right) \neq \emptyset\right) \Rightarrow (\mathcal{G}(f,X) = \emptyset). \tag{28}$$

Proof. Let $x_0 \in \mathcal{S}_w(f,X) \setminus \left(\mathcal{S}(f,X) \bigcup \left(\bigcap_{\tau>0} \mathcal{G}_{\tau\epsilon}(f,X)\right)\right)$. Applying (17), it follows that x_0 cannot belong to $\mathcal{G}(f,X)$.

Let us assume on the contrary that $\mathcal{G}(f,X)$ is non-empty and let $\hat{x} \in \mathcal{G}(f,X)$ be arbitrary but fixed. As $\hat{x} \neq x_0$ and $\mathcal{G}(f,X) \subseteq \mathcal{S}(f,X)$, we have that $\hat{x} \in \mathcal{S}(f,X)$.

We can assume without loss of generality that $f_1(x_0) \leq f_1(\hat{x})$ and $f_2(x_0) > f_2(\hat{x})$. Let $a := f_1(\hat{x}) - f_1(x_0) \geq 0$, $b := f_2(x_0) - f_2(\hat{x}) > 0$, $u := (f_1(x_0) - 1, f_2(\hat{x}) - 1)^\top$, and X^u, X^l, X^b be defined as follows:

$$\begin{aligned} X^u &=: \{x \in X | f_1(x) \geq u_1,\ f_2(x) \geq u_2\}, \\ X^l &=: \{x \in X | f_1(x) \leq u_1,\ f_2(x) \geq u_2\}, \\ X^b &=: \{x \in X | f_1(x) \geq u_1,\ f_2(x) \leq u_2\}. \end{aligned}$$

Note that, the set X^u is not empty as $x_0, \hat{x} \in X^u$, while the sets X^l and X^b can be empty.

The restriction of f to X^u is bounded below. Therefore, applying Theorem 3.5, it follows that

$$\bigcap_{\tau>0} \mathcal{G}_{\tau\epsilon}(f\restriction_{X^u}, X^u) = \mathcal{S}_w(f\restriction_{X^u}, X^u). \tag{29}$$

We will shows that $x_0 \in \bigcap_{\tau>0} \mathcal{G}_{\tau\epsilon}(f, X)$, and will arrive at a contradiction. To this end, let $\hat{\tau} > 0$ be arbitrary but fixed. As $x_0 \in \mathcal{S}_w(f, X)$, it follows that $x_0 \in \mathcal{S}_w(f\restriction_{X^u}, X^u)$. Taking into account (29), we have that $x_0 \in \mathcal{G}_{\hat{\tau}\epsilon}(f\restriction_{X^u}, X^u)$. From Definition 1.3, there exists an $\hat{M} > 0$, such that for all $(i, x) \in I \times X^u$ satisfying $f_i(x) < f_i(x_0) - \hat{\tau}\epsilon_i$, there exists a $j \in I$ such that $f_j(x_0) - \hat{\tau}\epsilon_j < f_j(x)$ and

$$\frac{f_i(x_0) - f_i(x) - \hat{\tau}\epsilon_i}{f_j(x) - f_j(x_0) + \hat{\tau}\epsilon_j} \leq \hat{M}. \tag{30}$$

Consequently, for the case of $x \in X^u$, the trade-off between $f(x_0)$ and $f(x)$ is bounded. As $x_0 \in \mathcal{S}_w(f, X) \setminus \mathcal{S}(f, X)$, two other cases are possible:

(1) $\boldsymbol{X^l = \emptyset}$: Consider an $x \in X \setminus X^u$ such that $f_2(x) < f_2(x_0) - \hat{\tau}\epsilon_2$. As can be seen in Figure 2, the point $x \in X^b$. As $f(X)$ is $\mathbb{R}^m_+$-closed, the curve $f\left(\mathcal{S}_w(f, X)\right)$, considered as function of f_1, is closed and non-increasing. Therefore, there exists a $\varrho > 0$ such that for every $\tilde{x} \in X^b$, we have $f_1(\tilde{x}) - f_1(\hat{x}) \geq \varrho$.
As $\hat{x} \in \mathcal{G}(f, X)$ and $x \in X^b$, it follows that $f_1(\hat{x}) < f_1(x)$, $f_2(\hat{x}) > f_2(x)$ and that there exists an $\tilde{M} > 0$ such that

$$\frac{f_2(\hat{x}) - f_2(x)}{f_1(x) - f_1(\hat{x})} \leq \tilde{M}. \tag{31}$$

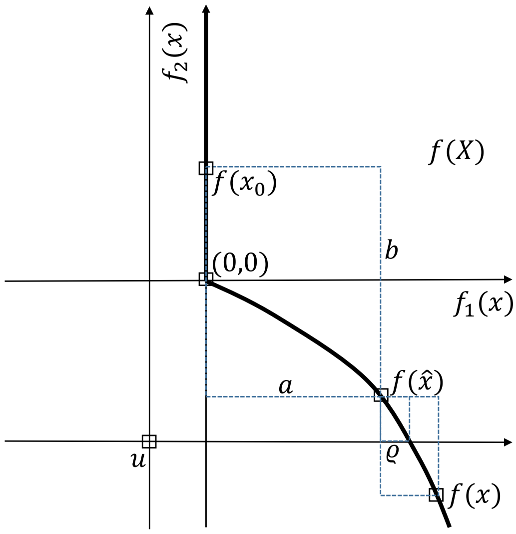

Figure 2. Illustration of case 1 in the proof of Proposition 3.6. The set $f(\mathcal{S}_w(f,X))$ comprises of the thick black curve and line, and is unbounded. The function f_1 is bounded below while the function f_2 is unbounded. If we shift the origin to u, then the first orthant would be X^u, while the second and fourth orthants would be X^l and X^b, respectively.

Using (31) and comparing the trade-off between $f(x_0)$ and $f(x)$, we obtain

$$\begin{aligned}\frac{f_2(x_0)-f_2(x)-\hat{\tau}\epsilon_2}{f_1(x)-f_1(x_0)+\hat{\tau}\epsilon_1} &\leq \frac{f_2(x_0)-f_2(x)}{f_1(x)-f_1(x_0)}\\ &= \frac{f_2(\hat{x})-f_2(x)+b}{f_1(x)-f_1(\hat{x})+a}\\ &= \frac{f_2(\hat{x})-f_2(x)}{f_1(x)-f_1(\hat{x})+a}+\frac{b}{f_1(x)-f_1(\hat{x})+a}\\ &\leq \frac{f_2(\hat{x})-f_2(x)}{f_1(x)-f_1(\hat{x})}+\frac{b}{\varrho+a}\\ &\leq \tilde{M}+\frac{b}{\varrho+a}.\end{aligned}$$

Hence, inequality (30) is satisfied for $x\in X^b$ by setting $\hat{M}$ as $\tilde{M}+\frac{b}{\varrho+a}$.

(2) $\boldsymbol{X^b=\emptyset}$: The proof follows along the lines of the first case. As in the first case, there is a value of $\hat{M}$ that satisfies inequality (30).

As a consequence of the above, we obtain that $x_0\in\mathcal{G}_{\hat{\tau}\epsilon}(f,X)$. As the choice of $\hat{\tau}$ was arbitrary, it follows that $x_0\in\bigcap_{\tau>0}\mathcal{G}_{\tau\epsilon}(f,X)$, and we thereby arrive at a contradiction. □

Proposition 3.6 could easily be tightened by replacing (28) by

$$\left(\mathcal{S}_w(f,X)\setminus\left(\bigcap_{\tau>0}\mathcal{G}_{\tau\epsilon}(f,X)\right)\neq\emptyset\right)\Rightarrow\left(\mathcal{G}(f,X)=\emptyset\right).$$

However, the idea of the proposition was to explicitly show that (28) holds for those weakly Pareto optimal points that are not Pareto optimal–these points can be *far* from the set of Pareto optimal points.

Let for sets $Y, Z \subseteq \mathbb{R}^n$, the Hausdoroff distance between Y and Z be denoted by $d_H(X,Y)$. Theorem 3.5 shows that the set of Geoffrion ϵ-proper solutions only converges to the set of weakly Pareto optimal points. The answers to $Q1$ and $Q2$ are therefore the same for $\mathbb{T}_2 = \mathcal{G}(f,X)$ and $\mathbb{T}_3 = \mathcal{S}(f,X)$. In general, the set of weakly Pareto optimal points can be much larger than the set of Geoffrion proper solutions. For such an multicriteria optimization problem, if f is Lipschitz continuous and bounded below, then $d_H\left(\mathcal{G}_{\tau\epsilon}(f,X), \mathcal{G}(f,X)\right)$ as a function of τ is bounded away from zero.

Theorem 3.7 is a comprehensive decomposition result for Geoffrion $(\hat{M},\epsilon)$-proper solutions. Theorems 3.3 and 3.7 show that for such solutions, i.e. by choosing $\mathbb{T}_1 = \mathcal{G}_{\hat{M},0}(f,X)$, one gets an affirmative answer to $Q1$ and equality in the case of $Q2$. Apart from the fact that the trade-offs for Geoffrion $(\hat{M},\epsilon)$-proper solutions are bounded, these answers to $Q1$ and $Q2$ makes these solutions more practical than Geoffrion ϵ-proper solutions and than ϵ-Pareto optimal solutions.

THEOREM 3.7 *The following set relations hold:*

I. $\mathcal{G}_{\hat{M},0}(f,X) = \bigcap_{\tau>0} \mathcal{G}_{\hat{M},\tau\epsilon}(f,X) = \bigcap_{M>\hat{M}} \mathcal{G}_{M,0}(f,X) = \bigcap_{M>\hat{M}} \bigcap_{\tau>0} \mathcal{G}_{M,\tau\epsilon}(f,X)$ *and*

II. $\mathcal{G}_{\hat{M},0}(f,X) = \bigcup_{M\leq\hat{M}} \bigcap_{\tau>0} \mathcal{G}_{M,\tau\epsilon}(f,X) = \bigcap_{\tau>0} \bigcup_{M\leq\hat{M}} \mathcal{G}_{M,\tau\epsilon}(f,X)$.

Proof. Part *I*. We first show that

$$\mathcal{G}_{\hat{M},0}(f,X) = \bigcap_{\tau>0} \mathcal{G}_{\hat{M},\tau\epsilon}(f,X) \tag{32}$$

holds. To this end, let us consider the case when both $\mathcal{G}_{\hat{M},0}(f,X)$ and $\bigcap_{\tau>0} \mathcal{G}_{\hat{M},\tau\epsilon}(f,X)$ are non-empty.

($\Rightarrow$) Let $x_0 \in \mathcal{G}_{\hat{M},0}(f,X)$. By setting $\epsilon = 0$ in Proposition 2.2, it follows that for each $i \in I$, the system $\Gamma(x_0, 0, i, \hat{M}, X)$, written as,

$$\begin{cases} -f_i(x_0) + f_i(x) < 0, \\ -f_i(x_0) + f_i(x) < \hat{M}(f_j(x_0) - f_j(x)) \quad \forall j \in I \setminus \{i\}, \\ x \in X, \end{cases}$$

is inconsistent. This shows that the system

$$\begin{cases} -f_i(x_0) + f_i(x) < -\tau\epsilon_i, \\ -f_i(x_0) + f_i(x) < \hat{M}(f_j(x_0) - f_j(x)) - \hat{M}\tau\epsilon_j - \tau\epsilon_i, \quad \forall j \in I \setminus \{i\}, \\ x \in X, \end{cases} \tag{33}$$

is inconsistent for every $\tau > 0$. It can be easily seen that (33) is the same as $\Gamma(x_0, \tau\epsilon, i, \hat{M}, X)$. Consequently, for every $\tau > 0$, $x_0 \in \mathcal{G}_{\hat{M},\tau\epsilon}(f,X)$ follows from Part 1 of Proposition 2.2. Hence $x_0 \in \bigcap_{\tau>0} \mathcal{G}_{\hat{M},\tau\epsilon}(f,X)$.

($\Leftarrow$) Let $x_0 \in \bigcap_{\tau>0} \mathcal{G}_{\hat{M},\tau\epsilon}(f,X)$. Applying Part 1 of Proposition 2.2 it follows that, for each $i \in I$ and every $\tau > 0$, the system $\Gamma(x_0, \tau\epsilon, i, \hat{M}, X)$, written as

$$\begin{cases} -f_i(x_0) + f_i(x) + \tau\epsilon_i < 0, \\ -f_i(x_0) + f_i(x) + \tau\epsilon_i < \hat{M}(f_j(x_0) - f_j(x) - \tau\epsilon_j) \ \forall j \in I \setminus \{i\}, \\ x \in X, \end{cases}$$

is inconsistent. Let $W = \mathbb{R}^m \setminus (-\mathrm{int}(\mathbb{R}^m_+))$ and, for each $i \in I$ and every $\tau > 0$, consider the vectors $F^i(\tau, x, \hat{M})$ defined by

$$F^i_j(\tau, x, \hat{M}) = \begin{cases} -f_i(x_0) + f_i(x) + \tau\epsilon & \text{if } j = 1; \\ -f_i(x_0) + f_i(x) + \tau\epsilon_i - \hat{M}(f_j(x_0) - f_j(x) - \tau\epsilon_j) & \text{if } j \in I \setminus \{1\}. \end{cases}$$

Since W is a closed cone, we have that, for all $i \in I$,

$$\lim_{\tau \to 0^+} F^i(\tau, x, \hat{M}) \in W.$$

This shows that, for each $i \in I$, the system

$$\begin{cases} -f_i(x_0) + f_i(x) < 0 \\ -f_i(x_0) + f_i(x) - \hat{M}(f_j(x_0) - f_j(x))) < 0, \\ x \in X, \end{cases} \tag{34}$$

is inconsistent. We see that (34) is the same as $\Gamma(x_0, 0, i, \hat{M}, X)$. Consequently, $x_0 \in \mathcal{G}_{\hat{M},0}(f, X)$ follows from Part 1 of Proposition 2.2.
The case when at least one of the sets $\mathcal{G}_{\hat{M},0}(f, X)$ and $\bigcap_{\tau>0} \mathcal{G}_{\hat{M},\tau\epsilon}(f, X)$ is empty is discussed next.
($\Rightarrow$) Let $\mathcal{G}_{\hat{M},0}(f, X) = \emptyset$ and assume on the contrary that $\bigcap_{\tau>0} \mathcal{G}_{\hat{M},\tau\epsilon}(f, X) \neq \emptyset$. There there exists $x_0 \in \bigcap_{\tau>0} \mathcal{G}_{\hat{M},\tau\epsilon}(f, X)$. From the proof above we see that $x_0 \in \mathcal{G}_{\hat{M},0}(f, X)$, which is a contradiction.
($\Leftarrow$) Let $\bigcap_{\tau>0} \mathcal{G}_{\hat{M},\tau\epsilon}(f, X) = \emptyset$ and assume on the contrary that $\mathcal{G}_{\hat{M},0}(f, X) \neq \emptyset$. Using similar arguments as in the last case, we deduce that this situation is not possible as well. This completes the proof of (32).
The proofs of

$$\mathcal{G}_{\hat{M},0}(f, X) = \bigcap_{M > \hat{M}} \mathcal{G}_{M,0}(f, X)$$

and

$$\mathcal{G}_{\hat{M},0}(f, X) = \bigcap_{M > \hat{M}} \bigcap_{\tau > 0} \mathcal{G}_{M,\tau\epsilon}(f, X)$$

follow along the lines of the proof of (32) by defining vectors $F^i_j(0, x, M)$ and $F^i_j(\tau, x, M)$, and noting that

$$\lim_{M \to \hat{M}^+} F^i_j(0, x, M), \ \lim_{\substack{\tau \to 0^+ \\ M \to \hat{M}^+}} F^i_j(\tau, x, M) \in W.$$

Part *II*. Applying Part *I* of this theorem, it follows that, for every $M > 0$, every

$\tau > 0$, and every $\epsilon > 0$,

$$\bigcap_{\tau > 0} \mathcal{G}_{M,\tau\epsilon}(f, X) = \mathcal{G}_{M,0}(f, X)$$

holds. Therefore, to prove

$$\mathcal{G}_{\hat{M},0}(f, X) = \bigcup_{M \leq \hat{M}} \bigcap_{\tau > 0} \mathcal{G}_{M,\tau\epsilon}(f, X) = \bigcap_{\tau > 0} \bigcup_{M \leq \hat{M}} \mathcal{G}_{M,\tau\epsilon}(f, X),$$

it is sufficient to show that, for every $\tau \geq 0$ and every $\epsilon > 0$,

$$\mathcal{G}_{\hat{M},\tau\epsilon}(f, X) = \bigcup_{M \leq \hat{M}} \mathcal{G}_{M,\tau\epsilon}(f, X).$$

Clearly, for every $\tau \geq 0$ and every $\epsilon > 0$, $\mathcal{G}_{\hat{M},\tau\epsilon}(f, X) \subseteq \bigcup_{M \leq \hat{M}} \mathcal{G}_{M,\tau\epsilon}(f, X)$ holds. The other side follows by noting that from the definition of $\mathcal{G}_{\hat{M},\tau\epsilon}(f, X)$, if $x_0 \in \mathcal{G}_{M,\tau\epsilon}(f, X)$ for some $M \leq \hat{M}$, then $x_0 \in \mathcal{G}_{\hat{M,\tau\epsilon}}(f, X)$ as well. This completes the proof of *II*. □

The next result summarizes the case when X is a finite set.

Theorem 3.8 *Let X be a finite set. Then,*

$$\mathcal{S}^{\epsilon}(f, X) = \mathcal{G}_{\epsilon}(f, X). \tag{35}$$

Furthermore, there exists a $\hat{\tau} > 0$ such that,

$$\mathcal{G}_{\hat{M},0}(f, X) = \mathcal{G}_{\hat{M},\hat{\tau}\epsilon}(f, X) \ \text{and} \tag{36}$$

$$\mathcal{S}^{w}(f, X) = \mathcal{G}_{\hat{\tau}\epsilon}(f, X). \tag{37}$$

Proof. As X is finite, for every $x_0 \in \mathcal{S}^{\epsilon}(f, X)$ the trade-off in Definition 1.3 is always bounded. Therefore $x_0 \in \mathcal{G}_{\epsilon}(f, X)$ as well. On the other hand, the set $\mathcal{G}_{\epsilon}(f, X)$ is always a subset of $\mathcal{S}^{\epsilon}(f, X)$. Consequently, we obtain (35).

Proof of (36). ($\Rightarrow$) Applying Part *I* of Theorem 3.7, it follows that, for every $\epsilon > 0$ and every $\hat{M} > 0$, there exists a $\bar{\tau} > 0$ such that

$$\mathcal{G}_{\hat{M},0}(f, X) \subseteq \mathcal{G}_{\hat{M},\bar{\tau}\epsilon}(f, X).$$

($\Leftarrow$) Let $X = \{x^1, x^2, \ldots, x^{\nu}\}$ for some $\nu \in \mathbb{N}$, and let us fix $\epsilon, \hat{M} > 0$. If the sets $\mathcal{G}_{\hat{M},0}(f, X)$ and X are the same, then the statement of the proposition holds trivially. We therefore assume that $\mathcal{G}_{\hat{M},0}(f, X) \neq X$.

Consider an $x^{\ell} \in X$ such that $x^{\ell} \notin \mathcal{G}_{\hat{M},0}(f, X)$. Applying Part 1 of Proposition 2.2, it follows that there exists an $i \in I$ and $x^k \in X$ such that

$$\begin{cases} -f_i(x^{\ell}) + f_i(x^k) < 0, \\ -f_i(x^{\ell}) + f_i(x^k) - \hat{M}(f_j(x^{\ell}) - f_j(x^k)) < 0 \ \ \forall j \in I \setminus \{i\}, \end{cases} \tag{38}$$

holds. Due to the strict inequalities in (38), there exists a $\tau^{\ell} > 0$, such that for all

$\tau \in [0, \tau^\ell]$,

$$\begin{cases} -f_i(x^\ell) + f_i(x^k) < -\tau\epsilon_i, \\ -f_i(x^\ell) + f_i(x^k) - \hat{M}(f_j(x^\ell) - f_j(x^k)) < -\tau\epsilon_i - \hat{M}\tau\epsilon_j \quad \forall j \in I \setminus \{i\}, \end{cases} \tag{39}$$

holds. By setting $\tilde{\tau} := \min\{\tau^1, \dots, \tau^\nu\}$, we see that, for every $x^\ell \in X \setminus \mathcal{G}_{\hat{M}}(f, X)$, there exists an $x^k \in X$ such that (39) remains satisfied for every $\tau \in [0, \tilde{\tau}]$. As $x^\ell \notin \mathcal{G}_{\hat{M},0}(f, X)$ and as (39) is the same as $\Gamma(x^\ell, \tau\epsilon, i, \hat{M}, X)$, applying Part **1** of Proposition 2.2, it follows that

$$\forall x^\ell \in X: \quad x^\ell \notin \mathcal{G}_{\hat{M},0}(f, X) \Rightarrow x^\ell \notin \mathcal{G}_{\hat{M},\tilde{\tau}\epsilon}(f, X). \tag{40}$$

Contraposition of (40) gives

$$\forall x^\ell \in X: \quad x^\ell \in \mathcal{G}_{\hat{M},\tilde{\tau}\epsilon}(f, X) \Rightarrow x^\ell \in \mathcal{G}_{\hat{M},0}(f, X). \tag{41}$$

Consequently, $\mathcal{G}_{\hat{M},\hat{\tau}\epsilon}(f, X) \subseteq \mathcal{G}_{\hat{M},0}(f, X)$ and $\mathcal{G}_{\hat{M},0}(f, X) = \mathcal{G}_{\hat{M},\hat{\tau}\epsilon}(f, X)$ follows by setting $\hat{\tau} := \min\{\tilde{\tau}, \bar{\tau}\}$.

Proof of(37). ($\Rightarrow$) As X is finite, the conditions in Theorem 3.5 are satisfied. Thus, $\bigcap_{\tau>0} \mathcal{G}_{\tau\epsilon}(f, X) = \mathcal{S}_w(f, X)$. Hence, for every $\epsilon > 0$, there exists a $\bar{\tau} > 0$ such that

$$\mathcal{S}_w(f, X) \subseteq \mathcal{G}_{\bar{\tau}\epsilon}(f, X).$$

($\Leftarrow$) Consider an $x^\ell \in X$ such that $x^\ell \notin \mathcal{S}_w(f, X)$. Consequently, an $x^k \in X$ such that

$$-f_i(x^\ell) + f_i(x^k) < 0 \quad \forall i \in I. \tag{42}$$

exists. Simple computation shows that (42) implies that (39) is satisfied for $i = 1$, for all $\hat{M}$ sufficiently large, and for all $\tau \in [0, \tau^\ell]$ with τ^ℓ sufficiently small. Applying Part **3** of Proposition 2.1 and proceeding in a way similar to the proof of (36), we obtain a $\check{\tau} > 0$ such that

$$\forall x^\ell \in X: \quad x^\ell \notin \mathcal{S}_w(f, X) \Rightarrow x^\ell \notin \mathcal{G}_{\check{\tau}\epsilon}(f, X). \tag{43}$$

Contraposition of (43) gives

$$\forall x^\ell \in X: \quad x^\ell \in \mathcal{G}_{\check{\tau}\epsilon}(f, X) \Rightarrow x^\ell \in \mathcal{S}_w(f, X). \tag{44}$$

Consequently, we obtain the desired result. □

Example 3.1 *Let $X^g \subset \mathbb{R}^2$ be the set of 50×50 grid points that are equispaced in the rectangle $[-1, 0.5]^2$. Let*

$$X := X^g \setminus \{x \in \mathbb{R}^2 | x_1, x_2 \leq 0, \|x\|_2 > 1\}$$

and $f = \mathrm{id}_X$, where id_X the identity function on X.

For $\epsilon = (0.1, 0.2)^\top$, the sets $\mathcal{G}_{1,\tau\epsilon}(f, X)$ and $\mathcal{S}^{\tau\epsilon}(f, X)$ converge to the sets $\mathcal{G}_{1,0}(f, X)$ and $\mathcal{S}_w(f, X)$, respectively (Figure 3). Relations (36) and (37) of Proposition 3.8 can be seen to hold for $\hat{\tau} = 1/16$ (Figure 3, bottom left).

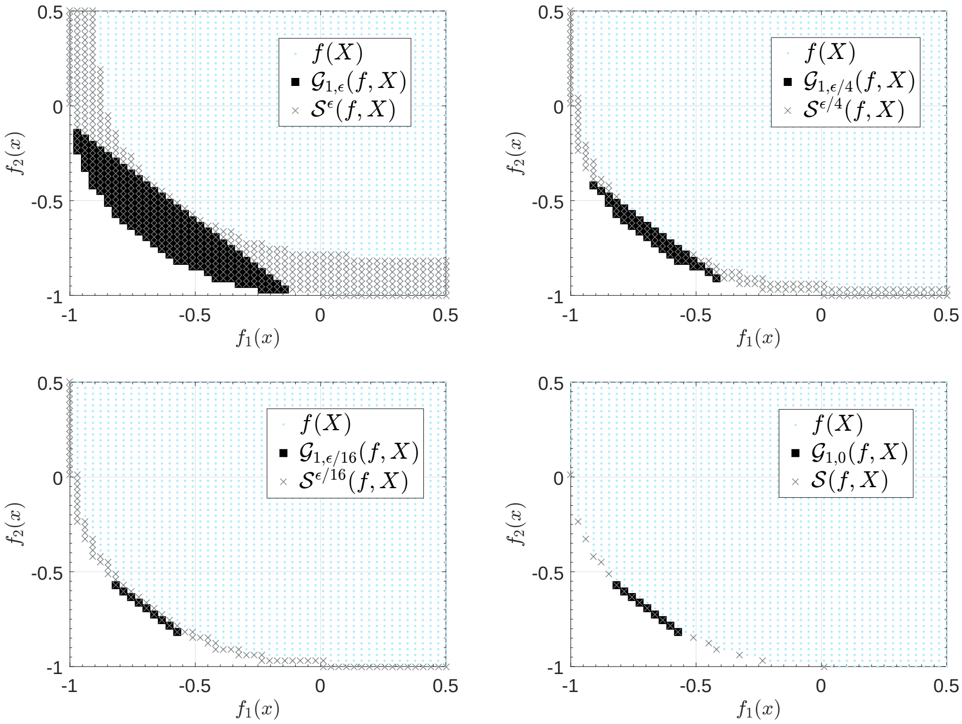

Figure 3. Convergence results on a discrete bi-criteria optimization problem with $\epsilon = (0.1, 0.2)^\top$. The bottom right figure shows the sets of exact solutions. The other figures show the convergence behavior of Geoffrion $(1, \tau\epsilon)$-proper solutions to Geoffrion $(1, 0)$-proper solutions for τ values of 1, $\frac{1}{4}$, and $\frac{1}{16}$. It can be seen that $\mathcal{S}^\epsilon(f,X)$ converges to $\mathcal{S}_w(f,X)$. As X is finite, for every $\epsilon \geq 0$, we have $\mathcal{S}^\epsilon(f,X) = \mathcal{G}_\epsilon(f,X)$. Therefore, $\mathcal{G}_{\tau\epsilon}(f,X)$ also converges to $\mathcal{S}_w(f,X)$. Due to $\mathcal{S}_w(f,X) \setminus \mathcal{S}(f,X)$ being non-empty, the Hausdoroff distance $d_H(\mathcal{G}_{\tau\epsilon}(f,X), \mathcal{G}(f,X))$ as a function of τ is bounded away from zero, while $d_H(\mathcal{G}_{1,\tau\epsilon}(f,X), \mathcal{G}_{1,0}(f,X))$ goes to zero.

Table 1. Answers to $Q1$ and $Q2$ for $\mathbb{T} = \Big(\mathcal{G}_{\hat{M},0}(f,X), \mathcal{G}(f,X), \mathcal{S}(f,X), \mathcal{S}_w(f,X), \mathcal{S}(f,X,C), \mathcal{S}_w(f,X,C)\Big)$.

$\mathbb{T}_i$	$\mathcal{G}_{\hat{M},0}(f,X)$	$\mathcal{G}(f,X)$	$\mathcal{S}(f,X)$	$\mathcal{S}_w(f,X)$	$\mathcal{S}(f,X,C)$	$\mathcal{S}_w(f,X,C)$
$Q1$: Does $(u^\tau \in \mathbb{T}_i^\tau$ for $\tau > 0) \wedge (u^\tau \to u) \Rightarrow u \in \mathbb{T}_i$?	Yes[a]	No[b], Yes[c]	No, Yes[c]	Yes[a]	No, Yes[c]	Yes[a]
$Q2$: What is the relation between $\mathbb{T}_i$ and $\bigcap_{\tau>0} \mathbb{T}_i^\tau$?	$=$	$\subseteq$[b]	$\subseteq$	$=$	$\subseteq$	$=$

[a] If X is closed and f continuous.
[b] If all the f_j's are bounded below.
[c] If $u^\tau \in \mathbb{W}^\tau$ for every $\tau > 0$.

A summary of the answers to $Q1$ and $Q2$ from Page 10 is presented in Table 1.

4. Concluding remarks

We have analyzed different ϵ-optimality notions for multicriteria optimization problems. Geoffrion $(\hat{M}, \epsilon)$-proper optimality seems to be the most useful notion due to the following reasons:

(1) A decision maker has a direct control over the trade-off bound–both for approximate (i.e. $\epsilon > 0$) and for exact (i.e. $\epsilon = 0$) solutions. This bound can be used to incorporate preferences of the decision maker. Although ϵ-Pareto optimality implies the existence of trade-offs between criteria and Geoffrion

ϵ-proper optimality implies the existence of an upper bound on trade-offs, Geoffrion $(\hat{M}, \epsilon)$-proper optimality is the one that explicitly uses an upper bound on the trade-offs.

(2) A sequence of Geoffrion $(\hat{M}, \epsilon)$-proper solutions converges to an exact Geoffrion $(\hat{M}, 0)$-proper solution under very mild conditions. Neither Geoffrion ϵ-proper not ϵ-Pareto optimal solutions have this convergence property. This convergence property has algorithmic implications for iterative, point-by-point search algorithms.

(3) The set of Geoffrion $(\hat{M}, \epsilon)$-proper solutions converges to the set of exact Geoffrion $(\hat{M}, 0)$-proper solution in the sense of $Q2$. As in (2), neither Geoffrion ϵ-proper not ϵ-Pareto optimal solutions have this convergence property. This convergence property has algorithmic implications for iterative, population based search algorithms.

Taken together, these findings suggest Geoffrion $(\hat{M}, \epsilon)$-proper optimality to be a practical optimality notion.

The results of this paper can be extended to a scenario where the trade-off bound depends on the criteria pair (i, j) involved in the trade-off, i.e. $M^I : I \times I \to \mathbb{R}_+$ instead of a constant $\hat{M}$. One could use this function M^I to define a Geoffrion (M^I, ϵ)-proper solution. For such solutions, it is easy to see that Theorem 3.2 remains true if $\hat{M}$ is substituted by the M^I. Furthermore, similar to Theorem 3.7, we can prove that

$$\mathcal{G}_{M^I,0}(f, X) = \bigcap_{\tau>0} \mathcal{G}_{M^I,\tau\epsilon}(f, X).$$

To further our research we intend to develop scalarization approaches and optimality conditions for Geoffrion $(\hat{M}, \epsilon)$-proper solutions. The rate of convergence in $Q1$ and an estimate of the Hausdoroff distance between approximate and exact solution sets for continuous multicriteria optimization problems are another important topics of future research.

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