The Stochastic Multistage Fixed Charge Transportation Problem: Worst-Case Analysis of the Rolling Horizon Approach

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Abstract

We introduce the *Stochastic multistage fixed charge transportation problem* in which a producer has to ship an uncertain load to a customer within a deadline. At each time period, a fixed transportation price can be paid to buy a transportation capacity. If the transportation capacity is used, the supplier also pays an uncertain unit transportation price. A unit inventory cost is charged for the quantity that remains to be sent. The aim is to determine the transportation capacities to buy and the quantity to send at each time period in order to minimize the expected total cost. We prove that this problem is NP-hard, we propose a multistage stochastic optimization model formulation, and we determine optimal policies for particular cases, having deterministic unit transportation prices or load. *Rolling horizon* is a classical heuristic approach for solving multistage stochastic programming models. Our aim is to provide the worst–case analysis of this approach, applied to this NP-hard problem and to polynomially solvable particular cases. Numerical results are also provided.

1 Introduction

The *Fixed charge transportation problem* is a generalization of the classical *Transportation problem*, where the transportation cost function is given by the sum of a fixed cost and a variable cost, proportional to the quantity sent from each source to each sink. This problem has several

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practical applications, like distribution, transportation, scheduling, location, allocation of launch vehicles to space missions, solid-waste management, process selection, teacher assignment. It has been studied in the past and in the last years (see Gray (1971), Kennington and Unger (1976), Adlakha et al. (2010), Papageorgiou et al. (2012), Agarwal and Aneja (2012) and Roberti et al. (2014) for exact algorithms, Jawahar and Balaji (2009), Raj and Rajendran (2012) and Buson et al. (2014) for heuristic algorithms and see Sheng and Yao (2012) for a case with uncertain variables).

In this paper, we introduce a stochastic version of this problem, referred to as the *Stochastic multistage fixed charge transportation problem*, in which a producer has to ship an uncertain load to a customer within a deadline. Transportation is outsourced, as in Bolduc et al. (2008), Chu (2005), Côté and Potvin (2009), Potvin and Naud (2011) and Stenger et al. (2013). Combinatorial auctions are typically used in transportation procurement to determine the best offer of transportation services (see Sheffi (2004)). We assume that, for each time period, the transportation company offering the best fixed transportation price for a given transportation capacity, is known. If the transportation capacity available at a time period is used, the supplier has also to pay an uncertain unit transportation price, for which a probability distribution is given. A unit inventory cost is charged for the quantity that remains to be sent at the end of the time period. The aim is to determine the transportation capacities to buy and the quantity to send at each time period in order to minimize the expected total cost. The deterministic counterpart of this problem can be viewed as the *Single-sink fixed charge transportation problem* studied in Herer et al. (1996), Görtz and Klose (2009), Christensen et al. (2013). In fact, each time period can be viewed as a different source from which a quantity can be sent to the sink. However, at each time period, we have to pay a unit inventory cost for the quantity not already sent.

We first formulate a multistage mixed-integer stochastic programming model (see Birge (1997) and Dupačová (2002) for a brief review of history and achievements of stochastic programming and for selected modeling issues concerning applications of multistage stochastic programs). Multistage stochastic mixed integer linear programs are among the most challenging optimization problems combining stochastic programs and discrete optimization problems (see Klein Haneveld and van der Vlerk (1999), Römisch and Schultz (2001), Sen (2005), Parija et al. (2004), Johnson et al. (2000), Pantuso et al. (2015) for some major results in this area). Exact solution methods are in general based on branch and bound type algorithms or branch and price methods, see Lulli and Sen (2004). Bounds and approximations for such a class of problems are provided in Maggioni et al. (2014, 2016), Maggioni and Pflug (2016).

After proving that this problem is NP-hard, we design exact polynomial time algorithms for the solution of two particular cases, having deterministic unit transportation price or load. Our main aim is to provide the worst–case analysis of the classical *Rolling horizon approach*, a heuristic approach frequently used to solve multistage stochastic programming models. In this approach, a policy is computed by optimal solving a sequence of stochastic programming models having a reduced time horizon. At each iteration, only the value of the first-stage variables is captured (we refer to Kusy and Ziemba (1986), Kouwenberg (2001), Guigues and Sagastizábal (2012), Silventea et al. (2015), for applications of this approach to different problems, to Chand et al. (2002) for a classified bibliography of the literature and to Bertocchi et al. (2006) for the choice of the time horizon, stages, methods for generating scenario trees). The comparison of the total cost of the *Rolling horizon approach* with the optimal total cost is often missing in the literature, as the optimal policy cannot be computed.

Worst-case analysis (see Gary and Johnson (1979)) is a useful tool to give this comparison in the worst case and, to the best of our knowledge, it has never been applied to the *Rolling horizon approach*. This analysis computes the value of the ratio between the total cost of the heuristic approach and the optimal total cost, in the worst case. Upper bounds on the total cost provided by the heuristic approach and lower bounds on the optimal total cost are used to prove the worst-case performance bound, that holds for any instance of the problem. Then, a worst-case instance, or a sequence of worst-case instances, are provided to show that the bound is tight, i.e. it is not overestimated. More formally, a heuristic approach $\mathcal{H}$, which gives a solution whose cost is $z^{\mathcal{H}}(I)$ on an instance I for which the optimal cost is $z^*(I)$, has a worst–case performance bound δ if $\frac{z^{\mathcal{H}}(I)}{z^*(I)} \leq \delta$, for any instance I. The ratio δ is tight if, for any $\delta' < \delta$, an instance I' exists for which $\frac{z^{\mathcal{H}}(I')}{z^*(I')} > \delta'$. Finally, we provide a systematic computational experiment that allows us to show the maximum dimension of the instances (in terms of number of stages) that can be solved by using a state-of-the-art solver, the sensitivity of the optimal total cost of the stochastic programming models solved at each iteration of the *Rolling horizon approach* with respect to increasing values of the reduced time horizon and, finally, the average performance of the *Rolling horizon approach* in a given set of problem instances.

The paper is organized as follows. In Section 2 the *Stochastic multistage fixed charge transportation problem* is formally described. In Section 3 a multistage stochastic programming model is formulated. In Section 4 we prove that this problem is NP-hard and provide exact polynomial time algorithms for the solution of the two particular cases with deterministic load or unit transportation prices. In Section

5 the worst-case analysis of the *Rolling horizon approach* is provided. In Section 6 the computational results are shown. Finally, Section 7 concludes the paper.

2 Problem Description

A producer has to ship an uncertain load to a customer within a deadline H. The load is composed of an uncertain number of units L, that is a random variable having discrete probability distribution $\mathcal{L}$ defined over the uncertainty set $\mathscr{U}_1 = \{L_{\min}, \ldots, L_{\max}\}$, where $0 < L_{\min} \leq L_{\max}$. A shipment can be performed at any of the discrete time periods $t \in T = \{0, 1, \ldots, H-1\}$, paying a fixed transportation price Q_t to buy a transportation capacity K_t and an uncertain unit transportation price P_t. The fixed transportation price Q_t is given. The unit transportation price is described by a discrete random variable having probability distributions $\mathcal{P}_t$ defined over the uncertainty set $\mathscr{U}_2 = \{m_2, \ldots, M_2\}$, where $0 < m_2 \leq M_2$. We assume that the probability distributions $\mathcal{P}_t$, $t \in T$, and $\mathcal{L}$ are independent. A typical case is when $\mathcal{P}_t$'s expected values $\mathbb{E}(P_t)$ are decreasing over time. The realization of the random variables in each time period is available at the end of the time period. A unit inventory cost h is paid for the quantity that remains to be sent at the end of time t. The aim is to determine, for each time period $t \in T$, if to buy or not the offered transportation capacity and the quantity to ship to the customer, in order to minimize the expected total cost.

3 A Multistage Stochastic Programming Formulation

In this section, we present a multistage stochastic programming formulation of the problem (see Birge and Louveaux (2011)). In order to proceed with numerical computations, it is useful to have a discretization of the underlying random processes of transportation prices and load to be shipped. If we assume that the transportation price $\boldsymbol{P}_{H-1} := (P_0, \ldots, P_t, \ldots, P_{H-1})$ and load $\boldsymbol{L}_{H-1} := (L_0, \ldots, L_0, \ldots, L_0)$ are random parameters evolving as discrete-time stochastic processes with finite support, then the information structure can be described in the form of a *scenario tree* $\mathscr{T}$. At each stage t, there is a discrete number of atoms (nodes) j_t, where a specific realization of the uncertain price P_t and of the load L_0 takes place. There are $H+1$ levels (stages) in the tree, that correspond to specific time periods. The final nodes j_H are called leaves ones. Each node, except the root, is connected to a unique node at stage $t-1$, called the ancestor node, and to nodes at

stage $t+1$, called the successors ones. Each non-leaf node j is the root of a subtree $\mathscr{T}(j)$. For each node j at stage t, we denote its ancestor with $a(j)$ and with $\pi_{a(j),j}$ the conditional probability of the random process in node j given its history up to the ancestor node $a(j)$. A scenario is a path through nodes from the root node to a leaf node. We indicate with $p(\omega)$ the probability of scenario ω passing through nodes $j_0, j_1, j_2, \ldots, j_H$ (where j_t, $t = 0, \ldots, H$ is the generic node at stage t) defined by $p(\omega) := \pi_{j_0,j_1} \cdot \pi_{j_2,j_3} \cdot \ldots \cdot \pi_{j_{H-1},j_H}$. We also indicate with π_j the probability of node j (at stage t): if node j at stage t is reachable through node j_0 at stage 1, node j_1 at stage 2, ..., node j_{t-1} at stage $t-1$, that is given by $\pi_j := \pi_{j_0,j_1} \cdot \pi_{j_1,j_2} \cdot \ldots \cdot \pi_{j_{t-1},j_t}$. Moreover, $\sum_{j=0}^{n_t} \pi_j = 1$ where n_t is the number of nodes at stage t. Let $\mathbf{P}_t(\omega)$ be the history of the ω-realization, $\omega \in \Omega$, of the transportation prices P_t up to stage t with expected value $\mathbb{E}(P_t) = \sum_{\omega \in \Omega} p(\omega) P_t(\omega)$. We denote with $L(\omega)$, the possible realizations of the load to be shipped within the deadline H in scenario $\omega \in \Omega$.
Let us now define the following notation:

Sets:

- $\Omega = \{\omega\}$: set of scenarios
- $T' = \{1, \ldots, H-1\}$: subset of discrete times

Deterministic Parameters:

- K_t: transportation capacity offered at time $t \in T$
- Q_t: fixed transportation price to buy the capacity K_t at time $t \in T$
- h: unit inventory cost

Stochastic Parameters:

- $P_t(\omega)$: realization of the unit transportation price at time $t \in T$ in scenario $\omega \in \Omega$
- $\mathbf{P}_t(\omega)$: the history of the ω-realization of the unit transportation price up to stage t
- $L(\omega)$: load to be shipped within the deadline H in scenario $\omega \in \Omega$
- $p(\omega)$: probability of scenario $\omega \in \Omega$

Variables:

- $y_0 \geq 0$: quantity shipped at stage $t = 0$ at the uncertain price $P_0(\omega)$
- $x_0 \in \{0, 1\}$: 1 if capacity K_0 is used at time 0, 0 otherwise
- $y_t(\omega) \geq 0$: quantity shipped at stage $t \in T'$ in scenario ω
- $x_t(\omega) \in \{0, 1\}$: 1 if capacity K_t is used at time $t \in T'$ in scenario ω, 0 otherwise.

The risk-neutral mixed integer linear multistage stochastic programming model is formulated as follows:

$$\min Q_0 \cdot x_0 + \sum_{\omega \in \Omega} p(\omega) \left\{ P_0(\omega) \cdot y_0 + \sum_{t \in T'} \left[P_t(\omega) \cdot y_t(\omega) + Q_t \cdot x_t(\omega) + h \left(L(\omega) - y_0 - \sum_{k=1}^{t-1} y_k(\omega) \right) \right] \right\}$$

$$\begin{aligned}
&\text{s.t. } y_{0+\sum_{t\in T'}} y_t(\omega) = L(\omega), \qquad \omega \in \Omega,\\
&y_0 \leq K_0 \cdot x_0,\\
&y_t(\omega) \leq K_t \cdot x_t(\omega), \quad t \in T', \ \omega \in \Omega,\\
&y_t(\omega')=y_t(\omega''), \forall \omega', \omega'' \text{ f.w. } (\mathbf{P}_t(\omega'), L(\omega')) = (\mathbf{P}_t(\omega''), L(\omega'')), \ t \in T',\\
&y_0 \in \mathbb{R}^+, \qquad (1)\\
&y_t(\omega) \in \mathbb{R}^+, \ t \in T', \ \omega \in \Omega,\\
&x_t(\omega')=x_t(\omega''), \forall \omega', \omega'' \text{ f.w. } (\mathbf{P}_t(\omega') L(\omega'))=(\mathbf{P}_t(\omega''), L(\omega'')), \ t \in T',\\
&x_0 \in \{0,1\},\\
&x_t(\omega) \in \{0,1\}, \qquad t \in T', \ \omega \in \Omega.
\end{aligned}$$

The first term in the objective function denotes the fixed transportation cost paid at time 0, while the second term the expected total cost of paying the unit transportation price $P_t(\omega)$, the fixed transportation price Q_t and the inventory cost h in each scenario ω at each time $t \in T'$. The first constraint guarantees that, for each scenario ω, the total quantity shipped to the customer within the deadline H is equal to $L(\omega)$. The second constraint guarantees that the quantity y_0 that can be sent at time 0 is not greater than the transportation capacity K_0 when it has been bought, while it is equal to 0 otherwise. The same is guaranteed for each time $t \in T'$ by the third type of constraints. Constraints four and seven represent the so-called non-anticipativity constraints on the decision variables y_t and x_t. All the other constraints define the decision variables of the problem. We denote the optimal total cost of model (1), referred to as Case 1), with z^*.

4 Computational Complexity and Polynomially Solvable Cases

In this section, we prove the computational complexity of model (1) and provide two polynomially solvable cases.

Theorem 1 *Model (1) is NP–hard.*

Proof Consider the set of instances such that the unit transportation price is deterministic and equal to 0, the unit inventory cost $h = 0$

and the load is deterministic, say L. Then, model (1) becomes

$$\min \sum_{t \in T} Q_t \cdot x_t$$

$$\begin{aligned} \text{s.t.} \sum_{t \in T} y_t &= L, \\ y_t &\leq K_t \cdot x_t, \quad t \in T, \\ y_t &\geq 0, \quad t \in T, \\ x_t &\in \{0,1\}, \qquad t \in T \end{aligned} \tag{2}$$

which is equivalent to the following *Min Cost 0–1 Knapsack Problem*:

$$\begin{aligned} \min \quad & \sum_{t \in T} Q_t \cdot x_t & \quad (3) \\ \text{s.t.} \quad & \sum_{t \in T} K_t \cdot x_t \geq L & \\ & x_t \in \{0,1\} \qquad t \in T & \end{aligned}$$

which is known to be NP-hard.

□

We now provide two polynomially solvable special cases, referred to as Case 2) and Case 3), under the assumption that the transportation capacity $K_t = \infty$ and the fixed price $Q_t = 0$. Similar results can be derived also for cases 2) and 3) with capacity constraints ($K_t < \infty$, $t \in T$) under the assumption that no fixed price is paid.

4.1 Case 2) Deterministic load (uncapacitated)

If we assume that the load is deterministic, i.e. $L(\omega) = L$, $\omega \in \Omega$, there are no constraints on the capacities, i.e. $K_t = \infty$, and the fixed price $Q_t = 0$, $t \in T$, then model (1) reduces to the following model:

$$\min \sum_{\omega \in \Omega} p(\omega) \left\{ P_0(\omega) y_0 + \sum_{t \in T'} \left[P_t(\omega) \cdot y_t(\omega) + h \left(L - y_0 - \sum_{k=1}^{t-1} y_k(\omega) \right) \right] \right\}$$

$$\begin{aligned} &\text{s.t. } y_0 + \sum_{t \in T'} y_t(\omega) = L, \qquad \omega \in \Omega, \\ &y_t(\omega') = y_t(\omega''), \qquad \forall\, \omega', \omega'' \text{ f.w. } \mathbf{P}_t(\omega') = \mathbf{P}_t(\omega''),\ t \in T', \\ &y_0 \in \mathbb{R}^+ \\ &y_t(\omega) \in \mathbb{R}^+, \qquad t \in T',\ \omega \in \Omega. \end{aligned} \tag{4}$$

Theorem 2 *The optimal total cost of model (4) is*

$$z^* = \min_{t \in T} \{\mathbb{E}(P_t) + ht\} L.$$

Therefore, an optimal policy can be computed in $O(H)$ time.

Proof We first note that in the two-stage case ($H = 1$), since the unique feasible solution is to send the overall quantity L at time 0, the optimal total cost z^* of model (4) is $\mathbb{E}(P_0)L$ and the thesis is verified.

We now prove the theorem by induction on the deadline H.

(Base Case) We consider the case of a three stage problem ($H = 2$). The model (4) reduces to the following model:

$$\min \quad \sum_{\omega \in \Omega} p(\omega) \left\{ P_0(\omega) y_0 + P_1(\omega) \cdot y_1(\omega) + h\,(L - y_0) \right\}$$

$$\begin{aligned} \text{s.t. } y_0 + y_1(\omega) &= L, && \omega \in \Omega, \\ y_1(\omega') &= y_1(\omega''), && \forall\, \omega', \omega'' \text{ for which } \mathbf{P}_1(\omega') = \mathbf{P}_1(\omega''), \\ y_0 &\in \mathbb{R}^+, \\ y_1(\omega) &\in \mathbb{R}^+,\ \omega \in \Omega. \end{aligned} \tag{5}$$

From the first type of constraints, we have

$$y_1(\omega) = L - y_0 \qquad \omega \in \Omega, \tag{6}$$

meaning that $y_1(\omega)$ is constant, say y_1, over all scenarios $\omega \in \Omega$. Therefore, the non-anticipativity constraints are unnecessary. Substituting $y_1 = L - y_0$ in the objective function and taking into account that $y_1 \geq 0$, the model (5) reduces to the following model:

$$\begin{aligned} \min \quad & \sum_{\omega \in \Omega} p(\omega) \left\{ P_0(\omega) y_0 + (L - y_0)(P_1(\omega) + h) \right\}, \\ \text{s.t. } y_0 &\leq L, \\ y_0 &\in \mathbb{R}^+ \end{aligned} \tag{7}$$

which is equivalent to

$$\begin{aligned} \min \quad & (\mathbb{E}(P_1) + h)L + (\mathbb{E}(P_0) - \mathbb{E}(P_1) - h) y_0 \\ \text{s.t. } y_0 &\leq L, \\ y_0 &\in \mathbb{R}^+. \end{aligned} \tag{8}$$

If $\min_{t \in \{0,1\}} \{\mathbb{E}(P_t) + ht\} = \mathbb{E}(P_0)$ then $y_0^* = L$ and the corresponding total cost $z^* = \mathbb{E}(P_0)L$; otherwise $y_0^* = 0$, with total cost $z^* = (\mathbb{E}(P_1) + h)L$ and the thesis is verified. Notice that in this case the stochastic model degenerates in the (deterministic) expected value model.

(Inductive step) We assume now, as induction hypothesis, that the thesis is verified for a model with deadline H. We need to prove that the thesis is also verified for a model with deadline $H+1$. If $\min_{t\in\{0,1,\ldots,H\}}\{\mathbb{E}(P_t)+ht\} = \mathbb{E}(P_H)+hH$ then $y^*_H(\omega) = L$, $\forall\omega \in \Omega$ and the corresponding total cost $z^* = (\mathbb{E}(P_H)+hH)L$. Otherwise, the thesis follows by the induction hypothesis.

$\square$

4.2 Case 3) Deterministic unit transportation price (uncapacitated)

If we assume that the unit transportation price is deterministic, i.e. $P_t(\omega) = P_t$, $\omega \in \Omega$, there are no constraints on the capacities, i.e. $K_t = \infty$, and the fixed price $Q_t = 0$, $t \in T$, then the model (1) reduces to the following model:

$$\min \sum_{\omega\in\Omega} p(\omega) \left\{ P_0 y_0 + \sum_{t\in T'} \left[P_t \cdot y_t(\omega) + h \left(L(\omega) - y_0 - \sum_{k=1}^{t-1} y_k(\omega) \right) \right] \right\}$$

$$\begin{aligned} &\text{s.t. } y_0 + \sum_{t\in T'} y_t(\omega) = L(\omega), \qquad \omega \in \Omega, \\ &y_t(\omega') = y_t(\omega''), \qquad \forall\, \omega', \omega'' \text{ f.w. } L(\omega') = L(\omega''),\ t \in T', \\ &y_0 \in \mathbb{R}^+, \\ &y_t(\omega) \in \mathbb{R}^+, \qquad t \in T',\ \omega \in \Omega. \end{aligned} \tag{9}$$

Theorem 3 *The optimal total cost of model (9) is*

$$z^* = \begin{cases} P_0 \cdot L_{min} + \min_{t\in T'}\{P_t + ht\}[\mathbb{E}(L) - L_{min}] & \textit{if } P_0 = \min_{t\in T}\{P_t + ht\} \\ \mathbb{E}(L) \min_{t\in T'}\{P_t + ht\} & \textit{otherwise.} \end{cases}$$

Therefore, an optimal policy can be computed in $O(H)$ time.

Proof We first note that the two-stage case ($H = 1$) is infeasible, since the unique possibility would be to send the stochastic quantity $L(\omega)$ at time 0 for all scenarios $\omega \in \Omega$, clearly impossible due to the non-anticipativity constraints which force the solution y_0 to be constant for all scenarios. For this reason we assume $H \geq 2$.

As before, we prove the theorem by induction on the deadline H.

(Base Case) We consider the case of a three stage problem ($H = 2$). The model (9) reduces to the following model:

$$\min \quad \sum_{\omega\in\Omega} p(\omega) \left\{ P_0 y_0 + P_1 y_1(\omega) + h\left(L(\omega) - y_0\right) \right\}$$

$$
\begin{aligned}
\text{s.t. } y_0 + y_1(\omega) &= L(\omega), \qquad \omega \in \Omega, && (10)\\
y_1(\omega') &= y_1(\omega''), \qquad \forall\, \omega', \omega'' \text{ for which } L(\omega') = L(\omega''),\\
y_0 &\in \mathbb{R}^+\\
y_1(\omega) &\in \mathbb{R}^+, \quad \omega \in \Omega.
\end{aligned}
$$

From the first type of constraints, we have

$$y_1(\omega) = L(\omega) - y_0 \qquad \omega \in \Omega. \tag{11}$$

Substituting in the objective function and taking into account that $y_1(\omega) \geq 0$, the model (10) reduces to the following model:

$$
\begin{aligned}
\min \quad & \sum_{\omega\in\Omega} p(\omega)\left\{P_0 y_0 + (P_1 + h)(L(\omega) - y_0)\right\},\\
\text{s.t. } y_0 &\leq L(\omega), \qquad \omega \in \Omega,\\
y_0 &\in \mathbb{R}^+
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
\min \quad & (P_1 + h)\mathbb{E}(L) + (P_0 - P_1 - h)y_0\\
\text{s.t. } y_0 &\leq L_{\min},\\
y_0 &\in \mathbb{R}^+.
\end{aligned}
$$

If $\min_{t\in\{0,1\}}\{P_t + ht\} = P_0$ then $y_0^* = L_{min}$, $y_1^*(\omega) = L(\omega) - L_{min}$, $\omega \in \Omega$, and the corresponding cost $z^* = P_0 L_{min} + (P_1+h)[\mathbb{E}(L) - L_{min}]$; otherwise $y_0^* = 0$ and $y_1^*(\omega) = L(\omega)$, with cost $z^* = \mathbb{E}(L)(P_1 + h)$ and the thesis is verified.

(Inductive step) We assume now, as induction hypothesis, that the thesis is verified for a model with deadline H. We need to prove that the thesis is also verified for a model with deadline $H+1$. If $\min_{t\in\{0,1,\dots,H\}}\{P_t + ht\} = P_H + hH$ then $y_H^*(\omega) = L(\omega)$, $\forall\omega \in \Omega$, and the corresponding total cost $z^* = \mathbb{E}(L)(P_H + hH)$. Otherwise, the thesis follows by the induction hypothesis.

□

5 Worst-Case Analysis of the *Rolling Horizon Approach*

In this section, we evaluate the worst-case performance of the classical *Rolling horizon approach* in solving multistage stochastic programs with finite time horizon. In this heuristic approach, a policy is computed by optimal solving a sequence of $(W + 1)$-stage stochastic programming models, where $W < H$ is the reduced time horizon of the

$(W+1)$-stage stochastic programming model. In particular, in the first step, the $(W+1)$-stage stochastic programming model defined on $t = 0, 1, \ldots, W$ is optimally solved and only the values of the first-stage decision variables x_0 and y_0 are captured as the decision $x_0^{(W+1)S}$ to buy or not the capacity and the quantity $y_0^{(W+1)S}$ to send at time 0 in the rolling horizon policy. In the second step, the $(W+1)$-stage stochastic programming model defined on $t = 1, 2, \ldots, W+1$ is optimally solved by setting the load equal to the residual quantity to be sent, given by $S(\omega) = L(\omega) - y_0$. Only the values of the new first-stage variables x_1 and y_1 are captured as the decision $x_1^{(W+1)S}(\omega)$ to buy or not the capacity and the quantity $y_1^{(W+1)S}(\omega)$ to send on each scenario $\omega \in \Omega$ at time 1 in the rolling horizon policy. This process is repeated until time $t = H - W$. After solving the last $(W+1)$-stage stochastic programming model, a W-stage stochastic programming model on $t = H-W+1, H-W+2, \ldots, H$, is solved. Then, a $(W-1)$-stage stochastic programming model on $t = H-W+2, H-W+3, \ldots, H$ is solved. The process is repeated until the 2-stage stochastic programming model defined on $t = H-1, H$ is solved (see Figure 1). In the following, we denote with $z^{(W+1)S}$ the total cost of the *Rolling horizon approach*.

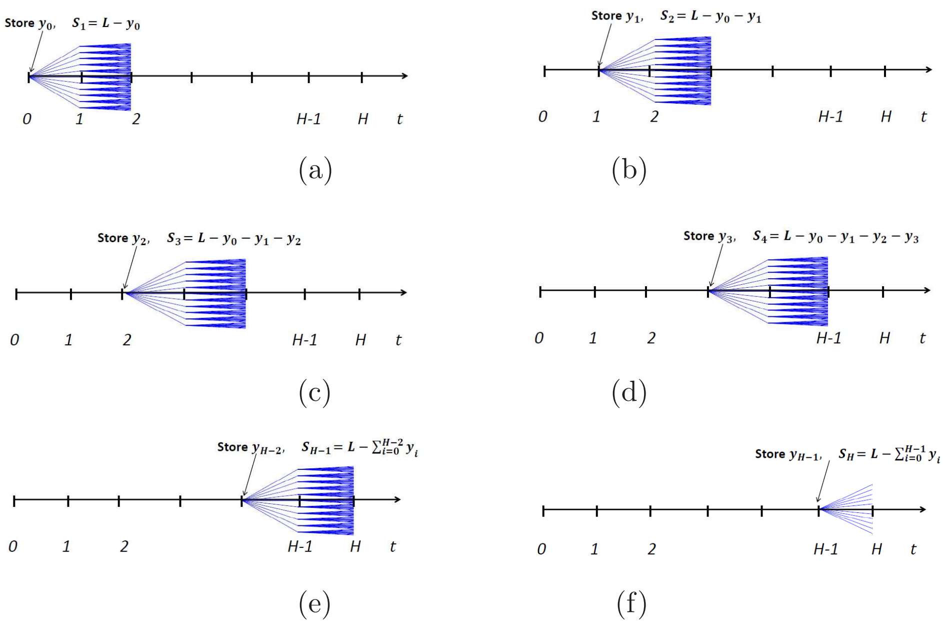

Figure 1: The *Rolling horizon approach* with $W = 2$ over $H+1$ stages. Follow the sequence (a)-(b)-(c)-(d)-(e)-(f).

We now prove the worst-case performance bound of *the Rolling horizon approach*. Similar results hold true also for cases 2) and 3) with capacity constraints ($K_t < \infty$, $t \in T$) and for reduced time horizon $W > 2$.

Case 1): stochastic load and unit transportation prices.

Remember that in this case any 2-stage stochastic programming model is infeasible. Therefore, we study the worst-case performance of the *Rolling horizon approach* with $W = 2$, based on the optimal solution of a sequence of 3-stage stochastic programming models. The following theorem holds.

Theorem 4 *In Case 1), there exists an instance such that* $\frac{z^{3S}}{z^*} \longrightarrow \infty$.

Proof Consider the following instance: Deterministic load $L = 1$; deadline $L = 4$; deterministic unit transportation prices $P_t = 0$, $t \in T$; transportation capacities $K_0 = K_1 = K_2 = \frac{L}{2}$ and $K_3 = L$; fixed transportation prices $Q_0 = Q_1 = Q_2 = 1$ and $Q_3 = \epsilon << 1$; unit inventory cost $h = \epsilon$.

Let us apply the *Rolling horizon approach* with $W = 2$. In the first step, the first three-stage stochastic programming model is solved. Since $Q_0 + h\frac{L}{2} < Q_1 + hL$, $x_0^* = 1$ and $y_0^* = \frac{L}{2}$. Therefore, in the rolling horizon policy, we have $x_0^{3S} = 1$ and $y_0^{3S} = \frac{L}{2}$. In the second step, since $Q_1 < Q_2 + h\frac{L}{2}$, $x_1^* = 1$ and $y_1^* = \frac{L}{2}$. Therefore, in the rolling horizon policy, we have $x_1^{3S} = 1$ and $y_1^{3S} = \frac{L}{2}$. All remaining variables are equal to 0. Therefore, the total cost is $z^{3S} = 2 + h\frac{L}{2} = 2 + \frac{\epsilon}{2}$.

The optimal total cost z^* is not greater than the cost of the following solution: $x_0 = x_1(\omega) = x_2(\omega) = 0$, $x_3(\omega) = 1$ and $y_0 = y_1(\omega) = y_2(\omega) = 0$ and $y_3(\omega) = L$, having total cost $Q_3 + 3hL = 4\epsilon$.

Therefore, in this instance

$$\frac{z^{3S}}{z^*} \geq \frac{2 + \frac{\epsilon}{2}}{4\epsilon} \to \infty \qquad \text{for } \epsilon \to 0.$$

□

Case 2): deterministic load (uncapacitated).

In this case, we start studying the *Rolling horizon approach* with $W = 1$, where a two-stage stochastic programming model is solved at each iteration, and then we study the case with $W = 2$. The following theorem holds.

Theorem 5 *In Case 2),* $\frac{z^{2S}}{z^*} \leq \frac{M_2}{m_2}$ *and the bound is tight.*

Proof Since the optimal solution of the two-stage stochastic programming model solved at time 0 is to send the overall quantity L at time 0, the policy provided by the *Rolling horizon approach* with $W = 1$ is $y_0^{2S} = L$ and $y_t^{2S}(\omega) = 0$, $t \in T'$, $\omega \in \Omega$. The corresponding total cost z^{2S} is $\mathbb{E}(P_0)L$, which is not greater than M_2L.

Let us now compute the worst-case performance bound. If the optimal policy is to send L at time 0, then $z^* = \mathbb{E}(P_0)L$ and therefore the worst-case performance bound is equal to 1. Otherwise, a lower bound on the optimal total cost z^* is m_2L. Therefore,

$$\frac{z^{2S}}{z^*} \leq \frac{M_2}{m_2}.$$

We now prove that the bound is tight. Consider the following instance: load $L = 1$; deadline $H = 2$; probability distribution at time 0: M_2 with probability 1; probability distribution at time 1: m_2 with probability 1; unit inventory cost $h = 0$.

The total cost of the *Rolling horizon approach* with $W = 1$ is $z^{2S} = M_2$, while the optimal total cost z^* is m_2. Therefore, in this instance

$$\frac{z^{2S}}{z^*} = \frac{M_2}{m_2}.$$

□

Consider now the *Rolling horizon approach* with $W = 2$, where a three-stage stochastic programming model is solved at each iteration. Let L_t be the residual quantity to ship at time t and Ω_t^W be the set of scenarios obtained by considering the probability distributions of the unit transportation prices at time t and $t+1$. Then, the three-stage stochastic programming model to solve at each time $t = 0, 1, \ldots, H-2$ is:

$$\min \quad \sum_{\omega \in \Omega_t^W} p(\omega)\left\{P_t(\omega)y_t + P_{t+1}(\omega)y_{t+1}(\omega)\right\} + h\left(L_t - y_t\right) \tag{12}$$

$$\begin{aligned} \text{s.t. } y_t + y_{t+1}(\omega) &= L_t, \qquad \omega \in \Omega_t^W, \\ y_t &\in \mathbb{R}^+, \\ y_{t+1}(\omega) &\in \mathbb{R}^+, \qquad \omega \in \Omega_t^W. \end{aligned}$$

Note that the non–anticipativity constraints are not needed due to the first type of constraints, as $y_{t+1}(\omega) = L_t - y_t$ is constant for all $\omega \in \Omega_t^W$. Replacing in the objective function and taking into account that $y_{t+1}(\omega) \geq 0$, this model can be written as follows:

$$\min \quad \sum_{\omega \in \Omega_t^W} p(\omega)\left\{P_t(\omega)y_t + (P_{t+1}(\omega) + h)(L_t - y_t)\right\} \tag{13}$$

$$\begin{array}{rcl} \text{s.t. } y_t & \leq & L_t, \\ y_t & \in & \mathbb{R}^+ \end{array}$$

which is equivalent to

$$\begin{array}{lrcl} \min & [\mathbb{E}(P_{t+1}) + h]L_t + [\mathbb{E}(P_t) - \mathbb{E}(P_{t+1}) - h]y_t \\ & \text{s.t. } y_t & \leq & L_t, \\ & y_t & \in & \mathbb{R}^+. \end{array} \tag{14}$$

Therefore, if $\mathbb{E}(P_t) \leq \mathbb{E}(P_{t+1}) + h$, then $y_t^* = L_t$, otherwise $y_t^* = 0$. This is the quantity $y_t^{3S}(\omega)$ to send at time t on each scenario $\omega \in \Omega$ in the policy provided by the *Rolling horizon approach* with $W = 2$. At time $H-1$ a two-stage stochastic programming model is solved. Since a shipment can be performed only at time $H-1$, the optimal solution is simply $y_{H-1}^* = L_{H-1}$. This is the quantity $y_{H-1}^{3S}(\omega)$ to send at time $H-1$ on each scenario $\omega \in \Omega$. Note that the overall quantity L is shipped in just one time $t \in T$. The following theorem holds.

Theorem 6 *In Case 2),* $\frac{z^{3S}}{z^*} \leq \frac{M_2}{m_2}$ *and the bound is tight.*

Proof We first compute an upper bound on the cost z^{3S} of the *Rolling horizon approach* with $W = 2$. Let τ be the time period having $y_\tau^* = L$ when the three-stage stochastic programing model on $\tau, \tau+1, \tau+2$ is solved (i.e. for $0 \leq \tau < H-1$) or when the two-stage stochastic programming model on $\tau, \tau+1$ is solved (i.e. for $\tau = H-1$). In order to have this solution, we need to have $\mathbb{E}(P_0) > \mathbb{E}(P_1) + h$, $\mathbb{E}(P_1) > \mathbb{E}(P_2) + h, \ldots, \mathbb{E}(P_{\tau-1}) > \mathbb{E}(P_\tau) + h$. The cost z^{3S} is $\mathbb{E}(P_\tau)L + h\tau L$, which is not greater than $(M_2 + h\tau)L$.

Since the optimal cost is equal to $\min_{t \in T} \{\mathbb{E}(P_t) + ht\} L$ (see Theorem 2), we have just two cases. In the first one, the policy is able to find the optimal cost and therefore the worst-case performance bound is equal to 1. In the second case, since $\mathbb{E}(P_0) > \mathbb{E}(P_1) + h$, $\mathbb{E}(P_1) > \mathbb{E}(P_2) + h, \ldots, \mathbb{E}(P_{\tau-1}) > \mathbb{E}(P_\tau) + h$, a lower bound on the optimal cost z^* is $(m_2 + h(\tau+1))L$, meaning that L is shipped at minimum cost m_2 at time $\tau + 1$. Therefore,

$$\frac{z^{3S}}{z^*} \leq \frac{(M_2 + h\tau)L}{(m_2 + h(\tau+1))L} \leq \frac{M_2}{m_2}.$$

We now prove that the bound is tight. Consider the following instance: load $L = 1$; deadline $H = 3$; probability distribution at time 0 and at time 1: M_2 with probability 1; probability distribution at time 2: m_2 with probability 1; unit inventory cost $h = \epsilon << 1$.

Let us apply the *Rolling horizon approach* with $W = 2$. In the first step, the first three-stage stochastic programming model is solved. Since $\mathbb{E}(P_0) < \mathbb{E}(P_1) + h$, $y_0^* = L$. Therefore, $y_0^{3S} = L$ and $z^{3S} = M_2$.

The optimal total cost z^* is not greater than the cost of the following solution: $y_0 = 0, y_1(\omega) = 0, y_2(\omega) = L$ having total cost $m_2 + 2\epsilon$. Therefore, in this instance

$$\frac{z^{3S}}{z^*} \geq \frac{M_2}{m_2 + 2\epsilon} \rightarrow \frac{M_2}{m_2} \qquad \text{for } \epsilon \rightarrow 0.$$

□

Case 3): deterministic unit transportation price (uncapacitated).

Remember that in this case any 2-stage stochastic programming model is infeasible. Therefore, we study the worst-case performance of the *Rolling horizon approach* with $W = 2$, based on the optimal solution of a sequence of 3-stage stochastic programming models. The following theorem holds.

Theorem 7 *In Case 3),* $\frac{z^{3S}}{z^*} \leq \max\left\{\frac{M_2}{m_2}, H-1\right\}$ *and the bound is tight.*

Proof Let us apply the *Rolling horizon approach* with $W = 2$. In the first step, the first three-stage stochastic program is solved. If $P_0 \leq P_1 + h$, then $y_0^* = L_{\min}$, otherwise, $y_0^* = 0$ (see Theorem 3). Therefore, in the rolling horizon policy, we have $y_0^{3S} = L_{\min}$ in the former case and $y_0^{3S} = 0$ in the latter case. Then, at each time $t < H - 2$, when the corresponding three-stage stochastic model is solved, we have two cases: 1) $\sum_{\tau=0}^{t-1} y_\tau^{3S} = 0$: if $P_t \leq P_{t+1} + h$ then $y_t^* = L_{\min}$, otherwise $y_t^* = 0$; therefore, in the rolling horizon policy, we have $y_t^{3S} = L_{\min}$ in the former case and $y_t^{3S} = 0$ in the latter case; 2) $\sum_{\tau=0}^{t-1} y_\tau^{3S} = L_{\min}$: since $\min_\omega \{L(\omega) - L_{\min}\} = 0$, then $y_t^* = 0$ and therefore $y_t^{3S} = 0$. At time $H - 2$, where the last three-stage stochastic model is solved, if $\sum_{\tau=0}^{H-3} y_\tau^{3S} = L_{\min}$, then $L(\omega) - L_{\min}$ is shipped at time $H - 1$ in each scenario ω, otherwise $L(\omega)$ is shipped at time $H - 1$ in each scenario ω. Therefore, $y_{H-1}^{3S} = L(\omega) - L_{\min}$ and $y_{H-1}^{3S} = L(\omega)$, respectively. In the former case, $z^{3S} \leq L_{\min}M_2 + [M_2 + (H-1)h][\mathbb{E}(L) - L_{\min}] = M_2\mathbb{E}(L) + (H-1)h[\mathbb{E}(L) - L_{\min}]$, while in the latter case $z^{3S} \leq [M_2 + (H-1)h]\mathbb{E}(L)$.

Let us now compute a lower bound on the optimal total cost z^*. If $P_0 = \min_{t \in T}\{P_t + ht\}$, then $z^* \geq m_2 L_{\min} + (m_2 + h)[\mathbb{E}(L) - L_{\min}] = m_2\mathbb{E}(L) + h[\mathbb{E}(L) - L_{\min}]$. Otherwise, $z^* \geq \mathbb{E}(L)(m_2 + h)$ (see Theorem 3).

Therefore, if $P_0 = \min_{t \in T}\{P_t + ht\}$, then

$$\frac{z^{3S}}{z^*} \leq \frac{M_2\mathbb{E}(L) + (H-1)h[\mathbb{E}(L) - L_{\min}]}{m_2\mathbb{E}(L) + h[\mathbb{E}(L) - L_{\min}]} \leq \max\left\{\frac{M_2}{m_2}, H-1\right\}.$$

Otherwise,

$$\frac{z^{3S}}{z^*} \leq \frac{[M_2 + (H-1)h]\mathbb{E}(L)}{(m_2+h)\mathbb{E}(L)} \leq \max\left\{\frac{M_2}{m_2}, H-1\right\}.$$

In order to prove that the bound is tight, consider the following two instances.
Instance 1: Unit transportation cost $P_t = \frac{1}{h}$ for all $t \in T$, where h is the unit inventory cost assumed to be greater than 0. Let us apply the *Rolling horizon approach* with $W = 2$. Since $P_0 < P_1 + h$, $y_0^* = L_{\min}$. Therefore, $y_0^{3S} = L_{\min}$ and $y_t^{3S}(\omega) = 0$, for $t = 1, 2, \dots, H-2$, while $y_{H-1}^{3S}(\omega) = L(\omega) - L_{\min}$. Therefore, $z^{3S} = \frac{1}{h}\mathbb{E}(L) + (H-1)h[\mathbb{E}(L) - L_{\min}]$. The optimal total cost z^* is not greater than the total cost of the solution in which $y_0 = L_{\min}$ and $y_1(\omega) = L(\omega) - L_{\min}$, that is $z^* \leq \frac{1}{h}\mathbb{E}(L) + h[\mathbb{E}(L) - L_{\min}]$. Therefore, in this instance:

$$\frac{z^{3S}}{z^*} \geq \frac{\frac{1}{h}\mathbb{E}(L) + (H-1)h[\mathbb{E}(L) - L_{\min}]}{\frac{1}{h}\mathbb{E}(L) + h[\mathbb{E}(L) - L_{\min}]} \to H-1 \qquad \text{for } h \to \infty.$$

Instance 2: Deadline $H = 3$, unit inventory cost $h = \epsilon << 1$, $L_{\min} = \epsilon$, unit transportation prices $P_0 = P_1 = m_2$ and $P_2 = M_2$. Let us apply the *Rolling horizon approach* with $W = 2$. Since $P_0 < P_1 + h$, $y_0^* = L_{\min}$. Therefore, $y_0^{3S} = L_{\min}$ and $y_1^{3S} = 0$,while $y_2^{3S}(\omega) = L(\omega) - L_{\min}$. Therefore, $z^{3S} = m_2 L_{\min} + (M_2 + 2h)[\mathbb{E}(L) - L_{\min}]$. The optimal total cost z^* is not greater than the total cost of the solution in which $y_0 = 0$ and $y_1(\omega) = L(\omega)$, that is $z^* \leq (m_2 + h)\mathbb{E}(L)$. Therefore, in this instance:

$$\frac{z^{3S}}{z^*} \geq \frac{m_2 L_{\min} + (M_2 + 2h)[\mathbb{E}(L) - L_{\min}]}{(m_2+h)\mathbb{E}(L)} \to \frac{M_2}{m_2} \qquad \text{for } \epsilon \to 0.$$

□

6 Numerical Results

In this section, our aim is three-fold. First, we aim at understanding the maximum dimension of the multistage stochastic programming models in term of stages that can be solved by a state-of-the-art solver. Second, we aim at understanding how sensitive are the optimal policies and the optimal total cost of the $(W+1)$-stage stochastic programming model with respect to the reduced time horizon W. Third, we aim at comparing the average performance of the *Rolling horizon approach* in a given set of instances with respect to the worst-case performance bounds provided in the previous section.

We use AMPL environment along CPLEX 12.5.1.0 solver to solve the stochastic programming models (see Valente et al. (2009)). All the computations were run on a 64-bit machine with 12 GB of RAM and a 2.90 GHz processor.

Our computational experiment is based on the following data:

1. Deadline H: up to 8 time periods.
2. Load L: In the cases 1) and 3), having stochastic load, the support of the probability distribution is the set of integer numbers in the interval $[L_{\min}, L_{\max}]$, with $L_{\min} = 8$ and $L_{\max} = 12$. We use a *Beta distribution* $B(\alpha, \beta)$, with $\alpha = 9$ and $\beta = 15$, having average load $\mathbb{E}(L) = 10.00875104$. In the Case 2), having deterministic laod, the load is $L = 10$.
3. Transportation capacities K_t: $K_0 = 6$, $K_1 = 7$, $K_2 = 4$, $K_3 = 6$, $K_4 = 9$. These values are such that $K_t < L$, $\forall t \in T$, and $\sum_{t \in T} K_t > L$.
4. Unit transportation prices P_t: in cases 1) and 2), having stochastic unit transportation prices, the support of the probability distribution at each time $t \in T$ is the set of integer numbers in the interval $[m_2, M_2]$, with $m_2 = 90$ and $M_2 = 100$. We use a *Beta Distribution* $B(\alpha_t, \beta_t)$ for each time period $t \in T$. The values of α_t and β_t are shown in Table 1. They are selected in such a way that the expected values $\mathbb{E}(P_t)$ of the probability distributions are decreasing over time, as shown in Table 2. In Case 3), having deterministic prices, the unit transportation price P_t at each time $t \in T$ is equal to $\mathbb{E}(P_t)$.
5. Fixed transportation prices Q_t to buy the full capacity K_t: they are generated to maintain a predefined ratio θ between the total variable cost $\mathbb{E}(P_t) \cdot K_t$ and the fixed cost Q_t. The instances are grouped into 2 classes characterized by $\theta = 0.2$ and 0.5. Similar results were obtained for $\theta = 0.1, 0.3, 0.4, 0.6, 0.7, 0.8, 0.9, 1$.
6. Unit inventory cost h: $0, 1, \ldots, 10$ in Case 1) and specific intervals, provided in the following, in cases 2) and 3).

t	0	1	2	3	4	5	6	7	8	9
α_t	9	15	6	5	9	8	3	5	1	1
β_t	10	20	10	10	20	20	10	20	10	20

Table 1: Values of α_t and β_t in the Beta distribution $B(\alpha_t, \beta_t)$

t	0	1	2	3	4	5	6	7	8	9
$\mathbb{E}(P_t)$	95.2368	94.7857	94.2500	93.8333	93.6034	93.3570	92.8060	92.5001	91.4914	91.1339

Table 2: Expected value of the unit transportation prices P_t

In order to describe the stochasticity of the unit transportation price over time (see Case 2)), we consider the scenario tree structures shown in Figures 2 and 3. The same structures are built in Case 1), but each node of the scenario trees is composed of the unit transportation price and the load. Notice that since the support of the uncertain load and prices are discrete, in our approach we solve the full stochastic programming problem with the complete scenario tree structure. No scenario reduction techniques are adopted. Figure

(a) (b)

Figure 2: Scenario tree structures respectively adopted for the (a) two-stage ($H = 1$) and (b) three-stage ($H = 2$) stochastic cases

(a) (b)

Figure 3: Scenario tree structures respectively adopted for the (a) four-stage ($H = 3$) and (b) five-stage ($H = 4$) stochastic cases

2(a) shows a two-stage tree with 11 branches from the root, resulting in $|\Omega| = 11$ scenarios and 12 nodes, while Figure 2(b) shows a three-stage tree with 11 branches from the root and 11 from each of the second-stage nodes resulting in $|\Omega| = 11^2 = 121$ scenarios and 133 nodes. Figure 3(a) shows a four-stage tree with 11 branches from the root and 11 from each of the second and third-stage nodes resulting in $|\Omega| = 11^3 = 1331$ scenarios and 1464 nodes. Finally, Figure 3(b) shows a five-stage tree with with 11 branches from the root, 11 from each of the second/third/fourth-stage nodes resulting in $|\Omega| = 11^4 = 14641$ scenarios and 16105 nodes.

6.1 Solving the multistage stochastic programming models

In this subsection, we provide statistics concerning the performance of a state-of-the-art solver (CPLEX) to find an optimal solution of the multistage stochastic programming models formulated in Sections 3 and 4 for the cases 1), 2) and 3). In particular, Tables 3-4-5 show the number of MIP simplex iterations and the CPU time (in seconds) in the cases 1), 2) and 3), respectively, required by CPLEX to find an optimal solution of the corresponding multistage stochastic programming model, when the number of stages increases.

	three-stage	four-stage	five-stage
MIP simplex iterations	0	1373	–
CPU time (s)	0.234001	80.4029	–

Table 3: Case 1): Summary statistics

	two-stage	three-stage	four-stage	five-stage	six-stage	seven-stage
MIP simplex iterations	0	0	65	488	3033	–
CPU time (s)	0.0156	0.0156	0.0936	0.655204	36.5666	–

Table 4: Case 2): Summary statistics

	three-stage	four-stage	five-stage	six-stage	seven-stage	eight-stage	nine-stage	ten-stage
MIP simplex it.	0	16	52	217	888	2943	9695	-
CPU time (s)	0.0156	0.0312002	0.0624004	0.265202	1.91881	23.681	221.568	-

Table 5: Case 3): Summary statistics

These results show that in Case 1) an optimal solution of the model (1), which is NP-hard, can be obtained just up to the four-stage stochastic programming model. Therefore, heuristic algorithms, like the *Rolling horizon approach*, are required. More interesting, even the polynomially solvable cases 2) and 3) can be solved just up to the six-stage and the nine-stage stochastic programming models, respectively. This gives additional value to the optimal policies provided in Section 4, that are able to solve any H-stage stochastic model in cases 2) and 3) in $O(H)$ time.

6.2 Analysis of the optimal policies and of the total costs

In this subsection, we show a sensitivity analysis of the optimal policies and of the total cost of the $(W+1)$-stage stochastic programming model with respect to the reduced time horizon W.

Case 1): stochastic load and unit transportation prices.

Table 6 and Figure 4 show the optimal value of the first-stage variable y_0 and the total cost in Case 1) for different values of the unit inventory cost $h = 0, \ldots, 10$, when the predefined ratio θ between the expected total variable cost $\mathbb{E}(P_t) \cdot K_t$ and the fixed cost Q_t is 0.2 and 0.5. We do not show the values of the variables $y_t(\omega)$ for $t > 0$ because they are different in different scenarios $\omega \in \Omega$. We just consider the cases with $W = 2$ (three-stage) and $W = 3$ (four-stage), as the optimal solution cannot be computed for larger numbers of stages (as previously shown in Table 3).

W	h	y_0	Total cost ($\theta = 0.2$)	Total cost ($\theta = 0.5$)
2	0	5	7166.289109	3426.525950
2	1	6	7171.159226	3431.396066
2	2	6	7175.24426	3435.481100
2	3	6	7179.329293	3439.566134
2	4	6	7183.414327	3443.651168
2	5	6	7187.499361	3447.736201
2	6	6	7191.584395	3451.821235
2	7	6	7195.669429	3455.906269
2	8	6	7199.754463	3459.991303
2	9	6	7203.839497	3464.076336
2	10	6	7207.924531	3468.16137
3	0	6	5971.764695	2946.929624
3	1	6	5979.117164	2954.282092
3	2	6	5986.469632	2961.634561
3	3	6	5993.822101	2968.98703
3	4	6	6001.174570	2976.339499
3	5	6	6008.527039	2983.691967
3	6	6	6015.879507	2991.044436
3	7	6	6023.231975	2998.396905
3	8	6	6030.584443	3005.749374
3	9	6	6037.936911	3013.101842
3	10	6	6045.289379	3020.454311

Table 6: Case 1): Optimal value of the first-stage variable y_0 and of the total cost with $\theta = 0.2$ and 0.5

The results show that, in all cases, the value of the optimal first-stage variable y_0 is to send a quantity equal to the capacity K_0, with exception of the case with $W = 2$ and $h = 0$ only, in which a lower quantity is sent at time 0. Moreover, the three-stage ($W = 2$) stochastic programming model is significantly more costly than the four-stage one ($W = 3$). In particular, the average percent increase of the total cost of the model with $W = 2$ with respect to the model with $W = 3$ is about 19% and 15% for $\theta = 0.2$ and 0.5, respectively.

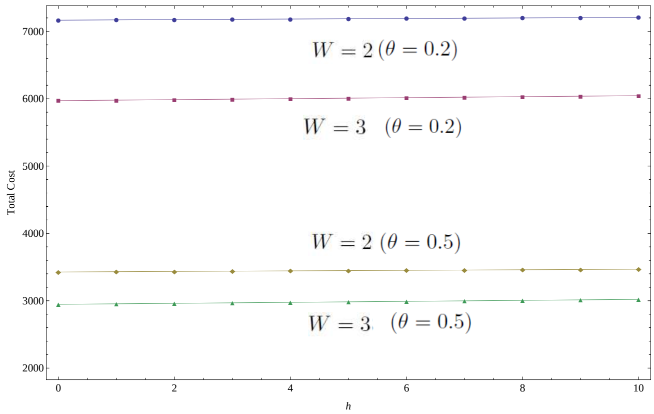

Figure 4: Case 1): Total cost against increasing values of h for different values of W, with $\theta = 0.2$ and 0.5

Case 2): deterministic load (uncapacitated).

Table 7 and Figure 5 show the optimal values of the variables y_0 and $y_t(\omega)$, $t > 0$, and the total cost in Case 2), having deterministic load $L = 10$, for different values of the inventory cost h, when the reduced time horizon W increases from 1 to 5.

W	h	$y_0,\ y_t(\omega) \neq 0,\ t > 0$	Total cost
1	$[0, \infty)$	$y_0 = L,\ \omega = 1, \ldots, 11$	$\mathbb{E}(P_0) \cdot L = 952.368$
2	$[0, 0.4511)$	$y_1(\omega) = L,\ \omega = 1, \ldots, 121$	$(\mathbb{E}(P_1) + h) \cdot L$
2	$[0.4511, \infty)$	$y_0 = L,\ \omega = 1, \ldots, 121$	$\mathbb{E}(P_0) \cdot L = 952.368$
3	$[0, 0.4934)$	$y_2(\omega) = L,\ \omega = 1, \ldots, 1331$	$(\mathbb{E}(P_2) + 2h) \cdot L$
3	$[0.4934, \infty)$	$y_0 = L,\ \omega = 1, \ldots, 1331$	$\mathbb{E}(P_0) \cdot L = 952.368$
4	$[0, 0.4167)$	$y_3(\omega) = L\ \omega = 1, \ldots, 14641$	$(\mathbb{E}(P_3) + 3h) \cdot L$
4	$[0.4167, 0.4934)$	$y_2(\omega) = L\ \omega = 1, \ldots, 14641$	$(\mathbb{E}(P_2) + 2h) \cdot L$
4	$[0.4934, \infty)$	$y_0 = L\ \omega = 1, \ldots, 14641$	$\mathbb{E}(P_0) \cdot L = 952.368$
5	$[0, 0.2299)$	$y_4(\omega) = L\ \omega = 1, \ldots, 161051$	$(\mathbb{E}(P_4) + 4h) \cdot L$
5	$[0.0.2299, 0.4167)$	$y_3(\omega) = L\ \omega = 1, \ldots, 161051$	$(\mathbb{E}(P_3) + 3h) \cdot L$
5	$[0.4167, 0.4934)$	$y_2(\omega) = L\ \omega = 1, \ldots, 161051$	$(\mathbb{E}(P_2) + 2h) \cdot L$
5	$[0.4934, \infty)$	$y_0 = L\ \omega = 1, \ldots, 161051$	$\mathbb{E}(P_0) \cdot L = 952.368$

Table 7: Case 2): Optimal value of the variable y_0 and $y_t(\omega)$, $t > 0$, and of the total cost

The results show that, in the simpler two-stage stochastic programming model ($W = 1$), the unique solution, irrespectively to the inventory cost values, is to ship the total quantity L at time 0, i.e. $y_0 = L$, with a total expected cost $\mathbb{E}(P_0) \cdot L = 952.368$, the highest one (see Table 7). For the three-stage stochastic programming model ($W = 2$), $y_0 = L$ only for $h \geq 0.4511$, while for $h < 0.4511$ it is more convenient to wait until the second stage. In the four-stage stochastic program-

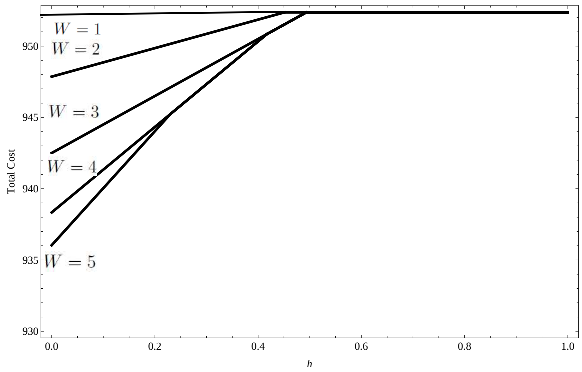

Figure 5: Case 2): Total cost against increasing values of h for different values of W

ming model ($W = 3$), the total quantity L is shipped at time 0, i.e. $y_0 = L$, if $\mathbb{E}(P_0) < \mathbb{E}(P_t) + ht, \forall\ t = 1, \ldots, H-1$, which is verified for $h \geq 0.4934$. On the contrary, when $h < 0.4934$, it is more convenient to wait until the third stage, with a total expected cost $(94.2500 + 2h)L$. The load is never shipped at the second stage since this requires that $\mathbb{E}(P_1) + h < \mathbb{E}(P_2) + 2h$, which is satisfied for $h > 0.5357$, but in such a range the total quantity L is shipped at time 0. Similarly, in the five-stage stochastic programming model ($W = 4$), the total quantity L is shipped at time 0, i.e. $y_0 = L$, for $h \geq 0.4934$. On the contrary, when $0.4167 \leq h < 0.4934$, it is more convenient to wait until the third stage, with a total expected cost of $L(94.2500 + 2h)$ and, when $h < 0.4167$, it is more convenient to wait until the fourth stage, with a total expected cost $10(93.8333 + 3h)$. Similar arguments can be applied to the six-stage stochastic programming model ($W = 5$). In conclusion, this analysis, based on different values of the reduced time horizon W, allows us to deduce that the same total cost is obtained for $h \geq 0.4934$. On the other hand, for $h < 0.4934$, the higher is the reduced time horizon W, the lower is the total cost to be paid. This gives a measure of the value of having more stages to ship the load. Note that all the results confirm the optimal policy derived in Theorem 2.

Case 3): deterministic unit transportation price (uncapacitated).

Table 8 and Figure 6 show the optimal values of the variables y_0 and $y_t(\omega)$, $t > 0$, and the total cost in Case 3), having deterministic unit transportation prices, for different values of the inventory cost h, when the reduced time horizon W increases from 2 to 5. The case with $W = 1$ is infeasible because it does not allow to satisfy the different loads in different scenarios.

W	h	y_0, $y_t(\omega) \neq 0$, $t > 0$	Total cost
2	$[0, 0.4511)$	$y_1(\omega) = L(\omega),\ \omega = 1, \ldots, 5$	$\mathbb{E}(L)(P_1 + h)$
2	$[0.4511, \infty)$	$y_0 = L_{\min}, y_1 = L(\omega) - L_{\min}$	$P_0 \cdot L_{\min} + (P_1 + h)(\mathbb{E}(L) - y_0)$
3	$[0, 0.4934)$	$y_2(\omega) = L(\omega),\ \omega = 1, \ldots, 5$	$\mathbb{E}(L)(P_2 + 2h)$
3	$[0.4934, 0.5357)$	$y_0 = L_{\min},\ y_2 = L(\omega) - L_{\min},\ \omega = 1, \ldots, 5$	$P_0 \cdot L_{\min} + (P_2 + 2h)(\mathbb{E}(L) - y_0)$
3	$[0.5357, \infty)$	$y_0 = L_{\min},\ y_1 = L(\omega) - L_{\min},\ \omega = 1, \ldots, 5$	$P_0 \cdot L_{\min} + (P_1 + h)(\mathbb{E}(L) - y_0)$
4	$[0, 0.4167)$	$y_3(\omega) = L(\omega),\ \omega = 1, \ldots, 5$	$\mathbb{E}(L)(P_3 + 3h)$
4	$[0.4167, 0.4934)$	$y_2(\omega) = L(\omega),\ \omega = 1, \ldots, 5$	$\mathbb{E}(L)(P_2 + 2h)$
4	$[0.4934, 0.5357)$	$y_0 = L_{\min},\ y_2 = L(\omega) - L_{\min},\ \omega = 1, \ldots, 5$	$P_0 \cdot L_{\min} + (P_2 + 2h)(\mathbb{E}(L) - y_0)$
4	$[0.5357, \infty)$	$y_0 = L_{\min},\ y_1 = L(\omega) - L_{\min},\ \omega = 1, \ldots, 5$	$P_0 \cdot L_{\min} + (P_1 + h)(\mathbb{E}(L) - y_0)$
5	$[0, 0.2299)$	$y_4(\omega) = L(\omega),\ \omega = 1, \ldots, 5$	$\mathbb{E}(L)(P_4 + 4h)$
5	$[0.2299, 0.4167)$	$y_3(\omega) = L(\omega),\ \omega = 1, \ldots, 5$	$\mathbb{E}(L)(P_3 + 3h)$
5	$[0.4167, 0.4934)$	$y_2(\omega) = L(\omega),\ \omega = 1, \ldots, 5$	$\mathbb{E}(L)(P_2 + 2h)$
5	$[0.4934, 0.5357)$	$y_0 = L_{\min},\ y_2 = L(\omega) - L_{\min},\ \omega = 1, \ldots, 5$	$P_0 \cdot L_{\min} + (P_2 + 2h)(\mathbb{E}(L) - y_0)$
5	$[0.5357, \infty)$	$y_0 = L_{\min},\ y_1 = L(\omega) - L_{\min},\ \omega = 1, \ldots, 5$	$P_0 \cdot L_{\min} + (P_1 + h)(\mathbb{E}(L) - y_0)$

Table 8: Case 3): Optimal value of the variables y_0 and $y_t(\omega)$, $t > 0$, and of the total cost

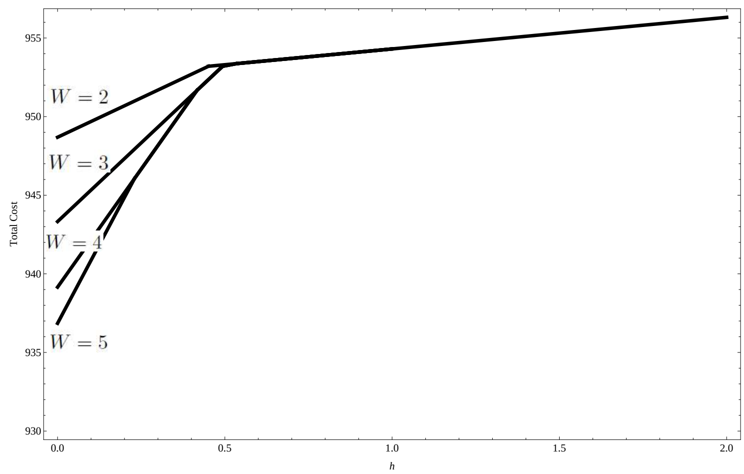

Figure 6: Case 3): Total cost against increasing values of h for different values of W

The results show that, in the case with $W = 2$, $L_{\min}$ is sent at time $t = 0$ and the remaining $L(\omega) - L_{\min}$ at time 1 if $P_0 \leq P_1 + h$, which is satisfied for $h \geq 0.4511$. Otherwise, the stochastic load $L(\omega)$ is sent at time 1. Analogous arguments apply for the cases with larger reduced time horizons $W = 3, \ldots, 5$. The results show that, for $h \geq 0.5357$, the optimal total costs are the same, irrespectively of the number of stages considered. On the other hand, for $h < 0.5357$, the larger is the reduced time horizon W, the lower is the total cost to be paid. Note that all the computational results shown in Table 8 confirm the optimal policy provided in Theorem 3.

6.3 The *Rolling horizon approach*

In this subsection, we evaluate the average performance of the *Rolling horizon approach* in a given set of instances.

In order to do this, we solve the original multistage problem by building a series of models with shorter horizon, which are easier to solve. This means that we solve a model with shorter time horizon, we store its first stage solution and step forward in time; then we solve the problem starting from the next time period again and we store its first stage solution. The process is repeated until we reach the end of the original time horizon H. The strategy that we use is to reduce the time horizon from $[t, H]$ to $[t, t+W]$, where W is a suitable short horizon and $t = 0, \ldots, H-W$. For our numerical analysis we create a multistage stochastic programming model with finite horizon, where uncertainties are captured using a scenario tree. Notice that, at each step of this procedure, we also need to make decisions over our planning horizon $t' = t, \ldots, t+W$, but the decisions we make at time $t' > t$ are purely for the purpose of making a better decision at time t.

Case 1): stochastic load and unit transportation prices.

Table 9 and Figure 7 report numerical results of the *Rolling horizon approach* in Case 1) for $W = 2$, in instances with deadline $H = 3$, in the cases with $\theta = 0.2$ and 0.5. The optimal policy and total cost of the four-stage problem ($H = 3$) are also reported for comparison.

The results show that the *Rolling horizon approach* gives an average percent cost increase with respect to the optimal total cost of about 51% and 74% in the cases with $\theta = 0.2$ and 0.5, respectively. This is manly due to the fact that, in the optimal policy, the quantity sent at time 1 is scenario dependent, while it is constant over all scenarios in the *Rolling horizon approach*, as shown in the top part of Table 9. However, the percent cost increase is not very affected by the value of the inventory cost h. Comparing these results with the corresponding worst-case analysis provided in Section 5 (where we showed that there is not a finite worst-case performance bound), we can conclude that in Case 1) the *Rolling horizon approach* can be very suboptimal. As a managerial insight, this implies that it is really important for the manager to evaluate the average performance of the *Rolling horizon approach* in the typical set of instances solved by the company to avoid

Rolling horizon approach with $W = 2$						
W	h	$y_0^{(W+1)S}$	$y_1^{(W+1)S}(\omega)$	$y_2^{(W+1)S}(\omega)$	Total cost ($\theta = 0.2$)	Total cost ($\theta = 0.5$)
2	0	5	3	$L(\omega) - 8$	8985.517079	5233.921842
2	1	6	2	$L(\omega) - 8$	9082.748763	5148.404484
2	2	6	2	$L(\omega) - 8$	9084.720448	5156.574552
2	3	6	2	$L(\omega) - 8$	9086.692132	5164.744619
2	4	6	2	$L(\omega) - 8$	9088.663817	5172.914687
2	5	6	2	$L(\omega) - 8$	9090.635501	5181.084754
2	6	6	2	$L(\omega) - 8$	9092.607185	5189.254822
2	7	6	2	$L(\omega) - 8$	9094.578870	5197.424889
2	8	6	2	$L(\omega) - 8$	9096.550554	5205.594957
2	9	6	2	$L(\omega) - 8$	9098.522238	5213.765024
2	10	6	2	$L(\omega) - 8$	9100.493923	5221.935092
Optimal policy						
H	h	y_0^*	$y_1^*(\omega)$	$y_2^*(\omega)$	Total cost ($\theta = 0.2$)	Total cost ($\theta = 0.5$)
3	0	6	–	–	5971.764695	2946.929624
3	1	6	–	–	5979.117164	2954.282092
3	2	6	–	–	5986.469632	2961.634561
3	3	6	–	–	5993.822101	2968.987030
3	4	6	–	–	6001.174570	2976.339499
3	5	6	–	–	6008.527039	2983.691967
3	6	6	–	–	6015.879507	2991.044436
3	7	6	–	–	6023.231975	2998.396905
3	8	6	–	–	6030.584443	3005.749374
3	9	6	–	–	6037.936911	3013.101842
3	10	6	–	–	6045.289379	3020.454311

Table 9: Case 1): Policies and total costs against different values of the inventory cost h for the *Rolling horizon approach* with $W = 2$ and for the optimal policy, in instances with deadline $H = 3$, in the cases with $\theta = 0.2$ and 0.5

that a blind application of this heuristic approach provides a very bad performance.

Table 10 and Figure 8 report optimal policies and total costs in Case 1) for the *Rolling horizon approach* with $W = 2$ and 3 in instances with deadline $H = 5$. Since the optimal policy is not available, we compare the results obtained by the *Rolling horizon approach* for $\theta = 0.2$ and 0.5.

The results show the value of using a problem with a larger number of stages in the *Rolling horizon approach*. In fact, the average percent cost increase in the total cost of the *Rolling horizon approach* with $W = 2$ with respect to $W = 3$ is about 40% and 15% in the case with $\theta = 0.2$ and 0.5, respectively. However, the percent cost increase is not very affected by the value of the inventory cost h.

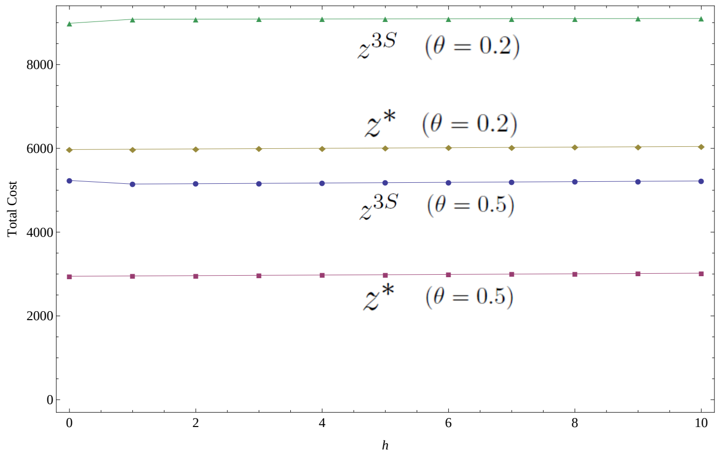

Figure 7: Case 1): Total cost against increasing value of the inventory cost h for the *Rolling horizon approach* with $W = 2$ and for the optimal policy in instances with deadline $H = 3$, in the cases with $\theta = 0.2$ and 0.5

W	h	$y_0^{(W+1)S}$	$y_1^{(W+1)S}(\omega)$	$y_2^{(W+1)S}(\omega)$	$y_3^{(W+1)S}(\omega)$	$y_4^{(W+1)S}(\omega)$	Total cost ($\theta = 0.2$)	Total cost ($\theta = 0.5$)
2	0	5	3	0	0	$L(\omega) - 8$	11467.95944	3426.52595
2	1	6	2	0	0	$L(\omega) - 8$	11470.32019	3431.396066
2	2	6	2	0	0	$L(\omega) - 8$	11471.89093	3435.4811
2	3	6	2	0	0	$L(\omega) - 8$	11473.46167	3439.566134
2	4	6	2	0	0	$L(\omega) - 8$	11475.03242	3443.651168
2	5	6	2	0	0	$L(\omega) - 8$	11476.60316	3447.736201
2	6	6	2	0	0	$L(\omega) - 8$	11478.1739	3451.821235
2	7	6	2	0	0	$L(\omega) - 8$	11479.74465	3455.906269
2	8	6	2	0	0	$L(\omega) - 8$	11481.31539	3459.991303
2	9	6	2	0	0	$L(\omega) - 8$	11482.88613	3464.076336
2	10	6	2	0	0	$L(\omega) - 8$	11484.45688	3468.16137
3	0	6	0	0	0	$L(\omega) - 6$	8155.058984	2946.929624
3	1	6	0	0	0	$L(\omega) - 6$	8158.629727	2954.282092
3	2	6	0	0	0	$L(\omega) - 6$	8162.200471	2961.634561
3	3	6	0	0	0	$L(\omega) - 6$	8165.771214	2968.98703
3	4	6	0	0	0	$L(\omega) - 6$	8169.341958	2976.339499
3	5	6	0	0	0	$L(\omega) - 6$	8172.912701	2983.691967
3	6	6	0	0	0	$L(\omega) - 6$	8176.483445	2991.044436
3	7	6	0	0	0	$L(\omega) - 6$	8180.054189	2998.396905
3	8	6	0	0	0	$L(\omega) - 6$	8183.624932	3005.749374
3	9	6	0	0	0	$L(\omega) - 6$	8187.195676	3013.101842
3	10	6	0	0	0	$L(\omega) - 6$	8190.766419	3020.454311

Table 10: Case 1): Policies and total costs against different values of the inventory cost h for the *Rolling horizon approach* with $W = 2$ and 3 in instances with deadline $H = 5$, in the cases with $\theta = 0.2$ and 0.5

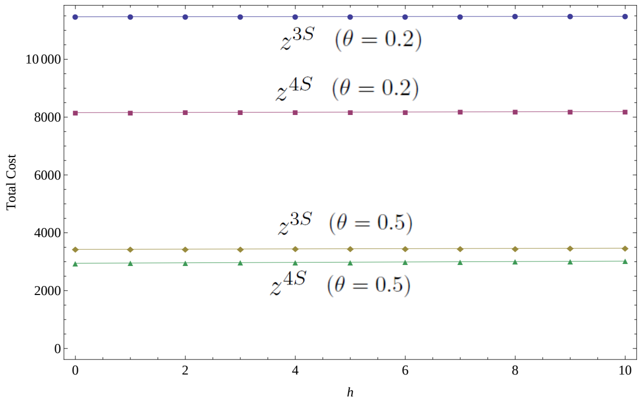

Figure 8: Case 1): Total cost against increasing values of the inventory cost h for the *Rolling horizon approach* with $W = 2$ and 3 in instances with deadline $H = 5$, in the cases with with $\theta = 0.2$ and 0.5

Case 2): deterministic load (uncapacitated).

Table 11 and Figure 9 show the numerical results obtained by applying the *Rolling horizon approach* in Case 2) for increasing values of the inventory cost h in instances with deadline $H = 5$. The optimal policy and total cost of the six-stage problem ($H = 5$) are also reported for comparison.

Rolling horizon approach with $W = 1, 2, 3, 4$							
W	h	$y_0^{(W+1)S}$	$y_1^{(W+1)S}(\omega)$	$y_2^{(W+1)S}(\omega)$	$y_3^{(W+1)S}(\omega)$	$y_4^{(W+1)S}(\omega)$	Total cost
1	$[0,\infty)$	L	0	0	0	0	$\mathbb{E}(P_0) \cdot L$
2	$[0, 0.2299)$	0	0	0	0	L	$(\mathbb{E}(P_4) + 4h) \cdot L$
2	$[0.2299, 0.4167)$	0	0	0	L	0	$(\mathbb{E}(P_3) + 3h) \cdot L$
2	$[0.4167, 0.4511)$	0	0	L	0	0	$(\mathbb{E}(P_2) + 2h) \cdot L$
2	$[0.4511, \infty)$	L	0	0	0	0	$\mathbb{E}(P_0) \cdot L$
3 /4	$[0, 0.2299)$	0	0	0	0	L	$(\mathbb{E}(P_4) + 4h) \cdot L$
3 /4	$[0.2299, 0.4167)$	0	0	0	L	0	$(\mathbb{E}(P_3) + 3h) \cdot L$
3 /4	$[0.4167, 0.4934)$	0	0	L	0	0	$(\mathbb{E}(P_2) + 2h) \cdot L$
3 /4	$[0.4934, \infty)$	L	0	0	0	0	$\mathbb{E}(P_0) \cdot L$
Optimal policy							
H	h	y_0^*	$y_1^*(\omega)$	$y_2^*(\omega)$	$y_3^*(\omega)$	$y_4^*(\omega)$	Total cost
5	$[0, 0.2299)$	0	0	0	0	L	$(\mathbb{E}(P_4) + 4h) \cdot L$
5	$[0.0.2299, 0.4167)$	0	0	0	L	0	$(\mathbb{E}(P_3) + 3h) \cdot L$
5	$[0.4167, 0.4934)$	0	0	L	0	0	$(\mathbb{E}(P_2) + 2h) \cdot L$
5	$[0.4934, \infty)$	L	0	0	0	0	$\mathbb{E}(P_0) \cdot L$

Table 11: Case 2): Policies and total costs against different values of the inventory cost h for the *Rolling horizon approach* with $W = 1, 2, 3, 4$ and for the optimal policy, in instances with deadline $H = 5$

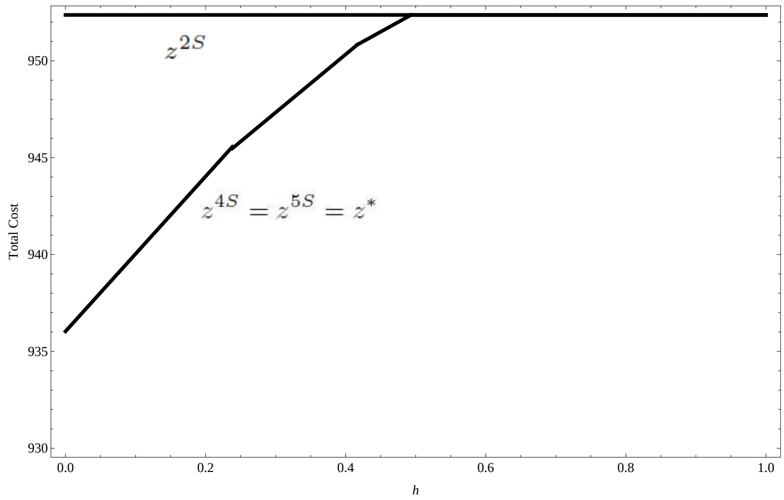

Figure 9: Case 2): Total cost against increasing value of the inventory cost h for the *Rolling horizon approach* with $W = 1, 2, 3, 4$ and for the optimal policy in instances with deadline $H = 5$

The results show that in the simpler *Rolling horizon approach* with $W = 1$, the policy, irrespectively to the inventory cost values, is to ship the total quantity L at time 0, i.e. $y_0^{2S} = L$ with a total cost $\mathbb{E}(P_0)L$. In the *Rolling horizon approach* with $W = 2$, L is sent at time 0 if $\mathbb{E}(P_0) \leq \mathbb{E}(P_1) + h$, i.e. $95.2368 \leq 94.7857 + h$, which is satisfied for $h \geq 0.4511$. Otherwise, if $h \leq 0.4511$, 0 units are sent at time 0. After storing the solution $y_0^{3S} = 0$ and solving the new three-stage stochastic programming model starting at time 1, L could be sent at time 1 if $\mathbb{E}(P_1) + h \leq \mathbb{E}(P_2) + 2h$, which is verified for $h \geq 0.535714$. However, this is not consistent with $h \leq 0.4511$. Consequently, $y_1^{3S}(\omega) = 0$. Then, a new three-stage stochastic programming model is solved: L is sent at time 2 if $\mathbb{E}(P_2) + 2h \leq \mathbb{E}(P_3) + 3h$, which is verified for $h \geq 0.4167$. Consequently, in the range $[0.4167, 0.4511)$, $y_2^{3S}(\omega) = L$. If $h \leq 0.4167$, $y_2^{3S}(\omega) = 0$ and a new three-stage stochastic programming model starting at time 3 is solved: L is sent at time 3 if $\mathbb{E}(P_3) + 3h \leq \mathbb{E}(P_4) + 4h$, which is verified for $h \geq 0.2299$. Consequently, in the range $[0.2299, 0.4167)$, $y_3^{3S}(\omega) = L$. Finally, if $h \leq 0.2299$ the load L is sent at time 4.

With similar arguments, the policies of the *Rolling horizon approach* with $W = 3, 4$ can be explained. Notice that these policies are optimal.

Figure 9 clearly shows that the ratio between the cost of the *Rolling horizon approach* and the optimal cost is always finite, as shown by the worst-case analysis (see Theorems 5 and 7), and that the maximum percent cost increase is obtained for $h = 0$, as in the worst-case analysis.

Case 3): deterministic unit transportation price (uncapacitated).

Table 12 and Figure 10 report numerical results of the *Rolling horizon approach* with $W = 2, 3, 4$ in the instances with deadline $H = 5$ for increasing values of the inventory cost h. The optimal policy and total cost are also reported for comparison.

Rolling horizon approach with $W = 2, 3, 4$							
W	h	$y_0^{(W+1)S}$	$y_1^{(W+1)S}(\omega)$	$y_2^{(W+1)S}(\omega)$	$y_3^{(W+1)S}(\omega)$	$y_4^{(W+1)S}(\omega)$	Total cost
2	$[0, 0.2299)$	0	0	0	0	$L(\omega)$	$(P_4{+}4h)\mathbb{E}(L)$
2	$[0.2299, 0.4167)$	0	0	0	$L_{\min}$	$L(\omega){-}L_{\min}$	$L_{\min}(P_3 + 3h){+}(P_4 + 4h)(\mathbb{E}(L){-}L_{\min})$
2	$[0.4167, 0.4511)$	0	0	$L_{\min}$	0	$L(\omega){-}L_{\min}$	$L_{\min}(P_2 + 2h){+}(P_4 + 4h)(\mathbb{E}(L){-}L_{\min})$
2	$[0.4511, \infty)$	$L_{\min}$	0	0	0	$L(\omega){-}L_{\min}$	$L_{\min}P_0{+}(P_4 + 4h)(\mathbb{E}(L){-}L_{\min})$
3	$[0, 0.2299)$	0	0	0	0	$L(\omega)$	$(P_4{+}4h)\mathbb{E}(L)$
3	$[0.2299, 0.4167)$	0	0	0	$L_{\min}$	$L(\omega){-}L_{\min}$	$L_{\min}(P_3 + 3h){+}(P_4 + 4h)(\mathbb{E}(L){-}L_{\min})$
3	$[0.4167, 0.4934)$	0	0	$L_{\min}$	0	$L(\omega){-}L_{\min}$	$L_{\min}(P_2 + 2h){+}(P_4 + 4h)(\mathbb{E}(L){-}L_{\min})$
3	$[0.4934, \infty)$	$L_{\min}$	0	0	0	$L(\omega){-}L_{\min}$	$L_{\min}P_0{+}(P_4 + 4h)(\mathbb{E}(L){-}L_{\min})$
4	$[0, 0.2299)$	0	0	0	0	$L(\omega)$	$(\mathbb{E}(P_4){+}4h)\mathbb{E}(L)$
4	$[0.2299, 0.4167)$	0	0	0	$L_{\min}$	$L(\omega){-}L_{\min}$	$L_{\min}(P_3 + 3h){+}(P_4 + 4h)(\mathbb{E}(L){-}L_{\min})$
4	$[0.4167, 0.4934)$	0	0	$L_{\min}$	0	$L(\omega){-}L_{\min}$	$L_{\min}(P_2 + 2h){+}(P_4 + 4h)(\mathbb{E}(L){-}L_{\min})$
4	$[0.4934, \infty)$	$L_{\min}$	0	0	0	$L(\omega){-}L_{\min}$	$L_{\min}P_0{+}(P_4 + 4h)(\mathbb{E}(L){-}L_{\min})$

Optimal policy							
H	h	y_0^*	$y_1^*(\omega)$	$y_{2^*}(\omega)$	$y_3^*(\omega)$	$y_4^*(\omega)$	Total cost
5	$[0, 0.2299)$	0	0	0	0	$L(\omega)$	$(P_4 + 4h)\mathbb{E}(L)$
5	$[0.2299, 0.4167)$	0	0	0	$L(\omega)$	0	$(P_3 + 3h)\mathbb{E}(L)$
5	$[0.4167, 0.4934)$	0	0	$L(\omega)$	0	0	$(P_2 + 2h)\mathbb{E}(L)$
5	$[0.4934, 0.5357)$	$L_{\min}$	0	$L(\omega) - L_{\min}$	0	0	$P_0 \cdot L_{\min}{+}(P_2 + 2h)(\mathbb{E}(L) - L_{\min})$
5	$[0.5357, \infty)$	$L_{\min}$	$L(\omega) - L_{\min}$	0	0	0	$P_0 \cdot L_{\min}{+}(P_1 + h)(\mathbb{E}(L) - L_{\min})$

Table 12: Case 3): Policies and total costs against different values of the inventory cost h for the *Rolling horizon approach* with $W = 2, 3, 4$ and for the optimal policy, in instances with deadline $H = 5$

Let us just analyze the policy provided by the *Rolling horizon approach* with $W = 4$: $L_{\min}$ is sent at time 0 if

$$\begin{cases} \mathbb{E}(P_0) \le \mathbb{E}(P_1) + h \\ \mathbb{E}(P_0) \le \mathbb{E}(P_2) + 2h \\ \mathbb{E}(P_0) \le \mathbb{E}(P_3) + 3h \end{cases}$$

which is satisfied for $h \ge 0.4934$. Therefore, $y_0^{5S} = L_{\min}$. We solve a new five-stage stochastic programming model starting at time 1. Since the remaining load $L(\omega) - L_{\min}$ is stochastic, due to the non-anticipativity constraints, it cannot be sent all at time 1. Since $\min_{\omega\in\Omega}(L(\omega) - L_{\min}) = 0$, $y_1^{5S}(\omega) = 0$ irrespectively of the prices in stages. Then, a

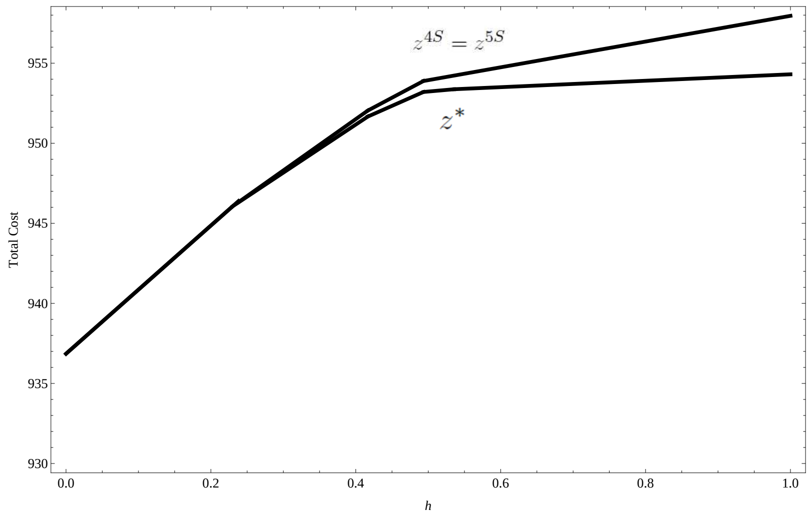

Figure 10: Case 3): Total cost against increasing value of the inventory cost h for the *Rolling horizon approach* with $W = 2, 3, 4$ and for the optimal policy in instances with deadline $H = 5$

new five-stage stochastic model is solved. Applying the same arguments as before, we have $y_2^{5S}(\omega) = 0$. With similar arguments, we derive that the residual stochastic load $L(\omega) - L_{\min}$ is shipped at the last available stage. Otherwise, if $h \leq 0.4934$, 0 units are sent at time 0. We store $y_0^{5S} = 0$ and solve a new five-stage stochastic programming model starting at time 1: $L_{\min}$ is sent at time 1 if the following system is satisfied:

$$\begin{cases} \mathbb{E}(P_1) + h \leq \mathbb{E}(P_2) + 2h \\ \mathbb{E}(P_1) + h \leq \mathbb{E}(P_3) + 3h \\ \mathbb{E}(P_1) + h \leq \mathbb{E}(P_4) + 4h \end{cases}$$

which is verified for $h \geq 0.535714$. However, since this is not consistent with the case $h \leq 0.4934$, then $y_1^{5S}(\omega) = 0$. A new four-stage stochastic programming model is solved: $L_{\min}$ is sent at time 2 if the following system is satisfied

$$\begin{cases} \mathbb{E}(P_2) + 2h \leq \mathbb{E}(P_3) + 3h \\ \mathbb{E}(P_2) + 2h \leq \mathbb{E}(P_4) + 4h \end{cases}$$

which is verified for $h \geq 0.4167$. Consequently, in the range $[0.4167, 0.4934)$, we store $y_2^{5S}(\omega) = L_{\min}$ and with similar arguments as before, the remaining load $L(\omega) - L_{\min}$ is shipped at the last available stage. If $h \leq 0.4167$, 0 units are sent at time 2. We store $y_2^{5S}(\omega) = 0$ and solve a new three-stage stochastic programming model starting at time 3: $L_{\min}$ is sent at time 3 if the following condition is satisfied: $\mathbb{E}(P_3)+3h \leq \mathbb{E}(P_4)+4h$ which is verified for $h \geq 0.2299$. Consequently, in the range [0.2299.4167), we store $y_3^{5S}(\omega) = L_{\min}$ and the stochastic remaining load $L(\omega) - L_{\min}$ is shipped at the last stage. Analogous arguments apply for the *Rolling horizon approach* with the other values of W.

Figure 10 clearly shows that the ratio between the total cost of the *Rolling horizon approach* and the optimal total cost increases with the inventory cost h. It can be easily checked that the ratio between these two costs tends to $H - 1 = 4$ for $h \to \infty$, as shown in the worst-case analysis.

7 Conclusions

The paper presents a worst-case analysis of *Rolling horizon approach* in stochastic programming applied to the *Stochastic multistage fixed charge transportation problem*. Theoretical results showed that the *Rolling horizon approach* can be very suboptimal in the worst case if used to solve the NP-hard problem, while finite bounds exist for the polynomially solvable cases, even if the value of the bounds can be different in different instances. Interesting results were also obtained by the computational experiment we carried out. First, we found that both the NP-hard problem and the polynomially solvable cases are very difficult to be solved by state-of-the-art solvers by considering the complete scenario tree. Therefore, it is really important to design heuristic algorithms to solve the NP-hard problem and to be able to design an exact polynomial time algorithm to solve the particular cases. Moreover, the computational results showed that the optimal total cost of the stochastic programming models solved at each iteration of the *Rolling horizon approach* significantly decreases increasing the value of the reduced time horizon. Finally, the computational results idenfied the cases in which even the average percent cost increase of the *Rolling horizon approach* with respect to the optimal policy can be high: mainly when the reduced time horizon in the subproblems is small and the unit transportation prices are low with respect to the fixed transportation costs.

References

Adlakha V, Kowalski K, Lev B (2010) A branching method for the fixed charge transportation problem. *Omega* 38(5):393–397.

Agarwal Y, Aneja Y (2012) Fixed-charge transportation problem: Facets of the projection polyhedron. *Operations Research* 60(3):638–654.

Bertocchi M, Moriggia V, Dupačová J (2006) Horizon and stages in applications of stochastic programming in finance. *Annals of Operations Research* 142(1):63–78.

Birge JR (1997) State-of-the-art-survey-stochastic programming: Computation and applications. *INFORMS Journal on Computing* 9(2):111–133.

Birge JR, Louveaux F (2011) *Introduction to stochastic programming* (Springer Science & Business Media).

Bolduc MC, Renaud J, Boctor F, Laporte G (2008) A perturbation metaheuristic for the vehicle routing problem with private fleet and common carriers. *Journal of the Operational Research Society* 59(6):776–787.

Buson E, Roberti R, Toth P (2014) A reduced-cost iterated local search heuristic for the fixed-charge transportation problem. *Operations Research* 62(5):1095–1106.

Chand S, Hsu VN, Sethi S (2002) Forecast, solution, and rolling horizons in operations management problems: a classified bibliography. *Manufacturing & Service Operations Management* 4(1):25–43.

Christensen TR, Andersen KA, Klose A (2013) Solving the single-sink, fixed-charge, multiple-choice transportation problem by dynamic programming. *Transportation Science* 47(3):428–438.

Chu CW (2005) A heuristic algorithm for the truckload and less-than-truckload problem. *European Journal of Operational Research* 165(3):657–667.

Côté JF, Potvin JY (2009) A tabu search heuristic for the vehicle routing problem with private fleet and common carrier. *European Journal of Operational Research* 198(2):464–469.

Dupačová J (2002) Applications of stochastic programming: achievements and questions. *European Journal of Operational Research* 140(2):281–290.

Gary MR, Johnson DS (1979) *Computers and Intractability: A Guide to the Theory of NP-completeness* (WH Freeman and Company, New York).

Görtz S, Klose A (2009) Analysis of some greedy algorithms for the single-sink fixed-charge transportation problem. *Journal of Heuristics* 15(4):331–349.

Gray P (1971) Technical note - exact solution of the fixed-charge transportation problem. *Operations Research* 19(6):1529–1538.

Guigues V, Sagastizábal C (2012) The value of rolling-horizon policies for risk-averse hydro-thermal planning. *European Journal of Operational Research* 217(1):129–140.

Herer YT, Rosenblatt M, Hefter I (1996) Fast algorithms for single-sink fixed charge transportation problems with applications to manufacturing and transportation. *Transportation Science* 30(4):276–290.

Jawahar N, Balaji A (2009) A genetic algorithm for the two-stage supply chain distribution problem associated with a fixed charge. *European Journal of Operational Research* 194(2):496–537.

Johnson EL, Nemhauser GL, Savelsbergh MW (2000) Progress in linear programming-based algorithms for integer programming: An exposition. *INFORMS Journal on Computing* 12(1):2–23.

Kennington J, Unger E (1976) A new branch-and-bound algorithm for the fixed-charge transportation problem. *Management Science* 22(10):1116–1126.

Klein Haneveld WK, van der Vlerk MH (1999) Stochastic integer programming:general models and algorithms. *Annals of Operations Research* 85(0):39–57.

Kouwenberg R (2001) Scenario generation and stochastic programming models for asset liability management. *European Journal of Operational Research* 134(2):279–292.

Kusy MI, Ziemba WT (1986) A bank asset and liability management model. *Operations Research* 34(3):356–376.

Lulli G, Sen S (2004) A branch-and-price algorithm for multistage stochastic integer programming with application to stochastic batch-sizing problems. *Management Science* 50(6):786–796.

Maggioni F, Allevi E, Bertocchi M (2014) Bounds in multistage linear stochastic programming. *Journal of Optimization Theory and Applications* 163(1):200–229.

Maggioni F, Allevi E, Bertocchi M (2016) Monotonic bounds in multistage mixed-integer stochastic programming. *Computational Management Science* 13(3):423–457.

Maggioni F, Pflug GC (2016) Bounds and approximations for multistage stochastic programs. *SIAM Journal on Optimization* 26(1):831–855.

Pantuso G, Fagerholt K, Wallace SW (2015) Solving hierarchical stochastic programs: Application to the maritime fleet renewal problem. *INFORMS Journal on Computing* 27(1):89–102.

Papageorgiou DJ, Toriello A, Nemhauser GL, Savelsbergh MW (2012) Fixed-charge transportation with product blending. *Transportation Science* 46(2):281–295.

Parija GR, Ahmed S, King AJ (2004) On bridging the gap between stochastic integer programming and mip solver technologies. *INFORMS Journal on Computing* 16(1):73–83.

Potvin JY, Naud MA (2011) Tabu search with ejection chains for the vehicle routing problem with private fleet and common carrier. *Journal of the Operational Research Society* 62(2):326–336.

Raj KAAD, Rajendran C (2012) A genetic algorithm for solving the fixed-charge transportation model: two-stage problem. *Computers & Operations Research* 39(9):2016–2032.

Roberti R, Bartolini E, Mingozzi A (2014) The fixed charge transportation problem: An exact algorithm based on a new integer programming formulation. *Management Science* 61(6):1275–1291.

Römisch W, Schultz R (2001) Multistage stochastic integer programs: An introduction. *Online optimization of large scale systems*, 581–600 (Springer).

Sen S (2005) Algorithms for stochastic mixed-integer programming models. K Aardal GN, Weismantel R, eds., *Discrete Optimization*, volume 12 of *Handbooks in Operations Research and Management Science*, 515 – 558 (Elsevier).

Sheffi Y (2004) Combinatorial auctions in the procurement of transportation services. *Interfaces* 34(4):245–252.

Sheng Y, Yao K (2012) Fixed charge transportation problem and its uncertain programming model. *Industrial Engineering and Management Systems* 11(2):183–187.

Silventea J, Kopanosb GM, Espuñaa A (2015) A rolling horizon stochastic programming framework for the energy supply and demand management in microgrids. *Computer aided chemical engineering* 37(2):2321–2326.

Stenger A, Vigo D, Enz S, Schwind M (2013) An adaptive variable neighborhood search algorithm for a vehicle routing problem arising in small package shipping. *Transportation Science* 47(1):64–80.

Valente C, Mitra G, Sadki M, Fourer R (2009) Extending algebraic modelling languages for stochastic programming. *INFORMS Journal on Computing* 21(1):107–122.