# Error bounds for nonlinear semidefinite optimization

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**Abstract**

In this paper, error bounds for nonlinear semidefinite optimization problem is considered. We assume the second order sufficient condition, the strict complementarity condition and the MFCQ condition at the KKT point. The nondegeneracy condition is not assumed in this paper. Therefore the Jacobian operator of the equality part of the KKT conditions is not assumed to be invertible. We derive lower bounds for the primal and dual distances to the solution set when the primal variable is close to the solution set. Then a global error bound of the dual distance to the solution set is obtained assuming the MFCQ condition and the strict complementarity condition. An error bound for the primal variable is given when the primal-dual pair is close to the solution set, and approximately satisfies the shifted complementarity condition along with the MFCQ condition and the second order sufficient condition. Finally we gather these results and obtain the upper and lower local error bounds for the primal-dual pair.

**Keywords** Nonlinear semidefinite optimization  Error bound  Eigenvalues

## 1 Introduction

In this paper, we consider the error bounds for the following nonlinear semidefinite optimization problem:

$$\begin{array}{lll} \text{minimize} & f(x), & x \in \mathbf{R}^n, \\ \text{subject to} & X(x) \succeq 0 & \end{array} \tag{1}$$

where the functions $f : \mathbf{R}^n \to \mathbf{R}$ and $X : \mathbf{R}^n \to \mathbf{S}^p$ are sufficiently smooth, and $\mathbf{S}^p$ denotes the set of $p$th-order real symmetric matrices. We also define $\mathbf{S}^p_+$ to denote the set of $p$th-order symmetric positive semidefinite matrices. By $X(x) \succeq 0$ and $X(x) \succ 0$, we mean that the matrix $X(x)$ is positive semidefinite and positive definite, respectively.

Error bounds in mathematical programming play important roles in the sensitivity analysis of optimal solutions and the convergence analysis of iterative methods. Since the seminal paper by Hoffman [5] in 1952 which gives a global error bound of the distance to the polyhedral convex set, extensive researches have been done. For example, his result

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is extended to the global error bound for the set defined by the infinite number of linear inequalities by Hu and Wang [6]. Various developments around extension to set defined by the convex system are summarized in Lewis and Pang [11], and Pang [12].

On the other hand, for the ordinary nonlinear optimization problems, the local error bound of the primal-dual distance to the solution set is obtained by Wright [18], and Hager and Gowda [4]. They derive perturbed problems, and show the current primal-dual pair satisfy the perturbed KKT conditions. Then it is possible to apply the classical works by Robinson [14, 15] to obtain local error bounds.

There are several researches on the error bounds of the distance to the semidefinite constraint set ([3],[1],[7]). However as far as the author aware, there are few researches on the error bound of the distance to the solution set itself. An exception is the work by Bonnans, Cominetti and Shapiro [2] which deals with the sensitivity analysis of nonlinear SDPs.

In this paper, the second order sufficient condition, the MFCQ condition and the strict complementarity condition are assumed for the proof of some error bounds. We will not assume the nondegeneracy condition in this paper. If we assume the nondegeneracy condition along with these conditions, it can be proved that the Jacobian operator of the equalities of the KKT conditions is left invertible [19]. Thus in this case, the estimate of error bounds becomes a rather trivial task.

Assuming the MFCQ condition and the strict complementarity condition, we will show that it is possible to prove the global error bound of the dual distance to the solution set by using the result of Lewis and Pang [11] on the error bound for the set defined by the convex system. Also we will show that, as in the ordinary NLPs described above, it is possible to derive the perturbed KKT conditions by using the eigenvalues of the relevant matrices. Then using the result by Hager and Gowda [4], we obtain the local error bound for the primal variables when the MFCQ condition and the second order sufficient condition are satisfied at the solution, and the primal-dual pair approximately satisfies the shifted complementarity condition. In the final section we summarize our results by gathering these bounds to obtain the lower and upper bounds for the primal-dual pair.

Throughout this paper, we assume that the functions $f$ and $X$ are twice continuously differentiable. We define the Lagrangian function of problem (1) by

$$L(x, Z) = f(x) - \langle X(x), Z \rangle \,,$$

where $Z \in \mathbf{S}^p$ is the Lagrange multiplier matrix (dual variable matrix) for the positive semidefiniteness constraint, and $\langle X(x), Z \rangle = \mathrm{tr}(X(x)Z)$. Define the matrices

$$A_i(x) = \frac{\partial X(x)}{\partial x_i}$$

for $i = 1, \ldots, n$. Then the Karush-Kuhn-Tucker (KKT) conditions for optimality of problem (1) are given by the following (see [16]):

$$\nabla_x L(x, Z) = 0, \tag{2}$$

$$\langle X(x), Z \rangle = 0, \; X(x) \succeq 0, Z \succeq 0. \tag{3}$$

Here $\nabla_x L(x, Z)$ is the gradient vector of the Lagrangian function given by

$$\nabla_x L(x, Z) = \nabla f(x) - \mathcal{A}^*(x) Z,$$

where $\mathcal{A}^*(x)$ is the operator such that for $Z$,

$$\mathcal{A}^*(x)Z = \nabla_x \langle X(x), Z\rangle = \begin{pmatrix} \langle A_1(x), Z\rangle \\ \vdots \\ \langle A_n(x), Z\rangle \end{pmatrix}.$$

For a given $X \in \mathbf{S}^p$, $\lambda_i(X), i = 1, ..., p$ denote the eigenvalues of $X$ in nonincreasing order. Similarly, for a given $Z \in \mathbf{S}^p$, $\varphi_i(Z), i = 1, ..., p$ denote the eigenvalues of $Z$ in nondecreasing order. From the Von Neumann-Theobald inequality

$$\mathrm{tr}(XZ) \geq \sum_{i=1}^{p} \lambda_i(X)\varphi_i(Z), \tag{4}$$

where the equality holds if and only if $X$ and $Z$ are simultaneously diagonalized, we note that the complementarity condition (3) can be written as

$$X(x)Z = 0,\ X(x) \succeq 0, Z \succeq 0, \tag{5}$$

or

$$\lambda_i(X(x))\varphi_i(Z) = 0,\ \lambda_i(X(x)) \geq 0, \varphi_i(Z) \geq 0,\ i = 1, ..., p. \tag{6}$$

Also note that condition (6) can be written as

$$\min\left(\lambda(X(x)), \varphi(Z)\right) = 0, \tag{7}$$

where $\min\left(\lambda(X(x)), \varphi(Z)\right) = \left(\min\left(\lambda_1(X(x)), \varphi_1(Z)\right), ..., \min\left(\lambda_p(X(x)), \varphi_p(Z)\right)\right)^T \in \mathbf{R}^p$.

It is known that there holds the Hoffman-Wielandt inequality for real symmetric matrices $X, X', Z$ an $Z'$:

$$\|\lambda(X) - \lambda(X')\| \leq \|X - X'\|, \|\varphi(Z) - \varphi(Z')\| \leq \|Z - Z'\|, \tag{8}$$

i.e., $\lambda$ and $\varphi$ are Lipschitzian with the above specified ordering. We use the $\ell_2$ norm for vectors, and the Frobenius norm for matrices in this paper.

In the following discussions, it is convenient to write the above optimality conditions as

$$\begin{pmatrix} \nabla_x L(x, Z) \\ X(x) \end{pmatrix} \in \begin{pmatrix} 0 \\ N(Z) \end{pmatrix}, \tag{9}$$

where $N(Z)$ is the set defined by

$$N(Z) = \begin{cases} \{M \in \mathbf{S}^p \,|M \succeq 0 \text{ and } MZ = 0\} & \text{if } Z \succeq 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Using the eigenvalue vectors $\lambda(X(x)) \in \mathbf{R}^p$ and $\varphi(Z) \in \mathbf{R}^p$, the condition $X(x) \in N(Z)$ implies

$$\lambda(X(x)) \in N'(\varphi(Z))$$

where

$$N'(\varphi(Z)) = \begin{cases} \left\{u \in \mathbf{R}^p \,\middle|u \geq 0 \text{ and } u^T\varphi(Z) = 0\right\} & \text{if } \varphi(Z) \geq 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

A point $x^* \in \mathbf{R}^n$ is said to be a KKT point of problem (1) if there exists a Lagrange multiplier matrix $Z$ such that $(x^*, Z)$ satisfies the KKT conditions (9). Let $\Lambda(x^*)$ denote the set of Lagrange multipliers such that $(x^*, Z)$, where $Z \in \Lambda(x^*)$, satisfies the KKT conditions.

The Mangasarian-Fromovitz constraint qualification (MFCQ) condition holds at a feasible point $x$ if there exists a vector $d \in \mathbf{R}^n$ such that

$$X(x) + \sum_{i=1}^{n} d_i A_i(x) \succ 0. \tag{10}$$

It can be shown that, if the MFCQ condition holds at a KKT point $x^*$, then the set $\Lambda(x^*)$ is bounded.

The set $C(x^*)$, the critical cone at $x^*$, is defined by

$$C(x^*) = \left\{ h \in \mathbf{R}^n \;\middle|\; \sum_{i=1}^{n} h_i A_i(x^*) \in T_{\mathbf{S}_+^p}(X(x^*)),\ \nabla f(x^*)^T h = 0 \right\},$$

where $T_{\mathbf{S}_+^p}(X(x^*)$ is the tangent cone of $\mathbf{S}_+^p$ at $X(x^*)$. Then the second order sufficient condition for local optimality of $x^*$ under the MFCQ condition is given by

$$\sup_{Z \in \Lambda(x^*)} h^T(\nabla_x^2 L(x^*, Z) + \hat{H}(x^*, Z))h > 0, \text{ for all } h \in C(x^*)\backslash\{0\}. \tag{11}$$

Here $\hat{H}(x, Z)$ is a matrix whose $(i, j)$ element is given by

$$(\hat{H}(x, Z))_{ij} = 2\mathrm{tr}(A_i(x)X(x)^\dagger A_j(x)Z)$$

and † denotes the Moore-Penrose generalized inverse.

It is said that a quadratic growth condition holds at a feasible point $x^*$ of problem (1) if there exists $c > 0$ such that the following inequality holds

$$f(x) \geq f(x^*) + c\|x - x^*\|^2 \tag{12}$$

for any feasible point $x$ in a neighborhood of $x^*$. The quadratic growth condition implies that $x^*$ is a strict local optimal solution (an isolated local optimal solution) of problem (1). If the MFCQ condition holds, then the quadratic growth condition holds if and only if the second order sufficient condition (11) is satisfied.

We will use the following block partition of matrices $X(x)$ and $Z$,

$$X(x) = \begin{pmatrix} X_B(x) & X_{BN}(x) \\ X_{BN}^T(x) & X_N(x) \end{pmatrix}, Z = \begin{pmatrix} Z_B & Z_{BN} \\ Z_{BN}^T & Z_N \end{pmatrix}, \tag{13}$$

where $X(x^*)$ is diagonalized to

$$X(x^*) = \begin{pmatrix} X_B^* & 0 \\ 0 & 0 \end{pmatrix}, X_B^* \succ 0, \tag{14}$$

if necessary. If $Z \in \Lambda(x^*)$, the conditions $X(x^*)Z = 0$ and $Z \succeq 0$ yield

$$Z = \begin{pmatrix} 0 & 0 \\ 0 & Z_N \end{pmatrix}, Z_N \succeq 0, \tag{15}$$

in this representation.

The strict complementarity condition holds at $x^*$ if there exists $Z^* \in \Lambda(x^*)$ such that

$$\operatorname{rank}(X(x^*)) + \operatorname{rank}(Z^*) = p.$$

In the above representation (15), the strict complementarity implies $Z_N^* \succ 0$.

Let $x \in \mathbf{R}^n$ and $Z \in \mathbf{S}^p$ be given, and define $\hat{Z} \in \mathbf{S}^p_+$ as a projection of $Z$ onto $\Lambda(x^*)$, i.e.,

$$\|Z - \hat{Z}\| = \inf\{\|Z - Z'\| \mid Z' \in \Lambda(x^*)\}. \tag{16}$$

As described above, we will be concerned with the estimation of the upper and lower bounds of the quantities $\operatorname{dist}(x, x^*) = \|x - x^*\|$ and $\operatorname{dist}(Z, \Lambda(x^*)) = \|Z - \hat{Z}\|$, or $\operatorname{dist}((x, Z), (x^*, \Lambda(x^*))) = \|x - x^*\| + \|Z - \hat{Z}\|$ under various assumptions. Instead of using positive constants $\beta_1, \beta_2, ...$ in the following proofs, we will use a generic positive constant $\beta$ that may have different values in different places.

# 2 Lower bounds

We first consider the lower bounds of $\operatorname{dist}(x, x^*)$, $\operatorname{dist}(Z, \Lambda(x^*))$ and $\operatorname{dist}((x, Z), (x^*, \Lambda(x^*)))$ that are not difficult to estimate.

**Theorem 1** *There exist a neighborhood $\mathcal{N}_x$ of $x^*$ and a constant $\gamma > 0$ with the property that for every $x \in \mathcal{N}_x$ and $Z \in \mathbf{S}^p$, we have*

$$\gamma^{-1} \|\nabla_x L(x, Z)\| \leq (1 + \|Z\|)\operatorname{dist}(x, x^*) + \operatorname{dist}(Z, \Lambda(x^*)),$$

$$\gamma^{-1} \|X(x)Z\| \leq \operatorname{dist}(x, x^*) \|Z\| + \operatorname{dist}(Z, \Lambda(x^*)),$$

*and further for every $x \in \mathcal{N}_x$ and $Z \in \mathbf{S}^p_+$, we have*

$$\gamma^{-1} \|\min(\lambda(X(x)), \varphi(Z))\| \leq \operatorname{dist}((x, Z), (x^*, \Lambda(x^*))).$$

**Proof.** Let $\mathcal{N}_x$ be sufficiently small, and thus $x$ be sufficiently close to $x^*$. Since $\nabla f(x^*) - \mathcal{A}^*(x^*)\hat{Z} = 0$, we have

$$\begin{aligned}
\|\nabla f(x) - \mathcal{A}^*(x)Z\| &= \left\|(\nabla f(x) - \mathcal{A}^*(x)Z) - \left(\nabla f(x^*) - \mathcal{A}^*(x^*)\hat{Z}\right)\right\| \\
&= \|(\nabla f(x) - \mathcal{A}^*(x)Z) - (\nabla f(x^*) - \mathcal{A}^*(x^*)Z) \\
&\quad - \left(\nabla f(x^*) - \mathcal{A}^*(x^*)\hat{Z}\right) + (\nabla f(x^*) - \mathcal{A}^*(x^*)Z)\Big\| \\
&\leq \|(\nabla f(x) - \mathcal{A}^*(x)Z) - (\nabla f(x^*) - \mathcal{A}^*(x^*)Z)\| \\
&\quad + \left\|- \left(\nabla f(x^*) - \mathcal{A}^*(x^*)\hat{Z}\right) + (\nabla f(x^*) - \mathcal{A}^*(x^*)Z)\right\| \\
&\leq \beta((1 + \|Z\|) \|x - x^*\| + \|Z - \hat{Z}\|),
\end{aligned}$$

for sufficiently large $\beta$. Similarly we obtain

$$\|X(x)Z\| = \|(X(x) - X(x^*))Z + X(x^*)(Z - \hat{Z})\| \leq \beta(\|x - x^*\| \, \|Z\| + \|Z - \hat{Z}\|).$$

Next we prove the third lower bound that uses eigenvalues. If $\min(\lambda_i(X(x)), \varphi_i(Z)) = \varphi_i(Z)$ and $\varphi_i(\hat{Z}) = 0$, we have

$$|\min(\lambda_i(X(x)), \varphi_i(Z))| = |\varphi_i(Z)| = \left|\varphi_i(Z) - \varphi_i(\hat{Z})\right| \leq \|Z - \hat{Z}\|.$$

The last inequality is due to the Hoffman-Wielandt theorem (8). If $\varphi_i(\hat{Z}) > 0$, then $\lambda_i(X(x^*)) = 0$, and we have

$$\begin{aligned} |\min(\lambda_i(X(x)), \varphi_i(Z))| &\leq |\lambda_i(X(x))| = |\lambda_i(X(x)) - \lambda_i(X(x^*))| \\ &\leq \|X(x) - X(x^*)\| \leq \beta \, \|x - x^*\| \, . \end{aligned}$$

Similarly if $\min(\lambda_i(X(x)), \varphi_i(Z)) = \lambda_i(X(x))$ and $\lambda_i(X(x^*)) = 0$, we obtain

$$|\min(\lambda_i(X(x)), \varphi_i(Z))| = |\lambda_i(X(x))| \leq \beta \, \|x - x^*\| \, .$$

If $\lambda_i(X(x^*)) > 0$, we have $\lambda_i(X(x)) > 0$ for sufficiently small $\mathcal{N}_x$, and $\varphi_i(\hat{Z}) = 0$. Then we obtain

$$|\min(\lambda_i(X(x)), \varphi_i(Z))| \leq |\varphi_i(Z)| \leq \|Z - \hat{Z}\|.$$

Thus these bounds for each $i = 1, ..., p$ yield

$$\|\min(\lambda(X(x)), \varphi(Z))\| \leq \beta(\|x - x^*\| + \|Z - \hat{Z}\|).$$

This completes the proof. □

This theorem also shows that if the right hand sides of the above inequalities are small, then the quantities on the left hand sides of the above inequalities are also small. This fact will be used for estimating the upper bounds in the following.

# 3 Global error bound for dual variable

In this section we consider an upper bound of the quantity $\|Z - \hat{Z}\|$ when $Z \in \mathbf{S}^p$ is given, where the point $\hat{Z} \in \Lambda(x^*)$ is the projection of $Z$ onto $\Lambda(x^*)$, and defined as follows:

$$\|Z - \hat{Z}\| = \inf \left\{ \|Z - Z'\| \, | Z' \in \mathbf{S}^p_+, \nabla f(x^*) - \mathcal{A}^*(x^*)Z' = 0, X(x^*)Z' = 0 \right\}. \qquad (17)$$

With the representation (14), we have

$$\hat{Z} = \begin{pmatrix} 0 & 0 \\ 0 & \hat{Z}_N \end{pmatrix}, \hat{Z}_N \succeq 0.$$

Since

$$\left\| \begin{pmatrix} Z_B & Z_{BN} \\ Z_{BN}^T & Z_N \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & Z'_N \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} Z_B & Z_{BN} \\ Z_{BN}^T & 0 \end{pmatrix} \right\|^2 + \|Z_N - Z'_N\|^2 ,$$

the definition of the distance in (17) implies

$$\|Z - \hat{Z}\| \leq \left\| \begin{pmatrix} Z_B & Z_{BN} \\ Z_{BN}^T & 0 \end{pmatrix} \right\| + \inf \left\{ \|Z_N - Z'_N\| \,\middle|\, Z'_N \in \mathbf{S}_+^{p'}, \frac{\partial f(x^*)}{\partial x_i} - \langle A_{Ni}(x^*), Z'_N \rangle = 0, i = 1, ..., n \right\} \tag{18}$$

where $p' = p - \text{rank}(X(x^*))$, and

$$A_{Ni}(x) = \frac{\partial X_N(x)}{\partial x_i}, i = 1, ..., n.$$

In order to estimate the distance from $Z_N \in \mathbf{S}^{p'}$ to the convex set defined by

$$\Lambda_N(x^*) = \left\{ Z'_N \in \mathbf{S}^{p'} \,\middle|\, Z'_N \succeq 0, \frac{\partial f(x^*)}{\partial x_i} - \langle A_{Ni}(x^*), Z'_N \rangle = 0, i = 1, ..., n \right\}, \tag{19}$$

in the proof of Lemma 1, we make use of the result of Lewis and Pang [11] which gives the global error bound of the distance to a set defined by a convex inequality system.

**Lemma 1** *Assume that the MFCQ condition and the strict complementarity condition hold at $x^*$. Then there exists a constant $\gamma > 0$ such that, for every $Z_N \in \mathbf{S}^{p'}$,*

$$\text{dist}(Z_N, \Lambda_N(x^*)) \leq \gamma \max \left\{ |\min\{0, \varphi_{\min}(Z_N)\}|, \left| \frac{\partial f(x^*)}{\partial x_i} - \langle A_{Ni}(x^*), Z_N \rangle \right|, i = 1, ..., n \right\}, \tag{20}$$

*where $\varphi_{\min}(Z_N)$ is the smallest eigenvalue of $Z_N$.*

**Proof.** Let us consider $\varphi_{\min}(\cdot)$ as a function of $p'(p'+1)/2$ independent elements of the matrix $Z'_N$. The convex set defined by $Z'_N \succeq 0$ is equivalent to the set defined by $\varphi_{\min}(Z'_N) \geq 0$ where $\varphi_{\min} : \mathbf{R}^{p'(p'+1)} \to \mathbf{R}$. We note that $-\varphi_{\min}$ is a closed proper convex function.

Then it is possible to utilize the result of Corollary 2 of Lewis and Pang [11]. The convex set $\mathcal{C}$ in Corollary 2 corresponds to the set defined by the equality conditions in the definition of $\Lambda_N(x^*)$, and the convex set defined by the inequality $f(x) \leq 0$ in Corollary 2 corresponds to $\varphi_{\min}(Z'_N) \geq 0$. By the strict complementarity condition, there exists a point $Z_N^* \succ 0$ ($\varphi_{\min}(Z_N^*) > 0$) that satisfies

$$\frac{\partial f(x^*)}{\partial x_i} - \langle A_{Ni}(x^*), Z_N^* \rangle = 0, i = 1, ..., n.$$

This shows the interiority condition assumed in Corollary 2 is satisfied. By the MFCQ condition, the set $\Lambda_N(x^*)$ is bounded. This implies the statement in Corollary 2 (d) is valid. Thus the statement of Corollary 2 (a) holds, and we have (20). □

We note that the above error estimate may also be proved using Corollary of Theorem 1 in Hu and Wang [6], if the equalities in (19) are converted to violated inequalities by appropriately changing the sign of each violated equality, and if the set $\varphi_{\min}(Z'_N) \geq 0$ is expressed by an infinite set of linear inequalities.

Since the above error bound is for the submatrix $Z_N$, it is necessary to derive a bound for the matrix $Z$ which does not depend on the particular representation.

**Theorem 2** *Assume that the MFCQ condition and the strict complementarity condition hold at $x^*$. Then there exist a neighborhood $\mathcal{N}_x$ of $x^*$ and a constant $\gamma > 0$ with the property that for every $x \in \mathcal{N}_x$ and $Z \in \mathbf{S}^p$, we have*

$$\mathrm{dist}(Z, \Lambda(x^*)) \leq \gamma \left( \|\nabla_x L(x, Z)\| + \|XZ\| + (1 + \|Z\|)\mathrm{dist}(x, x^*) + |\min\{0, \varphi_{\min}(Z)\}| \right), \tag{21}$$

*where $\varphi_{\min}(Z)$ is the smallest eigenvalue of $Z$.*

**Proof.** We need to estimate the value of each term in the right hand side of (18). Using the representations (13) and (14), write $X(x)Z$ as

$$X(x)Z = \begin{pmatrix} X_B(x)Z_B + X_{BN}(x)Z_{BN}^T & X_B Z_{BN} + X_{BN}(x)Z_N \\ X_{BN}^T(x)Z_B + X_N(x)Z_{BN}^T & X_{BN}^T(x)Z_{BN} + X_N(x)Z_N \end{pmatrix}.$$

Since $X_{BN}(x^*) = 0$, we have

$$\|X_{BN}(x)\| = \|X_{BN}(x) - X_{BN}(x^*)\| \leq \beta \|x - x^*\|. \tag{22}$$

Then from (22), we obtain

$$\begin{aligned} \|X(x)Z\| &\geq \|X_B(x)Z_{BN} + X_{BN}(x)Z_N\| \geq \|X_B(x)Z_{BN}\| - \|X_{BN}(x)Z_N\| \\ &\geq \left\|X_B(x)^{-1}\right\| \|Z_{BN}\| - \|X_{BN}(x)\| \|Z_N\| \\ &\geq \left\|X_B(x)^{-1}\right\| \|Z_{BN}\| - \beta \|Z_N\| \|x - x^*\|. \end{aligned} \tag{23}$$

Since $X_B(x^*) \succ 0$ and $x$ is sufficiently close to $x^*$, we have

$$\beta^{-1} \leq \left\|X_B(x)^{-1}\right\| \leq \beta. \tag{24}$$

Inequalities (23) and (24) imply

$$\|Z_{BN}\| \leq \beta(\|X(x)Z\| + \|Z_N\| \|x - x^*\|). \tag{25}$$

Similarly, we have from (22) and (25),

$$\begin{aligned} \|X(x)Z\| &\geq \left\|X_B(x)Z_B + X_{BN}(x)Z_{BN}^T\right\| \geq \|X_B(x)Z_B\| - \left\|X_{BN}(x)Z_{BN}^T\right\| \\ &\geq \left\|X_B(x)^{-1}\right\| \|Z_B\| - \beta \|x - x^*\| (\|X(x)Z\| + \|Z_N\| \|x - x^*\|)), \end{aligned}$$

and then we obtain

$$\|Z_B\| \leq \beta(\|X(x)Z\| + \|Z_N\| \|x - x^*\|^2). \tag{26}$$

Estimates (25) and (26) yield

$$\left\| \begin{pmatrix} Z_B & Z_{BN} \\ Z_{BN}^T & 0 \end{pmatrix} \right\| \leq \beta(\|X(x)Z\| + \|Z_N\| \|x - x^*\|). \tag{27}$$

Next we consider the second term in the right hand side of (18). Its upper bound is given by (20). Defining $\mathcal{A}_N^*(x)$ by

$$\mathcal{A}_N^*(x)Z_N = \begin{pmatrix} \langle A_{N1}(x), Z_N\rangle \\ \vdots \\ \langle A_{Nn}(x), Z_N\rangle \end{pmatrix},$$

we have

$$\begin{aligned} \|\mathcal{A}_N^*(x)Z_N - \mathcal{A}^*(x)Z\| &= \left\| \mathcal{A}^*(x) \begin{pmatrix} Z_B & Z_{BN} \\ Z_{BN}^T & 0 \end{pmatrix} \right\| \\ &= \sqrt{\sum_{i=1}^{p} \left\langle A_i(x), \begin{pmatrix} Z_B & Z_{BN} \\ Z_{BN}^T & 0 \end{pmatrix} \right\rangle^2} \\ &\leq \sqrt{\sum_{i=1}^{p} \|A_i(x)\|^2 \left\| \begin{pmatrix} Z_B & Z_{BN} \\ Z_{BN}^T & 0 \end{pmatrix} \right\|^2} \\ &\leq \beta \left\| \begin{pmatrix} Z_B & Z_{BN} \\ Z_{BN}^T & 0 \end{pmatrix} \right\|. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \|\nabla f(x^*) - \mathcal{A}_N^*(x^*)Z_N\| &= \|\nabla f(x) - \mathcal{A}_N^*(x)Z_N \\ &\quad -(\nabla f(x) - \mathcal{A}_N^*(x)Z_N - \nabla f(x^*) + \mathcal{A}_N^*(x^*)Z_N)\| \\ &\leq \|\nabla f(x) - \mathcal{A}_N^*(x)Z_N\| + \beta(1 + \|Z_N\|)\,\|x - x^*\| \\ &= \|\nabla f(x) - \mathcal{A}^*(x)Z - (\mathcal{A}_N^*(x)Z_N - \mathcal{A}^*(x)Z)\| \\ &\quad +\beta(1 + \|Z_N\|)\,\|x - x^*\| \\ &\leq \|\nabla f(x) - \mathcal{A}^*(x)Z\| + \beta \left\| \begin{pmatrix} Z_B & Z_{BN} \\ Z_{BN}^T & 0 \end{pmatrix} \right\| \\ &\quad +\beta(1 + \|Z_N\|)\,\|x - x^*\|. \end{aligned} \tag{28}$$

Finally, we consider the term $|\min\{0, \varphi_{\min}(Z_N)\}|$ in (20). Let $y_{\min} \in \mathbf{R}^{p'}$ be the normalized eigenvector of $Z_N$ corresponding to the eigenvalue $\varphi_{\min}(Z_N)$. Then we have

$$\begin{aligned} \varphi_{\min}(Z) &= \min_{y\in\mathbf{R}^p, \|y\|=1} y^T Z y \leq \begin{pmatrix} 0 \\ y_{\min} \end{pmatrix}^T \begin{pmatrix} Z_B & Z_{BN} \\ Z_{BN}^T & Z_N \end{pmatrix} \begin{pmatrix} 0 \\ y_{\min} \end{pmatrix} \\ &= y_{\min}^T Z_N y_{\min} = \varphi_{\min}(Z_N). \end{aligned}$$

Therefore we obtain

$$|\min\{0, \varphi_{\min}(Z)\}| \geq |\min\{0, \varphi_{\min}(Z_N)\}|. \tag{29}$$

Then from (18), (20), (27), (28) and (29), we obtain (21). □

# 4 Local error bound for primal variable

In this section, we consider the error bound of the primal variables. In the nonlinear programming problems:

$$\begin{array}{lll} \text{minimize} & f(x), & x \in \mathbf{R}^n, \\ \text{subject to} & g(x) \geq 0, & \end{array}$$

where $g : \mathbf{R}^n \to \mathbf{R}^m$, given a primal dual pair $(x, z) \in \mathbf{R}^n \times \mathbf{R}^m$ which is close to the solution set, Wright [18], and Hager and Gowda [4] derive perturbed problems of the above NLP problem, and show a pair $(x, \bar{z}) \in \mathbf{R}^n \times \mathbf{R}^m$ satisfies

$$\begin{pmatrix} \nabla_x L(x, \bar{z}) \\ g(x) + v \end{pmatrix} \in \begin{pmatrix} v_L \\ N'(\bar{z}) \end{pmatrix},$$

where $\bar{z}$ is close to $z$, and $v_L \in \mathbf{R}^n$ and $v \in \mathbf{R}^m$ are small in magnitude. Then they show it is possible to obtain error bounds of the pair $(x, z)$ using the works by Robinson [14, 15]. We will show that this kind of reasoning is also possible in our nonlinear SDP problems, if we make use of the eigenvalues of $X(x)$ and $Z$.

In the following we first consider the case when $x \in \mathbf{R}^n$ and $Z \in \mathbf{S}^p$ satisfy

$$X(x)Z = \mu I,$$

for $\mu > 0$, where $I \in \mathbf{R}^{p \times p}$ is an identity matrix. The above condition is known as the shifted complementarity condition, and is the part of the barrier KKT conditions in various interior point methods. We will show that if $x$ and $Z$ satisfy the above relation among others, then it is possible to obtain a local error bound for the primal variable in Lemma 3. The next lemma gives the formula for necessary derivatives which will be used later.

**Lemma 2** *Let $X(x)$ be analytic, and $Z_c \in \mathbf{S}^p$ and $\mu > 0$ be given. Define the function $h_{Z_c} : \mathbf{R}^n \longrightarrow \mathbf{R}$ by*

$$h_{Z_c}(x) = \sum_{i=1}^{p} \lambda_i(X(x))\varphi_i(Z_c).$$

*If $x_c \in \mathbf{R}^n$ and $Z_c \in \mathbf{S}^p$ satisfy*

$$X(x_c)Z_c = \mu I, \tag{30}$$

*then $h_{Z_c}(x)$ is analytic at $x_c$, and*

$$\begin{aligned} h_{Z_c}(x_c) &= \langle X(x_c), Z_c \rangle, & (31) \\ \nabla_x h_{Z_c}(x_c) &= \nabla_x \langle X(x_c), Z_c \rangle & (32) \end{aligned}$$

*hold.*

**Proof.** If condition (30) is satisfied at $x_c$, then we have $\lambda_i(X(x_c))\varphi_i(Z_c) = \mu, i = 1, ..., p$ by simultaneously diagonalizing $X(x_c)$ and $Z_c$. Therefore, relation (31) is obvious. In order to prove (32), we use the result given in Theorem 3.1 of Tsing, Fang and Verriest

[17] (see also Theorem 1.1 of Lewis [10]) for the derivative of a spectral function which has a required symmetry.

It is known that if $\lambda_i(X(x))$ is distinct from other eigenvalues, i.e., $\lambda_{i-1}(X(x)) > \lambda_i(X(x)) > \lambda_{i+1}(X(x))$, then $\lambda_i(X(x))$ is analytic at $x$ (Fact 1.2 in [17]). If $\varphi_{i'-1}(Z_c) < \varphi_{i'}(Z_c) = \varphi_{i'+1}(Z_c) = ... = \varphi_{i''}(Z_c) < \varphi_{i''+1}(Z_c)$, then $h_{Z_c}(x)$ is symmetric with respect to the indices $\{i', ..., i''\}$, i.e., $h_{Z_c}(x)$ is invariant with the permutation between these indices at any $x$. In this case, we have $\lambda_{i'-1}(X(x_c)) > \lambda_{i'}(X(x_c)) = ... = \lambda_{i''}(X(x_c)) > \lambda_{i''+1}(X(x_c))$, and the symmetry with respect to the indices $\{i', ..., i''\}$ holds at $x_c$. Then from Theorem 2.1 of [17], the function $h_{Z_c}(x)$ is analytic at $x_c$.

Now it is possible to use formula (3.7) in [17] for the derivatives of $h_{Z_c}(x)$ at $x_c$. We have

$$\nabla_x h_{Z_c}(x_c) = \sum_{i=1}^{p} \frac{\partial h_{Z_c}(x_c)}{\partial \lambda_i} u_i^T \nabla_x X(x_c) u_i = \sum_{i=1}^{p} \varphi_i(Z_c) u_i^T \nabla_x X(x_c) u_i,$$

where $u_i \in \mathbf{R}^p, i = 1, ..., p$ are orthonormal eigenvectors of $X(x_c)$ and $Z_c$ corresponding to $\lambda_i(X(x_c))$ and $\varphi_i(Z_c), i = 1, ..., p$ respectively. Define $U = (u_1, ..., u_p) \in \mathbf{R}^{p\times p}$. Since $U^T U = UU^T = I$, we have

$$\sum_{i=1}^{p} \varphi_i(Z_c) u_i^T \nabla_x X(x_c) u_i = \sum_{i=1}^{p} u_i^T \nabla_x X(x_c) Z_c u_i = \left\langle U^T \nabla_x X(x_c), Z_c U \right\rangle = \nabla_x \left\langle X(x_c), Z_c \right\rangle.$$

This completes the proof. □

Next we consider the eigenvalues $\lambda_i(X(x)), \varphi_i(Z)$ , $i = 1, ..., p$ when $(x, Z)$ is close to the solution set $(x^*, \Lambda(x^*))$. Assuming at least $\lambda_i(X(x))$ or $\varphi_i(Z)$ is nonnegative for each $i = 1, ..., p$, we construct vectors $v \in \mathbf{R}^p$ and $\bar{\varphi} \in \mathbf{R}^p$, that satisfy

$$(\lambda(X(x)) + v)^T \bar{\varphi} = 0, \lambda(X(x)) + v \geq 0, \bar{\varphi} \geq 0, \tag{33}$$

where $\|v\|$ and $\|\bar{\varphi} - \varphi(Z)\|$ are small. To this end, for each $i = 1, ..., p$, we consider two cases:

(i) If $\min(\lambda_i(X(x)), \varphi_i(Z)) = \varphi_i(Z)$, we let

$$\begin{array}{c} \lambda_i(X(x)) + v_i = \lambda_i(X(x)), v_i = 0, \\ \bar{\varphi}_i = -\min(\lambda_i(X(x)), \varphi_i(Z)) + \varphi_i(Z) = 0; \end{array} \tag{34}$$

(ii) Otherwise, $\min(\lambda_i(X(x)), \varphi_i(Z)) = \lambda_i(X(x))$ holds, and we let

$$\begin{array}{c} \lambda_i(X(x)) + v_i = 0, v_i = -\min(\lambda_i(X(x)), \varphi_i(Z)), \\ \bar{\varphi}_i = \varphi_i(Z). \end{array} \tag{35}$$

Thus we have (33), and confirm that

$$\lambda(X(x)) + v \in N'(\bar{\varphi}), \tag{36}$$

$$|v_i| + |\bar{\varphi}_i - \varphi_i(Z)| = |\min(\lambda_i(X(x)), \varphi_i(Z))|, i = 1, ..., p. \tag{37}$$

**Remark.** In the following proofs of lemmas and theorems, we define neighborhoods $\mathcal{N}$ and $\mathcal{N}'$ of $(x^*, \Lambda(x^*))$ in a usual sense. We say $\mathcal{N}$ is small when $\mathrm{dist}((x, Z), (x^*, \Lambda(x^*)))$ is small for all $(x, Z) \in \mathcal{N} \backslash (x^*, \Lambda(x^*))$, and similarly for $\mathcal{N}'$. □

**Remark.** We will estimate the quantity $\|\min(\lambda(X(x)), \varphi(Z))\|$ when $XZ = \mu I$ is satisfied. The fact $\lambda_i(X(x))\varphi_i(Z) = \mu$ implies $\min(\lambda_i(X(x)), \varphi_i(Z)) \le \mu^{1/2}$ in general, but if all the solutions are known to satisfy the strict complementarity condition, we have $\min(\lambda_i(X(x)), \varphi_i(Z)) \le \beta\mu$. Thus we will use the following relation

$$\|\min(\lambda(X(x)), \varphi(Z))\| \le \beta\mu^{\kappa/2}, \tag{38}$$

where $\kappa = 1$ is always valid, and $\kappa = 2$ can be valid if all the solutions in $(x^*, \Lambda(x^*))$ are known to be strictly complementary. □

**Lemma 3** *Assume that $X(x)$ is analytic, and that the second order sufficient condition and the MFCQ condition hold at $x^*$. Then there exist a neighborhood $\mathcal{N}'$ of $(x^*, \Lambda(x^*))$, and constants $\gamma > 0$ and $\bar{\mu} > 0$ with the property that for each $(x, Z) \in \mathcal{N}'$ and $\mu \in (0, \bar{\mu}]$ such that*

$$X(x)Z = \mu I, \quad X(x) \succ 0, Z \succ 0, \tag{39}$$

*holds, and*

$$X(x^*) + \sum_{i=1}^{n}(x_i - x_i^*)A_i(x^*) \succeq \varepsilon \|x - x^*\| I \tag{40}$$

*holds, where $\varepsilon > 0$ is a given constant, we have*

$$\mathrm{dist}(x, x^*) \le \gamma\left(\|\nabla_x L(x, Z)\| + \mu^{\kappa/2}\right). \tag{41}$$

**Proof.** Let $d \in \mathbf{R}^n$ be

$$d = \frac{x - x^*}{\|x - x^*\|},$$

and write $x$ as

$$x = x^* + \bar{t}d, \quad \bar{t} = \|x - x^*\|$$

and denote $f(x^* + td)$ by $f(t)$, and $X(x^* + td)$ by $X(t)$ for $t \in \mathbf{R}$. Because $X(t)$ depends on one parameter $t$, it is known that there exist appropriately ordered eigenvalues $\xi_i(X(t)), i = 1, ..., p$ and corresponding orthonormal eigenvectors $u_i(t) \in \mathbf{R}^p, i = 1, ..., p$ of $X(t)$ which are analytic ([13],[9],[8]). We understand that $\xi_i(X(t)), i = 1, ..., p$ are ordered so that $\xi_i(X(\bar{t})) = \lambda_i(X(\bar{t})), i = 1, ..., p$.

Since $x^*$ is a KKT point of problem (1) that satisfies the second order sufficient condition, $t = 0$ is an isolated local minimum of the following one-dimensional optimization problem:

$$\begin{array}{lll} \text{minimize} & f(t), & t \in \mathbf{R}, \\ \text{subject to} & \xi_i(X(t)) \ge 0, & i = 1, ..., p. \end{array} \tag{42}$$

Define $v_L$ by

$$v_L = f'(\bar{t}) - \sum_{i=1}^{p} \xi_i'(X(\bar{t}))\bar{\varphi}_i, \tag{43}$$

where $\bar{\varphi}_i, i = 1, ..., p$ are defined in (34) and (35), $f'(\bar{t})$ denotes the derivative of $f(t)$ with respect to $t$ at $t = \bar{t}$, and similarly for $\xi_i'(X(\bar{t})), i = 1, ..., p$. From (36), $(\bar{t}, \bar{\varphi})$ satisfy the KKT conditions of the problem:

$$\begin{array}{lll} \text{minimize} & f(t) - tv_L, & t \in \mathbf{R}, \\ \text{subject to} & \xi_i(X(t)) + v_i \ge 0, & i = 1, ..., p, \end{array} \tag{44}$$

where $v_i, i = 1, ...p$ are defined in (34) and (35), because the KKT conditions of (44) at $t = \bar{t}$ is

$$\begin{array}{c} f'(\bar{t}) - \sum_{i=1}^{p} \xi_i'(X(\bar{t}))\bar{\varphi}_i = v_L, \\ \xi(X(\bar{t})) + v \in N'(\bar{\varphi}). \end{array} \tag{45}$$

If $\lambda_i(X(\bar{t}))$ is distinct from all other eigenvalues for some $i$, $\xi_i(X(t)) = \lambda_i(X(t))$ in a sufficiently small neighborhood of $\bar{t}$. We have $\xi_i(X(t)) = \lambda_j(X(t)), i \neq j$ in this neighborhood only when $\lambda_i(X(\bar{t})) = \lambda_j(X(\bar{t}))$, which implies $\varphi_i(Z) = \varphi_j(Z)$. Therefore we have

$$\sum_{i=1}^{p} \lambda_i(X(t))\varphi_i(Z) = \sum_{i=1}^{p} \xi_i(X(t))\varphi_i(Z),$$

in a sufficiently small neighborhood of $\bar{t}$. Thus Lemma 2 yields

$$\sum_{i=1}^{p} \xi_i'(X(\bar{t}))\varphi_i(Z) = \langle X'(\bar{t}), Z \rangle .$$

Then we have

$$\sum_{i=1}^{p} \xi_i'(X(\bar{t}))\bar{\varphi}_i = \langle X'(\bar{t}), Z \rangle + \sum_{i=1}^{p} (\bar{\varphi}_i - \varphi_i(Z))\xi_i'(X(\bar{t})). \tag{46}$$

From (43) and (46), we have

$$\begin{array}{rcl} |v_L| & = & \left| f'(\bar{t}) - \langle X'(\bar{t}), Z \rangle - \sum_{i=1}^{p} (\bar{\varphi}_i - \varphi_i(Z))\xi_i'(X(\bar{t})) \right| \\ & \leq & \left\| d^T(\nabla f(x) - \mathcal{A}^*(x)Z) \right\| + \beta \left\| \bar{\varphi} - \varphi(Z) \right\| \\ & \leq & \left\| \nabla f(x) - \mathcal{A}^*(x)Z \right\| + \beta \left\| \bar{\varphi} - \varphi(Z) \right\| . \end{array}$$

Now from (37) and Theorem 1, we obtain

$$\begin{array}{rcll} |v_L| + \|v\| & \leq & \|\nabla f(x) - \mathcal{A}^*(x)Z\| + \|v\| + \beta \|\bar{\varphi} - \varphi(Z)\| & \\ & \leq & \|\nabla f(x) - \mathcal{A}^*(x)Z\| + \beta \|\min(\lambda(X(x)), \varphi(Z))\| & (47) \\ & \leq & \beta \mathrm{dist}((x, Z), (x^*, \Lambda(x^*))), & (48) \end{array}$$

and confirm that $|v_L| + \|v\|$ is small when $\mathcal{N}'$ is small.

Next we prove that the KKT conditions of problem (42) is satisfied at $t = 0$, and $\bar{\varphi}$ is close to the corresponding dual solution set. We first consider the case when there exists nonzero $\tilde{Z} \in \Lambda(x^*)$. From assumption (40), we have

$$X(0) + \bar{t}X'(0) \succeq \varepsilon\bar{t}I. \tag{49}$$

Since the complementarity condition $X(0)\tilde{Z} = X(x^*)\tilde{Z} = 0$ holds, we have

$$f'(0) = \left\langle X'(0), \tilde{Z} \right\rangle = \bar{t}^{-1} \left\langle (X(0) + \bar{t}X'(0)), \tilde{Z} \right\rangle . \tag{50}$$

Let $\tilde{Z} = Q\tilde{\Phi}Q^T$, where an orthogonal matrix $Q = (q_1, ..., q_p) \in \mathbf{R}^{p\times p}$ diagonalizes $\tilde{Z}$ to $\tilde{\Phi} = \mathrm{diag}(\tilde{\varphi}_1, ..., \tilde{\varphi}_p) \succeq 0$. Then we have

$$\begin{aligned}\left\langle (X(0) + \bar{t}X'(0)), \tilde{Z} \right\rangle &= \left\langle (X(0) + \bar{t}X'(0)), Q\tilde{\Phi}Q^T \right\rangle = \left\langle Q^T(X(0) + \bar{t}X'(0))Q, \tilde{\Phi} \right\rangle \\ &= \sum_{i=1}^{p} q_i^T(X(0) + \bar{t}X'(0))q_i\tilde{\varphi}_i \geq \varepsilon\bar{t}\sum_{i=1}^{p}\tilde{\varphi}_i,\end{aligned}$$

from (49) and $\tilde{\Phi} \neq 0$. Thus from (50), we obtain

$$f'(0) \geq \varepsilon \sum_{i=1}^{p} \tilde{\varphi}_i. \tag{51}$$

Next we note

$$\xi_i'(X(t)) = u_i^T(t)X'(t)u_i(t), i = 1, ..., p,$$

which can be proved by using the relations

$$\xi_i(X(t)) = u_i^T(t)X(t)u_i(t), (u_i^T(t)u_i(t))' = 0, i = 1, ..., p.$$

Then, from (49), we have

$$\xi_i(X(0)) + \bar{t}\xi_i'(X(0)) \geq \varepsilon\bar{t}, i = 1, ..., p. \tag{52}$$

This inequality yields

$$\xi_i'(X(0)) \geq \varepsilon, \tag{53}$$

for $i \in I_N = \{i\,|\xi_i(X(0)) = 0, i = 1, ..., p\}$. From (51) and (53), it is apparent that there exists $\phi \in \mathbf{R}^p$ such that

$$\begin{aligned} f'(0) - \sum_{i=1}^{p} \xi_i'(X(0))\phi_i &= 0, & (54)\\ \xi_i(X(0))\phi_i &= 0, \xi_i(X(0)) \geq 0, \phi_i \geq 0, i = 1, ..., p. & (55)\end{aligned}$$

Let $\Lambda_1$ be the set of dual variables that satisfy the above KKT conditions for problem (42). The MFCQ condition holds at $t = 0$ from (52). Also there exists $\phi \in \Lambda_1$ such that $\phi_i > 0$ for all $i \in I_N$, and therefore the strict complementarity condition is satisfied.

In order to estimate the distance between $\bar{\varphi}$ and $\Lambda_1$, we can apply the result of Hoffman [5] which states the global error bound for the distance to the boundary of the polyhedron defined by a linear system of inequalities, or Theorem 2 of this paper. Here we follow Hoffman's result. Thus the distance from $\bar{\varphi}$ to $\Lambda_1$ is estimated by

$$\mathrm{dist}(\bar{\varphi}, \Lambda_1) \leq \beta\left(\left|f'(0) - \sum_{i=1}^{p}\xi_i'(X(0))\bar{\varphi}_i\right| + \left|\sum_{i=1}^{p}\xi_i(X(0))\bar{\varphi}_i\right|\right).$$

We note that the constant $\beta$ is uniformly bounded when $x$ varies near $x^*$ because the coefficients $f'(0)$ and $\xi_i'(X(0)), i = 1, ..., p$ are uniformly bounded, and $f'(0)$ and $\xi_i'(X(0)), i \in$

$I_N \neq \emptyset$ are bounded away from zero. Noting that $|f'(\bar{t}) - f'(0)| \leq \beta\bar{t}$ and $|\xi_i'(\bar{t}) - \xi_i'(0)| \leq \beta\bar{t}, i = 1, ..., p$, we obtain

$$\begin{aligned} \text{dist}(\bar{\varphi}, \Lambda_1) &\leq \beta\left(\left|f'(\bar{t}) - \sum_{i=1}^{p} \xi_i'(X(\bar{t}))\bar{\varphi}_i\right| + \left|\sum_{i=1}^{p} \xi_i(X(\bar{t}))\bar{\varphi}_i\right| + \bar{t}\right) \\ &= \beta\left(|v_L| + \left|v^T\bar{\varphi}\right| + \bar{t}\right) \\ &\leq \beta \text{dist}((x, Z), (x^*, \Lambda(x^*))), \end{aligned}$$

where the last inequality is derived from (48). We conclude that $\bar{\varphi}$ is close to $\Lambda_1$ when there exists $0 \neq \tilde{Z} \in \Lambda(x^*)$ and $\mathcal{N}'$ is small.

The fact $f'(0) > 0$ implies $0 \notin \Lambda_1$. The second order sufficient condition is trivially holds for every element of $\Lambda_1$ because the condition $\xi_i'(X(0))h = 0$, for $i$ such that $\phi_i > 0$ gives $h = 0$, and thus we have $h(f''(0) - \sum_{i=1}^{p} \xi_i''(X(0))\phi_i)h \geq \sigma h^2, \sigma > 0$.

If $\Lambda(x^*) = \{0\}$, then $\nabla f(x^*) = 0$, and $f'(0) = 0$. This implies $0 \in \Lambda_1$. If $Z$ is close to $\Lambda(x^*) = \{0\}$, then $\varphi$ and $\bar{\varphi}$ is close to $0 \in \Lambda_1$. Because the quadratic growth condition holds, we have $f''(0) > 0$, and thus the second order sufficient condition holds for $0 \in \Lambda_1$.

Now it is possible to apply Lemma 2 of Hager and Gowda [4], because the KKT conditions (54) and (55) are satisfied at $t = 0$, the second order sufficient condition holds at $t = 0$ as above, $(\bar{t}, \bar{\varphi})$ is close to $(0, \Lambda_1)$ as above, and perturbed KKT condition (45) holds with small perturbations $v_L$ and $v$. Then from Lemma 2 of Hager and Gowda [4], (47) and (38), we obtain

$$\begin{aligned} \text{dist}(x, x^*) &= \bar{t} \leq \text{dist}((\bar{t}, \bar{\varphi}), (0, \Lambda_1)) \leq \beta(|v_L| + \|v\|) \\ &\leq \beta(\|\nabla f(x) - \mathcal{A}^*(x)Z\| + \mu^{\kappa/2}) \end{aligned}$$

This completes the proof. □

We note that the above lemma implies that the set of $x$ which satisfies the condition:

$$X(x^*) + \sum_{i=1}^{n}(x_i - x_i^*)A_i(x^*) \succeq \varepsilon\gamma\left(\|\nabla_x L(x, Z)\| + \mu^{\kappa/2}\right) I$$

is contained in the set of $x$ which satisfies (40), i.e., in the region where (57) holds. Thus we obtain the following variation of the previous lemma.

**Lemma 4** *Assume that $X(x)$ is analytic, and that the second order sufficient condition and the MFCQ condition hold at $x^*$. Then there exist a neighborhood $\mathcal{N}'$ of $(x^*, \Lambda(x^*))$, and constants $\gamma > 0$ and $\bar{\mu} > 0$ with the property that for each $(x, Z) \in \mathcal{N}'$ and $\mu \in (0, \bar{\mu}]$ such that*

$$X(x)Z = \mu I, \quad X(x) \succ 0, Z \succ 0,$$

*holds, and*

$$X(x^*) + \sum_{i=1}^{n}(x_i - x_i^*)A_i(x^*) \succeq \varepsilon\gamma\left(\|\nabla_x L(x, Z)\| + \mu^{\kappa/2}\right) I \tag{56}$$

*holds, where $\varepsilon > 0$ is a given constant, we have*

$$\text{dist}(x, x^*) \leq \gamma\left(\|\nabla_x L(x, Z)\| + \mu^{\kappa/2}\right). \tag{57}$$

□

In the following, we will describe theorems using the condition on the quantity $X(x^*)+\sum_{i=1}^{n}(x_i - x_i^*)A_i(x^*)$ as in Lemma 4 instead of using the condition as in Lemma 3 for the sake of brevity.

Lemmas 3 and 4 assume that $X(x)$ and $Z$ satisfy the condition $X(x)Z = \mu I$. We next consider how to relax this condition. In order to obtain the primal error bound, the dual variable $Z$ can be considered as an auxiliary one. For a given $x$ which is close to $x^*$, we define $Z$ by $Z = \mu X(x)^{-1}$. If $\mu X(x)^{-1}$ is close to the dual solution set $\Lambda(x^*)$, we can obtain the primal error bound from Lemma 3 or 4. We will show two ways of relaxing the condition in the following.

**Theorem 3** *Assume that $X(x)$ is analytic, and that the second order sufficient condition, the MFCQ condition and the strict complementarity condition hold at $x^*$. Then there exist a neighborhood $\mathcal{N}_x$ of $x^*$, and constants $\gamma > 0$ and $\bar{\mu} > 0$ with the property that for each $x \in \mathcal{N}_x$, and for each $\mu \in (0, \bar{\mu}]$ such that*

$$\left\|\nabla f(x) - \mu \mathcal{A}^*(x)X(x)^{-1}\right\| \leq M_c \mu^{1/2}, \tag{58}$$

*where $M_c > 0$ is a given constant, and*

$$X(x^*) + \sum_{i=1}^{n}(x_i - x_i^*)A_i(x^*) \succeq \varepsilon \mu^{\kappa/2}\gamma I,$$

*holds, where $\varepsilon > 0$ is a given constant, we have*

$$\mathrm{dist}(x, x^*) \leq \gamma \mu^{\kappa/2}. \tag{59}$$

**Proof**. Define $Z = \mu X(x)^{-1}$. If the right hand side of inequality (21) in Theorem 2 is small enough, then $\mu X(x)^{-1}$ is close to $\Lambda(x^*)$. Since $\|\nabla_x L(x, Z)\| \leq M_c \mu^{1/2}$ and $\|X(x)Z\| = p^{1/2}\mu$, we have

$$\mathrm{dist}(\mu X(x)^{-1}, \Lambda(x^*)) \leq \beta(\mu^{1/2} + \mu + \mathrm{dist}(x, x^*)),$$

from Theorem 2. For small enough $\bar{\mu} > 0$, $Z$ is close to $\Lambda(x^*)$, and from Lemma 4, we obtain (59). □

We note that the inequality in (58) can be written as

$$\|\nabla F_B(x)\| \leq M_c \mu^{1/2},$$

where the barrier function $F_B(x)$ is defined by

$$F_B(x) = f(x) - \mu \sum_{i=1}^{p} \log(\det X(x)).$$

The next theorem shows that it is possible to have a similar error bound by directly relaxing the shifted complementarity condition.

**Theorem 4** *Assume that $X(x)$ is analytic, and that the second order sufficient condition and the MFCQ condition hold at $x^*$. Then there exist a neighborhood $\mathcal{N}$ of $(x^*, \Lambda(x^*))$, and constants $\gamma > 0$ and $\bar{\mu} > 0$ with the property that for each $(x, Z) \in \mathcal{N}$, and for each $\mu \in (0, \bar{\mu}]$ such that*

$$\|X(x)Z - \mu I\| \leq M_c \mu^{1+\tau}, \quad \mu \leq M_\mu \lambda_{\min}(X(x)) \tag{60}$$

*where $M_c > 0, M_\mu > 0$ and $\tau > 0$ are given constants, and $\lambda_{\min}(X(x))$ is the minimum eigenvalue of $X(x)$, and*

$$X(x^*) + \sum_{i=1}^{n}(x_i - x_i^*)A_i(x^*) \succeq \varepsilon\gamma \left(\|\nabla_x L(x, Z)\| + \mu^{\kappa/2} + M_c\mu^\tau\right) I$$

*holds, where $\varepsilon > 0$ is a given constant, we have*

$$\operatorname{dist}(x, x^*) \leq \gamma \left(\|\nabla_x L(x, Z)\| + \mu^{\kappa/2} + M_c\mu^\tau\right).$$

**Proof**. We have

$$\begin{aligned}
\left\|\nabla f(x) - \mu\mathcal{A}^*(x)X(x)^{-1}\right\| &\leq \|\nabla f(x) - \mathcal{A}^*(x)Z\| + \left\|\mathcal{A}^*(x)(Z - \mu X(x)^{-1})\right\| \\
&\leq \|\nabla f(x) - \mathcal{A}^*(x)Z\| + \beta \left\|Z - \mu X(x)^{-1}\right\|.
\end{aligned}$$

The distance between $Z$ and $\mu X(x)^{-1}$ is estimated as

$$\begin{aligned}
\left\|Z - \mu X(x)^{-1}\right\| &\leq \mu^{-1} \left\|\mu X(x)^{-1}\right\| \|X(\mu)Z - \mu I\| \\
&\leq M_c\mu^\tau \left\|\mu X(x)^{-1}\right\| \\
&\leq p^{1/2} M_c \mu^\tau \mu \lambda_{\min}(X(x))^{-1} \\
&\leq p^{1/2} M_c M_\mu \mu^\tau.
\end{aligned} \tag{61}$$

If we choose sufficiently small $\mathcal{N}$, then from (61), we have $(x, \mu X(x)^{-1}) \in \mathcal{N}'$ where $\mathcal{N}'$ is defined in Lemma 4. Therefore from Lemma 4, (57) and (61), we obtain

$$\begin{aligned}
\operatorname{dist}(x, x^*) &\leq \beta(\left\|\nabla f(x) - \mu\mathcal{A}^*(x)X(x)^{-1}\right\| + \mu^{\kappa/2}) \\
&\leq \beta(\|\nabla f(x) - \mathcal{A}^*(x)Z\| + \mu^{\kappa/2} + \left\|Z - \mu X(x)^{-1}\right\|) \\
&\leq \beta(\|\nabla f(x) - \mathcal{A}^*(x)Z\| + \mu^{\kappa/2} + M_c\mu^\tau).
\end{aligned}$$

The theorem is proved. □

If $X(x)$ is a concave function, the inequality

$$X(x^*) + \sum_{i=1}^{n}(x_i - x_i^*)A_i(x^*) \succeq X(x)$$

holds, and assumption (56) can be expressed differently. Thus we have the following corollaries of the preceding theorems.

**Corollary 1** *Assume that $X(x)$ is concave and analytic, and that the second order sufficient condition, the MFCQ condition and the strict complementarity condition hold at $x^*$. Then there exist a neighborhood $\mathcal{N}_x$ of $x^*$, and constants $\gamma > 0$ and $\bar{\mu} > 0$ with the property that for each $x \in \mathcal{N}_x$, and for each $\mu \in (0, \bar{\mu}]$ such that*

$$\begin{aligned} \left\| \nabla f(x) - \mu \mathcal{A}^*(x) X(x)^{-1} \right\| &\leq M_c \mu^{1/2}, \\ \lambda_{\min}(X(x)) &\geq \varepsilon \gamma \mu^{\kappa/2}, \end{aligned}$$

*holds, where $M_c > 0$ and $\varepsilon > 0$ are given constants, and $\lambda_{\min}(X(x))$ is the minimum eigenvalue of $X(x)$, we have*

$$\operatorname{dist}(x, x^*) \leq \gamma \mu^{\kappa/2}.$$

**Corollary 2** *Assume that $X(x)$ is concave and analytic, and that the second order sufficient condition and the MFCQ condition hold at $x^*$. Then there exist a neighborhood $\mathcal{N}$ of $(x^*, \Lambda(x^*))$, and constants $\gamma > 0$ and $\bar{\mu} > 0$ with the property that for each $(x, Z) \in \mathcal{N}$, and for each $\mu \in (0, \bar{\mu}]$ such that*

$$\begin{aligned} \| X(x) Z - \mu I \| &\leq M_c \mu^{1+\tau}, \\ \lambda_{\min}(X(x)) &\geq \varepsilon \gamma \left( \| \nabla_x L(x, Z) \| + \mu^{\kappa/2} + M_c \mu^{\tau} \right), \end{aligned}$$

*holds, where $M_c > 0, \tau > 0$ and $\varepsilon > 0$ are given constants, and $\lambda_{\min}(X(x))$ is the minimum eigenvalue of $X(x)$, we have*

$$\operatorname{dist}(x, x^*) \leq \gamma \left( \| \nabla_x L(x, Z) \| + \mu^{\kappa/2} + M_c \mu^{\tau} \right).$$

In the last Corollary, the condition $\mu \leq M_\mu \lambda_{\min}(X(x))$ in (60) is removed since $\lambda_{\min}(X(x)) \geq \varepsilon \gamma \left( \| \nabla_x L(x, Z) \| + \mu^{\kappa/2} + M_c \mu^{\tau} \right) \geq \varepsilon \gamma \mu^{\kappa/2}$ is a stronger or equal condition.

# 5 Local error bounds for primal and dual variables

In this section, we summarize error bounds obtained in this paper by gathering the estimates from preceding sections. We first gather the results from Theorems 1, 2 and 3.

**Theorem 5** *Assume that $X(x)$ is analytic, and that the second order sufficient condition, the MFCQ condition and the strict complementarity condition hold at $x^*$. Then there exist a neighborhood $\mathcal{N}$ of $(x^*, \Lambda(x^*))$, and constants $\gamma > 0$ and $\bar{\mu} > 0$ with the property that for each $(x, Z) \in \mathcal{N}$, and for each $\mu \in (0, \bar{\mu}]$ such that*

$$\left\| \nabla f(x) - \mu \mathcal{A}^*(x) X(x)^{-1} \right\| \leq M_c \mu^{1/2}, \quad X(x) \succ 0,$$

*holds, where $M_c > 0$ is a given constant, and*

$$X(x^*) + \sum_{i=1}^{n} (x_i - x_i^*) A_i(x^*) \succeq \varepsilon \gamma \mu^{\kappa/2} I \tag{62}$$

*holds, where* $\varepsilon > 0$ *is a given constant, we have*

$$\begin{aligned}\gamma^{-1}\left(\|\nabla_x L(x,Z)\| + \|X(x)Z\|\right) &\leq \operatorname{dist}((x,Z),(x^*,\hat{Z}))\\ &\leq \gamma\left(\|\nabla_x L(x,Z)\| + \|X(x)Z\| + |\min\{0,\varphi_{\min}(Z)\}| + \mu^{\kappa/2}\right),\end{aligned}$$

*where* $\varphi_{\min}(Z)$ *is the smallest eigenvalue of* $Z$*.*

The next theorem is from the results of Theorems 1, 2 and 4. In this theorem we use the fact

$$\beta^{-1}\mu \leq \|X(x)Z\| \leq \beta\mu$$

which can be obtained from the assumption $\|X(x)Z - \mu I\| \leq M_c\mu^{1+\tau}$ below.

**Theorem 6** *Assume that* $X(x)$ *is analytic, and that the second order sufficient condition, the MFCQ condition and the strict complementarity condition hold at* $x^*$*. Then there exist a neighborhood* $\mathcal{N}$ *of* $(x^*, \Lambda(x^*))$*, and constants* $\gamma > 0$ *and* $\bar{\mu} > 0$ *with the property that for each* $(x,Z) \in \mathcal{N}$*, and for each* $\mu \in (0, \bar{\mu}]$ *such that*

$$\|X(x)Z - \mu I\| \leq M_c\mu^{1+\tau}, \quad \mu \leq M_\mu \lambda_{\min}(X(x)),$$

*holds, where* $M_c > 0, M_\mu > 0$ *and* $\tau > 0$ *are given constants, and*

$$X(x^*) + \sum_{i=1}^{n}(x_i - x_i^*)A_i(x^*) \succeq \varepsilon\gamma\left(\|\nabla_x L(x,Z)\| + \mu^{\kappa/2} + M_c\mu^\tau\right) I \qquad (63)$$

*holds, where* $\varepsilon > 0$ *is a given constant, we have*

$$\begin{aligned}\gamma^{-1}\left(\|\nabla_x L(x,Z)\| + \mu\right) &\leq \operatorname{dist}((x,Z),(x^*,\hat{Z}))\\ &\leq \gamma\left(\|\nabla_x L(x,Z)\| + |\min\{0,\varphi_{\min}(Z)\}| + \mu^{\kappa/2} + M_c\mu^\tau\right),\end{aligned}$$

*where* $\varphi_{\min}(Z)$ *is the smallest eigenvalue of* $Z$*.*

As mentioned in the previous section, conditions on the quantity $X(x^*) + \sum_{i=1}^{n}(x_i - x_i^*)A_i(x^*)$ can be represented differently when $X(x)$ is a concave function.

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