Ambiguous Risk Constraints with Moment and Unimodality Information

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Abstract

Optimization problems face random constraint violations when uncertainty arises in constraint parameters. Effective ways of controlling such violations include risk constraints, e.g., chance constraints and conditional Value-at-Risk (CVaR) constraints. This paper studies these two types of risk constraints when the probability distribution of the uncertain parameters is ambiguous. In particular, we assume that the distributional information consists of the first two moments of the uncertainty and a generalized notion of unimodality. We find that the ambiguous risk constraints in this setting can be recast as a set of second-order cone (SOC) constraints. In order to facilitate the algorithmic implementation, we also derive efficient ways of finding violated SOC constraints. Finally, we demonstrate the theoretical results via a computational case study on power system operations.

Key words: ambiguity, chance constraints, conditional Value-at-Risk, second-order cone representation, separation, golden section search.

1 Introduction

In an uncertain environment, optimization problems often involve making decisions before the uncertainty is realized. In this case, constraints, which may include security criteria and capacity restrictions, may face random violations. For example, we consider a constraint subject to uncertainty taking the form

$$a(x)^\top \xi \leq b(x), \tag{1}$$

where $x \in \{0,1\}^{n_\mathrm{B}} \times \mathbb{R}^{n-n_\mathrm{B}}$ represents an n-dimensional decision variable, $n_\mathrm{B} \in \{0, 1, \ldots, n\}$ represents the number of binary decisions, $a(x) : \mathbb{R}^n \to \mathbb{R}^T$ and $b(x) : \mathbb{R}^n \to \mathbb{R}$ represent two affine transformations of x, and $\xi \in \mathbb{R}^T$ represents a T-dimensional random vector defined on probability space $(\mathbb{R}^T, \mathcal{B}^T, \mathbb{P}_\xi)$ with Borel σ-algebra $\mathcal{B}^T$. An intuitive way of handling random violations of (1) employs chance constraints, which attempt to satisfy (1) with at least a pre-specified probability, i.e.,

$$\mathbb{P}_\xi\{a(x)^\top \xi \leq b(x)\} \geq 1 - \epsilon, \tag{2}$$

where $1 - \epsilon$ represents the confidence level of the chance constraint with ϵ usually taking a small value (e.g., 0.05 or 0.10; see, e.g., Charnes et al. (1958); Miller and Wagner (1965)). Dating back to the 1950s, chance constraints have been applied in a wide range of applications including power system operations (see, e.g., Ozturk et al. (2004); Wang et al. (2012)), production planning (see, e.g., Bookbinder and Tan (1988); Gade and Küçükyavuz (2013)), and chemical processing (see, e.g., Henrion et al. (2001); Henrion and Möller (2003)).

In practice, a decision maker is often interested in not only the violation probability of constraint (1), but also the violation magnitude if any. Indeed, chance constraint (2) offers no guarantees on the magnitude of $a(x)^\top \xi - b(x)$ when it is positive. This motivates an alternative risk measure called the conditional Value-at-Risk (CVaR) that examines the (right) tail of $a(x)^\mathsf{T}\xi - b(x)$. More precisely, the CVaR of a one-dimensional random variable χ with confidence level $1 - \epsilon \in (0, 1)$ is defined as

$$\text{CVaR}^\epsilon_{\mathbb{P}_\chi}(\chi) = \inf_{\beta \in \mathbb{R}} \left\{ \beta + \frac{1}{\epsilon} \mathbb{E}_\mathbb{P}[\chi - \beta]_+ \right\}, \tag{3}$$

where $\mathbb{P}_\chi$ represents the probability distribution of χ. When the infimum is attained in (3), β represents the Value-at-Risk of χ with confidence level $1-\epsilon$, that is, $\mathbb{P}_\chi\{\chi \leq \beta\} \geq 1-\epsilon$ (see Artzner et al. (1999); Rockafellar and Uryasev (2002)). As a consequence, $\text{CVaR}^\epsilon_{\mathbb{P}_\chi}(\chi)$ measures the conditional expectation of χ on its right ϵ-tail. Hence, chance constraint (2) is implied by the CVaR constraint

$$\text{CVaR}^\epsilon_{\mathbb{P}_\xi}(a(x)^\mathsf{T}\xi) \leq b(x). \tag{4}$$

A basic challenge to using risk constraints (2) and (4) is that complete information of probability distribution $\mathbb{P}_\xi$ may not be available. Under many circumstances, we only have structural knowledge of $\mathbb{P}_\xi$ (e.g., symmetry, unimodality, etc.) and possibly a series of historical data that can be considered as samples taken from the true (while ambiguous) distribution. As a result, the solution obtained from a risk-constrained model can be biased, i.e., sensitive to the $\mathbb{P}_\xi$ we employ in constraints (2) and (4), and hence perform poorly in out-of-sample tests. A natural way of addressing this challenge is to employ a set of plausible probability distributions, termed the ambiguity set, rather than a single estimate of $\mathbb{P}_\xi$.

1.1 Ambiguity Set with Unimodality Information

We consider an ambiguity set characterized by the first two moments of ξ and a structural requirement that $\mathbb{P}_\xi$ is unimodal in a generalized sense. By definition, if $T = 1$, then $\mathbb{P}_\xi$ is unimodal about 0 if function $F(z) := \mathbb{P}_\xi(\xi \leq z)$ is convex on $(-\infty, 0)$ and concave on $(0, \infty)$. If ξ admits a density function $f_\xi(z)$, then unimodality is equivalent to $f_\xi(z)$ being nondecreasing on $(-\infty, 0)$ and nonincreasing on $(0, \infty)$. In a multidimensional setting, i.e., if $T > 1$, an intuitive extension of this notion is that $f_\xi(zd)$ is nonincreasing on $(0, \infty)$ for all $d \in \mathbb{R}^T$ and $d \neq 0$. That is, the density function of ξ is nonincreasing along any ray emanating from the mode. The following definitions extend this intuitive notion to also cover the distributions that do not admit density functions.

Definition 1 *(Star-Unimodality; see Dharmadhikari and Joag-Dev (1988)) A set $S \subseteq \mathbb{R}^T$ is called* star-shaped *about* 0 *if, for all $\xi \in S$, the line segment connecting* 0 *and ξ is completely contained in S. A probability distribution $\mathbb{P}_\xi$ on $\mathbb{R}^T$ is called star-unimodal about* 0 *if it belongs to the closed convex hull of the set of all uniform distributions on sets in $\mathbb{R}^T$ which are star-shaped about* 0*.*

In this paper, we consider a more general notion than the star-unimodality as follows.

Definition 2 *(α-Unimodality; see Dharmadhikari and Joag-Dev (1988)) For any given $\alpha > 0$, a probability distribution $\mathbb{P}_\xi$ is called α-unimodal about* 0 *if function $G(z) := z^\alpha \mathbb{P}_\xi(S/z)$ is nondecreasing on $(0, \infty)$ for all Borel set $S \in \mathcal{B}^T$.*

If ξ admits a density function $f_\xi(z)$, then it can be shown that $\mathbb{P}_\xi$ is α-unimodal about 0 if and only if $z^{T-\alpha} f_\xi(zd)$ is nonincreasing on $(0, \infty)$ for all $d \in \mathbb{R}^T$ and $d \neq 0$ (see Dharmadhikari and Joag-Dev (1988); Van Parys et al. (2015b)). As compared to star-unimodal distributions, the density of an α-unimodal distribution can increase along rays emanating from the mode (e.g., when $\alpha > T$), but the increasing rate is controlled by α. Indeed, along any ray d, $f_\xi(zd)$ does not increase faster than $z^{\alpha - T}$ on $(0, \infty)$. Furthermore, when $\alpha = T$, $f_\xi(zd)$ is nonincreasing on $(0, \infty)$ for all d. This implies that α-unimodality reduces to star-unimodality when $\alpha = T$.

Given the first two moments of ξ and α-unimodality, we define the following ambiguity set

$$\mathcal{D}_\xi(\mu, \Sigma, \alpha) := \left\{ \mathbb{P}_\xi \in \mathcal{M}_T : \mathbb{E}_{\mathbb{P}_\xi}[\xi] = \mu, \ \mathbb{E}_{\mathbb{P}_\xi}[\xi\xi^\mathsf{T}] = \Sigma, \ \mathbb{P}_\xi \text{ is } \alpha\text{-unimodal about } 0 \right\}, \tag{5}$$

where $\mathcal{M}_T$ represents the set of all probability distributions on $(\mathbb{R}^T, \mathcal{B}^T)$, and μ and Σ represent the first and second moments of ξ, respectively. Without loss of generality, we assume that the mode of ξ is 0 in definition (5) and a general mode m can be modeled by shifting ξ to $\xi - m$. For notational brevity, we often refer to ambiguity set $\mathcal{D}_\xi$ with its dependency on parameters (μ, Σ, α) omitted. Based on $\mathcal{D}_\xi$, we consider an ambiguous chance constraint (ACC)

$$\inf_{\mathbb{P}_\xi \in \mathcal{D}_\xi} \mathbb{P}_\xi\{a(x)^\mathsf{T}\xi \leq b(x)\} \geq 1 - \epsilon, \tag{6}$$

that is, we wish to satisfy chance constraint (2) for all probability distributions $\mathbb{P}_\xi$ in ambiguity set $\mathcal{D}_\xi$. Similarly, we define an ambiguous CVaR constraint (AVC)

$$\sup_{\mathbb{P}_\xi \in \mathcal{D}_\xi} \text{CVaR}^\epsilon_{\mathbb{P}_\xi}(a(x)^\mathsf{T}\xi) \leq b(x), \tag{7}$$

which requires that CVaR constraint (4) is satisfied for all $\mathbb{P}_\xi$ in $\mathcal{D}_\xi$.

1.2 Relations to the Prior Work

In recent years, distributionally robust optimization (DRO) has become an important tool to handle distributional ambiguity in stochastic programs. Using concepts similar to ACC (6) and AVC (7), DRO aims to optimize or protect a system from the worst-case probability distribution, which belongs to a pre-specified ambiguity set. DRO was first introduced by Scarf (1958) as a minimax stochastic program for the classical newsvendor problem under an ambiguous demand with only moment information. Following this seminal work, moment information has been widely used for characterizing ambiguity sets in various DRO models (see, e.g., Bertsimas et al., 2010; Delage and Ye, 2010; Zymler et al., 2013b). A key merit of the DRO approach is that the model can often be recast as tractable convex programs such as semidefinite programs (SDPs) (see, e.g., Delage and Ye, 2010) or SOC programs (see, e.g., El Ghaoui et al., 2003). Recently, Wiesemann et al. (2014) successfully identified a class of ambiguity sets that lead to tractable convex program reformulations of general DRO models.

ACCs with moment information (and without structural information) have been well-studied in recent years (see, e.g., El Ghaoui et al., 2003; Wagner, 2008; Calafiore and El Ghaoui, 2006; Vandenberghe et al., 2007; Zymler et al., 2013a; Ahmed and Papageorgiou, 2013; Cheng et al., 2014; Hanasusanto et al., 2015). In particular, El Ghaoui et al. (2003), Wagner (2008), and Calafiore and El Ghaoui (2006) showed that the ACC can be recast as a SOC constraint if the ambiguity set is characterized by the first two moments of ξ. Later, Zymler et al. (2013a) showed that ACC and AVC are actually equivalent if the same ambiguity set is employed. Recently, Ahmed and Papageorgiou (2013) and Cheng et al. (2014) extended the analysis of ACC to the case when variable x involves binary (i.e., 0-1) decisions, and Hanasusanto et al. (2015) made significant progress on representing the ambiguous joint chance constraints in tractable forms. ACCs with information on the density function have also been studied (see, e.g., Erdoğan and Iyengar, 2006; Jiang and Guan, 2016; Esfahani and Kuhn, 2015).

In contrast, ACCs and AVCs with both moment and structural information have received less attention. Popescu (2005) considered general DRO models with ambiguity sets incorporating unimodality, symmetry, and convexity. Recently, by using the Choquet representation of α-unimodal distributions, Van Parys et al. (2015a) successfully derived SDPs to quantify the worst-case probability bound in ACC. Furthermore, based on both α-unimodality and γ-monotonicity, Van Parys et al. (2015b) extended the analysis to quantifying the worst-case expectation in AVC. The main focus of Van Parys et al. (2015a,b) is to evaluate the worst-case expectations in ACC or AVC for a given decision variable x. In contrast, we adjust x to satisfy ACC and AVC. In our prior work Li et al. (2016), we derived approximations of AVC. Here, we obtain an exact representation of AVC and derive tighter approximations than those in Li et al. (2016). To the best of our knowledge, our results on ACC are most related to Hanasusanto (2015) (in particular, Section 3.4.1), which employs a different ambiguity set that bounds the second moment of ξ by Σ instead of matching it as in (5). Furthermore, Hanasusanto (2015) derived a representation of ACC based on linear matrix inequalities (LMIs). In contrast, in this paper, we employ a different approach based on projection, which allows us to represent ACC with SOC constraints. When x involves binary decisions, SOC constraints are more computationally tractable than LMIs because many off-the-shelf commercial solvers (e.g., CPLEX and GUROBI) can directly handle mixed-integer SOC programs.

We summarize our main contributions as follows.

1. We derive equivalent reformulations of ACC (6) and AVC (7) using both moment and unimodality information. Both reformulations are SOC constraints and so can be efficiently handled in commercial solvers. Different from previous results in Zymler et al. (2013a), we find that ACC and AVC are not equivalent after incorporating the unimodality information.

2. Inspired by the separation approach (see, e.g., Nemhauser and Wolsey (1999)), we derive efficient ways for finding violated SOC constraints in the reformulations of ACC and AVC. The separation procedures can be used to accelerate the algorithmic implementation of ACC and AVC.

3. We derive conservative and relaxed approximations of ACC and AVC that are asymptotically tight. As demonstrated in the computational case study, these approximations help to provide high-quality bounds for the optimal objective value of the test instances.

The remainder of this paper is organized as follows. Section 2 represents ACC (6) as a set of SOC constraints. Section 3 represents AVC (7) as a set of SOC constraints. In both sections, we derive separation procedures for finding violated SOC constraints based on the golden section search. In Section 4, we analyze an extension of ACC and AVC to incorporate the linear unimodality in the ambiguity set. We present a computational case study in Section 5.

2 Representation of the Ambiguous Chance Constraint

We show that ACC (6) can be recast as second-order cone (SOC) constraints. To this end, we first simplify the computation of the left-hand side of (6), i.e., $\inf_{\mathbb{P}_\xi \in \mathcal{D}_\xi} \mathbb{P}_\xi\{a(x)^\mathsf{T}\xi \leq b(x)\}$, by projecting random vector ξ on $\mathbb{R}$ and considering a one-dimensional random variable ζ. We summarize this projection result in the following proposition, whose proof relies on the representation of α-unimodal random vectors in Dharmadhikari and Joag-Dev (1988).

Proposition 1 *Define scalars $\mu_1 = a(x)^\mathsf{T}\mu$, $\Sigma_1 = a(x)^\mathsf{T}\Sigma a(x)$, and ambiguity set $\mathcal{D}_1 = \{\mathbb{P}_\zeta \in \mathcal{M}_1 : \mathbb{E}_{\mathbb{P}_\zeta}[\zeta] = \mu_1,\ \mathbb{E}_{\mathbb{P}_\zeta}[\zeta^2] = \Sigma_1,\ \mathbb{P}_\zeta$ is α-unimodal about $0\}$. Then*

$$\inf_{\mathbb{P}_\xi \in \mathcal{D}_\xi} \mathbb{P}_\xi\{a(x)^\mathsf{T}\xi \leq b(x)\} \;=\; \inf_{\mathbb{P}_\zeta \in \mathcal{D}_1} \mathbb{P}_\zeta\{\zeta \leq b(x)\}. \tag{8}$$

Proof: Theorem 3.5 in Dharmadhikari and Joag-Dev (1988) states that a random vector $X \in \mathbb{R}^m$ is α-unimodal if and only if there exists a random vector $Z \in \mathbb{R}^m$ such that $X = U^{1/\alpha} Z$, where U is uniform in $(0,1)$ and independent of Z.

First, pick any ξ such that $\mathbb{P}_\xi \in \mathcal{D}_\xi$. Then, there exists Z_ξ such that $\xi = U^{1/\alpha} Z_\xi$. We define $\zeta = a(x)^\mathsf{T}\xi$. It follows that ζ is α-unimodal because $\zeta = a(x)^\mathsf{T}(U^{1/\alpha} Z_\xi) = U^{1/\alpha}(a(x)^\mathsf{T} Z_\xi)$. Furthermore, $\mathbb{E}_{\mathbb{P}_\zeta}[\zeta] = \mu_1$ and $\mathbb{E}_{\mathbb{P}_\zeta}[\zeta^2] = \Sigma_1$. Hence, $\mathbb{P}_\zeta \in \mathcal{D}_1$, and so $\inf_{\mathbb{P}_\xi \in \mathcal{D}_\xi} \mathbb{P}_\xi\{a(x)^\mathsf{T}\xi \le b(x)\} \ge \inf_{\mathbb{P}_\zeta \in \mathcal{D}_1} \mathbb{P}_\zeta\{\zeta \le b(x)\}$.

Second, pick any ζ such that $\mathbb{P}_\zeta \in \mathcal{D}_1$. Then, there exists a Z_ζ such that $\zeta = U^{1/\alpha} Z_\zeta$. It follows that $\mathbb{E}[Z_\zeta] = (\frac{\alpha+1}{\alpha})\mu_1$ and $\mathbb{E}[Z_\zeta^2] = (\frac{\alpha+2}{\alpha})\Sigma_1$. Based on Theorem 1 in Popescu (2007), there exists a $Z_\xi \in \mathbb{R}^T$ such that $Z_\zeta = a(x)^\mathsf{T} Z_\xi$, $\mathbb{E}[Z_\xi] = (\frac{\alpha+1}{\alpha})\mu$, and $\mathbb{E}[Z_\xi Z_\xi^\mathsf{T}] = (\frac{\alpha+2}{\alpha})\Sigma$. We define $\xi = U^{1/\alpha} Z_\xi$. It follows that ξ is α-unimodal, and furthermore $\mathbb{E}_{\mathbb{P}_\xi}[\xi] = (\frac{\alpha}{\alpha+1})\mathbb{E}[Z_\xi] = \mu$ and $\mathbb{E}_{\mathbb{P}_\xi}[\xi\xi^\mathsf{T}] = (\frac{\alpha}{\alpha+2})\mathbb{E}[Z_\xi Z_\xi^\mathsf{T}] = \Sigma$. Therefore, $\mathbb{P}_\xi \in \mathcal{D}_\xi$, and so $\inf_{\mathbb{P}_\xi \in \mathcal{D}_\xi} \mathbb{P}_\xi\{a(x)^\mathsf{T}\xi \le b(x)\} \le \inf_{\mathbb{P}_\zeta \in \mathcal{D}_1} \mathbb{P}_\zeta\{\zeta \le b(x)\}$. ∎

Next, we compute the worst-case probability $\inf_{\mathbb{P}_\zeta \in \mathcal{D}_1} \mathbb{P}_\zeta\{\zeta \le b(x)\}$, for which we make the following two assumptions in the remainder of this section.

Assumption 1 $\left(\frac{\alpha+2}{\alpha}\right)\Sigma \succ \left(\frac{\alpha+1}{\alpha}\right)^2 \mu\mu^\mathsf{T}$.

Assumption 2 *Constraint $a(x)^\mathsf{T}\xi \le b(x)$, and so constraint $\zeta \le b(x)$ as well, is satisfied when ξ takes the value of its mode 0. That is, $b(x) \ge 0$.*

Assumption 1 is standard in the literature and ensures that $\mathcal{D}_\xi \neq \emptyset$ (see, e.g., Van Parys et al. (2015a)). Assumption 2 is reasonable and satisfied in most practical situations. In fact, given ACC (6) and Proposition 1, we observe that Assumption 2 holds if $\mathbb{P}_\zeta\{\zeta \le 0\} < 1 - \alpha$ for each $\mathbb{P}_\zeta \in \mathcal{D}_1$, i.e., if the distributions in $\mathcal{D}_1$ are not extremely negative-skewed. To represent ACC (6), we show an equivalent reformulation of $\inf_{\mathbb{P}_\zeta \in \mathcal{D}_1} \mathbb{P}_\zeta\{\zeta \le b(x)\}$ in the following proposition that also sheds light on the worst-case probability distribution.

Proposition 2 *Define $\mu_0 = \left(\frac{\alpha+1}{\alpha}\right)\mu_1$ and $\Sigma_0 = \left(\frac{\alpha+2}{\alpha}\right)\Sigma_1$. Then, $\inf_{\mathbb{P}_\zeta \in \mathcal{D}_1} \mathbb{P}_\zeta\{\zeta \le b(x)\}$ is equivalent to the optimal objective value of optimization problem*

$$\min_{p_1,p_2,z_1,z_2} \quad p_1 + \left(\frac{b(x)}{z_2}\right)^\alpha p_2 \tag{9a}$$

$$\text{s.t.} \quad p_1 + p_2 = 1, \tag{9b}$$

$$p_1 z_1 + p_2 z_2 = \mu_0, \tag{9c}$$

$$p_1 z_1^2 + p_2 z_2^2 = \Sigma_0, \tag{9d}$$

$$p_1, p_2 \ge 0, \quad z_1 \in \mathbb{R}, \quad z_2 \ge b(x). \tag{9e}$$

Proof: First, we rewrite $\inf_{\mathbb{P}_\zeta \in \mathcal{D}_1} \mathbb{P}_\zeta\{\zeta \le b(x)\}$ as a functional optimization problem as follows:

$$\min_{\mathbb{P}_\zeta} \ \mathbb{P}_\zeta\{\zeta \le b(x)\} \tag{10a}$$

$$\text{s.t. } \ \mathbb{E}_{\mathbb{P}_\zeta}[\zeta] = \mu_1, \tag{10b}$$

$$\mathbb{E}_{\mathbb{P}_\zeta}[\zeta^2] = \Sigma_1, \tag{10c}$$

$$\mathbb{E}_{\mathbb{P}_\zeta}[1] = 1, \tag{10d}$$

$$\mathbb{P}_\zeta \text{ is } \alpha\text{-unimodal}, \tag{10e}$$

where constraints (10b)–(10c) describe the two moments of ζ, and constraint (10d) ensures that $\mathbb{P}_\zeta$ is a probability distribution. Using Theorem 3.5 in Dharmadhikari and Joag-Dev (1988), since $\mathbb{P}_\zeta$ is α-unimodal, there exists a random variable Z such that $\zeta = U^{1/\alpha}Z$, where U is uniform in $(0,1)$ and independent of Z. It follows that $\mathbb{E}_{\mathbb{P}_\zeta}[\zeta] = \mathbb{E}[U^{1/\alpha}]\mathbb{E}_{\mathbb{P}_Z}[Z] = (\frac{\alpha}{\alpha+1})\mathbb{E}_{\mathbb{P}_Z}[Z]$ and $\mathbb{E}_{\mathbb{P}_\zeta}[\zeta^2] = \mathbb{E}[U^{2/\alpha}]\mathbb{E}_{\mathbb{P}_Z}[Z^2] = (\frac{\alpha}{\alpha+2})\mathbb{E}_{\mathbb{P}_Z}[Z^2]$. Furthermore,

$$\begin{aligned}
\mathbb{P}_\zeta\{\zeta \le b(x)\} &= \mathbb{P}\{U^{1/\alpha}Z \le b(x)\} \\
&= \int_{z=-\infty}^{+\infty} \mathbb{P}\{U^{1/\alpha}z \le b(x)\} d\mathbb{P}_Z(z) \\
&= \int_{z=-\infty}^{b(x)} 1 \ d\mathbb{P}_Z(z) + \int_{z=b(x)}^{+\infty} \mathbb{P}\left\{U^{1/\alpha} \le \frac{b(x)}{z}\right\} d\mathbb{P}_Z(z) &\quad (11a)\\
&= \int_{z=-\infty}^{b(x)} 1 \ d\mathbb{P}_Z(z) + \int_{z=b(x)}^{+\infty} \left(\frac{b(x)}{z}\right)^\alpha d\mathbb{P}_Z(z) \\
&= \int_{z=-\infty}^{+\infty} \left[\frac{b(x)}{\max\{z, b(x)\}}\right]^\alpha d\mathbb{P}_Z(z), &\quad (11b)
\end{aligned}$$

where equality (11a) is because $U^{1/\alpha}z \le b(x)$ when $z \le b(x)$ (note that $b(x) \ge 0$ due to Assumption 2), and in (11b) we designate that $0/0 = 1$ in case $b(x) = 0$. Hence, problem (10a)–(10e) can be recast as

$$\min_{\mathbb{P}_Z} \ \mathbb{E}_{\mathbb{P}_Z}\left[\frac{b(x)}{\max\{Z, b(x)\}}\right]^\alpha \tag{12a}$$

$$\text{s.t. } \ \mathbb{E}_{\mathbb{P}_Z}[Z] = \mu_0, \tag{12b}$$

$$\mathbb{E}_{\mathbb{P}_Z}[Z^2] = \Sigma_0, \tag{12c}$$

$$\mathbb{E}_{\mathbb{P}_Z}[1] = 1. \tag{12d}$$

Second, we take the dual of problem (12a)–(12d) to obtain

$$\max_{\pi,\lambda,\gamma} \ \mu_0\pi + \Sigma_0\lambda + \gamma \tag{13a}$$

$$\text{s.t. } \ \lambda z^2 + \pi z + \gamma \le \left[\frac{b(x)}{\max\{z, b(x)\}}\right]^\alpha, \quad \forall z \in \mathbb{R}, \tag{13b}$$

where dual variables π, λ, and γ are associated with primal constraints (12b)–(12d), respectively. Meanwhile, dual constraints (13b) are associated with primal variable $\mathbb{P}_Z$. Strong duality holds between problems (12a)–(12d) and (13a)–(13b) due to Assumption 1 (see Proposition 3.4 in Shapiro (2000)). As $[b(x)/\max\{z, b(x)\}]^\alpha \leq 1$, we can rewrite constraints (13b) as

$$\lambda z_1^2 + \pi z_1 + \gamma - 1 \leq 0, \quad \forall z_1 \in \mathbb{R}, \tag{13c}$$

$$\lambda z_2^2 + \pi z_2 - \left(\frac{b(x)}{z_2}\right)^\alpha + \gamma \leq 0, \quad \forall z_2 \geq b(x). \tag{13d}$$

Third, from constraints (13c), we observe that $\lambda \leq 0$. It follows that functions $\lambda z_1^2 + \pi z_1 + \gamma - 1$ and $\lambda z_2^2 + \pi z_2 - (b(x)/z_2)^\alpha + \gamma$ are concave in z_1 and z_2, respectively. Furthermore, we note that constraint (13c) is the robust counterpart of an uncertain constraint $\lambda \tilde{z}_1^2 + \pi \tilde{z}_1 + \gamma \leq 1$ containing random variable $\tilde{z}_1 \in \mathbb{R}$, and likewise constraint (13d) is the robust counterpart of an uncertain constraint $\lambda \tilde{z}_2^2 + \pi \tilde{z}_2 - (b(x)/\tilde{z}_2)^\alpha + \gamma \leq 0$ containing random variable $\tilde{z}_2 \geq b(x)$. According to the "primal worst equals dual best" result in Beck and Ben-Tal (2009) (see Theorem 4.1), problem (13a)–(13b) has the same optimal objective value as the optimistic dual problem (9a)–(9e). ■

Remark 1 *Suppose that* $(p_1^*, p_2^*, z_1^*, z_2^*)$ *is an optimal solution to problem* (9a)–(9e)*. From the proof of Proposition 2, we observe that problem* (9a)–(9e) *is solved for the worst-case probability distribution of a random variable* Z *such that* $\zeta = U^{1/\alpha}Z$*, where* U *is uniform on* $(0, 1)$ *and independent of* Z*. It follows that the worst-case distribution* $\mathbb{P}_\zeta^*$ *attaining* $\inf_{\mathbb{P}_\zeta \in \mathcal{D}_1} \mathbb{P}_\zeta\{\zeta \leq b(x)\}$ *is a mixture in the form* $\mathbb{P}_\zeta^* = p_1^*\mathbb{P}_\zeta^1 + p_2^*\mathbb{P}_\zeta^2$*, where, for* $i = 1, 2$*,* $\mathbb{P}_\zeta^i$ *is defined on the interval connecting* 0 *and* z_i^* *(i.e.,* $[0, z_i^*]$ *or* $[z_i^*, 0]$*, depending on the sign of* z_i^**) and* $\mathbb{P}_\zeta^i\{|\zeta| \leq t|z_i^*|\} = t^\alpha$ *for all* $t \in [0, 1]$*.*

Finally, we reformulate ACC (6) by analyzing problem (9a)–(9e). We summarize the main result of this section in the following theorem.

Theorem 1 *ACC* (6) *is equivalent to a set of SOC constraints*

$$\sqrt{\frac{1 - \epsilon - \tau^{-\alpha}}{\epsilon}}||\Lambda a(x)|| \leq \tau b(x) - \left(\frac{\alpha + 1}{\alpha}\right)\mu^\mathsf{T} a(x), \quad \forall \tau \geq \left(\frac{1}{1 - \epsilon}\right)^{1/\alpha}, \tag{14}$$

where $\Lambda := [(\frac{\alpha+2}{\alpha})\Sigma - (\frac{\alpha+1}{\alpha})^2\mu\mu^\mathsf{T}]^{1/2}$*.*

Proof: We analyze the solutions to problem (9a)–(9e) and identify all possible solutions (p_1, p_2, z_1, z_2) that satisfy constraints (9b)–(9e). To this end, we analyze the following two cases.
Case 1. If $\mu_0 \leq b(x)$, then we parameterize z_2 by defining $z_2 = \tau b(x)$ for $\tau \geq 1$. Accordingly, we parameterize all solutions (p_1, p_2, z_1, z_2) that satisfy constraints (9b)–(9e) by τ as follows:

$$p_1 = \frac{(\tau b(x) - \mu_0)^2}{(\tau b(x) - \mu_0)^2 + \Sigma_0 - \mu_0^2}, \quad p_2 = \frac{\Sigma_0 - \mu_0^2}{(\tau b(x) - \mu_0)^2 + \Sigma_0 - \mu_0^2}, \tag{15a}$$

$$z_1 = \mu_0 - \frac{\Sigma_0 - \mu_0^2}{\tau b(x) - \mu_0}, \text{ and } z_2 = \tau b(x). \tag{15b}$$

Note that, for each $\tau \geq 1$, (p_1, p_2, z_1, z_2) satisfies constraints (9e) because $p_1, p_2 \geq 0$ and $z_2 = \tau b(x) \geq b(x)$. Then, problem (9a)–(9e) can be recast as

$$\min_{\tau \geq 1} \ \frac{(\tau b(x) - \mu_0)^2 + \tau^{-\alpha}(\Sigma_0 - \mu_0^2)}{(\tau b(x) - \mu_0)^2 + \Sigma_0 - \mu_0^2}.$$

Hence, ACC (6), i.e., $\inf_{\mathbb{P}_\zeta \in \mathcal{D}_1} \mathbb{P}_\zeta\{\zeta \leq b(x)\} \geq 1 - \epsilon$, can be recast as

$$\frac{(\tau b(x) - \mu_0)^2 + \tau^{-\alpha}(\Sigma_0 - \mu_0^2)}{(\tau b(x) - \mu_0)^2 + (\Sigma_0 - \mu_0^2)} \geq 1 - \epsilon, \quad \forall \tau \geq 1.$$

After simple transformations, this is equivalent to

$$(\tau b(x) - \mu_0)^2 \geq \left(\frac{1 - \epsilon - \tau^{-\alpha}}{\epsilon}\right)(\Sigma_0 - \mu_0^2), \quad \forall \tau \geq 1. \tag{16}$$

As $(\tau b(x) - \mu_0)^2 \geq 0$, we can assume $\tau \geq (1/(1-\epsilon))^{1/\alpha}$ without loss of generality. Furthermore, because $\tau b(x) - \mu_0 \geq 0$ for all $\tau \geq 1$, we can rewrite constraints (16) as (14), using the definitions of μ_0 and Σ_0.

Case 2. If $\mu_0 > b(x)$, then we parameterize z_2 by defining $z_2 = \tau b(x)$ for $\tau \geq 1$. For all $\tau \geq \mu_0/b(x)$, because $z_2 \geq \mu_0$, we parameterize (p_1, p_2, z_1, z_2) by τ as in (15a)–(15b). Similar to Case 1, ACC (6) can be recast as

$$\tau b(x) - \mu_0 \geq \sqrt{\left(\frac{1 - \epsilon - \tau^{-\alpha}}{\epsilon}\right)_+} \sqrt{\Sigma_0 - \mu_0^2}, \quad \forall \tau \geq \frac{\mu_0}{b(x)}. \tag{17a}$$

For all $1 \leq \tau < \mu_0/b(x)$, because $b(x) \leq z_2 < \mu_0$, we parameterize (p_1, p_2, z_1, z_2) by τ as follows:

$$p_1 = \frac{(\mu_0 - \tau b(x))^2}{(\mu_0 - \tau b(x))^2 + \Sigma_0 - \mu_0^2}, \quad p_2 = \frac{\Sigma_0 - \mu_0^2}{(\mu_0 - \tau b(x))^2 + \Sigma_0 - \mu_0^2},$$
$$z_1 = \mu_0 + \frac{\Sigma_0 - \mu_0^2}{\mu_0 - \tau b(x)}, \text{ and } z_2 = \tau b(x).$$

Then, because $\mu_0 > \tau b(x)$, ACC (6) can be recast as

$$\mu_0 - \tau b(x) \geq \sqrt{\left(\frac{1 - \epsilon - \tau^{-\alpha}}{\epsilon}\right)_+} \sqrt{\Sigma_0 - \mu_0^2}, \quad \forall 1 \leq \tau < \frac{\mu_0}{b(x)}. \tag{17b}$$

Combining inequalities (17a)–(17b) and the fact that $(1 - \epsilon - \tau^{-\alpha})/\epsilon > 0$ if and only if $\tau > [1/(1-\epsilon)]^{1/\alpha}$, we have $\mu_0/b(x) \leq [1/(1-\epsilon)]^{1/\alpha}$ because otherwise, when $\tau = \mu_0/b(x)$, the left-hand side of (17a) equals zero while the right-hand side is strictly positive. It follows that inequalities (17b) are equivalent to $\mu_0 - \tau b(x) \geq 0$ for all $1 \leq \tau < \mu_0/b(x)$ and so redundant, and inequalities (17a) are equivalent to (14), using the definitions of μ_0 and Σ_0. ∎

In computation, directly replacing ACC with constraints (14) involves an infinite number of SOC constraints and so is computationally intractable. An alternative approach is by separation, i.e., (i) obtain a solution $\hat{x}$ from a relaxed formulation, (ii) find a $\hat{\tau} \geq (1/(1-\epsilon))^{1/\alpha}$ such that $\hat{x}$ violates the corresponding SOC constraint (14), and (iii) incorporate the violated SOC constraint to strengthen the formulation. We discuss how to efficiently conduct Step (ii) of the separation approach, which is equivalent to solving the following problem:

Separation Problem 1: Given $\hat{x}$, does there exist a $\hat{\tau} \geq (1/(1-\epsilon))^{1/\alpha}$ such that $\hat{x}$ violates constraints (14)?

In the following proposition, we show that Separation Problem 1 can be solved by conducting a golden section search on the real line. This search is computationally efficient.

Proposition 3 *Define $\hat{\mu}_0 = (\frac{\alpha+1}{\alpha})\mu^\mathsf{T} a(\hat{x})$ and $\hat{\Sigma}_0 = (\frac{\alpha+2}{\alpha}) a(\hat{x})^\mathsf{T} \Sigma a(\hat{x})$. We have the following:*

1. *If $a(\hat{x}) = 0$, then constraints (14) are always satisfied;*

2. *If $a(\hat{x}) \neq 0$ and $b(\hat{x}) = 0$, then $\hat{\tau} = +\infty$;*

3. *If $a(\hat{x}) \neq 0$ and $b(\hat{x}) > 0$, then $\hat{\tau}$ equals the minimizer of the one-dimensional problem*

$$\min_{\tau \geq (1/(1-\epsilon))^{1/\alpha}} \ (b(\hat{x})\tau - \hat{\mu}_0)^2 - \left(\frac{1-\epsilon-\tau^{-\alpha}}{\epsilon}\right)(\hat{\Sigma}_0 - \hat{\mu}_0^2), \tag{18}$$

   whose objective function is strongly convex and can be minimized via a golden section search in the interval $\left[(1/(1-\epsilon))^{1/\alpha},\ \hat{\mu}_0/b(\hat{x}) + \alpha(1-\epsilon)^{(\alpha+1)/\alpha}(\hat{\Sigma}_0 - \hat{\mu}_0^2)/(2\epsilon b(\hat{x})^2)\right]$.

Proof: First, if $a(\hat{x}) = 0$, then constraints (14) reduce to $\tau b(x) \geq 0$ for all $\tau \geq (1/(1-\epsilon))^{1/\alpha}$, which always holds due to Assumption 2. Second, if $a(\hat{x}) \neq 0$ and $b(\hat{x}) = 0$, then constraints (14) reduce to

$$\sqrt{\frac{1-\epsilon-\tau^{-\alpha}}{\epsilon}}||\Lambda a(\hat{x})|| \leq -\left(\frac{\alpha+1}{\alpha}\right)\mu^\mathsf{T} a(\hat{x}), \quad \forall \tau \geq \left(\frac{1}{1-\epsilon}\right)^{1/\alpha}.$$

As the left-hand side of the above inequality is increasing in τ, constraints (14) are violated if and only if they are violated at $\hat{\tau} = +\infty$. Third, if $a(\hat{x}) \neq 0$ and $b(\hat{x}) > 0$, then constraints (14) are satisfied if and only if

$$\left[\sqrt{\frac{1-\epsilon-\tau^{-\alpha}}{\epsilon}}||\Lambda a(\hat{x})||\right]^2 \leq \left[\tau b(\hat{x}) - \left(\frac{\alpha+1}{\alpha}\right)\mu^\mathsf{T} a(\hat{x})\right]^2, \quad \forall \tau \geq \left(\frac{1}{1-\epsilon}\right)^{1/\alpha}$$

because both sides of constraints (14) are nonnegative. By the definitions of $\hat{\mu}_0$ and $\hat{\Sigma}_0$, this is equivalent to $(b(\hat{x})\tau - \hat{\mu}_0)^2 - [(1-\epsilon-\tau^{-\alpha})/\epsilon](\hat{\Sigma}_0 - \hat{\mu}_0^2) \geq 0$ for all $\tau \geq (1/(1-\epsilon))^{1/\alpha}$. It follows that the Separation Problem 1 can be answered by checking constraints (14) at the optimal solution $\hat{\tau}$ of problem (18).

Finally, we denote the objective function of problem (18) as $H(\tau)$. It follows that

$$\begin{aligned} H'(\tau) &= 2b(\hat{x})(b(\hat{x})\tau - \hat{\mu}_0) - \left(\frac{\alpha}{\epsilon}\right)(\hat{\Sigma}_0 - \hat{\mu}_0^2)\tau^{-\alpha-1}, \\ H''(\tau) &= \left(\frac{\alpha^2+\alpha}{\epsilon}\right)(\hat{\Sigma}_0 - \hat{\mu}_0^2)\tau^{-\alpha-2}. \end{aligned}$$

As $H''(\tau) > 0$ for all $\tau \geq (1/(1-\epsilon))^{1/\alpha}$, $H(\tau)$ is strongly convex and can be minimized via a golden section search. More specifically, if $H'((1/(1-\epsilon))^{1/\alpha}) \geq 0$, then $(1/(1-\epsilon))^{1/\alpha}$ is optimal to problem (18). Otherwise, if $H'((1/(1-\epsilon))^{1/\alpha}) < 0$, then problem (18) is optimized at $\hat{\tau}$ such that $H'(\hat{\tau}) = 0$. It follows that $2b(\hat{x})(b(\hat{x})\hat{\tau} - \hat{\mu}_0) = (\alpha/\epsilon)(\hat{\Sigma}_0 - \hat{\mu}_0^2)\hat{\tau}^{-\alpha-1}$. Since $\hat{\tau} \geq (1/(1-\epsilon))^{1/\alpha}$, we have

$$\begin{aligned} 2b(\hat{x})^2\hat{\tau} &\leq 2b(\hat{x})\hat{\mu}_0 + \left(\frac{\alpha}{\epsilon}\right)(\hat{\Sigma}_0 - \hat{\mu}_0^2)(1-\epsilon)^{(\alpha+1)/\alpha} \\ \Rightarrow \hat{\tau} &\leq \frac{\hat{\mu}_0}{b(\hat{x})} + \frac{\alpha(1-\epsilon)^{(\alpha+1)/\alpha}}{2\epsilon b(\hat{x})^2}(\hat{\Sigma}_0 - \hat{\mu}_0^2). \end{aligned}$$

Hence, the golden section search can be restricted to the interval $[(1/(1-\epsilon))^{1/\alpha}, \hat{\mu}_0/b(\hat{x}) + \alpha(1-\epsilon)^{(\alpha+1)/\alpha}(\hat{\Sigma}_0 - \hat{\mu}_0^2)/(2\epsilon b(\hat{x})^2)]$ without loss of optimality. ■

2.1 Approximations of the Ambiguous Chance Constraint

Before closing this section, we derive relaxed and conservative approximations of ACC (6) by using a finite number of SOC constraints. First, based on the exact representation (14) that involves all $\tau \in \big[[1/(1-\epsilon)]^{1/\alpha}, \infty\big)$, we obtain a relaxed approximation by only involving a finite number of τ. We summarize this approximation in the following proposition, whose proof is immediate and so omitted.

Proposition 4 *For given integer $K \geq 1$ and real numbers $[1/(1-\epsilon)]^{1/\alpha} \leq n_1 < n_2 < \cdots < n_K \leq \infty$, ACC (6) implies the SOC constraints*

$$\sqrt{\frac{1-\epsilon-n_k^{-\alpha}}{\epsilon}}\,||\Lambda a(x)|| \leq n_k b(x) - \left(\frac{\alpha+1}{\alpha}\right)\mu^{\mathsf{T}} a(x), \quad \forall k = 1, \ldots, K. \tag{19}$$

Second, we obtain a conservative approximation by approximating the left-hand sides of the inequalities (14) by using a piece-wise linear function of τ.

Proposition 5 *Given integer $K \geq 2$ and real numbers $[1/(1-\epsilon)]^{1/\alpha} = n_1 < n_2 < \cdots < n_K = \infty$, we define a piece-wise linear function containing $(K-1)$ pieces:*

$$g(\tau) = \min_{k=2,\ldots,K}\left\{\sqrt{\frac{1}{\epsilon(1-\epsilon-n_k^{-\alpha})}}\left[\left(\frac{\alpha n_k^{-\alpha-1}}{2}\right)\tau + 1 - \epsilon - \left(1+\frac{\alpha}{2}\right)n_k^{-\alpha}\right]\right\}.$$

Then, $g(\tau) \geq \sqrt{(1-\epsilon-\tau^{-\alpha})/\epsilon}$ *for all* $\tau \geq [1/(1-\epsilon)]^{1/\alpha}$*. Furthermore, denote* $m_1 = [1/(1-\epsilon)]^{1/\alpha}$ *and let* $m_2 < \cdots < m_{K-1}$ *represent the* $(K-2)$ *breakpoints of function* $g(\tau)$*, i.e.,*

$$m_k = \frac{(1-\epsilon)\left(1-\sqrt{\frac{1-\epsilon-n_k^{-\alpha}}{1-\epsilon-n_{k+1}^{-\alpha}}}\right) + \left(1+\frac{\alpha}{2}\right)\left(n_{k+1}^{-\alpha}\sqrt{\frac{1-\epsilon-n_k^{-\alpha}}{1-\epsilon-n_{k+1}^{-\alpha}}} - n_k^{-\alpha}\right)}{\left(\frac{\alpha}{2}\right)\left(n_{k+1}^{-\alpha-1}\sqrt{\frac{1-\epsilon-n_k^{-\alpha}}{1-\epsilon-n_{k+1}^{-\alpha}}} - n_k^{-\alpha-1}\right)}, \quad \forall m = 2, \ldots, K-1.$$

Then, ACC (6) *is implied by the SOC constraints*

$$g(m_k)\big|\big|\Lambda a(x)\big|\big| \leq m_k b(x) - \left(\frac{\alpha+1}{\alpha}\right)\mu^\top a(x), \quad \forall k = 1, \ldots, K-1. \tag{20}$$

Proof: Denote $h(\tau) = (1-\epsilon-\tau^{-\alpha})/\epsilon$. Then, the first derivative $h'(\tau) = \left(\frac{\alpha\tau^{-\alpha-1}}{2}\right)\sqrt{\frac{1}{\epsilon(1-\epsilon-\tau^{-\alpha})}}$ and the tangent of $h(\tau)$ at n_k is

$$\sqrt{\frac{1}{\epsilon(1-\epsilon-n_k^{-\alpha})}}\left[\left(\frac{\alpha n_k^{-\alpha-1}}{2}\right)\tau + 1 - \epsilon - \left(1+\frac{\alpha}{2}\right)n_k^{-\alpha}\right]$$

for all $k = 2, \ldots, K$. It follows that $g(\tau) \geq h(\tau)$ for all $\tau \geq [1/(1-\epsilon)]^{1/\alpha}$ because $h(\tau)$ is concave on the interval $\big[[1/(1-\epsilon)]^{1/\alpha}, \infty\big)$. Hence, ACC (6) is implied by

$$g(\tau)\big|\big|\Lambda a(x)\big|\big| \leq \tau b(x) - \left(\frac{\alpha+1}{\alpha}\right)\mu^\top a(x), \quad \forall \tau \geq [1/(1-\epsilon)]^{1/\alpha}. \tag{21}$$

Furthermore, given x, as the left-hand side of (21) is piece-wise linear in τ and the right-hand side of (21) is linear in τ, inequalities (21) hold if and only if they hold at the breakpoints of $g(\tau)$. Therefore, ACC (6) is implied by constraints (20). ∎

Remark 2 *In computation, we can use the conservative approximation* (20) *to find near-optimal solutions. More specifically, suppose that we employ the separation approach to solve problem* $\min\{c(x) : x \in X, \ x$ *satisfies* $(6)\}$ *and have finished the first* K *iterations. Then, from these iterations, we obtain a lower bound* c_L^K *of the optimal objective value and* $\hat{\tau}_1, \ldots, \hat{\tau}_K$ *by iteratively solving Separation Problem 1. By letting* $n_1 = [1/(1-\epsilon)]^{1/\alpha}$*,* $n_{K+2} = \infty$*, and* $n_k = \hat{\tau}_{k-1}$ *for all* $k = 2, \ldots, K+1$*, we obtain an upper bound* c_U^K *of the optimal objective value by solving problem* $\min\{c(x) : \ x \in X, \ x$ *satisfies* (20) *based on* $n_1, \ldots, n_{K+2}\}$*, whose optimal solution is denoted* x_K^**. If* $(c_U^K - c_L^K)/c_L^K$ *is small enough, then we can stop the iterations and output* x_K^* *as a near-optimal solution.*

3 Representation of the Ambiguous CVaR Constraint

To recast AVC (7) as SOC constraints, we adopt a similar method to that described in Section 2. Again, we project random vector ξ on $\mathbb{R}$ and consider a one-dimensional random variable ζ. We summarize this result in the following proposition and omit the proof due to its similarity to that of Proposition 1.

Proposition 6 *The following equality holds:*

$$\sup_{\mathbb{P}_\xi \in \mathcal{D}_\xi} CVaR^{\epsilon}_{\mathbb{P}_\xi}(a(x)^{\mathsf{T}}\xi) \;=\; \sup_{\mathbb{P}_\zeta \in \mathcal{D}_1} CVaR^{\epsilon}_{\mathbb{P}_\zeta}(\zeta).$$

We compute $\sup_{\mathbb{P}_\zeta \in \mathcal{D}_1} \text{CVaR}^{\epsilon}_{\mathbb{P}_\zeta}(\zeta)$ by observing that $\mathbb{P}_\zeta$ is α-unimodal and so there exists a random variable Z such that $\zeta = U^{1/\alpha}Z$, where U is uniform in $(0,1)$ and independent of Z (see Theorem 3.5 in Dharmadhikari and Joag-Dev (1988)). We summarize this computation in the following proposition, and note that it can also be obtained by following Theorem 2.1 in Van Parys et al. (2015b).

Proposition 7 *The following equality holds:*

$$\sup_{\mathbb{P}_\zeta \in \mathcal{D}_1} CVaR^{\epsilon}_{\mathbb{P}_\zeta}(\zeta) \;=\; \inf_{\beta \in \mathbb{R}} \left\{ \beta + \frac{1}{\epsilon} \sup_{\mathbb{P}_Z \in \mathcal{D}(\mu_0, \Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[f(Z)] \right\},$$

where $\mathcal{D}(\mu_0, \Sigma_0) := \{\mathbb{P}_Z \in \mathcal{M}_1 : \mathbb{E}_{\mathbb{P}_Z}[Z] = \mu_0, \mathbb{E}_{\mathbb{P}_Z}[Z^2] = \Sigma_0\}$ *and* $f(Z) = \mathbb{1}[\beta < 0]f_-(Z) + \mathbb{1}[\beta \geq 0]f_+(Z)$*, where*

$$f_+(z) \;=\; \begin{cases} 0 & \text{if } z \leq \beta \\ \left(\frac{\alpha}{\alpha+1}\right) z - \beta + \left(\frac{\beta}{\alpha+1}\right)\left(\frac{\beta}{z}\right)^{\alpha} & \text{if } z > \beta \end{cases}, \text{ and } f_-(z) \;=\; \begin{cases} -\left(\frac{\beta}{\alpha+1}\right)\left(\frac{\beta}{z}\right)^{\alpha} & \text{if } z < \beta \\ \left(\frac{\alpha}{\alpha+1}\right) z - \beta & \text{if } z \geq \beta \end{cases}.$$

Proof: First, based on the definition of CVaR, we have

$$\begin{aligned} \sup_{\mathbb{P}_\zeta \in \mathcal{D}_1} \text{CVaR}^{\epsilon}_{\mathbb{P}_\zeta}(\zeta) \;&=\; \sup_{\mathbb{P}_\zeta \in \mathcal{D}_1} \inf_{\beta} \left\{ \beta + \frac{1}{\epsilon}\mathbb{E}_{\mathbb{P}_\zeta}[\zeta - \beta]_+ \right\} \\ &=\; \inf_{\beta} \left\{ \beta + \frac{1}{\epsilon} \sup_{\mathbb{P}_\zeta \in \mathcal{D}_1} \mathbb{E}_{\mathbb{P}_\zeta}[\zeta - \beta]_+ \right\} \end{aligned} \tag{22}$$

where the switch of $\inf_\beta$ and $\sup_{\mathbb{P}_\zeta}$ in (22) follows from the Sion's minimax theorem (see Sion (1958)) by observing that $\beta + \frac{1}{\epsilon}\mathbb{E}_{\mathbb{P}_\zeta}[\zeta - \beta]_+$ is convex in β and concave (actually affine) in $\mathbb{P}_\zeta$.

Second, based on the representation $\zeta = U^{1/\alpha}Z$ (see Theorem 3.5 in Dharmadhikari and Joag-Dev (1988)), we obtain that $\mathbb{E}_{\mathbb{P}_Z}[Z] = (\frac{\alpha+1}{\alpha})\mathbb{E}_{\mathbb{P}_\zeta}[\zeta] = \mu_0$, $\mathbb{E}_{\mathbb{P}_Z}[Z^2] = (\frac{\alpha+2}{\alpha})\mathbb{E}_{\mathbb{P}_\zeta}[\zeta^2] = \Sigma_0$, and

$$\begin{aligned} \mathbb{E}_{\mathbb{P}_\zeta}[\zeta - \beta]_+ \;&=\; \mathbb{E}_{\mathbb{P}_Z}[U^{1/\alpha}Z - \beta]_+ \\ &=\; \int_{z=-\infty}^{+\infty} \int_{u=0}^{1} \left[u^{1/\alpha}z - \beta\right]_+ du\; d\mathbb{P}_Z(z). \end{aligned}$$

It follows that, when $\beta \leq 0$,

$$\mathbb{E}_{\mathbb{P}_\zeta}[\zeta - \beta]_+ \;=\; \int_{z=-\infty}^{\beta} \int_{u=0}^{(\beta/z)^\alpha} \left(u^{1/\alpha}z - \beta\right) du\; d\mathbb{P}_Z(z) + \int_{z=\beta}^{+\infty} \int_{u=0}^{1} \left(u^{1/\alpha}z - \beta\right) du\; d\mathbb{P}_Z(z)$$

$$
\begin{aligned}
&= \int_{z=-\infty}^{\beta} \left(-\frac{1}{\alpha+1}\right)\left(\frac{\beta^{\alpha+1}}{z^{\alpha}}\right) d\mathbb{P}_Z(z) + \int_{z=\beta}^{+\infty} \left[\left(\frac{\alpha}{\alpha+1}\right) z - \beta\right] d\mathbb{P}_Z(z) \\
&= \mathbb{E}_{\mathbb{P}_Z}[f_-(Z)],
\end{aligned}
$$

and, when $\beta > 0$,

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}_\zeta}[\zeta - \beta]_+ &= \int_{z=\beta}^{+\infty} \int_{u=(\beta/z)^{\alpha}}^{1} \left(u^{1/\alpha} z - \beta\right) du \; d\mathbb{P}_Z(z) \\
&= \int_{z=\beta}^{+\infty} \left[\left(\frac{\alpha}{\alpha+1}\right) z - \beta + \left(\frac{1}{\alpha+1}\right)\left(\frac{\beta^{\alpha+1}}{z^{\alpha}}\right)\right] d\mathbb{P}_Z(z) \\
&= \mathbb{E}_{\mathbb{P}_Z}[f_+(Z)].
\end{aligned}
$$

■

Proposition 7 indicates that computing $\sup_{\mathbb{P}_\zeta \in \mathcal{D}_1} \text{CVaR}^{\epsilon}_{\mathbb{P}_\zeta}(\zeta)$ can be difficult because it needs to evaluate the worst-case expectation of a nonlinear function $f(z)$, i.e., $\sup_{\mathbb{P}_Z \in \mathcal{D}(\mu_0, \Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[f(Z)]$. To obtain a computable form, we first present two structural properties of $f(z)$. Lemma 1 proposes two approximations of $f(z)$ from above (termed $f_{\text{U}}(z)$) and below (termed $f_{\text{L}}(z)$), respectively. Both $f_{\text{U}}(z)$ and $f_{\text{L}}(z)$ are convex and consist of two linear pieces. Furthermore, Lemma 2 represents convex functions $f_+(z)$ and $f_-(z)$ by the supporting hyperplanes of their epigraphs.

Lemma 1 *Define* $f_U(z) = \left(\frac{\alpha}{\alpha+1}\right)(z-\beta)_+ + \left(\frac{1}{\alpha+1}\right)(-\beta)_+$ *and* $f_L(z) = \left[\left(\frac{\alpha}{\alpha+1}\right) z - \beta\right]_+$. *Then,* $f_L(z) \leq f(z) \leq f_U(z)$ *for all* $z \in \mathbb{R}$.

Proof: First, we prove $f_{\text{L}}(z) \leq f(z)$ by discussing the following four cases:

1. If $z < \beta < 0$, then $0 \leq (\beta/z) \leq 1$ and $(-\beta) \geq 0$. It follows that $f(z) = -(\beta/(\alpha+1))\,(\beta/z)^{\alpha} \geq 0$. Additionally, define $H(z) := -(\beta/(\alpha+1))\,(\beta/z)^{\alpha}$ and then $H(z)$ is a convex function of z on interval $(-\infty, \beta]$. It follows that $H(z) \geq H'(\beta)(z-\beta) + H(\beta)$, i.e.,

$$
-\left(\frac{\beta}{\alpha+1}\right)\left(\frac{\beta}{z}\right)^{\alpha} \geq \left(\frac{\alpha}{\alpha+1}\right)(z-\beta) + \left(-\frac{\beta}{\alpha+1}\right) = \left(\frac{\alpha}{\alpha+1}\right) z - \beta,
$$

   where the inequality is because $H'(z) = (\alpha/(\alpha+1))(\beta/z)^{\alpha+1}$ and $H(\beta) = (-\beta/(\alpha+1))$. Hence, $-(\beta/(\alpha+1))\,(\beta/z)^{\alpha} \geq [(\frac{\alpha}{\alpha+1})z - \beta]_+$, i.e., $f(z) \geq f_{\text{L}}(z)$.

2. If $\beta < 0$ and $z \geq \beta$, then $(\frac{\alpha}{\alpha+1})z - \beta \geq 0$. It follows that $f_{\text{L}}(z) = (\frac{\alpha}{\alpha+1})z - \beta = f(z)$.

3. If $\beta \geq 0$ and $z \leq \beta$, then $(\frac{\alpha}{\alpha+1})z - \beta < 0$. It follows that $f_{\text{L}}(z) = 0 = f(z)$.

4. If $z > \beta \geq 0$, then $(\beta/z) \geq 0$. It follows that $f(z) = (\alpha/(\alpha+1))z - \beta + (\beta/(\alpha+1))(\beta/z)^{\alpha} \geq (\frac{\alpha}{\alpha+1})z - \beta$. Additionally, as $-z < -\beta \leq 0$, from Case 1 we have

$$
-\left(\frac{-\beta}{\alpha+1}\right)\left(\frac{-\beta}{-z}\right)^{\alpha} \geq \left(\frac{\alpha}{\alpha+1}\right)(-z) - (-\beta).
$$

   In other words, $(\alpha/(\alpha+1))z - \beta + (\beta/(\alpha+1))(\beta/z)^{\alpha} \geq 0$. Hence, $(\alpha/(\alpha+1))z - \beta + (\beta/(\alpha+1))(\beta/z)^{\alpha} \geq [(\frac{\alpha}{\alpha+1})z - \beta]_+$, i.e., $f(z) \geq f_{\text{L}}(z)$.

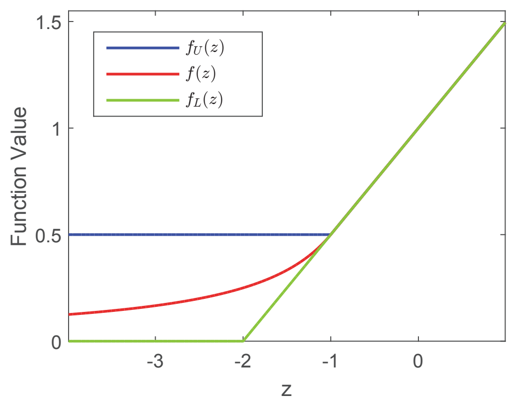
Second, we prove $f(z) \leq f_{\mathrm{U}}(z)$ by discussing the following four cases:

1. If $z < \beta < 0$, then $0 \leq (\beta/z) \leq 1$ and $(-\beta) \geq 0$. It follows that $f(z) = -(\beta/(\alpha+1))\,(\beta/z)^{\alpha} \leq (\frac{1}{\alpha+1})(-\beta) \leq f_{\mathrm{U}}(z)$.

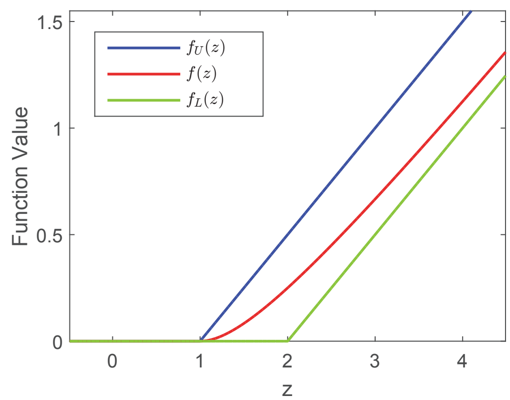
2. If $\beta < 0$ and $z \geq \beta$, then $(z-\beta)_+ = z - \beta$ and $(-\beta)_+ = -\beta$. It follows that $f_{\mathrm{U}}(z) = (\frac{\alpha}{\alpha+1})(z-\beta)_+ + (\frac{1}{\alpha+1})(-\beta)_+ = (\frac{\alpha}{\alpha+1})z - \beta = f(z)$.

3. If $\beta \geq 0$ and $z \leq \beta$, then $f(z) = 0 \leq f_{\mathrm{U}}(z)$.

4. If $z > \beta \geq 0$, then $0 \leq (\beta/z) < 1$ and $(z-\beta)_+ = z - \beta$. It follows that $f(z) = (\alpha/(\alpha+1))z - \beta + (\beta/(\alpha+1))(\beta/z)^{\alpha} \leq (\alpha/(\alpha+1))z - \beta + \beta/(\alpha+1) = f_{\mathrm{U}}(z)$. ■



(a) $\beta = -1$ and $\alpha = 1$



(b) $\beta = 1$ and $\alpha = 1$

Figure 1: Examples of function $f(z)$ and its approximations $f_{\mathrm{U}}(z)$ and $f_{\mathrm{L}}(z)$

Fig. 1a and 1b present examples of function $f(z)$ and its approximations $f_{\mathrm{U}}(z)$ and $f_{\mathrm{L}}(z)$. It can also be shown that $f_{\mathrm{U}}(z) \to f(z)$ and $f_{\mathrm{L}}(z) \to f(z)$ everywhere as $\alpha \to \infty$, i.e., the approximation errors shrink to zero.

Lemma 2 *The following two equalities hold:*

$$f_+(z) \;=\; \sup_{k \geq 1}\Big\{\Big(\frac{\alpha}{\alpha+1}\Big)(1-k^{-\alpha-1})z - (1-k^{-\alpha})\beta\Big\} \tag{23a}$$

when $\beta \geq 0$, *and*

$$f_-(z) \;=\; \sup_{k \geq 1}\Big\{\Big(\frac{\alpha}{\alpha+1}\Big)k^{-\alpha-1}z - k^{-\alpha}\beta\Big\}. \tag{23b}$$

when $\beta \leq 0$. *Furthermore,* $f_-(z) = f_L(z) \leq f_+(z)$ *for all* $z \in \mathbb{R}$ *when* $\beta \geq 0$ *and* $f_+(z) = f_L(z) \leq f_-(z)$ *for all* $z \in \mathbb{R}$ *when* $\beta \leq 0$.

Proof: First, we suppose that $\beta \geq 0$ and pick a $z_0 \geq \beta$. The first derivative of $f_+(z)$ at z_0 is $f'_+(z)\big|_{z=z_0} = (\frac{\alpha}{\alpha+1})\big[1 - (\frac{\beta}{z_0})^{\alpha+1}\big]$. It follows that the supporting hyperplane of the epigraph $\{(y,z) \in \mathbb{R}^2 : y \geq f_+(z)\}$ at z_0 is $y \geq (\frac{\alpha}{\alpha+1})\big[1 - (\frac{\beta}{z_0})^{\alpha+1}\big]z - \big[1 - (\frac{\beta}{z_0})^{\alpha}\big]\beta$. Hence, $f_+(z) = \sup_{z_0 \geq \beta}\big\{(\frac{\alpha}{\alpha+1})\big[1 - (\frac{\beta}{z_0})^{\alpha+1}\big]z - \big[1 - (\frac{\beta}{z_0})^{\alpha}\big]\beta\big\}$ for all $z \geq \beta$ because $f_+(z)$ is convex. Furthermore, as $f_+(z) = 0$ when $z \leq \beta$ and $(\frac{\alpha}{\alpha+1})\big[1 - (\frac{\beta}{z_0})^{\alpha+1}\big]z - \big[1 - (\frac{\beta}{z_0})^{\alpha}\big]\beta = 0$ when $z_0 = \beta$, we have $f_+(z) = \sup_{z_0 \geq \beta}\big\{(\frac{\alpha}{\alpha+1})\big[1 - (\frac{\beta}{z_0})^{\alpha+1}\big]z - \big[1 - (\frac{\beta}{z_0})^{\alpha}\big]\beta\big\}$ for all $z \in \mathbb{R}$. Rewriting $z_0 = k\beta$ for $k \geq 1$ leads to representation (23a). The proof of representation (23b) is similar and so omitted.

Second, we suppose that $\beta \geq 0$ and define $f_+^k(z) = (\frac{\alpha}{\alpha+1})(1 - k^{-\alpha-1})z - (1 - k^{-\alpha})\beta$ for all $k \geq 1$. Then, $f_+(z) = \sup_{k\geq 1}\{f_+^k(z)\}$ and $f_-(z) = \sup_{k\geq 1}\{(\frac{\alpha}{\alpha+1})z - \beta - f_+^k(z)\} = (\frac{\alpha}{\alpha+1})z - \beta - \inf_{k\geq 1}\{f_+^k(z)\}$. We prove that $\inf_{k\geq 1}\{f_+^k(z)\} = -\big[(\frac{\alpha}{\alpha+1})z - \beta\big]_-$ by discussing the following two cases:

1. When $z \leq (\frac{\alpha+1}{\alpha})\beta$, we have $z \leq (\frac{\alpha+1}{\alpha})k\beta$ as $k \geq 1$ and $\beta \geq 0$. It follows that $(\frac{\alpha}{\alpha+1})(-k^{-\alpha-1})z + k^{-\alpha}\beta \geq 0$ and so $f_+^k(z) = (\frac{\alpha}{\alpha+1})(1 - k^{-\alpha-1})z - (1 - k^{-\alpha})\beta \geq (\frac{\alpha}{\alpha+1})z - \beta$ for all $k \geq 1$. Hence, $\inf_{k\geq 1}\{f_+^k(z)\} \geq (\frac{\alpha}{\alpha+1})z - \beta$. In addition, by letting $k \to +\infty$, we have $f_+^k(z) \to (\frac{\alpha}{\alpha+1})z - \beta$. Therefore, $\inf_{k\geq 1}\{f_+^k(z)\} = (\frac{\alpha}{\alpha+1})z - \beta$ when $z \leq (\frac{\alpha+1}{\alpha})\beta$.

2. When $z \geq (\frac{\alpha+1}{\alpha})\beta$, we have $(1 - k^{-\alpha-1})z \geq (1 - k^{-\alpha})(\frac{\alpha+1}{\alpha})\beta$ because $\beta \geq 0$ and $1 - k^{-\alpha-1} \geq 1 - k^{-\alpha} \geq 0$. It follows that $f_+^k(z) = (\frac{\alpha}{\alpha+1})(1 - k^{-\alpha-1})z - (1 - k^{-\alpha})\beta \geq 0$ for all $k \geq 1$. Hence, $\inf_{k\geq 1}\{f_+^k(z)\} \geq 0$. In addition, by letting $k = 1$, we have $f_+^k(z) = 0$. Therefore, $\inf_{k\geq 1}\{f_+^k(z)\} = 0$ when $z \leq (\frac{\alpha+1}{\alpha})\beta$.

It follows that $f_-(z) = (\frac{\alpha}{\alpha+1})z - \beta + \big[(\frac{\alpha}{\alpha+1})z - \beta\big]_- = \big[(\frac{\alpha}{\alpha+1})z - \beta\big]_+$. Hence, by Lemma 1, $f_-(z) = f_{\mathrm{L}}(z) \leq f_+(z)$ for all $z \in \mathbb{R}$ when $\beta \geq 0$. The proof of $f_+(z) = f_{\mathrm{L}}(z) \leq f_-(z)$ when $\beta \leq 0$ is similar and so omitted. ∎

We are now ready to derive a reformulation of the worst-case expectation $\sup_{\mathbb{P}_Z \in \mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[f(Z)]$. We summarize this result in the following theorem.

Theorem 2 *For $\beta \in \mathbb{R}$, $\sup_{\mathbb{P}_Z \in \mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[f(Z)] = \frac{1}{2}\max\{E_+, E_-\}$, where*

$$S_{k,\mu_0,\Sigma_0,\beta} = \sqrt{\left[(1 - k^{-\alpha})\beta - \Big(\frac{\alpha}{\alpha+1}\Big)(1 - k^{-\alpha-1})\mu_0\right]^2 + \Big(\frac{\alpha}{\alpha+1}\Big)^2(1 - k^{-\alpha-1})^2(\Sigma_0 - \mu_0^2)},$$

$$E_+ = \sup_{k\geq 1}\left\{S_{k,\mu_0,\Sigma_0,\beta} - (1 - k^{-\alpha})\beta + \Big(\frac{\alpha}{\alpha+1}\Big)(1 - k^{-\alpha-1})\mu_0\right\}, \quad \textit{and} \tag{24a}$$

$$E_- = \sup_{k\geq 1}\left\{S_{k,\mu_0,\Sigma_0,\beta} - (1 + k^{-\alpha})\beta + \Big(\frac{\alpha}{\alpha+1}\Big)(1 + k^{-\alpha-1})\mu_0\right\}. \tag{24b}$$

Proof: To avoid clutter, throughout this proof, we assume that $\Sigma_0 > \mu_0^2$ and $\beta \neq 0$. The degenerate cases with $\Sigma_0 = \mu_0^2$ or $\beta = 0$ can be easily verified. First, we suppose that $\beta > 0$ and define $f_+^k(z) = (\frac{\alpha}{\alpha+1})(1-k^{-\alpha-1})z - (1-k^{-\alpha})\beta$ for $k \ge 1$. Then, $f(Z) = f_+(Z)$ by Proposition 7 and $f_+(z) = \sup_{k\ge1}\{f_+^k(z)\}$ by Lemma 2. It follows that $\sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[f(Z)] = \sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[\sup_{k\ge1}\{f_+^k(Z)\}]$. We make the following observation on switching the order of two supremum operators.

Observation 1 *For $\beta \in \mathbb{R}$, we have*

$$\sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}\left[\sup_{k\ge1}\left\{f_+^k(Z)\right\}\right] = \sup_{k\ge1}\left\{\sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}\left[f_+^k(Z)\right]_+\right\}.$$

Proof of Observation 1: First, for all $k \ge 1$, it is clear that $\sup_{k\ge1}\{f_+^k(Z)\} \ge [f_+^k(Z)]_+$ because $\sup_{k\ge1}\{f_+^k(z)\} = f_+(z) \ge 0$ for all $z \in \mathbb{R}$. It follows that $\sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[\sup_{k\ge1}\{f_+^k(Z)\}] \ge \sup_{k\ge1}\{\sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[f_+^k(Z)]_+\}$. We now show the opposite, i.e., $\sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[\sup_{k\ge1}\{f_+^k(Z)\}] \le \sup_{k\ge1}\{\sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[f_+^k(Z)]_+\}$. When $\beta \le 0$, this clearly holds because

$$\begin{aligned}\sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}\left[\sup_{k\ge1}\left\{f_+^k(Z)\right\}\right] &= \sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}\left[f_{\mathrm{L}}(Z)\right] \\ &= \lim_{k\to\infty}\left\{\sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}\left[f_+^k(Z)\right]_+\right\} \\ &\le \sup_{k\ge1}\left\{\sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}\left[f_+^k(Z)\right]_+\right\},\end{aligned}$$

where the first equality follows from Lemma 2 and the second equality follows from the definition of $f_+^k(z)$. Hence, we focus on the case when $\beta > 0$ in the remainder of this proof.

Second, we present $\sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[\sup_{k\ge1}\{f_+^k(Z)\}]$ as the following optimization problem:

$$\begin{aligned}(\mathrm{P}): \quad v_P = \max_{\mathbb{P}_Z}\ & \mathbb{E}_{\mathbb{P}_Z}[f_+(Z)] \\ \text{s.t.}\ & \mathbb{E}_{\mathbb{P}_Z}[Z] = \mu_0, \\ & \mathbb{E}_{\mathbb{P}_Z}[Z^2] = \Sigma_0, \\ & \mathbb{E}_{\mathbb{P}_Z}[1] = 1,\end{aligned}$$

whose dual is

$$\begin{aligned}(\mathrm{D}): \quad v_D = \min_{p,q,r}\ & \mu_0 p + \Sigma_0 q + r \\ \text{s.t.}\ & qz^2 + pz + r \ge f_+(z), \quad \forall z \in \mathbb{R}. \qquad (25)\end{aligned}$$

Strong duality holds between (P) and (D) due to Assumption 1 (see Proposition 3.4 in Shapiro (2000)), i.e., $v_P = v_D$. Furthermore, by Lemma 3.1 in Shapiro (2000), there exists a worst-case probability distribution (i.e., an optimal solution to (P)) with a finite support of at most 3 points. That is, there exists $m \in \{1,2,3\}$, $(z_1^*,\ldots,z_m^*) \in \mathbb{R}^m$, and $(\pi_1^*,\ldots,\pi_m^*) \in \mathbb{R}_+^m$ such that $\sum_{i=1}^m \pi_i^* z_i^* = \mu_0$, $\sum_{i=1}^m \pi_i^*(z_i^*)^2 = \Sigma_0$, and $\sum_{i=1}^m \pi_i^* = 1$. Denoting an optimal solution to (D) by

(p^*, q^*, r^*), we claim that $q^*(z_i^*)^2 + p^* z_i^* + r^* = f_+(z_i^*)$ for all $i = 1, \ldots, m$, i.e., constraint (25) holds at equality at points $z_1^*, \ldots, z_m^*$. Indeed, if this claim fails to hold, then we have

$$v_P = \sum_{i=1}^{m} \pi_i^* f_+(z_i^*) < \sum_{i=1}^{m} \pi_i^* [q^*(z_i^*)^2 + p^* z_i^* + r^*] = q^* \Sigma_0 + p^* \mu_0 + r^* = v_D, \qquad (26)$$

where the inequality follows from constraint (25), and the second equality follows from the definitions of $(z_1^*, \ldots, z_m^*)$ and $(\pi_1^*, \ldots, \pi_m^*)$. As inequality (26) violates the strong duality, the claim holds. In addition, it can be shown that $f_+(z)$ and any quadratic function $qz^2 + pz + r$ satisfying constraint (25) intersect at most once in interval $(-\infty, \beta]$ and at most once in interval $[\beta, \infty)$. It follows that $m \leq 2$, and so $m = 2$ because $\Sigma_0 > \mu_0^2$. Without loss of generality, we assume that $z_1^* \in (-\infty, \beta]$ and $z_2^* \in [\beta, \infty)$.

Third, we define $k^* = z_2^*/\beta$ and consider function $[f_+^{k^*}(z)]_+$ that is tangent to $f_+(z)$ at z_1^* and z_2^* by Lemma 2. Hence, $qz^2 + pz + r \geq [f_+^{k^*}(z)]_+$ for all $z \in \mathbb{R}$ with equality holding only at z_1^* and z_2^*. Consider the primal and dual formulations of $\sup_{\mathbb{P}_Z \in \mathcal{D}(\mu_0, \Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[f_+^{k^*}(Z)]_+$ as follows:

$$\begin{aligned}
(\mathrm{P}_{k^*}): \quad v_P^{k^*} = \max_{\mathbb{P}_Z} \;& \mathbb{E}_{\mathbb{P}_Z}[f_+^{k^*}(Z)]_+ \\
\text{s.t.} \;& \mathbb{E}_{\mathbb{P}_Z}[Z] = \mu_0, \\
& \mathbb{E}_{\mathbb{P}_Z}[Z^2] = \Sigma_0, \\
& \mathbb{E}_{\mathbb{P}_Z}[1] = 1,
\end{aligned}$$

$$\begin{aligned}
(\mathrm{D}_{k^*}): \quad v_D^{k^*} = \min_{p,q,r} \;& \mu_0 p + \Sigma_0 q + r \\
\text{s.t.} \;& qz^2 + pz + r \geq [f_+^{k^*}(z)]_+, \quad \forall z \in \mathbb{R}.
\end{aligned}$$

It is clear that the pair (z_1^*, z_2^*) and (π_1^*, π_2^*) provide a primal feasible solution to (P_{k^*}), and (p^*, q^*, r^*) is a dual feasible solution to (D_{k^*}) because $f_+(z) \geq [f_+^{k^*}(z)]_+$ for all $z \in \mathbb{R}$. Meanwhile, these two solutions share the same objective function value because $\sum_{i=1}^2 \pi_i^* [f_+^{k^*}(z_i^*)]_+ = \sum_{i=1}^2 \pi_i^* f_+(z_i^*) = \mu_0 p^* + \Sigma_0 q^* + r^*$, where the first equality follows from the definition of $[f_+^{k^*}(z)]_+$ and the second equality is due to $v_P = v_D$. It follows that strong duality holds between (P_{k^*}) and (D_{k^*}) and $\sup_{\mathbb{P}_Z \in \mathcal{D}(\mu_0, \Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[\sup_{k \geq 1}\{f_+^k(Z)\}] = \sup_{\mathbb{P}_Z \in \mathcal{D}(\mu_0, \Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[f_+^{k^*}(Z)]_+$. Therefore, $\sup_{\mathbb{P}_Z \in \mathcal{D}(\mu_0, \Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[\sup_{k \geq 1}\{f_+^k(Z)\}] \leq \sup_{k \geq 1}\{\sup_{\mathbb{P}_Z \in \mathcal{D}(\mu_0, \Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[f_+^k(Z)]_+\}$ and so the proof is completed. ∎

(Proof of Theorem 2 continued) By Observation 1, we have

$$\begin{aligned}
& \sup_{\mathbb{P}_Z \in \mathcal{D}(\mu_0, \Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[f(Z)] \\
= \;& \sup_{k \geq 1} \left\{ \sup_{\mathbb{P}_Z \in \mathcal{D}(\mu_0, \Sigma_0)} \mathbb{E}_{\mathbb{P}_Z} \left[\left(\frac{\alpha}{\alpha+1} \right) (1 - k^{-\alpha-1}) Z - (1 - k^{-\alpha}) \beta \right]_+ \right\} \\
= \;& \sup_{k \geq 1} \left\{ \left(\frac{\alpha}{\alpha+1} \right) (1 - k^{-\alpha-1}) \sup_{\mathbb{P}_Z \in \mathcal{D}(\mu_0, \Sigma_0)} \mathbb{E}_{\mathbb{P}_Z} \left[Z - \left(\frac{\alpha+1}{\alpha} \right) \left(\frac{1 - k^{-\alpha}}{1 - k^{-\alpha-1}} \right) \beta \right]_+ \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{k\geq 1}\Bigg\{\Big(\frac{\alpha}{\alpha+1}\Big)(1-k^{-\alpha-1})\Big(\frac{1}{2}\Big)\Bigg[\sqrt{\Big[\Big(\frac{\alpha+1}{\alpha}\Big)\Big(\frac{1-k^{-\alpha}}{1-k^{-\alpha-1}}\Big)\beta-\mu_0\Big]^2+(\sigma_0-\mu_0^2)} \\
&\qquad -\Big(\frac{\alpha+1}{\alpha}\Big)\Big(\frac{1-k^{-\alpha}}{1-k^{-\alpha-1}}\Big)\beta+\mu_0\Bigg]\Bigg\} \qquad (27) \\
&= \frac{1}{2}E_+,
\end{aligned}
$$

where equality (27) follows from Observation 2 presented in Appendix A.

Second, we suppose that $\beta < 0$. Then, $f(Z) = f_-(Z)$ by Proposition 7. It follows that

$$
\begin{aligned}
&\sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[f(Z)] \\
&= \sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}\Big[\sup_{k\geq 1}\Big\{\Big(\frac{\alpha}{\alpha+1}\Big)k^{-\alpha-1}Z-k^{-\alpha}\beta\Big\}\Big] \qquad (28a) \\
&= \sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}\Big[\sup_{k\geq 1}\Big\{\max\Big\{\Big(\frac{\alpha}{\alpha+1}\Big)k^{-\alpha-1}Z-k^{-\alpha}\beta,\ \Big(\frac{\alpha}{\alpha+1}\Big)Z-\beta\Big\}\Big\}\Big] \qquad (28b) \\
&= \sup_{k\geq 1}\Big\{\sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}\Big[\max\Big\{\Big(\frac{\alpha}{\alpha+1}\Big)k^{-\alpha-1}Z-k^{-\alpha}\beta,\ \Big(\frac{\alpha}{\alpha+1}\Big)Z-\beta\Big\}\Big]\Big\} \qquad (28c) \\
&= \sup_{k\geq 1}\Big\{\Big(\frac{\alpha}{\alpha+1}\Big)k^{-\alpha-1}\mu_0-k^{-\alpha}\beta+\sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[f_+^k(Z)]_+\Big\} \qquad (28d) \\
&= \frac{1}{2}E_-, \qquad (28e)
\end{aligned}
$$

where equality (28a) follows from Lemma 2, equality (28b) is because $(\frac{\alpha}{\alpha+1})k^{-\alpha-1}z - k^{-\alpha}\beta = (\frac{\alpha}{\alpha+1})z - \beta$ when $k = 1$, equality (28c) is parallel to Observation 1 and can be similarly proved, and equality (28d) follows from the definition of $f_+^k(z)$.

Finally, it remains to prove that $E_+ \geq E_-$ when $\beta > 0$ and $E_+ \leq E_-$ when $\beta < 0$. Due to the similarity of proof, we only show the former case, i.e., when $\beta > 0$. To that end, we note that the equalities (28b)–(28e) are independent of the sign of β and so still hold when $\beta > 0$. It follows that $\frac{1}{2}E_- = \sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[f_-(Z)]$ when $\beta > 0$. Similarly, we have $\frac{1}{2}E_+ = \sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[f_+(Z)]$. But $f_-(z) \leq f_+(z)$ for all $z \in \mathbb{R}$ by Lemma 2, and so $\frac{1}{2}E_- \leq \frac{1}{2}E_+$ when $\beta > 0$. ■

Theorem 2 leads to an equivalent reformulation of AVC (7). We summarize the main result of this section in the following theorem.

Theorem 3 *AVC* (7) *is equivalent to a set of SOC constraints*

$$
\left\|\begin{bmatrix}(1-k^{-\alpha})\beta-(1-k^{-\alpha-1})\mu^\mathsf{T}a(x) \\ \left(\frac{\alpha}{\alpha+1}\right)(1-k^{-\alpha-1})\Lambda a(x)\end{bmatrix}\right\| \leq 2\epsilon b(x)-(1-k^{-\alpha-1})\mu^\mathsf{T}a(x)+(1-k^{-\alpha}-2\epsilon)\beta, \quad (29a)
$$

$$\left\| \begin{bmatrix} (1-k^{-\alpha})\beta - (1-k^{-\alpha-1})\mu^\top a(x) \\ \left(\frac{\alpha}{\alpha+1}\right)(1-k^{-\alpha-1})\Lambda a(x) \end{bmatrix} \right\| \le 2\epsilon b(x) - (1+k^{-\alpha-1})\mu^\top a(x) + (1+k^{-\alpha}-2\epsilon)\beta, \quad (29b)$$

for all $k \geq 1$.

Proof: The conclusion follows from Theorem 2 by the definition of μ_0, Λ, and that

$$S_{k,\mu_0,\Sigma_0,\beta} = \left\| \begin{bmatrix} (1-k^{-\alpha})\beta - (1-k^{-\alpha-1})\mu^\top a(x) \\ \left(\frac{\alpha}{\alpha+1}\right)(1-k^{-\alpha-1})\Lambda a(x) \end{bmatrix} \right\|, \quad \forall k \geq 1. \qquad \blacksquare$$

In computation, directly replacing AVC with constraints (29a)–(29b) requires an infinite number of SOC constraints and is so computationally intractable. Like what we described for ACC in Section 2, we adopt the separation approach and solve the following problem:

Separation Problem 2: Given $\hat{\beta}$ and $\hat{x}$, does there exist a $\hat{k}$ such that $(\hat{\beta}, \hat{x})$ violate constraints (29a)–(29b)?

In the following proposition, we show that Separation Problem 2 can be solved by conducting a golden section search on the real line. This search is computationally efficient.

Proposition 8 *Define $\hat{\mu}_0 = (\frac{\alpha+1}{\alpha})\mu^\top a(\hat{x})$, $\hat{\Sigma}_0 = (\frac{\alpha+2}{\alpha})a(\hat{x})^\top \Sigma a(\hat{x})$. We have the following:*

1. *If $\hat{\beta} = 0$, then $\hat{k} = \infty$;*

2. *If $\hat{\beta} \neq 0$ and $\hat{\Sigma}_0 = \hat{\mu}_0^2$, then $\hat{k} = \max\{\hat{\mu}_0/\hat{\beta}, 1\}$;*

3. *If $\hat{\beta} \neq 0$ and $\hat{\Sigma}_0 > \hat{\mu}_0^2$, then $\hat{k}$ equals the unique root of equation*

$$2\left[\left(\frac{\alpha+1}{\alpha}\right)\left(\frac{1-k^{-\alpha}}{1-k^{-\alpha-1}}\right) - \mu_\beta\right] = (k-\mu_\beta) - \frac{\Gamma_\beta}{(k-\mu_\beta)} \quad (30)$$

lying within the interval $\left[1+\sqrt{(1-\mu_\beta)^2+\Gamma_\beta},\ 1+1/\alpha+\sqrt{(1-\mu_\beta+1/\alpha)^2+\Gamma_\beta}\right]$, where $\mu_\beta = \hat{\mu}_0/\hat{\beta}$ and $\Gamma_\beta = (\hat{\Sigma}_0 - \hat{\mu}_0^2)/\hat{\beta}^2$.

Proof: For a given $(\hat{\beta}, \hat{x})$, solving Separation Problem 2 is equivalent to finding $\sup_{\mathbb{P}_Z \in \mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[f(Z)]$, i.e., $1/2\max\{E_+, E_-\}$ defined in Theorem 2. First, if $\hat{\beta} = 0$, then

$$\begin{aligned} S_{k,\hat{\mu}_0,\hat{\Sigma}_0,\hat{\beta}} &= \sqrt{\left[\left(\frac{\alpha}{\alpha+1}\right)(1-k^{-\alpha-1})\hat{\mu}_0\right]^2 + \left(\frac{\alpha}{\alpha+1}\right)^2(1-k^{-\alpha-1})^2(\hat{\Sigma}_0-\hat{\mu}_0^2)} \\ &= \left(\frac{\alpha}{\alpha+1}\right)(1-k^{-\alpha-1})\sqrt{\hat{\Sigma}_0}. \end{aligned}$$

It follows that

$$\frac{1}{2}E_+ = \frac{1}{2}\sup_{k\geq 1}\left\{\left(\frac{\alpha}{\alpha+1}\right)(1-k^{-\alpha-1})\sqrt{\hat{\Sigma}_0} + \left(\frac{\alpha}{\alpha+1}\right)(1-k^{-\alpha-1})\hat{\mu}_0\right\}$$

$$= \frac{1}{2} \sup_{k \geq 1} \left\{ \left(\frac{\alpha}{\alpha+1} \right) (1 - k^{-\alpha-1}) \left(\sqrt{\hat{\Sigma}_0} + \hat{\mu}_0 \right) \right\} \tag{31a}$$

$$= \frac{1}{2} \left(\frac{\alpha}{\alpha+1} \right) \left(\sqrt{\hat{\Sigma}_0} + \hat{\mu}_0 \right), \tag{31b}$$

where equality (31b) is because $\sqrt{\hat{\Sigma}_0} + \hat{\mu}_0 \geq 0$ and so $k = \infty$ maximizes (31a). Additionally,

$$\frac{1}{2} E_- = \frac{1}{2} \sup_{k \geq 1} \left\{ \left(\frac{\alpha}{\alpha+1} \right) (1 - k^{-\alpha-1}) \sqrt{\hat{\Sigma}_0} + \left(\frac{\alpha}{\alpha+1} \right) (1 + k^{-\alpha-1}) \hat{\mu}_0 \right\}$$

$$= \frac{1}{2} \left(\frac{\alpha}{\alpha+1} \right) \sup_{k \geq 1} \left\{ \left(\sqrt{\hat{\Sigma}_0} + \hat{\mu}_0 \right) + k^{-\alpha-1} \left(\hat{\mu}_0 - \sqrt{\hat{\Sigma}_0} \right) \right\} \tag{31c}$$

$$= \frac{1}{2} \left(\frac{\alpha}{\alpha+1} \right) \left(\sqrt{\hat{\Sigma}_0} + \hat{\mu}_0 \right), \tag{31d}$$

where equality (31d) is because $\hat{\mu}_0 - \sqrt{\hat{\Sigma}_0} \leq 0$ and so $k = \infty$ maximizes (31c). Summing up the above two cases, we have $\hat{k} = \infty$ if $\hat{\beta} = 0$.

Second, if $\hat{\beta} \neq 0$ and $\hat{\Sigma}_0 = \hat{\mu}_0^2$, then $S_{k, \hat{\mu}_0, \hat{\Sigma}_0, \hat{\beta}} = |(1 - k^{-\alpha})\hat{\beta} - (\frac{\alpha}{\alpha+1})(1 - k^{-\alpha-1})\hat{\mu}_0|$. It follows that

$$\frac{1}{2} E_+ = \frac{1}{2} \sup_{k \geq 1} \left\{ \left| (1 - k^{-\alpha})\hat{\beta} - \left(\frac{\alpha}{\alpha+1} \right) (1 - k^{-\alpha-1}) \hat{\mu}_0 \right| - (1 - k^{-\alpha})\hat{\beta} + \left(\frac{\alpha}{\alpha+1} \right) (1 - k^{-\alpha-1}) \hat{\mu}_0 \right\}$$

$$= \sup_{k \geq 1} \left\{ \left[\left(\frac{\alpha}{\alpha+1} \right) (1 - k^{-\alpha-1}) \hat{\mu}_0 - (1 - k^{-\alpha})\hat{\beta} \right]_+ \right\} \tag{32a}$$

$$= f_+(\hat{\mu}_0), \tag{32b}$$

where equality (32b) results from Lemma 2 and so $k = \max\{\hat{\mu}_0/\hat{\beta}, 1\}$ maximizes (32a). Meanwhile,

$$\frac{1}{2} E_- = \frac{1}{2} \sup_{k \geq 1} \left\{ \left| (1 - k^{-\alpha})\hat{\beta} - \left(\frac{\alpha}{\alpha+1} \right) (1 - k^{-\alpha-1}) \hat{\mu}_0 \right| - (1 + k^{-\alpha})\hat{\beta} + \left(\frac{\alpha}{\alpha+1} \right) (1 + k^{-\alpha-1}) \hat{\mu}_0 \right\}$$

$$= \sup_{k \geq 1} \left\{ \max \left\{ \left(\frac{\alpha}{\alpha+1} \right) \hat{\mu}_0 - \hat{\beta}, \ \left(\frac{\alpha}{\alpha+1} \right) k^{-\alpha-1} \hat{\mu}_0 - k^{-\alpha} \hat{\beta} \right\} \right\} \tag{32c}$$

$$= f_-(\hat{\mu}_0), \tag{32d}$$

where equality (32d) results from Lemma 2 and so $k = \max\{\hat{\mu}_0/\hat{\beta}, 1\}$ maximizes (32c). Summing up the above two cases, we have $\hat{k} = \max\{\hat{\mu}_0/\hat{\beta}, 1\}$ if $\hat{\beta} \neq 0$ and $\hat{\Sigma}_0 = \hat{\mu}_0^2$.

Third, suppose that $\hat{\beta} \neq 0$ and $\hat{\Sigma}_0 > \hat{\mu}_0^2$. As the case when $\hat{\beta} < 0$ can be similarly derived, we focus on the case when $\hat{\beta} > 0$. In this case, solving Separation Problem 2 is equivalent to finding the maximizer of optimization problem (24a) that defines E_+. To this end, we let $F(k)$ represent the objective function of (24a), i.e., $F(k) := S_{k, \hat{\mu}_0, \hat{\Sigma}_0, \hat{\beta}} - (1 - k^{-\alpha})\hat{\beta} + (\frac{\alpha}{\alpha+1})(1 - k^{-\alpha-1})\hat{\mu}_0$. It follows that

$$F'(k) = \alpha \beta k^{-\alpha-2} \left\{ \frac{\left[\left(\frac{\alpha+1}{\alpha} \right) \left(\frac{1 - k^{-\alpha}}{1 - k^{-\alpha-1}} \right) - \mu_\beta \right] (k - \mu_\beta) + \Gamma_\beta}{\sqrt{\left[\left(\frac{\alpha+1}{\alpha} \right) \left(\frac{1 - k^{-\alpha}}{1 - k^{-\alpha-1}} \right) - \mu_\beta \right]^2 + \Gamma_\beta}} - (k - \mu_\beta) \right\}.$$

We prove that $F(k)$ is unimodal, and in particular, $F(k)$ is nondecreasing on $[1, \hat{k}]$ and nonincreasing on $[\hat{k}, \infty)$, where $\hat{k}$ represents the root of equation (30). The conclusion of this proposition then follows because $\hat{k}$ is the maximizer of $F(k)$ on $[1, \infty)$. To that end, it suffices to show that (i) $\lim_{k \to 1^+} F'(k) > 0$, (ii) there exists a $k \in [1, \infty)$ such that $F'(k) < 0$, and (iii) $\hat{k}$ is the unique root of equation $F'(k) = 0$. We show (i)–(iii) as follows.

(i) As $\lim_{k \to 1^+} \{\frac{1-k^{-\alpha}}{1-k^{-\alpha-1}}\} = \frac{\alpha}{\alpha+1}$ and $\Gamma_\beta > 0$, we have $\lim_{k \to 1^+} F'(k) = \sqrt{(1-\mu_\beta)^2 + \Gamma_\beta} - (1 - \mu_\beta) > 0$.

(ii) We have

$$\frac{F'(k)}{\alpha\beta k^{-\alpha-2}} = \left(\frac{\left(\frac{\alpha+1}{\alpha}\right)\left(\frac{1-k^{-\alpha}}{1-k^{-\alpha-1}}\right) - \mu_\beta}{\sqrt{\left[\left(\frac{\alpha+1}{\alpha}\right)\left(\frac{1-k^{-\alpha}}{1-k^{-\alpha-1}}\right) - \mu_\beta\right]^2 + \Gamma_\beta}} - 1 \right)(k - \mu_\beta) + \frac{\Gamma_\beta}{\sqrt{\left[\left(\frac{\alpha+1}{\alpha}\right)\left(\frac{1-k^{-\alpha}}{1-k^{-\alpha-1}}\right) - \mu_\beta\right]^2 + \Gamma_\beta}}.$$

As $\frac{1-k^{-\alpha}}{1-k^{-\alpha-1}} \in \left[1, \frac{\alpha+1}{\alpha}\right]$ and

$$\frac{\left(\frac{\alpha+1}{\alpha}\right)\left(\frac{1-k^{-\alpha}}{1-k^{-\alpha-1}}\right) - \mu_\beta}{\sqrt{\left[\left(\frac{\alpha+1}{\alpha}\right)\left(\frac{1-k^{-\alpha}}{1-k^{-\alpha-1}}\right) - \mu_\beta\right]^2 + \Gamma_\beta}} - 1 \ < \ 0,$$

there exists a sufficiently large k such that $F'(k) < 0$.

(iii) We consider the roots of equation $F'(k) = 0$. As $F'(k) = 0$ is equivalent to

$$\left(1 - \frac{\left(\frac{\alpha+1}{\alpha}\right)\left(\frac{1-k^{-\alpha}}{1-k^{-\alpha-1}}\right) - \mu_\beta}{\sqrt{\left[\left(\frac{\alpha+1}{\alpha}\right)\left(\frac{1-k^{-\alpha}}{1-k^{-\alpha-1}}\right) - \mu_\beta\right]^2 + \Gamma_\beta}} \right)(k - \mu_\beta) = \frac{\Gamma_\beta}{\sqrt{\left[\left(\frac{\alpha+1}{\alpha}\right)\left(\frac{1-k^{-\alpha}}{1-k^{-\alpha-1}}\right) - \mu_\beta\right]^2 + \Gamma_\beta}},$$

any root k satisfies $k - \mu_\beta > 0$ because $\Gamma_\beta > 0$. The above equation can be further simplified to equation (30) and so any roots k of equation (30) also satisfy $F'(k) = 0$. We now prove the uniqueness of the root. We note that the first derivative of $2\big[(\frac{\alpha+1}{\alpha})(\frac{1-k^{-\alpha}}{1-k^{-\alpha-1}}) - \mu_\beta\big]$, i.e., the left-hand-side of equation (30), is always less than 1. Meanwhile, the first derivative of $(k - \mu_\beta) - \frac{\Gamma_\beta}{(k-\mu_\beta)}$, i.e., the right-hand-side of equation (30), is always greater than 1. Furthermore, $2\big[(\frac{\alpha+1}{\alpha})(\frac{1-k^{-\alpha}}{1-k^{-\alpha-1}}) - \mu_\beta\big] \in [2(1-\mu_\beta), 2(\frac{\alpha+1}{\alpha} - \mu_\beta)]$, while the range of function $(k - \mu_\beta) - \frac{\Gamma_\beta}{(k-\mu_\beta)}$ is $(-\infty, \infty)$ for $k \in (\mu_\beta, \infty)$. It follows that the two sides of equation (30) can meet only once, i.e., this equation has a unique root.

Finally, we provide lower and upper bounds of root $\hat{k}$. As $\frac{1-k^{-\alpha}}{1-k^{-\alpha-1}} \in \left[1, \frac{\alpha+1}{\alpha}\right]$, we have $2(1-\mu_\beta) \le (\hat{k} - \mu_\beta) - \frac{\Gamma_\beta}{(\hat{k}-\mu_\beta)} \le 2(\frac{\alpha+1}{\alpha} - \mu_\beta)$. It follows that $\hat{k} \in \Big[1 + \sqrt{(1-\mu_\beta)^2 + \Gamma_\beta},\ 1 +$

$1/\alpha + \sqrt{(1-\mu_\beta + 1/\alpha)^2 + \Gamma_\beta}\Big].$ ■

3.1 Approximations of the Ambiguous CVaR Constraint

Before closing this section, we derive approximations of AVC (7). First, in the following proposition, we present a conservative approximation based on $f_{\mathrm{U}}(z)$ and a relaxed one based on $f_{\mathrm{L}}(z)$, both of which are in the form of SOC constraints.

Proposition 9 *AVC (7) is implied by SOC constraints*

$$\left\|\begin{bmatrix}\beta - \left(\frac{\alpha+1}{\alpha}\right)\mu^{\mathsf{T}} a(x) \\ \Lambda a(x)\end{bmatrix}\right\| \le \left[\frac{2\epsilon(\alpha+1)}{\alpha}\right] b(x) - \left[\frac{2\epsilon(\alpha+1)}{\alpha} - 1\right]\beta - \left(\frac{\alpha+1}{\alpha}\right)\mu^{\mathsf{T}} a(x), \tag{33a}$$

$$\left\|\begin{bmatrix}\beta - \left(\frac{\alpha+1}{\alpha}\right)\mu^{\mathsf{T}} a(x) \\ \Lambda a(x)\end{bmatrix}\right\| \le \left[\frac{2\epsilon(\alpha+1)}{\alpha}\right] b(x) - \left[\frac{(2\epsilon-1)(\alpha+1)-1}{\alpha}\right]\beta - \left(\frac{\alpha+1}{\alpha}\right)\mu^{\mathsf{T}} a(x). \tag{33b}$$

Furthermore, AVC (7) implies SOC constraint

$$\left\|\begin{bmatrix}\left(\frac{\alpha+1}{\alpha}\right)\beta - \left(\frac{\alpha+1}{\alpha}\right)\mu^{\mathsf{T}} a(x) \\ \Lambda a(x)\end{bmatrix}\right\| \le \left[\frac{2\epsilon(\alpha+1)}{\alpha}\right] b(x) - \left[\frac{(2\epsilon-1)(\alpha+1)}{\alpha}\right]\beta - \left(\frac{\alpha+1}{\alpha}\right)\mu^{\mathsf{T}} a(x). \tag{33c}$$

Proof: First, based on Propositions 6–7 and Lemma 1, AVC (7) is implied by constraint $\beta + \frac{1}{\epsilon}\sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[f_{\mathrm{U}}(Z)] \le b(x)$. Furthermore, we have

$$\begin{aligned}
\sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[f_{\mathrm{U}}(Z)] &= \sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}\left[\left(\frac{\alpha}{\alpha+1}\right)[Z-\beta]_+ + \left(-\frac{\beta}{\alpha+1}\right)_+\right] \\
&= \left(-\frac{\beta}{\alpha+1}\right)_+ + \left(\frac{\alpha}{\alpha+1}\right) \sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[Z-\beta]_+ \\
&= \left(-\frac{\beta}{\alpha+1}\right)_+ + \left(\frac{\alpha}{\alpha+1}\right)\left(\frac{1}{2}\right)\left[\sqrt{(\beta-\mu_0)^2 + (\Sigma_0-\mu_0^2)} - \beta + \mu_0\right],
\end{aligned}$$

where the last equality is due to Observation 2 presented in Appendix A. It follows that AVC (7) is implied by

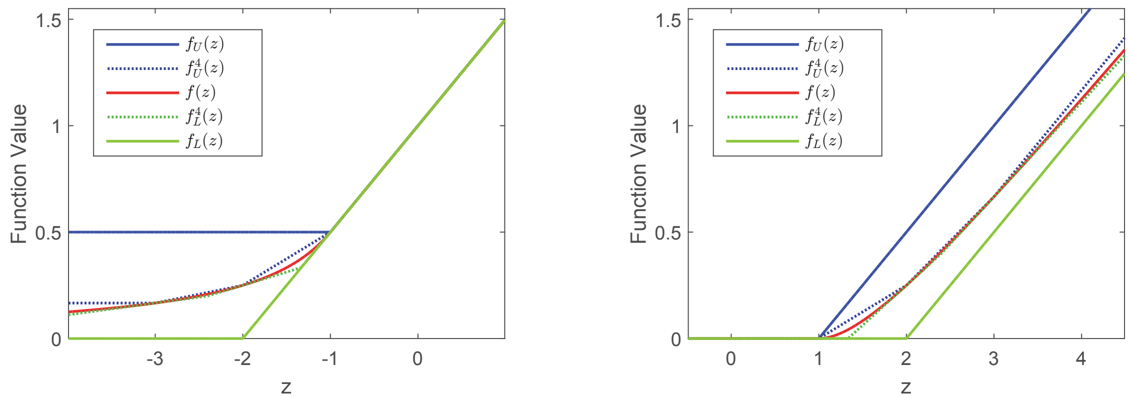
$$\begin{aligned}
&\beta + \left(\frac{1}{\epsilon}\right)\left\{\left(-\frac{\beta}{\alpha+1}\right)_+ + \left(\frac{\alpha}{\alpha+1}\right)\left(\frac{1}{2}\right)\left[\sqrt{(\beta-\mu_0)^2 + (\Sigma_0-\mu_0^2)} - \beta + \mu_0\right]\right\} \le b(x) \\
\Leftrightarrow\ &\sqrt{(\beta-\mu_0)^2 + (\Sigma_0-\mu_0^2)} \le \left[\frac{2\epsilon(\alpha+1)}{\alpha}\right] b(x) - \left[\frac{2\epsilon(\alpha+1)}{\alpha} - 1\right]\beta - \left(\frac{2}{\alpha}\right)(-\beta)_+ - \mu_0.
\end{aligned}$$

This is equivalent to constraints (33a)–(33b) by the definition of μ_0 and observing that

$$\sqrt{(\beta-\mu_0)^2 + (\Sigma_0-\mu_0^2)} = \left\|\begin{bmatrix}\beta - \left(\frac{\alpha+1}{\alpha}\right)\mu^{\mathsf{T}} a(x) \\ \Lambda a(x)\end{bmatrix}\right\|.$$

Second, based on Propositions 6–7 and Lemma 1, AVC (7) implies constraint $\beta + \frac{1}{\epsilon}\sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[f_{\mathrm{L}}(Z)] \le b(x)$. Furthermore, we have

$$\sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[f_{\mathrm{L}}(Z)] = \sup_{\mathbb{P}_Z\in\mathcal{D}(\mu_0,\Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}\left[\left(\frac{\alpha}{\alpha+1}\right)Z - \beta\right]_+$$



(a) $\beta = -1$ and $\alpha = 1$ (b) $\beta = 1$ and $\alpha = 1$

Figure 2: K-piece approximations of $f(z)$ with $K = 4$, $n_1 = 1$, $n_2 = 2$, $n_3 = 3$, and $n_4 = \infty$

$$= \left(\frac{\alpha}{\alpha+1}\right) \sup_{\mathbb{P}_Z \in \mathcal{D}(\mu_0, \Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}\left[Z - \left(\frac{\alpha+1}{\alpha}\right)\beta\right]_+$$

$$= \left(\frac{\alpha}{\alpha+1}\right)\left(\frac{1}{2}\right)\left[\sqrt{\left(\left(\frac{\alpha+1}{\alpha}\right)\beta - \mu_0\right)^2 + (\Sigma_0 - \mu_0^2)} - \left(\frac{\alpha+1}{\alpha}\right)\beta + \mu_0\right],$$

where the last equality is due to Observation 2. It follows that AVC (7) implies

$$\beta + \left(\frac{1}{\epsilon}\right)\left(\frac{\alpha}{\alpha+1}\right)\left(\frac{1}{2}\right)\left[\sqrt{\left(\left(\frac{\alpha+1}{\alpha}\right)\beta - \mu_0\right)^2 + (\Sigma_0 - \mu_0^2)} - \left(\frac{\alpha+1}{\alpha}\right)\beta + \mu_0\right] \leq b(x)$$

$$\Leftrightarrow \sqrt{\left(\left(\frac{\alpha+1}{\alpha}\right)\beta - \mu_0\right)^2 + (\Sigma_0 - \mu_0^2)} \leq \left[\frac{2\epsilon(\alpha+1)}{\alpha}\right] b(x) - \left[\frac{(2\epsilon-1)(\alpha+1)}{\alpha}\right]\beta - \mu_0.$$

This is equivalent to constraints (33c) by the definition of μ_0 and observing that

$$\sqrt{\left(\left(\frac{\alpha+1}{\alpha}\right)\beta - \mu_0\right)^2 + (\Sigma_0 - \mu_0^2)} = \left\|\begin{bmatrix}\left(\frac{\alpha+1}{\alpha}\right)\beta - \left(\frac{\alpha+1}{\alpha}\right)\mu^\top a(x) \\ \Lambda a(x)\end{bmatrix}\right\|. \qquad \blacksquare$$

Second, we derive tighter approximations of AVC (7) based on tighter approximations of function $f(z)$. Note that both $f_{\mathrm{U}}(z)$ and $f_{\mathrm{L}}(z)$ approximate $f(z)$ based on two linear pieces (see Fig. 2a–2b). We generalize $f_{\mathrm{U}}(z)$ and $f_{\mathrm{L}}(z)$ by defining K-piece approximations as follows.

Definition 3 *Given integer $K \geq 3$ and real numbers $1 = n_1 < n_2 < \cdots < n_K = \infty$, we define $f_U^K(z) = \mathbb{1}[\beta < 0] f_{U-}^K(z) + \mathbb{1}[\beta \geq 0] f_{U+}^K(z)$ and $f_L^K(z) = \mathbb{1}[\beta < 0] f_{L-}^K(z) + \mathbb{1}[\beta \geq 0] f_{L+}^K(z)$, where*

$$f_{U+}^K(z) = \max\left\{0, \max_{k=1,\dots,K-1}\left\{\left[\left(\frac{\alpha}{\alpha+1}\right) + \frac{n_{k+1}^{-\alpha} - n_k^{-\alpha}}{(\alpha+1)(n_{k+1}-n_k)}\right] z + \left[\frac{n_{k+1}n_k^{-\alpha} - n_k n_{k+1}^{-\alpha}}{(\alpha+1)(n_{k+1}-n_k)} - 1\right]\beta\right\}\right\},$$

$$f_{U-}^K(z) = \max\left\{\left(\frac{\alpha}{\alpha+1}\right)z - \beta, \max_{k=1,\dots,K-1}\left\{\left[\frac{n_k^{-\alpha} - n_{k+1}^{-\alpha}}{(\alpha+1)(n_{k+1}-n_k)}\right] z - \left[\frac{n_{k+1}n_k^{-\alpha} - n_k n_{k+1}^{-\alpha}}{(\alpha+1)(n_{k+1}-n_k)}\right]\beta\right\}\right\},$$

$$f_{L+}^K(z) \;=\; \max_{k=1,\dots,K}\left\{\Big(\frac{\alpha}{\alpha+1}\Big)(1-n_k^{-\alpha-1})z-(1-n_k^{-\alpha})\beta\right\}, \quad \textit{and}$$
$$f_{L-}^K(z) \;=\; \max_{k=1,\dots,K}\left\{\Big(\frac{\alpha}{\alpha+1}\Big)n_k^{-\alpha-1}z-n_k^{-\alpha}\beta\right\}.$$

We note that $f_{\mathrm{U}}^K(z)$ is the linear interpolation of points $\{(n_k, f(n_k\beta))\}_{k=1,\dots,K}$ and $f_{\mathrm{L}}^K(z)$ is the pointwise maximum of the tangents of $f(z)$ at these points (see Fig. 2a–2b). Due to the convexity of $f(z)$, it follows that $f_{\mathrm{U}}^K(z)$ and $f_{\mathrm{L}}^K(z)$ are convex, and $f_{\mathrm{L}}^K(z) \le f(z) \le f_{\mathrm{U}}^K(z)$. Furthermore, we observe that $f_{\mathrm{L}+}^K(z) \le f_+(z)$ by definition. Based on Lemma 2, $f_{\mathrm{L}+}^K(z) \le f_{\mathrm{L}}(z) \le f_{\mathrm{L}-}^K(z)$ when $\beta < 0$. Similarly, we have $f_{\mathrm{L}-}^K(z) \le f_{\mathrm{L}+}^K(z)$ when $\beta \ge 0$. It follows that $f_{\mathrm{L}}^K(z) = \max\{f_{\mathrm{L}+}^K(z), f_{\mathrm{L}-}^K(z)\}$. We formalize and extend this observation to $f_{\mathrm{U}}^K(z)$ in the following lemma.

Lemma 3 *We have $f_L^K(z) = \max\{f_{L+}^K(z), f_{L-}^K(z)\}$ for all $z \in \mathbb{R}$. Furthermore, $f_{U+}^K(z) \le f(z)$ when $\beta < 0$ and $f_{U-}^K(z) \le f(z)$ when $\beta \ge 0$. It follows that $f_U^K(z) = \max\{f_{U+}^K(z), f_{U-}^K(z)\}$.*

Proof: We first show that $f_{\mathrm{U}+}^K(z) \le f_{\mathrm{L}}(z) = \big[\big(\frac{\alpha}{\alpha+1}\big)z-\beta\big]_+$ when $\beta < 0$. Assuming this is true, we have $f_{\mathrm{U}+}^K(z) \le f(z)$ based on Lemma 2. To this end, for all $k = 1,\dots,K-1$, we observe that $\big(\frac{\alpha}{\alpha+1}\big) + \frac{n_{k+1}^{-\alpha}-n_k^{-\alpha}}{(\alpha+1)(n_{k+1}-n_k)} \le \frac{\alpha}{\alpha+1}$ because $n_{k+1} > n_k$ and $n_{k+1}^{-\alpha} - n_k^{-\alpha} < 0$. We next show that $f_{\mathrm{U}+}^K\big(\frac{\alpha+1}{\alpha}\beta\big) < 0$, which follows from the following chain of equivalences:

$$\begin{aligned}
&\Big[\Big(\frac{\alpha}{\alpha+1}\Big)+\frac{n_{k+1}^{-\alpha}-n_k^{-\alpha}}{(\alpha+1)(n_{k+1}-n_k)}\Big]\Big(\frac{\alpha+1}{\alpha}\Big)\beta+\Big[\frac{n_{k+1}n_k^{-\alpha}-n_kn_{k+1}^{-\alpha}}{(\alpha+1)(n_{k+1}-n_k)}-1\Big]\beta \;<\; 0\\
\Leftrightarrow\;& \frac{n_{k+1}^{-\alpha}-n_k^{-\alpha}}{\alpha(n_{k+1}-n_k)}+\frac{n_{k+1}n_k^{-\alpha}-n_kn_{k+1}^{-\alpha}}{(\alpha+1)(n_{k+1}-n_k)} \;>\; 0\\
\Leftrightarrow\;& n_{k+1}^{\alpha}(\alpha n_{k+1}-\alpha-1) \;>\; n_k^{\alpha}(\alpha n_k-\alpha-1),
\end{aligned}$$

where the last line holds because function $g(y) := y^\alpha(\alpha y-\alpha-1)$ is increasing when $y > 1$. Indeed, $g'(y) = (\alpha^2+\alpha)y^{\alpha-1}(y-1) > 0$ as $y > 1$.

Second, we show that $f_{\mathrm{U}-}^K(z) \le f_{\mathrm{L}}(z) = \big[\big(\frac{\alpha}{\alpha+1}\big)z-\beta\big]_+$ when $\beta \ge 0$. Assuming this is true, we have $f_{\mathrm{U}-}^K(z) \le f(z)$ based on Lemma 2. To that end, for all $k = 1,\dots,K-1$, we observe that $\frac{n_k^{-\alpha}-n_{k+1}^{-\alpha}}{(\alpha+1)(n_{k+1}-n_k)} \le \frac{\alpha}{\alpha+1}$, which follows from the following equivalence:

$$\frac{n_k^{-\alpha}-n_{k+1}^{-\alpha}}{(\alpha+1)(n_{k+1}-n_k)} \le \frac{\alpha}{\alpha+1} \quad\Leftrightarrow\quad n_{k+1}^{-\alpha}+\alpha n_{k+1} \;>\; n_k^{-\alpha}+\alpha n_k,$$

where the right-hand side holds because function $h(y) := y^{-\alpha}+\alpha y$ is increasing when $y > 1$. Indeed, $h'(y) = \alpha(1-y^{-\alpha-1}) > 0$ as $y > 1$. We next show that $f_{\mathrm{U}-}^K\big(\frac{\alpha+1}{\alpha}\beta\big) < 0$, which follows from the following chain of equivalences:

$$\begin{aligned}
&\Big[\frac{n_k^{-\alpha}-n_{k+1}^{-\alpha}}{(\alpha+1)(n_{k+1}-n_k)}\Big]\Big(\frac{\alpha+1}{\alpha}\Big)\beta-\Big[\frac{n_{k+1}n_k^{-\alpha}-n_kn_{k+1}^{-\alpha}}{(\alpha+1)(n_{k+1}-n_k)}\Big]\beta \;<\; 0\\
\Leftrightarrow\;& (\alpha+1)(n_k^{-\alpha}-n_{k+1}^{-\alpha}) \;<\; \alpha(n_{k+1}n_k^{-\alpha}-n_kn_{k+1}^{-\alpha})
\end{aligned}$$

$$\Leftrightarrow \; n_{k+1}^{\alpha}(\alpha n_{k+1} - \alpha - 1) \; > \; n_k^{\alpha}(\alpha n_k - \alpha - 1),$$

where the last line has been shown above. ∎

In the following proposition, we present conservative approximations based on $f_{\mathrm{U}}^K(z)$ and relaxed ones based on $f_{\mathrm{L}}^K(z)$, both of which are in the form of linear matrix inequalities. We note that these approximations are asymptotically tight as K grows to infinity. We omit the proof here because it follows from the standard duality approach. Interested readers are referred to El Ghaoui et al. (2003) and Zymler et al. (2013a).

Proposition 10 *Define $(T+1)\times(T+1)$ matrix $\Omega := \begin{bmatrix} \left(\frac{\alpha+2}{\alpha}\right)\Sigma & \left(\frac{\alpha+1}{\alpha}\right)\mu \\ \left(\frac{\alpha+1}{\alpha}\right)\mu^\top & 1 \end{bmatrix}$. Then, for given integer $K \geq 3$ and real numbers $1 = n_1 < n_2 < \cdots < n_K = \infty$, AVC (7) is satisfied if there exists a symmetric matrix $M_U \in \mathbb{R}^{(T+1)\times(T+1)}$ such that*

$$\beta + \frac{1}{\epsilon} M_U \cdot \Omega \;\leq\; b(x), \quad M_U \;\succeq\; 0, \quad M_U \;\succeq\; \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)a(x) \\ \frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)a(x)^\top & -\beta \end{bmatrix},$$

$$M_U \;\succeq\; \begin{bmatrix} 0 & \frac{1}{2}\left[\left(\frac{\alpha}{\alpha+1}\right) + \frac{n_{k+1}^{-\alpha} - n_k^{-\alpha}}{(\alpha+1)(n_{k+1}-n_k)}\right]a(x) \\ \frac{1}{2}\left[\left(\frac{\alpha}{\alpha+1}\right) + \frac{n_{k+1}^{-\alpha} - n_k^{-\alpha}}{(\alpha+1)(n_{k+1}-n_k)}\right]a(x)^\top & \left[\frac{n_{k+1}n_k^{-\alpha} - n_k n_{k+1}^{-\alpha}}{(\alpha+1)(n_{k+1}-n_k)} - 1\right]\beta \end{bmatrix}, \quad \forall k = 1,\ldots,K-1,$$

$$M_U \;\succeq\; \begin{bmatrix} 0 & \frac{1}{2}\left[\frac{n_k^{-\alpha} - n_{k+1}^{-\alpha}}{(\alpha+1)(n_{k+1}-n_k)}\right]a(x) \\ \frac{1}{2}\left[\frac{n_k^{-\alpha} - n_{k+1}^{-\alpha}}{(\alpha+1)(n_{k+1}-n_k)}\right]a(x)^\top & -\left[\frac{n_{k+1}n_k^{-\alpha} - n_k n_{k+1}^{-\alpha}}{(\alpha+1)(n_{k+1}-n_k)}\right]\beta \end{bmatrix}, \quad \forall k = 1,\ldots,K-1,$$

where $\cdot$ represents the Frobenius product of matrices. Furthermore, AVC (7) implies that there exists a symmetric matrix $M_L \in \mathbb{R}^{(T+1)\times(T+1)}$ such that

$$\beta + \frac{1}{\epsilon} M_L \cdot \Omega \;\leq\; b(x), \quad M_L \;\succeq\; 0, \quad M_L \;\succeq\; \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)a(x) \\ \frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)a(x)^\top & -\beta \end{bmatrix},$$

$$M_L \;\succeq\; \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)(1 - n_k^{-\alpha-1})a(x) \\ \frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)(1 - n_k^{-\alpha-1})a(x)^\top & -(1 - n_k^{-\alpha})\beta \end{bmatrix}, \quad \forall k = 1,\ldots,K-1,$$

$$M_L \;\succeq\; \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)n_k^{-\alpha-1}a(x) \\ \frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)n_k^{-\alpha-1}a(x)^\top & -n_k^{-\alpha}\beta \end{bmatrix}, \quad \forall k = 1,\ldots,K-1.$$

Similar to what we mention in Remark 2, we can incorporate the conservative approximation presented in Proposition 10 to find near-optimal solutions when solving a problem involving AVC (7).

4 Extension to Linear Unimodality

In this section, we consider an extension of ACC (6) and AVC (7) based on a related structural property called *linear unimodality*.

Definition 4 *(Linear Unimodality; see Dharmadhikari and Joag-Dev (1988)) A probability distribution $\mathbb{P}_\xi$ is called linear unimodal about* 0 *if for all $a \in \mathbb{R}^T$, the linear combination $a^\top \xi$ is univariate unimodal about* 0.

Analogous to (5), we define the alternative ambiguity set based on linear unimodality as

$$\mathcal{D}_\xi^{\mathrm{LU}}(\mu, \Sigma) := \left\{ \mathbb{P}_\xi \in \mathcal{M}_T : \mathbb{E}_{\mathbb{P}_\xi}[\xi] = \mu,\ \mathbb{E}_{\mathbb{P}_\xi}[\xi\xi^\mathsf{T}] = \Sigma,\ \mathbb{P}_\xi \text{ is linear unimodal about } 0 \right\}. \tag{34}$$

We now show an equivalence between ambiguity sets $\mathcal{D}_\xi^{\mathrm{LU}}(\mu, \Sigma)$ and $\mathcal{D}_\xi(\mu, \Sigma, \alpha)$ with $\alpha = 1$. It follows that all results derived in Sections 2–3, with α set to be 1, remain valid under $\mathcal{D}_\xi^{\mathrm{LU}}(\mu, \Sigma)$.

Proposition 11 *For any Borel measurable function $h : \mathbb{R} \to \mathbb{R}$, we have*

$$\inf_{\mathbb{P}_\xi \in \mathcal{D}_\xi(\mu,\Sigma,1)} \mathbb{E}_{\mathbb{P}_\xi}\big[h(a(x)^\top \xi)\big] \ = \inf_{\mathbb{P}_\xi \in \mathcal{D}_\xi^{LU}(\mu,\Sigma)} \mathbb{E}_{\mathbb{P}_\xi}\big[h(a(x)^\top \xi)\big].$$

Proof: By Theorem 3.5 in Dharmadhikari and Joag-Dev (1988), a random variable X is 1-unimodal if and only if there exists a random variable Z such that $X = UZ$, where U is uniform in $(0, 1)$ and independent of Z.

First, pick any ξ such that $\mathbb{P}_\xi \in \mathcal{D}_\xi(\mu, \Sigma, 1)$. As $a^\top \xi$ is univariate 1-unimodal for all $a \in \mathbb{R}^T$ because $\mathbb{P}_\xi$ is 1-unimodal, $\mathbb{P}_\xi \in \mathcal{D}_\xi^{\mathrm{LU}}(\mu, \Sigma)$. It follows that $\mathcal{D}_\xi(\mu, \Sigma, 1) \subseteq \mathcal{D}_\xi^{\mathrm{LU}}(\mu, \Sigma)$ and so $\inf_{\mathbb{P}_\xi \in \mathcal{D}_\xi(\mu,\Sigma,1)} \mathbb{E}_{\mathbb{P}_\xi}[h(a(x)^\top \xi)] \geq \inf_{\mathbb{P}_\xi \in \mathcal{D}_\xi^{\mathrm{LU}}(\mu,\Sigma)} \mathbb{E}_{\mathbb{P}_\xi}[h(a(x)^\top \xi)]$.

Second, pick any ξ such that $\mathbb{P}_\xi \in \mathcal{D}_\xi^{\mathrm{LU}}(\mu, \Sigma)$. Then, $\zeta := a(x)^\top \xi$ is 1-unimodal because $\mathbb{P}_\xi$ is linear unimodal. Hence, there exists a Z_ζ such that $\zeta = UZ_\zeta$. It follows that $\mathbb{E}[Z_\zeta] = 2\mu_1$ and $\mathbb{E}[Z_\zeta^2] = 3\Sigma_1$. Based on Theorem 1 in Popescu (2007), there exists a $Z_\xi \in \mathbb{R}^T$ such that $Z_\zeta = a(x)^\mathsf{T} Z_\xi$, $\mathbb{E}[Z_\xi] = 2\mu$, and $\mathbb{E}[Z_\xi Z_\xi^\mathsf{T}] = 3\Sigma$. It follows that UZ_ξ is 1-unimodal, and meanwhile $\mathbb{E}_{\mathbb{P}_\xi}[UZ_\xi] = \frac{1}{2}\mathbb{E}[Z_\xi] = \mu$ and $\mathbb{E}_{\mathbb{P}_\xi}[(UZ_\xi)(UZ_\xi)^\mathsf{T}] = \frac{1}{3}\mathbb{E}[Z_\xi Z_\xi^\mathsf{T}] = \Sigma$. Furthermore, $a(x)^\top \xi = a(x)^\top (UZ_\xi)$. Therefore, the probability distribution of UZ_ξ belongs to $\mathcal{D}_\xi(\mu, \Sigma, 1)$, and so $\inf_{\mathbb{P}_\xi \in \mathcal{D}_\xi(\mu,\Sigma,1)} \mathbb{E}_{\mathbb{P}_\xi}[h(a(x)^\top \xi)] \leq \inf_{\mathbb{P}_\xi \in \mathcal{D}_\xi^{\mathrm{LU}}(\mu,\Sigma)} \mathbb{E}_{\mathbb{P}_\xi}[h(a(x)^\top \xi)]$. ■

5 Computational Case Study

In this section, we evaluate the theoretical results derived in Sections 2–3 based on a risk-constrained economic dispatch (RCED) problem in power system operation. We present a nominal RCED model as follows:

$$\min_{g,d,r^{\mathrm{U}},r^{\mathrm{D}}} \quad \sum_{i \in \mathcal{I} \setminus \mathcal{I}_{\mathrm{R}}} \left[c_{i2} g_i^2 + c_{i1} g_i + c_i^{\mathrm{R}} (r_i^{\mathrm{U}} + r_i^{\mathrm{D}}) \right] \tag{35a}$$

$$\text{s.t.} \quad \sum_{i \in \mathcal{I} \setminus \mathcal{I}_\text{R}} g_i + \sum_{i \in \mathcal{I}_\text{R}} f_i = \sum_{b=1}^{B} L_b, \tag{35b}$$

$$r_i = -\Big(\sum_{i \in \mathcal{I}_\text{R}} w_i\Big) d_i, \quad \forall i \in \mathcal{I} \setminus \mathcal{I}_\text{R}, \tag{35c}$$

$$\sum_{i \in \mathcal{I} \setminus \mathcal{I}_\text{R}} d_i \ = \ 1, \tag{35d}$$

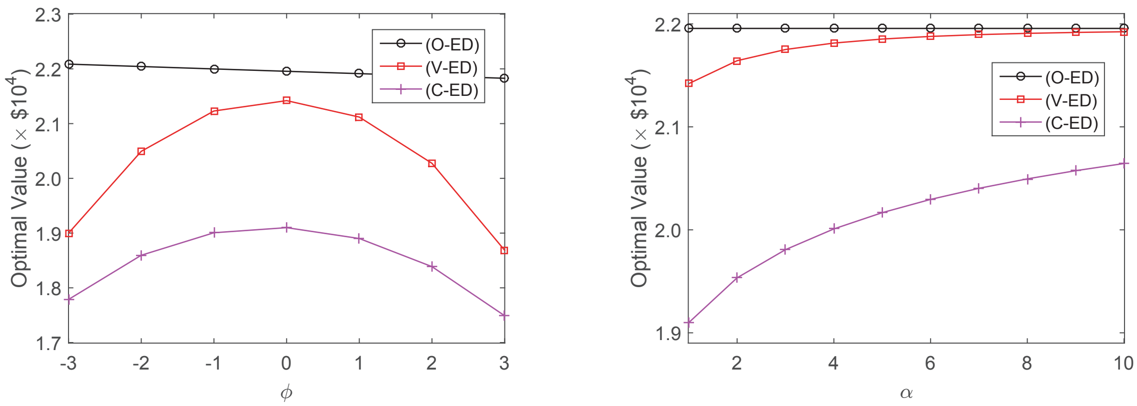
$$- r_i^\text{D} \le r_i \le r_i^\text{U}, \quad \forall i \in \mathcal{I} \setminus \mathcal{I}_\text{R}, \tag{35e}$$

$$g_i^\text{MIN} \le g_i + r_i \le g_i^\text{MAX}, \quad \forall i \in \mathcal{I} \setminus \mathcal{I}_\text{R}, \tag{35f}$$

$$- C_\ell \le \sum_{b=1}^{B} D_\ell^b \Big[\sum_{i \in \mathcal{G}_b} (g_i + r_i) + \sum_{i \in \mathcal{H}_b} (f_i + w_i) - L_b\Big] \le C_\ell, \quad \forall \ell \in \mathcal{L}, \tag{35g}$$

where B represents the number of buses in the power system, $\mathcal{I}$ represents the set of generating units (conventional and renewable), $\mathcal{I}_\text{R}$ represents the set of renewable units, $\mathcal{L}$ represents the set of transmission lines, $\mathcal{G}_b$ represents the set of conventional units at bus b, $\mathcal{H}_b$ represents the set of renewable units at bus b, c_{i2} and c_{i1} represent cost parameters of conventional unit i, c_i^R represents the unit cost for up/down reserve capacity of conventional unit i, L_b represents the load at bus b, and C_ℓ represents the capacity of transmission line ℓ. For each renewable unit $i \in \mathcal{I}_\text{R}$, f_i and w_i represent the forecasted power output and the forecast error, respectively. For each conventional unit $i \in \mathcal{I} \setminus \mathcal{I}_\text{R}$, g_i and r_i represent the planned generation amount and the adjustment amount, respectively, and d_i represents the portion of total generation-load mismatch to be offset by this unit (see, e.g., Vrakopoulou et al., 2013; Bienstock et al., 2014). Constraint (35b) describes the power balance requirement for generation and loads (we assume that the loads are deterministic), constraints (35c) describe the proportional distribution of mismatches, constraint (35d) requires that all proportions sum up to be 1, constraints (35e) limit the adjustment amount by the reserve capacities r^U and r^D, constraints (35f) bound the generation amount by the generation capacity, and constraints (35g) describe the transmission capacity limits based on the dc approximation where D_ℓ^b maps power injections to power flows (see, e.g., Bergen and Vittal (1999) and Gómez-Expósito et al. (2008)).

Our case study uses the IEEE 30-bus system (Zimmerman et al., 2011). We increase all electricity loads by 50% and add two wind farms at buses 5 and 22. The forecasted power output from each wind farm is 30MW. The transmission line between buses 1 and 2 has a capacity of 30MW, while all other line flows are unconstrained. Other cost and capacity coefficients are reported in Table 1. We assume random forecast errors and describe the uncertainty by an uncorrelated random vector $w := [w_1, w_2]^\top$ with mean μ_w and covariance matrix $\Gamma_\text{w} = \text{diag}(9, 9)$. Additionally, we assume that w is α-unimodal about $[0, 0]^\top$. To handle random violations of constraints (35e)–(35g), we replace them by ACC (6) and AVC (7), and term the resultant RCED model (C-ED) and (V-ED), respectively. For example, in (C-ED), we replace constraints (35e)



(a) Optimal Value vs. ϕ (b) Optimal Value vs. α

Figure 3: Optimal values of (O-ED), (C-ED), and (V-ED) with various ϕ and α

by $\inf_{\mathbb{P}_{\mathrm{w}}\in\mathcal{D}_{\mathrm{w}}}\{d_i\sum_{i\in\mathcal{I}_{\mathrm{R}}} w_i \le r_i^{\mathrm{D}}\} \ge 1-\alpha$ and $\inf_{\mathbb{P}_{\mathrm{w}}\in\mathcal{D}_{\mathrm{w}}}\{-d_i\sum_{i\in\mathcal{I}_{\mathrm{R}}} w_i \le r_i^{\mathrm{U}}\} \ge 1-\alpha$, where $\mathcal{D}_{\mathrm{w}} = \{\mathbb{P}_{\mathrm{w}} \in \mathcal{M}_2 : \mathbb{E}_{\mathbb{P}_{\mathrm{w}}}[w] = \mu_{\mathrm{w}}, \mathbb{E}_{\mathbb{P}_{\mathrm{w}}}[ww^\top] = \mu_{\mathrm{w}}\mu_{\mathrm{w}}^\top + \mathrm{diag}(9,9), \mathbb{P}_{\mathrm{w}} \text{ is } \alpha\text{-unimodal about } 0\}$. In contrast, in (V-ED), we replace constraints (35e) by $\sup_{\mathbb{P}_{\mathrm{w}}\in\mathcal{D}_{\mathrm{w}}} \mathrm{CVaR}^{\epsilon}_{\mathbb{P}_{\mathrm{w}}}\left(d_i\sum_{i\in\mathcal{I}_{\mathrm{R}}} w_i\right) \le r_i^{\mathrm{D}}$ and $\sup_{\mathbb{P}_{\mathrm{w}}\in\mathcal{D}_{\mathrm{w}}} \mathrm{CVaR}^{\epsilon}_{\mathbb{P}_{\mathrm{w}}}\left(-d_i\sum_{i\in\mathcal{I}_{\mathrm{R}}} w_i\right) \le r_i^{\mathrm{U}}$. Lastly, when the requirement of α-unimodality is relaxed from $\mathcal{D}_{\mathrm{w}}$, (C-ED) and (V-ED) become equivalent and we term this model (O-ED).

Conventional Unit	Bus Index	c_{i1} (\$/MW)	c_{i2} (\$/MW^2)	c_i^{R} (\$/MW)	g_i^{MIN} (MW)	g_i^{MAX} (MW)
1	1	20	0.04	200	0	360
2	2	40	0.25	400	0	140
3	5	40	0.01	400	0	100
4	8	40	0.01	400	0	100
5	11	40	0.01	400	0	100
6	13	40	0.01	400	0	100

Table 1: Coefficients of the Case Study

By using (O-ED) as a benchmark, we test (C-ED) and (V-ED) under various selections of μ_{w} and α values. First, we fix $\alpha = 1$ and let $\mu_{\mathrm{w}} = \phi[1,1]^\top$ with $\phi \in \{-3,-2,\ldots,3\}$. We report the optimal objective values of the three models in Fig. 3a. From this figure, we observe that the optimal value of (O-ED) is consistently larger than that of (V-ED), which is consistently larger than that of (C-ED). This demonstrates that incorporating α-unimodality makes the RCED model less conservative and hence decreases the cost of economic dispatch. Meanwhile, unlike in (O-ED), ACC (6) and AVC (7) are not equivalent when α-unimodality is incorporated in the ambiguity set. Furthermore, we observe that the discrepancy between (O-ED) and (C-ED)/(V-ED) amplifies as ϕ deviates from 0. This indicates that α-unimodality plays a more important role in $\mathcal{D}_{\mathrm{w}}$ as the difference between μ_{w} and the mode increases.

Second, we fix $\mu_{\rm w} = [0, 0]^\top$ and let α increase from 1 to 10. We report the optimal objective values of the three models in Fig. 3b. From this figure, we observe that the discrepancy between (O-ED) and (C-ED)/(V-ED) shrinks as α grows. This is as expected because the requirement of α-unimodality weakens as α grows. Although not shown in this figure, the convergence of (V-ED) to (O-ED) takes place when $\alpha \geq 40$, while the convergence of (C-ED) takes place when $\alpha \geq 10^4$. The slow convergence indicates that unimodality information can significantly influence the structure of $\mathcal{D}_{\rm w}$ and the worst-case probability distribution.

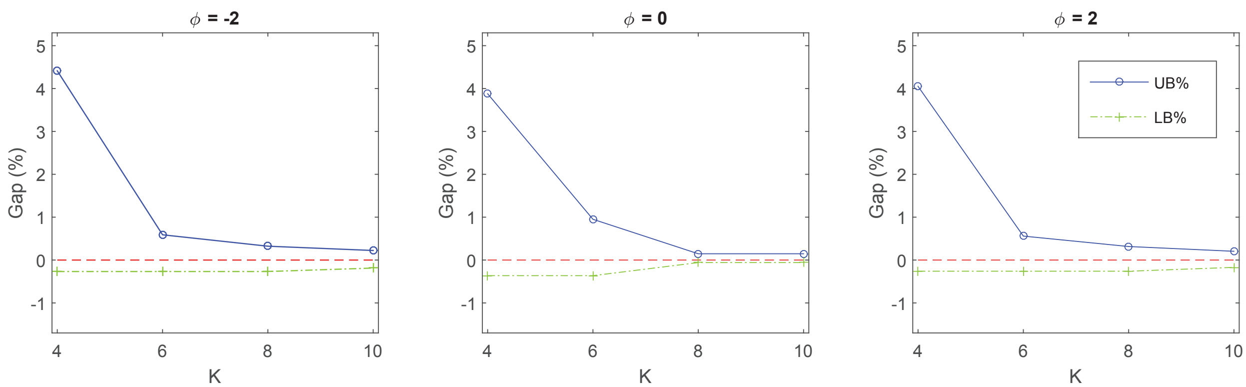


Figure 4: Gaps between the Optimal Objective Value and the Relaxed and Conservative Approximations of (C-ED)

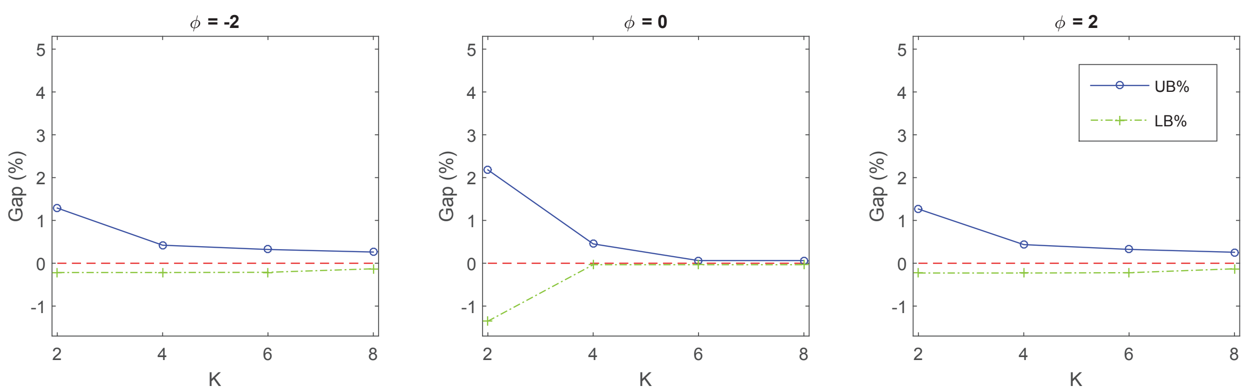


Figure 5: Gaps between the Optimal Objective Value and the Relaxed and Conservative Approximations of (V-ED)

Third, we let $\alpha = 1$, $\mu_{\rm w} = \phi[1, 1]^\top$ with $\phi \in \{-2, 0, 2\}$, and evaluate the tightness of the approximations of ACC and AVC derived in Propositions 4–5 and Proposition 10, respectively. In Fig. 4, we report the gap between the optimal objective value $v^*_{\text{(C-ED)}}$ of (C-ED) and the upper bound $v_{\rm UB}$ obtained from the conservative approximation, and the gap between $v^*_{\text{(C-ED)}}$ and the lower bound $v_{\rm LB}$ obtained from the relaxed approximation, for $K \in \{4, 6, 8, 10\}$. The gaps are obtained by computing $\text{UB\%} = (v_{\rm UB} - v^*_{\text{(C-ED)}})/v^*_{\text{(C-ED)}} \times 100\%$ and $\text{LB\%} = (v^*_{\text{(C-ED)}} - v_{\rm LB})/v^*_{\text{(C-ED)}} \times 100\%$. Similarly, in Fig. 5, we report the gap between the optimal objective value $v^*_{\text{(V-ED)}}$ of (V-ED) and those of its K-piece approximations with $K \in \{2, 4, 6, 8\}$. From Fig. 4–5, we observe that the gaps quickly shrink as K increases and the approximations become near-optimal (e.g., UB%+LB% $< 1\%$) when $K \geq 8$.

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Appendix A

For random variable Z and constant $\beta \in \mathbb{R}$, we make the following observation on the worst-case expectation $\sup_{\mathbb{P}_Z \in \mathcal{D}(\mu_0, \Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[Z - \beta]_+$. Note that this observation can be made following the derivations in Scarf (1958), and we present a proof below for completeness.

Observation 2 *Given $\beta \in \mathbb{R}$, we have*

$$\sup_{\mathbb{P}_Z \in \mathcal{D}(\mu_0, \Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[Z - \beta]_+ \;=\; \frac{1}{2}\left[\sqrt{(\beta - \mu_0)^2 + (\Sigma_0 - \mu_0^2)} - \beta + \mu_0\right].$$

Proof: We represent $\sup_{\mathbb{P}_Z \in \mathcal{D}(\mu_0, \Sigma_0)} \mathbb{E}_{\mathbb{P}_Z}[Z - \beta]_+$ as the following optimization problem

$$\begin{aligned} v_P \;=\; \max_{\mathbb{P}_Z} \;\; & \mathbb{E}_{\mathbb{P}_Z}[Z - \beta]_+ \\ \text{(P)} \qquad \text{s.t.} \;\; & \mathbb{E}_{\mathbb{P}_Z}[Z] = \mu_0, \\ & \mathbb{E}_{\mathbb{P}_Z}[Z^2] = \Sigma_0, \\ & \mathbb{E}_{\mathbb{P}_Z}[1] = 1, \end{aligned}$$

whose dual is

$$\begin{aligned} v_D \;=\; \min_{q,p,r} \;\; & \mu_0 p + \Sigma_0 q + r \\ \text{(D)} \qquad \text{s.t.} \;\; & qz^2 + pz + r \geq [z - \beta]_+, \quad \forall z \in \mathbb{R}. \end{aligned}$$

The weak duality between (P) and (D), i.e., $v_D \leq v_P$, holds because $\mu_0 p + \Sigma_0 q + r = \mathbb{E}_{\mathbb{P}_Z}[qZ^2 + pZ + r] \leq \mathbb{E}_{\mathbb{P}_Z}[Z - \beta]_+$ for any feasible solution (q, p, r) to (D) and feasible solution $\mathbb{P}_Z$ to (P). Now we prove the strong duality by constructing two feasible solutions to (P) and (D), respectively, that have the same objective value. On the one hand, the primal solution $\hat{\mathbb{P}}_Z$ is supported on two points z_1 and z_2 with probability masses p_1 and p_2, respectively, where $\Delta = \sqrt{(\beta - \mu_0)^2 + (\Sigma_0 - \mu_0^2)}$ and

$$p_1 = \frac{\beta - \mu_0 + \Delta}{2\Delta}, \; p_2 = \frac{\mu_0 - \beta + \Delta}{2\Delta}, \; z_1 = \beta - \Delta, \text{ and } z_2 = \beta + \Delta.$$

We have $p_1, p_2 \geq 0$ because $\Delta \geq |\beta - \mu_0|$. Meanwhile, we have

$$p_1 z_1 + p_2 z_2 \;=\; \frac{(\beta - \mu_0 + \Delta)(\beta - \Delta)}{2\Delta} + \frac{(\mu_0 - \beta + \Delta)(\beta + \Delta)}{2\Delta} \;=\; \mu_0,$$

and

$$\begin{aligned} p_1 z_1^2 + p_2 z_2^2 &= \frac{(\beta - \mu_0 + \Delta)(\beta - \Delta)^2}{2\Delta} + \frac{(\mu_0 - \beta + \Delta)(\beta + \Delta)^2}{2\Delta} \\ &= \frac{(\beta - \mu_0)\left[(\beta - \Delta)^2 - (\beta + \Delta)^2\right] + \Delta\left[(\beta - \Delta)^2 + (\beta + \Delta)^2\right]}{2\Delta} \\ &= -\beta^2 + 2\mu_0\beta + \Delta^2 \;=\; -\beta^2 + 2\mu_0\beta + (\beta - \mu_0)^2 + (\Sigma_0 - \mu_0^2) \;=\; \Sigma_0. \end{aligned}$$

Hence, $\hat{\mathbb{P}}_Z$ is feasible to (P). On the other hand, the dual solution $(\hat{q}, \hat{p}, \hat{r})$ is such that

$$\hat{q} = \frac{1}{4\Delta},\ \hat{p} = \frac{\Delta - \beta}{2\Delta},\ \text{and}\ \hat{r} = \frac{(\Delta - \beta)^2}{4\Delta}.$$

Hence, $\hat{q}z^2 + \hat{p}z + \hat{r} = \frac{1}{4\Delta}(z + \Delta - \beta)^2$. It follows that $\hat{q}z^2 + \hat{p}z + \hat{r} \geq 0$ for all $z \in \mathbb{R}$. Meanwhile, $(\hat{q}z^2 + \hat{p}z + \hat{r}) - (z - \beta) = \frac{1}{4\Delta}(z - \beta - \Delta)^2 \geq 0$, i.e., $\hat{q}z^2 + \hat{p}z + \hat{r} \geq z - \beta$. Thus, $\hat{q}z^2 + \hat{p}z + \hat{r} \geq [z - \beta]_+$ and so $(\hat{q}, \hat{p}, \hat{r})$ is feasible to (D).

Finally, the primal objective value associated with $\hat{\mathbb{P}}_Z$ is $p_2(z_2 - \beta) = \frac{(\mu_0 - \beta + \Delta)\Delta}{2\Delta} = \frac{1}{2}(\Delta - \beta + \mu_0)$. Meanwhile, the dual objective value associated with $(\hat{q}, \hat{p}, \hat{r})$ is

$$\begin{aligned} \mu_0\left(\frac{\Delta - \beta}{2\Delta}\right) + \Sigma_0\left(\frac{1}{4\Delta}\right) + \frac{(\Delta - \beta)^2}{4\Delta} &= \frac{\Delta^2 + (\beta^2 - 2\mu_0\beta + \mu_0^2) + (\Sigma_0 - \mu_0^2) + 2\mu_0\Delta - 2\Delta\beta}{4\Delta} \\ &= \frac{2\Delta^2 + 2\mu_0\Delta - 2\Delta\beta}{4\Delta} \;=\; \frac{1}{2}(\Delta - \beta + \mu_0), \end{aligned}$$

which coincides with the primal objective value associated with $\hat{\mathbb{P}}_Z$. ∎

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