

Closed-form solutions for worst-case law invariant risk measures with application to robust portfolio optimization *

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Abstract

Worst-case risk measures refer to the calculation of the largest value for risk measures when only partial information of the underlying distribution is available. For the popular risk measures such as Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR), it is now known that their worst-case counterparts can be evaluated in closed form when only the first two moments are known for the underlying distribution. We show in this paper that somewhat surprisingly similar closed-form solutions also exist for the general class of law invariant coherent risk measures, which consists of spectral risk measures as special cases that are arguably the most important extensions of CVaR. We characterize the worst-case distributions that offer great intuition related to one's choice of risk spectrum. As applications of the closed-form results, new formulas are derived for tight bounds on higher order risk measures, as well as robust portfolio optimization models.

1 Introduction

Measuring how risky a random loss is often requires the knowledge of its probability distribution. The industry standard measure of risk, Value-at-Risk (VaR), for example, reports the risk level of a random loss by calculating an extremal quantile of its distribution. Another measure of risk, Conditional Value-at-Risk (CVaR), which has emerged as the most popular alternative to replace VaR as industry standard, calculates the average loss exceeding an extremal quantile to indicate the riskiness of a random loss. The problem however of implementing both of these measures and any other distribution-based risk measure is that in most practices the exact form of distribution is often lacking and only sample data is available for estimating the distribution, which is inevitably prone to sampling error.

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This issue has motivated the development of worst-case risk measures where the goal is to determine the worst-possible risk level over a set of candidate distributions that captures the uncertainty of distribution. Worst-Case Value-at-Risk (WCVaR) was first studied by El Ghaoui et al. (2003) [12], who considered a set of candidate distributions described by the first two moments and showed how the worst-possible VaR value can be calculated for the set. One of the most notable results of El Ghaoui et al. (2003) [12] is perhaps the closed-form solution for WCVaR. The closed-form expression remarkably resembles the risk measure of weighted mean-standard deviation, and hence provides useful insight into how WCVaR can be minimized. It turns out that a closed-form expression also exists for Worst-Case Conditional Value-at-Risk (WCCVaR) when the set of candidate distributions is described by the first two moments (see Chen et al. (2011)[8], Natarajan et al. (2010)[15]), and the expression is identical to the one for WCVaR. Alternative formulations of worst-case risk measures can also be found in the literature of distributionally robust optimization (DRO) (see for example [7, 9, 15, 23, 24, 25]). Most of these works focus on deriving tractable convex or conic programs for computing the worst-case values (and finding the corresponding robust solutions).

Our work is motivated by the insight gained from the closed-form solutions of the WCVaR and WCCVaR. Given the elegance of the closed form, it is natural to wonder if the closed-form result is just a consequence of the relatively simple structure of VaR and CVaR, or it can be found also for alternative risk measures with more sophisticated structure. On the top of the list of more sophisticated risk measures is the class of spectral risk measures that plays an essential role in both theory and practice. They were first introduced by Acerbi (2002) [1] who attempted to generalize CVaR (and VaR) so that a more realistic description of risk-aversion can be made over a spectrum of CVaRs (VaRs). Later, it became clear that this class of measures is equivalent to the class of distortion risk measures that have applications in insurance [16, 17]. It is also known that spectral risk measures satisfy most, if not all, desirable properties (see Section 2) that have been postulated by the modern risk theory ([1, 4, 11, 13]). A more surprising finding however is that any risk measure that satisfies these properties, also known as law invariant coherent risk measures, can be represented through spectral risk measures (see [13, 21]). We study in this paper both the case of Worst-Case Spectral Risk Measure (WCSRm) and Worst-Case Law-Invariant Coherent Risk Measure (WCLICRM). Our finding is that despite their seeming complexity, both can be boiled down to a closed-form expression when only the first two moments are known for the underlying distribution. This also provides the opportunity to identify the worst-case distributions, to derive new tight bounds, and to formulate new robust optimization models.

This article is organized as follows. In Section 2, we provide the results related to the closed-form solutions for WCSRm and WCLICRM. We then comment in Section 3 how the results can be applied in the context of robust portfolio optimization.

2 Analytical Results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and Z denote a random variable with its distribution F_Z , i.e. $Z : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathfrak{R}$ and $F_Z(t) := \mathbb{P}(Z \leq t)$. The space of random variables is contained in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. We begin by recalling the following definition of spectral risk measure.

Definition 1. (Spectral risk measure [1]) Given a random variable Z , let F_Z^{-1} denote its general inverse cdf function, i.e. $F_Z^{-1}(\alpha) := \inf\{q \mid F_Z(q) \geq \alpha\}$. The function

$$\rho_\phi(Z) := \int_0^1 \phi(\alpha) F_Z^{-1}(\alpha) d\alpha$$

is called a spectral risk measure parameterized by ϕ , if $\phi \in \mathcal{A}$, where

$$\mathcal{A} := \left\{ \phi : [0, 1) \rightarrow \mathfrak{R}_{\geq 0} \mid \int_0^1 \phi(\alpha) d\alpha = 1, \phi: \text{right-continuous, monotonically nondecreasing} \right\}.$$

The density function $\phi \in \mathcal{A}$ is also called an ‘‘admissible’’ risk spectrum.

To understand why spectral risk measures play an central role in the modern theory of risk measures [4, 11, 13], we shall review the following definition about law invariant coherent risk measures, which satisfy most properties postulated by the theory about what a reasonable risk measure should satisfy.

Definition. (Law invariant coherent risk measures) Any risk measure $\rho : L^2(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathfrak{R}$ that satisfies

- 1) Monotonicity: $\rho(Z_1) \leq \rho(Z_2)$ for any $Z_1 \leq Z_2$ almost surely;
- 2) Convexity : $\rho((1 - \lambda)Z_1 + \lambda Z_2) \leq (1 - \lambda)\rho(Z_1) + \lambda\rho(Z_2)$, $0 \leq \lambda \leq 1$;
- 3) Translation invariance : $\rho(Z + c) = \rho(Z) + c$, $c \in \mathfrak{R}$;
- 4) Positive homogeneity : $\rho(\lambda Z) = \lambda\rho(Z)$, $\lambda \geq 0$;
- 5) Law invariance : $\rho(Z_1) = \rho(Z_2)$ if $F_{Z_1} \equiv F_{Z_2}$.

is said to be a law invariant coherent risk measure.

Interestingly, despite its generality, there is a close link between this general class of risk measures and spectral risk measures, namely that the former can always be represented through the latter via a supremum representation.

Theorem 1. (Kusuoka representation (see [13, 21]) Any law invariant coherent risk measure $\rho : L^2(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathfrak{R}$ has the following representation

$$\rho(Z) = \sup_{\phi \in \Phi} \rho_\phi(Z),$$

where $\Phi \subseteq L^1[0, 1)$ denotes a set of admissible spectrums.

While the industry-standard risk measure, Value-at-Risk (VaR) is known to violate the convexity condition, Conditional Value-at-Risk (CVaR) proposed to replace VaR is the most notable example of the above class of risk measures.

Definition 2. Given a tail probability $\epsilon \in (0, 1)$ of a distribution F_Z , $(1 - \epsilon)$ -Value-at-Risk (VaR) is defined by the calculation of $\min\{q \mid F_Z(q) \geq 1 - \epsilon\}$, whereas $(1 - \epsilon)$ -Conditional Value-at-Risk (CVaR) in its original form is defined by $\min_t\{t + \frac{1}{\epsilon}\mathbb{E}[(Z - t)^+]\}$.

It is known that $(1 - \epsilon)$ -CVaR admits also the form of a spectral risk measure with a spectrum ϕ taking the form $\phi(\alpha) := \frac{1}{\epsilon}\mathbf{1}_{[1-\epsilon, 1]}(\alpha)$. One of the key results about VaR and CVaR that motivates the development of this paper is that their worst-case counterparts can be evaluated in closed-form. Recall the following definition of a worst-case risk measure. For simplicity, from here on the integral $\int_{-\infty}^{\infty}$ may be written as \int only.

Definition 3. For any law invariant risk measure ρ , given its definition, we can interchange $\rho(Z)$ and $\rho(F_Z)$ to indicate that it is essentially a function of distribution functions. Given a pair of mean and standard deviation (μ, σ) , let

$$\Theta(\mu, \sigma) := \{F_Z : (-\infty, \infty) \rightarrow \mathfrak{R}_{\geq 0} \mid \int dF_Z = 1, \int z dF_Z = \mu, \int z^2 dF_Z = \mu^2 + \sigma^2\},$$

i.e. the set of distributions that match the specified moments. The worst-case counterpart of a risk measure ρ admits the following optimization representation

$$\rho^\uparrow(\mu, \sigma) := \sup_{F_Z \in \Theta(\mu, \sigma)} \rho(F_Z).$$

Proposition 1. *The worst-case $(1 - \epsilon)$ -VaR (WCVaR) (Theorem 1 in [12]) and the worst-case $(1 - \epsilon)$ -CVaR (WCCVaR) (Theorem 2.9 in [8], Proposition 3.3 in [15]) can both be calculated by the closed-form*

$$\rho_\epsilon^\uparrow(\mu, \sigma) = \mu + \sigma \sqrt{\frac{1 - \epsilon}{\epsilon}}.$$

The goal of this paper is to study worst-case Law Invariant Coherent Risk Measures (WCLICRM) in its full generality, and in light of Theorem 1, we may write it as

$$\rho_\Phi^\uparrow(\mu, \sigma) := \sup_{F_Z \in \Theta(\mu, \sigma)} \{\sup_{\phi \in \Phi} \rho_\phi(F_Z)\}.$$

As a special case of WCLICRM, we define also the worst-case Spectral Risk Measures (WCSRMs) when a single spectrum ϕ is considered

$$\rho_\phi^\uparrow(\mu, \sigma) := \sup_{F_Z \in \Theta(\mu, \sigma)} \rho_\phi(F_Z). \tag{1}$$

We present the main result in two steps. We first show in the following theorem that WCSRMs can be reduced to a closed-form of a weighted sum of mean and standard deviation. Thereafter, the result of WCLICRM will follow fairly straightforwardly. Our proof requires the following duality theorem for moment problems.

Lemma 1. (Duality Theorem for moment problems [22]) Consider the following set of functions $\mathcal{D} := \{F : (-\infty, \infty) \rightarrow \mathfrak{R}_{\geq 0} \mid \mathbb{E}_F[Z^r] = m_r, r = 0, \dots, 2k\}$, where $m_0 = 1$ and (m_1, \dots, m_{2k}) represents the first $2k$ -moments. The following inequality always holds, i.e. weak duality

$$\sup_{F \in \mathcal{D}} E_F[\phi] \leq \inf_{\lambda_0, \dots, \lambda_{2k}} \left\{ \sum_{r=0}^{2k} \lambda_r m_r : \sum_{r=0}^{2k} \lambda_r z^r \geq \phi(z), \forall z \right\},$$

and the equality holds, i.e. strong duality, when (m_0, \dots, m_{2k}) is an interior point of the set of $\{(\mathbb{E}_{F_Z}[Z^0], \dots, \mathbb{E}_{F_Z}[Z^{2k}]) \mid F_Z : (-\infty, \infty) \rightarrow \mathfrak{R}_{\geq 0}\}$.

Theorem 2. Any worst-case spectral risk measure (WCSR) can be evaluated in closed-form:

$$\rho_\phi^\uparrow(\mu, \sigma) = \mu + \sigma \kappa, \quad \kappa := \sqrt{\|\phi\|_2^2 - 1} \quad (2)$$

and is infinite if $\phi \notin L^2[0, 1]$. In the case of $(1-\epsilon)$ -WCCVaR, we have $\|\phi\|_2^2 = \frac{1}{\epsilon}$. Moreover, the corresponding worst case distribution F is characterized by

$$F^{-1}(\beta) = \left(\mu - \frac{\sigma}{\kappa}\right) + \frac{\sigma}{\kappa} \phi(\beta^-), \quad \beta \in (0, 1),$$

where $\phi(\beta^-) := \lim_{\alpha \rightarrow \beta^-} \phi(\alpha)$ and $F^{-1}(\beta) = \mu$ if $\kappa = 0$ (i.e. $\|\phi\|_2^2 = 1$). In the case of $(1-\epsilon)$ -WCCVaR, we have

$$F^{-1}(\beta) = \begin{cases} \mu - \sigma \sqrt{\frac{\epsilon}{1-\epsilon}} & , 0 < \beta \leq 1 - \epsilon \\ \mu + \sigma \sqrt{\frac{1-\epsilon}{\epsilon}} & , 1 - \epsilon < \beta < 1 \end{cases}.$$

Proof. We begin by first applying the result of Proposition 3.2 in [2] which states that $\rho_\phi(F_Z)$ can be equivalently formulated as

$$\rho_\phi(F_Z) = \min_{\psi} \int [\phi(0)z + g(z; \psi)] dF_Z,$$

where $\psi : (0, 1) \rightarrow \mathfrak{R}$ and $g(z; \psi) := \int_0^1 [(1-\alpha)\psi(\alpha) + (z - \psi(\alpha))^+] d\phi(\alpha)$, and that the optimal solution ψ^* would satisfy $\psi^*(\alpha) \in [F_Z^{-1}(\alpha), F_Z^{-1}(\alpha)^+]^1$ over $\alpha \in \text{supp}(\phi) \cap (0, 1)$, i.e. the support of the measure defined by ϕ , and can take arbitrary values otherwise. Due to the latter, we can assume without loss of generality that $\psi \in L^2(0, 1)$ (since $F_Z^{-1} \in L^2(0, 1)$) and impose the constraint $\psi \in \Psi^\nearrow$, where Ψ^\nearrow denotes the set of non-decreasing functions over $(0, 1)$. This will facilitate the rest of the proof.

In what follows, we first prove that the closed-form (2) is an upper bound of (1), and then we characterize the worst-case distribution that actually attains the bound. Applying max-min inequality, we have

$$(1) = \sup_{F_Z \in \Theta(\mu, \sigma)} \min_{\psi \in \Psi^\nearrow} \int [\phi(0)z + g(z; \psi)] dF_Z \leq \min_{\psi \in \Psi^\nearrow} \sup_{F_Z \in \Theta(\mu, \sigma)} \int [\phi(0)z + g(z; \psi)] dF_Z.$$

¹ $F_Z^{-1}(\alpha)^+ := \inf\{q \mid F_Z(q) > \alpha\}$

Applying the duality theorem in Lemma 1 to the above right-hand-side, we have the following minimization problem which in turn bounds from above the original problem (1)

$$\min_{\psi \in \Psi^{\nearrow}} \inf_{\lambda_0, \lambda_1, \lambda_2} \lambda_0 + \mu \lambda_1 + (\mu^2 + \sigma^2) \lambda_2 \quad (3)$$

$$\text{subject to } \lambda_0 + z \lambda_1 + z^2 \lambda_2 \geq \phi(0)z + g(z; \psi), \quad \forall z. \quad (4)$$

To further reduce the above problem, we claim that given any fixed $\psi \in \Psi^{\nearrow}$, the function $g(z; \psi)$ can be equivalently written as the following function

$$h(z; \psi) := \sup_{\beta \in (0,1)} \int_0^1 [(1 - \alpha)\psi(\alpha) + \mathbf{1}_{(0,\beta]}(\alpha)(z - \psi(\alpha))]d\phi(\alpha),$$

where $\mathbf{1}_{(0,\beta]}(\alpha) = 1$ for $\alpha \in (0, \beta]$ and 0 otherwise.

Consider the set $I(z) := \{\alpha \in (0, 1) \mid \psi(\alpha) \leq z\}$, which is an interval that either takes the form $(0, \gamma)$ or $(0, \gamma]$, $\gamma \in (0, 1)$ due to the monotonicity of ψ . We can rewrite $g(z; \psi)$ into

$$g(z; \psi) = \int_0^1 [(1 - \alpha)\psi(\alpha) + \mathbf{1}_{I(z)}(\alpha)(z - \psi(\alpha))]d\phi(\alpha).$$

We show only that $h(z; \psi) \leq g(z; \psi)$, since the other direction is immediate by definition. Let us suppose the opposite that $h(z; \psi) > g(z; \psi)$, which implies that $\exists \beta' \in (0, 1)$ such that

$$\begin{aligned} & \int_0^1 [(1 - \alpha)\psi(\alpha) + \mathbf{1}_{(0,\beta']}(\alpha)(z - \psi(\alpha))]d\phi(\alpha) > \int_0^1 [(1 - \alpha)\psi(\alpha) + \mathbf{1}_{I(z)}(\alpha)(z - \psi(\alpha))]d\phi(\alpha) \\ \Rightarrow & \int_{(0,\beta']} (z - \psi(\alpha))d\phi(\alpha) > \int_{I(z)} (z - \psi(\alpha))d\phi(\alpha), \end{aligned}$$

which is a contradiction if $\beta' \in I(z)$, since $z - \psi(\alpha) \geq 0$ over $I(z)$ and $(0, \beta'] \subseteq I(z)$, and is also a contradiction if $\beta' \notin I(z)$, since it would imply $\exists \beta' \in (0, 1) \setminus I(z)$ such that $z - \psi(\beta') > 0$, which contradicts the definition of $I(z)$.

Hence, we can replace $g(z; \psi)$ by $h(z; \psi)$ in the constraint (4), and with some additional rearrangement of terms we arrive at

$$(\lambda_0 - \int_0^1 [(1 - \alpha) - \mathbf{1}_{(0,\beta]}(\alpha)]\psi(\alpha)d\phi(\alpha)) + (\lambda_1 - \phi(\beta))z + \lambda_2 z^2 \geq 0, \quad \forall z, \forall \beta \in (0, 1). \quad (5)$$

For any fixed β , we seek the minimum value of the left-hand-side of the above inequality over $z \in \mathfrak{R}$, which is an elementary minimization problem of a univariate quadratic function. It is bounded below if and only if $\lambda_2 \geq 0$. By replacing the left-hand-side by the formula known for the optimal value of a quadratic function, we have the following equivalent formulation:

$$(\lambda_0 - \varphi(\beta)) - \frac{(\lambda_1 - \phi(\beta))^2}{4\lambda_2} \geq 0, \quad \forall \beta \in (0, 1),$$

where $\lambda_2 \geq 0$ and $\varphi(\beta) := \int_0^1 [(1 - \alpha) - \mathbf{1}_{(0,\beta]}(\alpha)]\psi(\alpha)d\phi(\alpha)$.

The optimization problem (3) with the above reformulated constraints can be further reformulated into

$$\begin{aligned}
& \inf_{\psi \in \Psi^{\nearrow}, \lambda_1, \lambda_2 \geq 0} \sup_{\beta \in (0,1)} \left\{ \frac{(\lambda_1 - \phi(\beta))^2}{4\lambda_2} + \varphi(\beta) + \mu\lambda_1 + (\mu^2 + \sigma^2)\lambda_2 \right\} \\
\Rightarrow & \inf_{\psi \in \Psi^{\nearrow}, \lambda_1, \lambda_2 \geq 0} \sup_{\beta \in (0,1)} \left\{ \frac{\lambda_1^2}{4\lambda_2} - \frac{\lambda_1\phi(\beta)}{2\lambda_2} + \frac{\phi(\beta)^2}{4\lambda_2} + \varphi(\beta) + \mu\lambda_1 + (\mu^2 + \sigma^2)\lambda_2 \right\} \\
\Rightarrow & \inf_{\psi \in \Psi^{\nearrow}, q, r \geq 0} \sup_{\beta \in (0,1)} \left\{ \phi(\beta)^2 r + \phi(\beta)q + \varphi(\beta) \right\} + \frac{q^2}{4r} + \mu\left(\frac{-q}{2r}\right) + (\mu^2 + \sigma^2)\frac{1}{4r} \\
\Rightarrow & \inf_{\psi \in \Psi^{\nearrow}, q, r \geq 0} \sup_{\beta \in (0,1)} \left\{ \phi(\beta)^2 r + \phi(\beta)q + \varphi(\beta) \right\} + \frac{(q - \mu)^2 + \sigma^2}{4r}
\end{aligned}$$

where $r = \frac{1}{4\lambda_2}$ and $q = \frac{-\lambda_1}{2\lambda_2}$ is applied in the third line.

By introducing dummy variables $s, t \in \mathfrak{R}$, we have the following equivalent formulation

$$\inf_{\psi \in \Psi^{\nearrow}, q, r, s, t} s + \int_0^1 (1 - \alpha)\psi(\alpha)d\phi(\alpha) + t \tag{6}$$

$$\int_0^1 \mathbf{1}_{(0,\beta]}(\alpha)\psi(\alpha)d\phi(\alpha) - \phi(\beta)^2 r - \phi(\beta)q + s \geq 0, \forall \beta \in (0, 1) \tag{7}$$

$$4rt \geq (q - \mu)^2 + \sigma^2 \tag{8}$$

$$r \geq 0, \tag{9}$$

where the second constraint (8) can be recast as a second order cone constraint

$$((q - \mu, \sigma, r - t), r + t) \in \mathcal{Q}_4, \tag{10}$$

where $\mathcal{Q}_4 := \{(u, t) \mid u \in \mathfrak{R}^3, t \in \mathfrak{R}, \|u\|_2 \leq t\}$, which is self-dual (see, e.g. [3]).

To solve the above problem, we relax first the constraint $\psi \in \Psi^{\nearrow}$ and will verify later that there exists an optimal solution for the relaxed problem that satisfies the constraint. We now proceed by deriving the dual of the relaxed problem. Assuming that $\phi \in L^2[0, 1)$ we define the dual variable for (7) by $f \in L^\infty(0, 1)^2$. In addition, let $y \in \mathfrak{R}^4$ denote the dual variables corresponding to the second order cone constraint (10). We can write the Lagrange

²Since (7) is in $L^1(0, 1)$ given that $\psi \in L^2(0, 1)$ and $\phi \in L^2[0, 1)$.

function as follows, where $x := (\psi, q, r, s, t)$,

$$\begin{aligned}
L(x, f, y) &= \int_0^1 (1 - \alpha)\psi(\alpha)d\phi(\alpha) + s + t - \int_0^1 \left[\int_0^\beta \psi(\alpha)d\phi(\alpha) - \phi(\beta)^2 r - \phi(\beta)q + s \right] f(\beta)d\beta \\
&\quad - y_0(q - \mu) - y_1(\sigma) - y_2(r - t) - y_3(r + t) \\
&= \int_0^1 (1 - \alpha)\psi(\alpha)d\phi(\alpha) - \int_0^1 \left[\int_\alpha^1 f(\beta)d\beta \right] \psi(\alpha)d\phi(\alpha) \\
&\quad + s(1 - \int_0^1 f(\beta)d\beta) + t(1 + y_2 - y_3) + r(\int_0^1 \phi(\beta)^2 f(\beta)d\beta - y_2 - y_3) \\
&\quad + q(\int_0^1 \phi(\beta)f(\beta)d\beta - y_0) + y_0\mu - y_1\sigma,
\end{aligned}$$

where in the second line of equality the second term is obtained by interchanging the order of integration. The dual problem $\sup_{f,y} \inf_x L(x, f, y)$ reduces to the following problem

$$\sup_{f,y} \mu y_0 - \sigma y_1 \tag{11}$$

$$\text{subject to } (1 - \alpha) - \int_\alpha^1 f(\beta)d\beta = 0, \forall \alpha \in \text{supp}(\phi) \cap (0, 1) \tag{12}$$

$$\int_0^1 f(\beta)d\beta = 1 \tag{13}$$

$$f(\beta) \in \mathfrak{R}_{\geq 0}, \forall \beta \in (0, 1) \tag{14}$$

$$y_2 - y_3 = -1 \tag{15}$$

$$y_2 + y_3 \leq \int_0^1 \phi(\beta)^2 f(\beta)d\beta \tag{16}$$

$$y_0 = \int_0^1 \phi(\beta)f(\beta)d\beta \tag{17}$$

$$\sqrt{y_0^2 + y_1^2 + y_2^2} \leq y_3. \tag{18}$$

Given that weak duality always holds, we can derive that strong duality holds if there exists a pair of solutions satisfying the following conditions

$$\text{(primal feasible)}(7) \sim (9)$$

$$\text{(dual feasible)}(12) \sim (18), \text{ and,}$$

$$\text{(complementary condition)} y_0(q - \mu) + y_1(\sigma) + y_2(r - t) + y_3(r + t) = 0,$$

$$\begin{aligned}
&\int_0^1 \left[\int_0^\beta \psi(\alpha)d\phi(\alpha) - \phi(\beta)^2 r - \phi(\beta)q + s \right] f(\beta)d\beta = 0, \tag{19} \\
&\int_0^1 \phi^2(\beta)f(\beta)d\beta - y_2 - y_3 = 0,
\end{aligned}$$

where the complementary condition is obtained by requiring that the solution has to be a saddle point of the Lagrange function.

It turns out that with some careful observation, we can actually identify the following pair of primal and dual solutions that satisfy the above conditions

$$\begin{aligned} s^* &= r^* \phi^2(0) + q^* \phi(0), \\ \psi^*(\alpha) &= 2r^*(\phi(\alpha^-) + \frac{1}{2}h(\alpha)) + q^*, \quad h(\alpha) = \phi(\alpha) - \phi(\alpha^-), \\ q^* &= \mu + \sigma \sqrt{\int_0^1 \phi^2(\beta) d\beta - 1} - \sigma \int_0^1 \phi^2(\beta) d\beta \sqrt{(\int_0^1 \phi^2(\beta) d\beta - 1)^{-1}}, \end{aligned} \quad (20)$$

$$r^* = \frac{\sigma}{2} \sqrt{(\int_0^1 \phi^2(\beta) d\beta - 1)^{-1}}, \quad (21)$$

$$t^* = \frac{\sigma}{2} \int_0^1 \phi^2(\beta) d\beta \sqrt{(\int_0^1 \phi^2(\beta) d\beta - 1)^{-1}},$$

$$f^*(\beta) = 1, \quad \beta \in (0, 1),$$

$$(y_0^*, y_1^*, y_2^*, y_3^*) = \left(1, -\sqrt{\int_0^1 \phi^2(\beta) d\beta - 1}, \frac{\int_0^1 \phi^2(\beta) d\beta - 1}{2}, \frac{\int_0^1 \phi^2(\beta) d\beta + 1}{2} \right),$$

where $\int_0^1 \phi^2(\beta) d\beta > 1$, and the case $\int_0^1 \phi^2(\beta) d\beta = 1$ is trivial.

Once the solution can be identified, verifying its feasibility can be simply done by direct substitution. To verify the second complementary condition (19), given that $f^*(\beta) > 0$, $\forall \beta \in (0, 1)$, it can be reduced to (see e.g. [20])

$$\int_0^\beta \psi(\alpha) d\phi(\alpha) - \phi^2(\beta)r - \phi(\beta)q + s = 0, \quad \forall \beta \in (0, 1) \quad (22)$$

which can be satisfied by the above s^*, r^*, q^* and ψ^* . Given that other substitutions are fairly straightforward but tedious, we omit their details for brevity by noting only that one will find all constraints except $r \geq 0$ are binding constraints with the substitutions.

As the result, given that ϕ is non-decreasing, we have $\psi^*(\alpha) = r^*(\phi(\alpha) + \phi(\alpha^-)) + q^*$ (with $r^* \geq 0$) is non-decreasing. This also implies that the relaxed problem of (7) is tight since there exists a solution in (7) that attains the optimal value of the relaxed problem. Substituting y_0^* and y_1^* into the objective function (11), we have the closed form result.

Now, we prove that the worst-case distribution F can be fully characterized by

$$F^{-1}(\alpha) = 2\phi(\alpha^-)r^* + q^*.$$

First, we reduce the objective function (6) by substituting into it the above identified solution, which gives us (after re-arranging the terms)

$$s^* + t^* + \int_0^1 (1 - \alpha)\psi^*(\alpha) d\phi(\alpha) = 2r^* \int_0^1 \phi^2(\alpha) d\alpha + q^*. \quad (23)$$

We can then verify that the worst-case distribution indeed reaches this bound since

$$\int_0^1 F^{-1}(\alpha)\phi(\alpha)d\alpha = \int_0^1 (2\phi(\alpha^-)r^* + q^*)\phi(\alpha)d\alpha = 2r^* \int_0^1 \phi^2(\alpha)d\alpha + q^*.$$

Moreover, we can also verify that based on F

$$\begin{aligned}\mathbb{E}[Z] &= \int_0^1 F^{-1}(\alpha)d\alpha = 2r^* + q^* = \mu, \\ \mathbb{E}[Z^2] &= \int_0^1 (F^{-1}(\alpha))^2d\alpha = 4r^{*2} \int_0^1 \phi^2(\beta)d\beta + 4r^*q^* + q^{*2} = \mu^2 + \sigma^2.\end{aligned}$$

By substituting the closed form q^* and r^* into F^{-1} we have the final closed form result. The result of $(1 - \epsilon)$ -WCCVaR follows immediately with $\phi(\alpha^-) := \frac{1}{\epsilon}\mathbf{1}_{(1-\epsilon,1)}(\alpha)$ and $\|\phi\|_2^2 = \frac{1}{\epsilon}$.

Finally, in the case of $\phi \notin L^2[0, 1)$, we may construct a sequence of $F_i^{-1}(\alpha) := 2\phi^i(\alpha^-)r_i^* + q_i^*$ where $\phi^i(\alpha) := \frac{1}{p_i} \min\{\phi(\alpha), u_i\} \in L^2[0, 1]$ with $p_i = \int_0^1 \min\{\phi(\alpha), u_i\}d\alpha$ and r_i^* and q_i^* are (20) and (21) with ϕ replaced by ϕ^i . We thus have

$$\int F_i^{-1}(\alpha)\phi(\alpha)d\alpha \geq p_i \int F_i^{-1}(\alpha)\phi^i(\alpha)d\alpha = p_i(\mu + \sigma\sqrt{\|\phi^i\|_2^2 - 1}) \rightarrow \infty$$

as $u_i \rightarrow \infty$.³ □

There are two reasons why the worst-case distributions identified above should be compelling. Firstly, they can take any form of distribution bounded from below (since ϕ is so), which is less restrictive (and more realistic) than the existing ones in the literature (see, e.g. [19]). Secondly, they are intuitive, since they essentially share the same characteristics with ϕ except with different location and scale parameter.

Example 1. (WCSRМ with a power risk spectrum) To demonstrate the practical value of the above closed-form formula, we consider here a situation where an investor who uses a power risk spectrum $\phi(p) = kp^{k-1}$, $k \geq 1^4$ (see [6] for its property) is uncertain about if the future asset price would follow a geometric brownian motion (GBM) or a Merton's jump diffusion process (see Lo (1987) [14] for a similar example on option pricing). Their continuous compound returns $r_i(t)$ are respectively defined by, i.e. $S_t = S_0e^{r_i(t)}$ where S_t denotes the price at time t ,

$$\text{(Geometric brownian motion)} \quad r_1(t) = \mu_1 t + \sigma_1 \sqrt{t}Z,$$

$$\text{(Merton's jump diffusion)} \quad r_2(t) = \mu_2 t + \sigma_2 \sqrt{t}Z_b + \sum_{i=0}^{N(t)} Z_i,$$

³To be precise, since $\lim_{i \rightarrow \infty} \|\phi^i\|_2^2 = \lim_{i \rightarrow \infty} \frac{1}{p_i^2} \int \min\{\phi(\alpha), u_i\}^2 d\alpha \geq \lim_{i \rightarrow \infty} \int \min\{\phi(\alpha), u_i\}^2 d\alpha = \int \phi^2 d\alpha = \infty$ (monotone convergence theorem), we have for any $i > i'$, where i' is such that $\mu + \sigma\sqrt{\|\phi^{i'}\|_2^2 - 1} \geq 0$, $p_i(\mu + \sigma\sqrt{\|\phi^i\|_2^2 - 1}) \geq p_{i'}(\mu + \sigma\sqrt{\|\phi^{i'}\|_2^2 - 1}) \rightarrow \infty$ as $i \rightarrow \infty$.

⁴ $\|\phi\|_2^2 = k^2/(2k - 1)$.

where $Z \sim \mathcal{N}(0, 1)$ and $Z_b \sim \mathcal{N}(0, 1)$, $Z_i \sim \mathcal{N}(\beta, \delta^2)$, $N(t) \sim \text{Poisson}(\lambda t)$. When only the mean and variance ($\hat{\mu}, \hat{\sigma}^2$) can be estimated, any of the above processes that matches ($\hat{\mu}, \hat{\sigma}^2$), i.e. $\mu_1 = \mu_2 + \beta\lambda = \hat{\mu}$ and $\sigma_1^2 = \sigma_2^2 + (\beta^2 + \delta^2)\lambda = \hat{\sigma}^2$ may be realized. Needless to say, it is infeasible to enumerate all the processes and perform risk measurement. In Figure 1(a), we provide 1) the upper bounds based on the closed-form formula, 2) the largest risk estimates based on a grid search over the parameter space of jump diffusion processes, and 3) the risk estimate based on a GBM. Different degrees of risk aversion (the larger the k , the more averse), and different time horizons (t) are considered, where the latter can also be used to infer the sensitivity to volatility levels (since volatility increases in time). In Figure 1(b), we further present the corresponding worst-case distributions (in the case $t = 1$) and also the physical distributions of $r_i(t)$ of GBM and the corresponding jump diffusion process.

Although our search over jump diffusion processes cannot be exhaustive, one can already see the significant gaps between the estimates based on the jump diffusion processes and the GBMs. On the other hand, the closed-form bounds are convincingly close to the largest estimates we found, which strongly supports its practical usefulness, i.e. a far more efficient method to estimate the potential (model) risk. One can also see in Figure 1(b) the concave shape of the worst-case cdfs due to the convexity of the power risk spectrum, and as k increases both the left-end support and the weights are shifted to the right.

We are now ready to present the result for the general case of worst-case law invariant coherent risk measures, which can be straightforwardly obtained from Theorem 1, the result of Theorem 2, and the fact that $\sigma\sqrt{\cdot}$ is an increasing function.

Theorem 3. *Given its Kusuoka representation $\rho = \sup_{\phi \in \Phi} \rho_\phi$, any worst-case law invariant coherent risk measure can be evaluated in closed-form*

$$\rho_\Phi^\uparrow(\mu, \sigma) = \mu + \sigma \sqrt{\sup_{\phi \in \Phi} \|\phi\|_2^2 - 1}.$$

Based on the above theorem, we can prove for the first time the tight bounds for law invariant coherent risk measures that have been covered in the recent literature of law invariant risk measures (see e.g. [18]).

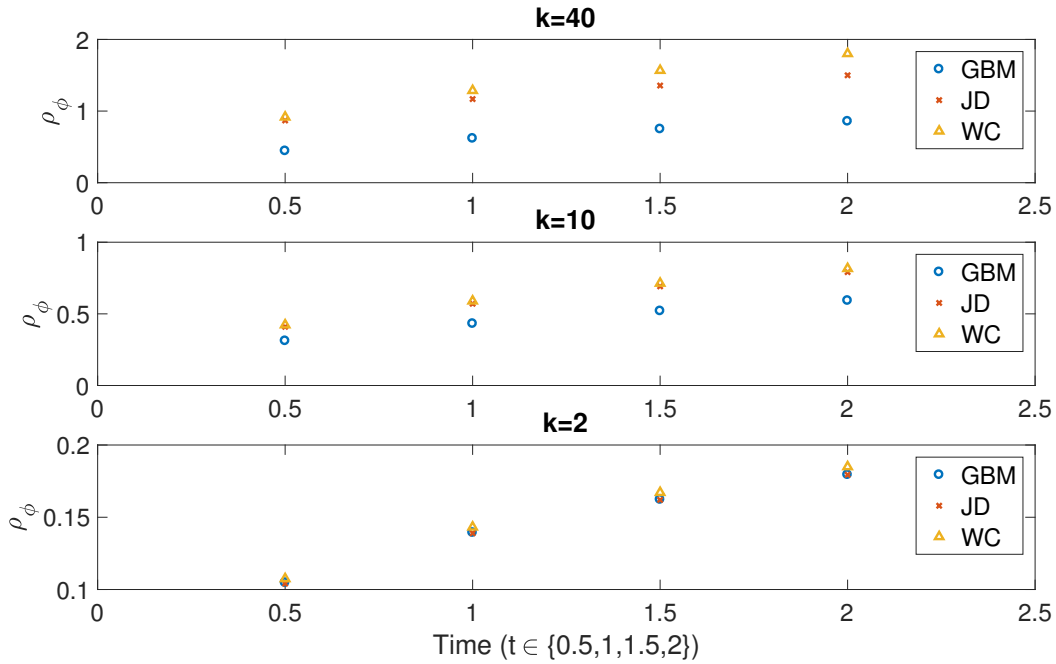
Corollary 1. *(Higher order risk measures) The worst-case counterpart of the higher-order risk measures, i.e.*

$$\rho(Z) := \inf_{t \in \mathbb{R}} \{t + c \cdot \|(Z - t)^+\|_p\},$$

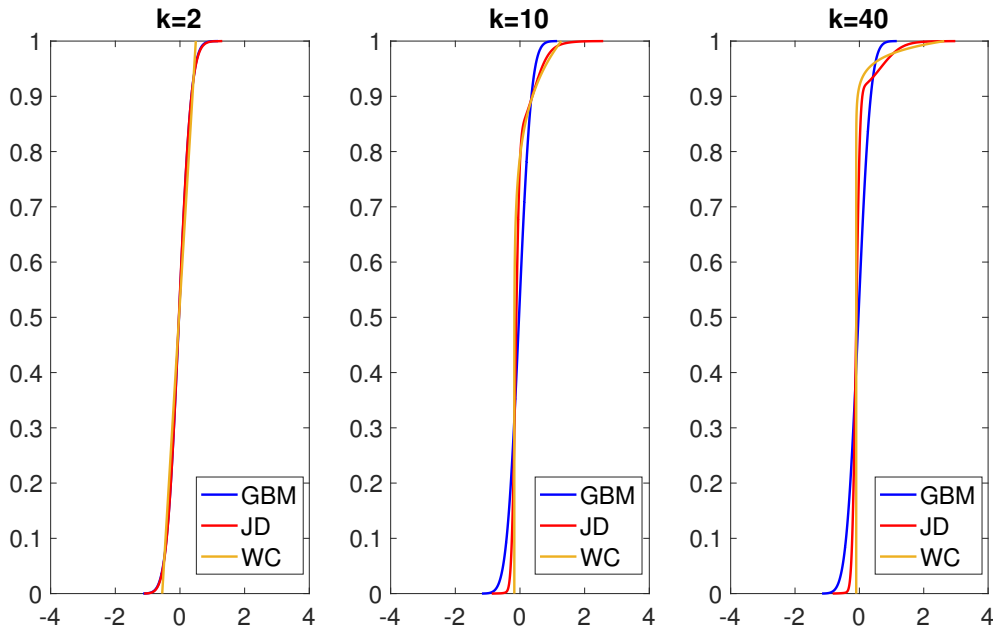
where $c \geq 1$ and $p \geq 1$, is finite when $1 \leq p \leq 2$ and can be computed by $\mu + \sigma\sqrt{c^p - 1}$. When $p = 1$, it reduces to the worst-case CVaR.

Proof. Based on the Kusuoka representation of the risk measures (see e.g. [10, 18]), we have

$$\Phi := \{\phi \mid \phi \in \mathcal{A}, \|\phi\|_q \leq c\},$$



(a) Risk estimates of ρ_ϕ based on Geometric Brownian Motion (GBM), Merton's jump diffusion (JD), and the worst-case bound (WC) for different time horizons t and different risk aversion k .



(b) The cdfs of the worst case distributions for $k \in \{2, 10, 40\}$ and physical return distributions of Geometric brownian motion (GBM) and Merton's jump diffusion (JD) at time $t = 1$.

Figure 1: Comparison of risk estimates and worst-case distributions

where $\frac{1}{p} + \frac{1}{q} = 1$. We can derive the upper bound of $\|\phi\|_2$ by applying Holder's interpolation inequality, i.e. $\|\phi\|_{p_\theta} \leq \|\phi\|_{p_1}^\theta \cdot \|\phi\|_{p_0}^{1-\theta}$, for $\theta \in [0, 1]$ and $\frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_0}$. Namely, by setting $p_\theta = 2$, $p_1 = q$, $p_0 = 1$ and $\theta = \frac{p}{2} \leq 1$, we have

$$\|\phi\|_2 \leq \|\phi\|_q^{\frac{p}{2}} \cdot \|\phi\|_1^{1-\frac{p}{2}} = c^{\frac{p}{2}}, \quad p \leq 2. \quad (24)$$

Moreover, the bound is in fact tight since we can always construct a stepwise function

$$\hat{\phi}(p) = \alpha \cdot \mathbf{1}_{[1-\frac{1}{\alpha}, 1]},$$

where $\alpha = c^p$, that attains the bound, i.e. $\|\hat{\phi}\|_2 = \alpha^{\frac{1}{2}} = c^{\frac{p}{2}}$. It is straightforward to verify that $\hat{\phi} \in \mathcal{A}$ and $\|\hat{\phi}\|_q = c$. Hence, we have $\sup_{\phi \in \Phi} \|\phi\|_2^2 = c^p$.

The case $p > 2$ is unbounded, since for any fixed $q^* < 2$ we can always construct $\bar{\phi}(\alpha) := (\frac{c}{f(q^*)})^{\frac{1}{\sqrt{1-\alpha}}}$, where $f(q) = \|\frac{1}{\sqrt{1-\alpha}}\|_q = (\frac{2}{2-q})^{\frac{1}{q}}$, that gives $\|\bar{\phi}\|_{q^*} = (\frac{c}{f(q^*)})^{\frac{1}{\sqrt{1-\alpha}}} \| \frac{1}{\sqrt{1-\alpha}} \|_{q^*} = (\frac{c}{f(q^*)}) f(q^*) = c$, but $\|\bar{\phi}\|_2 = \frac{c}{f(q^*)} f(2) = \infty$. \square

Corollary 2. (*Higher order semideviation*) *The worst-case counterpart of higher-order semideviation, i.e.*

$$\rho(Z) := \mathbb{E}[Z] + \lambda \|(Z - \mathbb{E}[Z])^+\|_p,$$

where $p \geq 1$ and $0 \leq \lambda \leq 1$, is finite when $1 \leq p \leq 2$ and can be computed by $\mu + \sigma \lambda (\frac{p^{1/2}}{2^{1/p}}) (\frac{(2-p)^{1/p}}{(2-p)^{1/2}})$ when $1 \leq p < 2$, and $\mu + \sigma \lambda$ when $p = 2$.

Proof. We can identify the following set Φ based on the discussion of the Kusuoka representation given in [18]

$$\Phi := \{\phi \mid \phi = (1 - \frac{\lambda}{\|\sigma\|_q}) + \frac{\lambda}{\|\sigma\|_q} \sigma, \sigma \in \mathcal{A}\}.$$

Observe now that for any fixed $\sigma \in \mathcal{A}$ (and thus fixed ϕ), we can always construct a stepwise function

$$\sigma^e = \|\sigma\|_q^p \cdot \mathbf{1}_{[1-\frac{1}{\|\sigma\|_q^p}, 1]}$$

that satisfies $\|\sigma^e\|_q = \|\sigma\|_q$, $\sigma^e \in \mathcal{A}$, and $\int_0^1 \sigma^e(t)^2 dt = \|\sigma\|_q^p \geq \int_0^1 \sigma(t)^2 dt$ for $p \leq 2$, where the last inequality is due to Holder's interpolation inequality (see (24) for clarity). Given these, we have the following for $p \leq 2$

$$\begin{aligned} \|\phi\|_2^2 &= \int_0^1 ((1 - \frac{\lambda}{\|\sigma\|_q}) + \frac{\lambda}{\|\sigma\|_q} \sigma(t))^2 dt \\ &= (1 - \frac{\lambda}{\|\sigma\|_q})^2 + 2(1 - \frac{\lambda}{\|\sigma\|_q}) (\frac{\lambda}{\|\sigma\|_q}) + (\frac{\lambda}{\|\sigma\|_q})^2 \int_0^1 \sigma(t)^2 dt \\ &\leq (1 - \frac{\lambda}{\|\sigma^e\|_q})^2 + 2(1 - \frac{\lambda}{\|\sigma^e\|_q}) (\frac{\lambda}{\|\sigma^e\|_q}) + (\frac{\lambda}{\|\sigma^e\|_q})^2 \int_0^1 \sigma^e(t)^2 dt \\ &= 1 + \lambda^2 (\frac{c^p - 1}{c^2}), \end{aligned} \quad (25)$$

where $c := \|\sigma^e\|_q \geq 1$. It thus suffices to maximize $f(c) := \frac{c^p-1}{c^2}$ over $c \geq 1$.

For $p = 2$, the function $f(c)$ is strictly increasing in c and $f(c) \rightarrow 1$ as $c \rightarrow \infty$ and hence $\sup_{\phi \in \Phi} \|\phi\|_2^2 = 1 + \lambda^2$. For $1 \leq p < 2$, setting the first-order derivative of $f(c)$ equal to zero, we have $c^* = (\frac{2}{2-p})^{\frac{1}{p}}$, which is the only solution over $c \geq 1$. Given that $f(c)$ is increasing over $[1, c^*]$, we conclude c^* maximizes the function. Substituting c^* into (25) and simplifying the resulting expression, we obtain $\sup_{\phi} \|\phi\|_2^2 = 1 + \lambda^2 (\frac{p}{2-p}) (\frac{2}{2-p})^{-\frac{2}{p}}$. Finally, the case $p > 2$ is unbounded since there exists σ^e such that $\|\phi\|_2^2 \rightarrow \infty$ (as evidenced by (25) where $f(c)$ is unbounded for $p > 2$). \square

We wrap up this section by presenting a natural generalization of WCSRM when higher order moments are available, which requires the use of semi-definite programming (SDP).

Proposition 2. *The worst-case spectral risk measures with higher order moments, i.e. $\sup_{F_Z \in \mathcal{D}} \rho_{\phi}(F_Z)$ with \mathcal{D} as defined in Lemma 1, can be formulated as the following Continuous Semidefinite Programming (CSDP) problem if the interior point condition in Lemma 1 holds*

$$\begin{aligned} \min_{\lambda_0, \dots, \lambda_{2k}, \psi \in \Psi^{\nearrow}} \quad & \sum_{r=0}^{2k} \lambda_r m_r + \int_0^1 (1-\alpha) \psi(\alpha) d\phi(\alpha) \\ \text{subject to} \quad & (\lambda_0 + \int_0^{\beta} \psi(\alpha) d\phi(\alpha), \lambda_1 - \phi(\beta), \lambda_2, \dots, \lambda_r) \in \mathcal{M}_k, \forall \beta \in (0, 1), \end{aligned} \quad (26)$$

where $\mathcal{M}_k := \{(y_0, \dots, y_r) \mid y_r = \sum_{i,j:i+j=r} x_{ij}, r = 0, \dots, 2k, [x_{ij}]_{i,j=0,\dots,k} \succeq 0\}$ and $\succeq 0$ refers to the left-hand-side matrix being positive semidefinite.

In the case that ϕ is a stepwise function⁵, i.e. $\phi := \sum_{i=1}^n \tilde{\phi}_i \mathbf{1}_{[\alpha_{i-1}, \alpha_i]}$ where $0 = \alpha_0 < \dots < \alpha_n = 1$ and $0 < \tilde{\phi}_i < \tilde{\phi}_{i+1}$, (26) reduces to a finite dimensional semi-definite program, i.e.

$$\begin{aligned} \min_{\lambda_0, \dots, \lambda_{2k}, \xi: \xi_0 \leq \dots \leq \xi_{n-1}} \quad & \sum_{r=0}^{2k} \lambda_r m_r + \sum_{i=1}^n (1 - \alpha_{i-1}) \xi_{i-1} (\tilde{\phi}_i - \tilde{\phi}_{i-1}) \\ \text{subject to} \quad & (\lambda_0 + \sum_{i=1}^j \xi_{i-1} (\tilde{\phi}_i - \tilde{\phi}_{i-1}), \lambda_1 - \tilde{\phi}_j, \lambda_2, \dots, \lambda_r) \in \mathcal{M}_k, j = 1, \dots, n. \end{aligned}$$

Proof. The formulation (26) follows (5) in the proof of Theorem 2, where the nonnegativity constraint of a polynomial function (now is with any even order) can be reformulated as a SDP constraint (using 21.a. on p. 157 in [5]). In the case that ϕ is stepwise, the constraint (5) becomes

$$(\lambda_0 - \sum_{i=1}^n (1 - \alpha_{i-1}) \psi(\alpha_{i-1}) (\tilde{\phi}_i - \tilde{\phi}_{i-1})) + \lambda_1 z + \lambda_2 z^2 \geq \tilde{\phi}_j z - \sum_{i=1}^j \psi(\alpha_{i-1}) (\tilde{\phi}_i - \tilde{\phi}_{i-1}), \forall z,$$

⁵Note that $p = \int_0^1 \phi(\alpha) d\alpha$ does not have to be 1 here, in which case we can always first normalize it and multiply later on the final result with p .

for any $\beta \in [\alpha_{j-1}, \alpha_j)$, $j = 1, \dots, n$, where $\tilde{\phi}_0 := 0$. This implies that it suffices to consider finitely many constraints over $\beta \in \{\alpha_{j-1}\}$, $j = 1, \dots, n$ and finitely many variables $\xi_{i-1} := \psi(\alpha_{i-1})$, $i = 1, \dots, n$. \square

Although in general the CSDP formulation might be difficult to solve, in the case of a bounded spectrum $\phi \in L^\infty[0, 1]$ we can always find two stepwise functions ϕ_N^- and ϕ_N^+ such that $\phi_N^- \leq \phi \leq \phi_N^+$ and $\phi_N^-, \phi_N^+ \rightarrow \phi$ as $N \rightarrow \infty$, i.e. taking $\alpha_i - \alpha_{i-1} \rightarrow 0$. It is not difficult to verify that their worst-case counterparts follow the same ordering relations. Hence, this provides a computationally tractable means to estimate the (upper and lower bound of) WCSRM in the case of higher moments.

3 Application to robust portfolio optimization

We comment in this last section on how the closed-form result can be applied in the context of robust portfolio optimization. It is known that by applying the general projection property in Popescu (2007) [19], one can always reduce a robust portfolio optimization problem by the following equivalency relation

$$\min_{x \in \mathcal{X}} \sup_{F_R \sim (\mu_R, \Sigma_R)} \rho(-R^\top x) \Leftrightarrow \min_{x \in \mathcal{X}} \sup_{F_Z \in \Theta(-\mu_R^\top x, \sqrt{x^\top \Sigma_R x})} \rho(Z),$$

where in the left-hand-side a portfolio $R^\top x$ is minimized over a polyhedral set \mathcal{X} for its worst-case risk over all multivariate distributions of random returns R having mean μ_R and covariance Σ_R . As a result, we can immediately apply the closed-form formula to the right-hand-side problem, which leads to the following second order cone program over x

$$\min_{x \in \mathcal{X}} -\mu_R^\top x + \sqrt{x^\top \Sigma_R x} \sqrt{\sup_{\phi \in \Phi} \|\phi\|_2^2 - 1}.$$

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