Joint rectangular geometric chance constrained programs

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Abstract

This paper discusses joint rectangular geometric chance constrained programs. When the stochastic parameters are elliptically distributed and pairwise independent, we present a reformulation of the joint rectangular geometric chance constrained programs. As the reformulation is not convex, we propose new convex approximations based on variable transformation together with piecewise linear approximation method. Our results show that the approximations are tight.

Keywords: Geometric optimization, Joint probabilistic constraint, Variable transformation, Piecewise linear approximation.

1. Introduction

A rectangular geometric program can be formulated as

$$(GP) \quad \min_{t\in\mathbb{R}^M_{++}} \quad g_0(t)$$
$$\text{s.t.} \quad \alpha_k \le g_k(t) \le \beta_k, \; k = 1, \dots, K,$$

where $\alpha_k, \beta_k \in \mathbb{R}$, $k = 1, \dots, K$ and

$$g_k(t) = \sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k}, \; k = 0, \dots, K. \tag{1}$$

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Preprint submitted to Elsevier *October 28, 2016*

Usually, $c_i^k \prod_{j=1}^M t_j^{a_{ij}^k}$ is called a monomial where c_i^k, $i = 1, \ldots, I_k$, $k = 0, \ldots, K$, are nonnegative and $g_k(t)$, $k = 0, \ldots, K$, are called posynomials.

We require that $0 < \alpha_k < \beta_k, \forall k = 1, \ldots, K$. When $\alpha_k \leq 0$, $k = 1, \ldots, K$, the rectangular geometric program is equivalent to geometric programs discussed in [6, 17, 2].

Both geometric programs and rectangular geometric programs are not convex with respect to t. However, geometric programs are convex with respect to $\{r : r_j = \log\ t_j,\ j = 1, \ldots, M\}$. Hence, interior point method can be efficiently used to solve geometric programs. To the best of our knowledge, there is no possible variable transformation method to derive a convex equivalent reformulation.

Stochastic geometric programming is used to model geometric problems when some of the parameters are not known precisely. Stochastic geometric programs with individual probabilistic constraints are discussed in [5] and [19] where the authors showed that an individual probabilistic constraint is equivalent to several deterministic constraints involving posynomials and common additional slack variables. In this case, the parameters $a_{ij}^k, \forall k, i, j$, are deterministic and $c_i^k, \forall k, i$, are uncorrelated normally distributed random variables. Liu et al. [13] discussed stochastic geometric programs with joint probabilistic constraints and proposed tractable approximations by using piecewise linear functions and the sequential convex optimization algorithm under the same assumption as in [5].

When $a_{ij}^k \in \{0, 1\}, \forall k, i, j$ and $\sum_j a_{ij}^k = 1, \forall k, i$, stochastic geometric programs are equivalent to stochastic linear programs. The latter were first considered by Miller and Wagner [14]. Prékopa [18] proposed the concept of log-concavity and showed the convexity of joint chance constraint problems under log-concavity distributions, e.g., the normal distribution. Later, the concepts of r-concave function and r-decreasing function are proposed by Dentcheva et al. [4] and Henrion and Strugarek [10], respectively, which can be used to show the convexity of chance constraint problems under discrete distributions and with dependence between different rows. For the rectangular case, Van Ackooij et al. [20] discussed the joint separable rectangular chance constrained problems for the underlying normally distributed random vector, and provided a derivative formula for probabilities of rectangles.

However, for complex distributions, the joint chance constraint problems are nonconvex and difficult to compute. Hence, approximation methods are widely used. Nemirovski and Shapiro [15] and Luedtke and Ahmed [12] proposed a mix-integer programming approach for chance constraint problems by using the sample average approximation method. Nemirovski and Shapiro [16] proposed the Bernstein approximation approach for the chance constrained problems, which is convex and tractable. Ben-tal et al. [1] showed that conditional value-at-risk function can be used to construct convex approximations for chance constrained problems. Cheng and Lisser [3] proposed a piecewise linear approximation approach for normally distributed linear programs with joint probabilistic constraints. This piecewise linear approach is improved by Liu et al. [13], and used for joint geometric chance constrained programming under the normal distribution. To the best of our knowledge, there is no research on the joint rectangular geometric chance constrained programs.

In this paper, we consider the following joint rectangular geometric chance constrained programs

$$(SGP) \quad \min_{t\in\mathbb{R}^M_{++}} \quad \mathbb{E}\left[\sum_{i=1}^{I_0} c_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0}\right] \tag{2}$$

$$\text{s.t.} \quad \mathbb{P}\left(\alpha_k \leq \sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \leq \beta_k,\ k=1,\dots,K\right) \geq 1-\epsilon. \tag{3}$$

where $1-\epsilon$ is a prespecified probability with $\epsilon < 0.5$.

We introduce the program under the elliptical distribution assumption in Section 2. In Section 3, new convex approximations are proposed based on variable transformation together with piecewise linear approximation method.

2. Elliptically distributed stochastic geometric problems

In this paper, we consider the joint rectangular geometric chance constrained programs under the elliptical distribution assumption.

Assumption 1. *We suppose that the coefficients a_{ij}^k, $k=1,\dots,K$, $i=1,\dots,I_k$, $j=1,\dots,M$, are deterministic and the parameter $c^k=[c_1^k,c_2^k,\dots,c_{I_k}^k]$ follows a multivariate elliptical distribution $Ellip_{I_k}(\mu^k,\Gamma^k,\varphi_k)$ with $\mu^k=[\mu_1^k,\mu_2^k,\dots,\mu_{I_k}^k]^\top \geq 0$, and $\Gamma^k=\{\sigma_{i,p}^k,\ i,p=1,\dots,I_k\}$ positive definite, $k=1,\dots,K$. Moreover, we assume that c^k, $k=1,\dots,K$ are pairwise independent.*

Definition 1. *A L-dimensional random vector ξ follows an elliptical distribution $Ellip_L(\mu, \Gamma, \varphi)$ if its characteristic function is given by $\mathbb{E}e^{iz^\top c} = e^{iz^\top \mu}\varphi(z^\top \Gamma z)$ where φ is the characteristic generator function, μ is the location parameter, and Γ is the scale matrix.*

Elliptical distributions include normal distribution with $\varphi(t) = \exp\{-\frac{1}{2}t\}$, student's t distribution with $\varphi(t)$ varying with its degree of freedom [11], Cauchy distribution with $\varphi(t) = \exp\{-\sqrt{t}\}$, Laplace distribution with $\varphi(t) = (1 + \frac{1}{2}t)^{-1}$, and logistic distribution with $\varphi(t) = \frac{2\pi\sqrt{t}}{e^{\pi\sqrt{t}} - e^{-\pi\sqrt{t}}}$. The mean value of an elliptical distribution $Ellip_L(\mu, \Gamma, \varphi)$ is μ, and its covariance matrix is $\frac{E(r^2)}{\text{rank}(\Gamma)}\Gamma$, where r is the random radius [7].

Proposition 1. *If a L-dimensional random vector ξ follows an elliptical distribution $Ellip_L(\mu, \Gamma, \varphi)$, then for any $(L \times N)$-matrix A and any N-vector b, $A\xi + b$ follows an N-dimensional elliptical distribution $Ellip_N(A\mu + b, A\Gamma A^\top, \varphi)$.*

Moreover, we have some restrictions on ϵ as follows.

Assumption 2. *We assume that*

- $\frac{\phi'_{\varphi_k}(\Phi^{-1}_{\varphi_k}(1-\epsilon))}{\phi_{\varphi_k}(\Phi^{-1}_{\varphi_k}(1-\epsilon))}\Phi^{-1}_{\varphi_k}(1-\epsilon) < -1,\ k = 1, \ldots, K,$

- $(\Phi^{-1}_{\varphi_k}(1-\epsilon))^2\sigma^k_{i,p} - \mu^k_i\mu^k_p \geq 0,\ i, p = 1, \ldots, I_k,\ k = 1, \ldots, K,$

- $2\sigma^k_{i,p}\left(1 - \frac{\phi'_{\varphi_k}(\Phi^{-1}_{\varphi_k}(z))}{\phi_{\varphi_k}(\Phi^{-1}_{\varphi_k}(z))}\Phi^{-1}_{\varphi_k}(z)\right)\left((\Phi^{-1}_{\varphi_k}(z))^2\sigma^k_{i,p} - \mu^k_i\mu^k_p\right) - (2\sigma^k_{i,p}\Phi^{-1}_{\varphi_k}(z))^2 \geq 0,\ 1-\epsilon \leq z \leq 1,\ i, p = 1, \ldots, I_k,\ k = 1, \ldots, K.$

Here, $\Phi_\varphi(\cdot)$ and $\phi_\varphi(\cdot)$ are the distribution function and the density function of an univariate standard elliptical distribution $Ellip_1(0, I, \varphi)$, where I is the identity matrix. $\Phi^{-1}_\varphi(\cdot)$ is the inverse function of $\Phi_\varphi(\cdot)$, i.e., the quantile of the standard elliptical distribution, and $\phi'_\varphi(\cdot)$ is the first order derivative of $\phi_\varphi(\cdot)$

Theorem 1. *Given Assumption 1, the joint rectangular geometric chance constrained programs (SGP) can be equivalently reformulated as*

(SGP_r)

$$\min_{t \in \mathbb{R}^M_{++}} \quad \sum_{i=1}^{I_0} \mu_i^0 \prod_{j=1}^{M} t_j^{a_{ij}^0} \tag{4}$$

$$\text{s.t.} \quad \Phi^{-1}_{\varphi_k}(z_k^+) \sqrt{\sum_{i=1}^{I_k} \sum_{p=1}^{I_k} \sigma_{i,p}^k \prod_{j=1}^{M} t_j^{a_{ij}^k + a_{pj}^k}} - \sum_{i=1}^{I_k} \mu_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \leq -\alpha_k, \ k = 1, \dots, K, \tag{5}$$

$$\Phi^{-1}_{\varphi_k}(z_k^-) \sqrt{\sum_{i=1}^{I_k} \sum_{p=1}^{I_k} \sigma_{i,p}^k \prod_{j=1}^{M} t_j^{a_{ij}^k + a_{pj}^k}} + \sum_{i=1}^{I_k} \mu_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \leq \beta_k, \ k = 1, \dots, K, \tag{6}$$

$$z_k^+ + z_k^- - 1 \geq y_k, \ 0 \leq z_k^+, z_k^- \leq 1, \ k = 1, \dots, K, \tag{7}$$

$$\prod_{k=1}^{K} y_k \geq 1 - \epsilon, \ 0 \leq y_k \leq 1, \ k = 1, \dots, K. \tag{8}$$

Proof. As c_i^k are pairwise independent, constraint (3) is equivalent to

$$\prod_{k=1}^{K} \mathbb{P}\left(\alpha_k \leq \sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \leq \beta_k \right) \geq 1 - \epsilon. \tag{9}$$

By introducing auxiliary variables $y_k \in \mathbb{R}_+, \ k = 1, \dots, K$, (9) can be equivalently written as

$$\mathbb{P}\left(\alpha_k \leq \sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \leq \beta_k \right) \geq y_k, \ k = 1, \dots, K, \tag{10}$$

and

$$\prod_{k=1}^{K} y_k \geq 1 - \epsilon, \ 0 \leq y_k \leq 1, \ k = 1, \dots, K. \tag{11}$$

(10) is also equivalent to

$$\mathbb{P}\left(\sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \geq \alpha_k \right) + \mathbb{P}\left(\sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \leq \beta_k \right) - 1 \geq y_k, \ k = 1, \dots, K. \tag{12}$$

Let $z_k^+, z_k^- \in \mathbb{R}_+,\ k = 1, \ldots, K$, be two additional auxiliary variables. Constraint (12) can be equivalently expressed by

$$\mathbb{P}\left(\sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \geq \alpha_k\right) \geq z_k^+,\ k = 1, \ldots, K, \tag{13}$$

$$\mathbb{P}\left(\sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} \leq \beta_k\right) \geq z_k^-,\ k = 1, \ldots, K, \tag{14}$$

$$z_k^+ + z_k^- - 1 \geq y_k,\ 0 \leq z_k^+, z_k^- \leq 1, k = 1, \ldots, K. \tag{15}$$

From proposition 1, we know that $\sum_{i=1}^{I_k} c_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k}$ follows an elliptical distribution $Ellip_1(\sum_{i=1}^{I_k} \mu_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k}, \sum_{i=1}^{I_k} \sum_{p=1}^{I_k} \sigma_{i,p}^k \prod_{j=1}^{M} t_j^{a_{ij}^k + a_{pj}^k}, \varphi_k)$. By the quantile transformation, constraint (13) is equivalent to

$$-\sum_{i=1}^{I_k} \mu_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} + \Phi_{\varphi_k}^{-1}(z_k^+) \sqrt{\sum_{i=1}^{I_k} \sum_{p=1}^{I_k} \sigma_{i,p}^k \prod_{j=1}^{M} t_j^{a_{ij}^k + a_{pj}^k}} \leq -\alpha_k,\ k = 1, \ldots, K,$$

and constraint (14) is equivalent to

$$\sum_{i=1}^{I_k} \mu_i^k \prod_{j=1}^{M} t_j^{a_{ij}^k} + \Phi_{\varphi_k}^{-1}(z_k^-) \sqrt{\sum_{i=1}^{I_k} \sum_{p=1}^{I_k} \sigma_{i,p}^k \prod_{j=1}^{M} t_j^{a_{ij}^k + a_{pj}^k}} \leq \beta_k,\ k = 1, \ldots, K.$$

This gives the equivalent reformulation of the joint constrained problem. As $c^k \sim Ellip_{I_k}(\mu^k, \Gamma^k, \varphi_k)$, its expected value is μ^k. Hence, from the additivity property of the expectation operator, we can get the equivalent reformulation of the objective function. □

In (SGP_r), both constraints (5) and (6) are nonconvex constraints. In the next section, we propose inner and outer convex approximations.

3. Convex approximations of constraints (5) and (6)

3.1. Convex approximations of constraint (5)

We first denote

$$w^k = \left[\prod_{j=1}^{M} t_j^{a_{1j}^k}, \ldots, \prod_{j=1}^{M} t_j^{a_{I_k j}^k}\right],\ k = 1, \ldots, K.$$

Constraint (5) can be reformulated as

$$-(\mu^k)^\top w^k + \Phi_{\varphi_k}^{-1}(z_k^+) \sqrt{(w^k)^\top \Gamma_k w^k} \leq -\alpha_k,\ k = 1, \ldots, K. \tag{16}$$

As we assume that $\epsilon \leq 0.5$ and c^k follows a symmetric distribution, it is easy to see that $(\mu^k)^\top w^k - \alpha_k \geq 0$. Hence, (16) is equivalent to

$$(\Phi_{\varphi_k}^{-1}(z_k^+))^2((w^k)^\top \Gamma_k w^k) \leq ((\mu^k)^\top w^k - \alpha_k)^2,\ k = 1, \ldots, K,$$

which can be reformulated as

$$(w^k)^\top \big((\Phi_{\varphi_k}^{-1}(z_k^+))^2 \Gamma_k - \mu^k (\mu^k)^\top\big) w^k + 2\alpha_k (\mu^k)^\top w^k \leq \alpha_k^2,\ k = 1, \ldots, K. \quad (17)$$

As $w^k = \left[\prod_{j=1}^M t_j^{a_{1j}^k}, \ldots, \prod_{j=1}^M t_j^{a_{I_k j}^k}\right],\ k = 1, \ldots, K,$ constraint (17) is equivalent to

$$2\alpha_k \sum_{i=1}^{I_k} \mu_i^k \prod_{j=1}^M t_j^{a_{ij}^k} + \sum_{i=1}^{I_k} \sum_{p=1}^{I_k} ((\Phi_{\varphi_k}^{-1}(z_k^+))^2 \sigma_{i,p}^k - \mu_i^k \mu_p^k) \prod_{j=1}^M t_j^{a_{ij}^k + a_{pj}^k} \leq \alpha_k^2,$$
$$k = 1, ...K. \quad (18)$$

From (7) and (8), we know that $z_k^+ \geq 1-\epsilon \geq 0.5$. Moreover, we know from Assumption 2 that $(\Phi_{\varphi_k}^{-1}(z_k^+))^2 \sigma_{i,p}^k - \mu_i^k \mu_p^k \geq (\Phi_{\varphi_k}^{-1}(1-\epsilon))^2 \sigma_{i,p}^k - \mu_i^k \mu_p^k \geq 0$, for all $z_k^+ \in [1-\epsilon, 1)$, $i, p = 1, \ldots, I_k$, $k = 1, \ldots, K$. Hence, given Assumption 2, we can apply the standard variable transformation $r_j = \log(t_j),\ j = 1, \ldots, M$, to (18). Therefore, we have an equivalent formulation of (18)

$$2\alpha_k \sum_{i=1}^{I_k} \mu_i^k \exp\left\{\sum_{j=1}^M a_{ij}^k r_j\right\} + \sum_{i=1}^{I_k} \sum_{p=1}^{I_k} \exp\left\{\sum_{j=1}^M (a_{ij}^k r_j + a_{pj}^k r_j)\right.$$
$$\left. + \log((\Phi_{\varphi_k}^{-1}(z_k^+))^2 \sigma_{i,p}^k - \mu_i^k \mu_p^k)\right\} \leq \alpha_k^2,\ k = 1, \ldots, K. \quad (19)$$

Proposition 2. *Given Assumption 2,* $f_{i,p,k}(z_k^+) = \log((\Phi_{\varphi_k}^{-1}(z_k^+))^2 \sigma_{i,p}^k - \mu_i^k \mu_p^k)$ *is monotone increasing and convex for* $z_k^+ \in [1-\epsilon, 1)$, $i, p = 1, \ldots, I_k$, $k = 1, \ldots, K$.

Proof. From the continuity and differentiability of $\Phi_{\varphi_k}^{-1}$, we know that $f_{i,p,k}(z_k^+)$

is differential and its first and second orders derivatives are

$$\begin{aligned}
f'_{i,p,k}(z_k^+) &= \frac{df_{i,p,k}(z_k^+)}{dz_k^+} = \frac{2\sigma_{i,p}^k \Phi_{\varphi_k}^{-1}(z_k^+)}{\phi_{\varphi_k}(\Phi_{\varphi_k}^{-1}(1-\epsilon))((\Phi_{\varphi_k}^{-1}(z_k^+))^2\sigma_{i,p}^k - \mu_i^k\mu_p^k)} \\
f''_{i,p,k}(z_k^+) &= \frac{d^2 f_{i,p,k}(z_k^+)}{d(z_k^+)^2} \\
&= \frac{2\sigma_{i,p}^k \left(1 - \frac{\phi'_{\varphi_k}(\Phi_{\varphi_k}^{-1}(z_k^+))}{\phi_{\varphi_k}(\Phi_{\varphi_k}^{-1}(z_k^+))}\Phi_{\varphi_k}^{-1}(z_k^+)\right)\left((\Phi_{\varphi_k}^{-1}(z_k^+))^2\sigma_{i,p}^k - \mu_i^k\mu_p^k\right) - \left(2\sigma_{i,p}^k\Phi_{\varphi_k}^{-1}(z_k^+)\right)^2}{(\phi_{\varphi_k}(\Phi_{\varphi_k}^{-1}(z_k^+)))^2((\Phi_{\varphi_k}^{-1}(z_k^+))^2\sigma_{i,p}^k - \mu_i^k\mu_p^k)^2}.
\end{aligned}$$

From Assumption 2, we know that $\frac{df_{i,p,k}(z_k^+)}{dz_k^+} \geq 0$ and $f_{i,p,k}(z_k^+)$ is monotone increasing for $z_k^+ \in [1-\epsilon, 1)$, $i, p = 1, \ldots, I_k$, $k = 1, \ldots, K$. From Assumption 2, we know that $\frac{d^2 f_{i,p,k}(z_k^+)}{d(z_k^+)^2} \geq 0$ and $f_{i,p,k}(z_k^+)$ is convex for $z_k^+ \in [1-\epsilon, 1)$, $i, p = 1, \ldots, I_k$, $k = 1, \ldots, K$.

□

Thanks to the convexity and the monotonicity of $f_{i,p,k}(z_k^+)$, we use the piecewise linear approximation method to find an inner approximation of $f_{i,p,k}(z_k^+)$ [13]. Then, we propose a piecewise linear approximation method to find an outer approximation of $f_{i,p,k}(z_k^+)$.

We choose S different linear functions:

$$F_{s,i,p,k}^L(z_k^+) = d_{s,i,p,k}z_k^+ + b_{s,i,p,k}, \; s = 1, \ldots, S,$$

which are the tangent segments of $f_{i,p,k}(z_k^+)$ at given points in $[1-\epsilon, 1)$, e.g., $\xi_1, \xi_2, \ldots, \xi_S$. Here, we choose $\xi_S = 1 - \delta$, where δ is a very small positive real number. We have

$$d_{s,i,p,k} = f'_{i,p,k}(\xi_s)$$

and

$$b_{s,i,p,k} = f_{i,p,k}(\xi_s) - f'_{i,p,k}(\xi_s)\xi_s, \; s = 1, \ldots, S.$$

Then, we use the piecewise linear function

$$F_{i,p,k}^L(z_k^+) = \max_{s=1,\ldots,S} F_{s,i,p,k}^L(z_k^+),$$

to approximate $f_{i,p,k}(z_k^+)$.

Proposition 3. *Given Assumption 2,* $F^L_{i,p,k}(z_k^+) \leq f_{i,p,k}(z_k^+),\ \forall z_k^+ \in [1-\epsilon, 1)$.

Proof. The proof can be drawn from the convexity of $f_{i,p,k}(z_k^+)$ shown in Proposition 2. □

We use the piecewise linear function $F^L_{i,p,k}(z_k^+)$ to replace $\log((\Phi^{-1}_{\varphi_k}(z_k^+))^2\sigma^k_{i,p} - \mu_i^k\mu_p^k)$ in (19). Hence, we have the following convex approximation of constraint (5):

$$\begin{cases} 2\alpha_k \sum_{i=1}^{I_k} \mu_i^k \exp\left\{ \sum_{j=1}^{M} a_{ij}^k r_j \right\} + \sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \exp\left\{ \sum_{j=1}^{M} (a_{ij}^k r_j + a_{pj}^k r_j) \right. \\ \qquad \left. + \, \omega^L_{i,p,k} \right\} \leq \alpha_k^2,\ k = 1, \ldots, K, \\ d_{s,i,p,k} z_k^+ + b_{s,i,p,k} \leq \omega^L_{i,p,k},\ s = 1, \ldots, S,\ i, p = 1, \ldots, I_k,\ k = 1, \ldots, K. \end{cases} \tag{20}$$

As $\log((\Phi^{-1}_{\varphi_k}(z_k^+))^2\sigma^k_{i,p} - \mu_i^k\mu_p^k)$ is convex, (20) provides an inner approximation.

To get an outer approximation of the function $f_{i,p,k}(z_k^+)$, we sort $1 - \epsilon, \xi_1, \xi_2, \ldots, \xi_S$ in increasing order and denote the sorted array $\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_{S+1}$. The segments

$$F^U_{s,i,p,k}(z_k^+) = \tilde{d}_{s,i,p,k} z_k^+ + \tilde{b}_{s,i,p,k},\ s = 1, \ldots, S,$$

form a piecewise linear function

$$F^U_{i,p,k}(z_k^+) = \max_{s=1,\ldots,S} F^U_{s,i,p,k}(z_k^+).$$

Here,

$$\tilde{d}_{s,i,p,k} = \frac{f_{i,p,k}(\tilde{\xi}_{s+1}) - f_{i,p,k}(\tilde{\xi}_s)}{\tilde{\xi}_{s+1} - \tilde{\xi}_s}$$

and

$$\tilde{b}_{s,i,p,k} = -\tilde{d}_{s,i,p,k}\tilde{\xi}_s + f_{i,p,k}(\tilde{\xi}_s),\ s = 1, \ldots, S.$$

Using the piecewise linear function $F^U_{i,p,k}(z^+_k)$ leads to the following convex approximation of constraint (5):

$$\begin{cases} 2\alpha_k \sum_{i=1}^{I_k} \mu_i^k \exp\left\{\sum_{j=1}^{M} a_{ij}^k r_j\right\} + \sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \exp\left\{\sum_{j=1}^{M}(a_{ij}^k r_j + a_{pj}^k r_j)\right. \\ \qquad \left. + \omega^U_{i,p,k}\right\} \le \alpha_k^2,\ k = 1,\dots,K, \\ \tilde{d}_{s,i,p,k} z_k^+ + \tilde{b}_{s,i,p,k} \le \omega^U_{i,p,k},\ s = 1,\dots,S,\ i,p = 1,\dots,I_k,\ k = 1,\dots,K. \end{cases} \tag{21}$$

As $\log((\Phi^{-1}_{\varphi_k}(z^+_k))^2\sigma^k_{i,p} - \mu^k_i\mu^k_p)$ is convex, (21) provides an outer approximation.

3.2. Convex approximation of constraint (6)

In constraint (6), the terms $\sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k}\sigma^k_{i,p}\prod_{j=1}^M t_j^{a^k_{ij}+a^k_{pj}}}$ and $\sum_{i=1}^{I_k}\mu^k_i\prod_{j=1}^M t_j^{a^k_{ij}}$ are convex with respect to $r_j = \log(t_j)$, $j = 1,\dots,M$.

From [10], we know that $\Phi^{-1}_{\varphi_k}(z^-_k)$ is convex with respect to z^-_k on $[1-\epsilon, 1)$, if ϕ_{φ_k} is 0-decreasing with some threshold $t^*(0) > 0$, and $\epsilon < 1 - \Phi_{\varphi_k}(t^*(0))$. The definition of r-decreasing and $t^*(0)$ can be found in [10]. Hence, given some conditions, constraint (6) is a biconvex constraint on $[1-\epsilon, 1)$. One can use the sequential convex approach to solve this problem. However, $\Phi^{-1}_{\varphi_k}(z^-_k)$ cannot be expressed analytically, it is still not easy to compute the optimal z^-_k with fixed t_j. In this paper, we use the piecewise linear approximation method proposed in [13], and its modified approximation method to find tight lower and upper bounds of (6).

We first make the standard variable transformation $r_j = \log(t_j)$, $j = 1,\dots,M$ to (6) in order to get an equivalent formulation of (6)

$$\sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k}\exp\left\{\sum_{j=1}^{M}(a^k_{ij}r_j + a^k_{pj}r_j) + 2\log(\Phi^{-1}_{\varphi_k}(z^-_k))\right\}} + \sum_{i=1}^{I_k}\mu^k_i\exp\left\{\sum_{j=1}^{M}a^k_{ij}r_j\right\} \le \beta_k,\ k = 1,\dots,K. \tag{22}$$

Lemma 1. *Given Assumption 2,* $\log(\Phi^{-1}_{\varphi_k}(z^-_k))$ *is monotone increasing and convex on* $[1-\epsilon, 1)$.

Proof. The first order and second order derivatives of $\log(\Phi_{\varphi_k}^{-1}(z_k^-))$ are

$$\frac{d(\log(\Phi_{\varphi_k}^{-1}(z_k^-)))}{dz_k^-} = \frac{1}{\Phi_{\varphi_k}^{-1}(z_k^-)\phi_{\varphi_k}(\Phi_{\varphi_k}^{-1}(z_k^-))},$$

and

$$\frac{d^2(\log(\Phi_{\varphi_k}^{-1}(z_k^-)))}{d(z_k^-)^2} = -\frac{1+\Phi_{\varphi_k}^{-1}(z_k^-)\frac{\phi'_{\varphi_k}(\Phi_{\varphi_k}^{-1}(z_k^-))}{\phi_{\varphi_k}(\Phi_{\varphi_k}^{-1}(z_k^-))}}{(\Phi_{\varphi_k}^{-1}(z_k^-)\phi_{\varphi_k}(\Phi_{\varphi_k}^{-1}(z_k^-)))^2}.$$

According to Assumption 2, $\frac{\phi'_{\varphi_k}(\Phi_{\varphi_k}^{-1}(1-\epsilon))}{\phi_{\varphi_k}(\Phi_{\varphi_k}^{-1}(1-\epsilon))}\Phi_{\varphi_k}^{-1}(1-\epsilon) < -1$, $k = 1, \ldots, K$. As $\frac{\phi'_{\varphi_k}(\Phi_{\varphi_k}^{-1}(x))}{\phi_{\varphi_k}(\Phi_{\varphi_k}^{-1}(x))}\Phi_{\varphi_k}^{-1}(x)$ is decreasing for $x \in [1-\epsilon, 1)$, we have $1+\Phi_{\varphi_k}^{-1}(z_k^-)\frac{\phi'_{\varphi_k}(\Phi_{\varphi_k}^{-1}(z_k^-))}{\phi_{\varphi_k}(\Phi_{\varphi_k}^{-1}(z_k^-))} < 0$, for $z_k^- \in [1-\epsilon, 1)$, $k = 1, \ldots, K$. Hence, the second order derivative is larger than or equal to 0, and $\log(\Phi_{\varphi_k}^{-1}(z_k^-))$ is convex on $[1-\epsilon, 1)$, $k = 1, \ldots, K$. Finally, the monotonicity follows directly from the non-negativeness of the first order derivative.

□

From the convexity and monotonicity, we can use the piecewise linear approximation methods introduced in the last section to find tight piecewise linear approximations for $\log(\Phi_{\varphi_k}^{-1}(z_k^-))$. We choose S different linear functions:

$$G_{s,k}^L(z_k^-) = l_{s,k}z_k^- + q_{s,k},\ \ s = 1, \ldots, S,$$

which are the tangent segments of $\log(\Phi_{\varphi_k}^{-1}(z_k^-))$ at $\xi_1, \xi_2, \ldots, \xi_S$, respectively. We have

$$l_{s,k} = \frac{1}{\phi_{\varphi_k}(\Phi_{\varphi_k}^{-1}(\xi_s))}$$

and

$$q_{s,k} = \Phi_{\varphi_k}^{-1}(\xi_s) - \frac{\xi_s}{\phi_{\varphi_k}(\Phi_{\varphi_k}^{-1}(\xi_s))},\ \ s = 1, \ldots, S.$$

Then, we use the piecewise linear function

$$G_k^L(z_k^-) = \max_{s=1,\ldots,S} G_{s,k}^L(z_k^-),$$

to approximate $\log(\Phi_{\varphi_k}^{-1}(z_k^-))$, and derive the following convex approximation of (22):

$$\begin{cases} \displaystyle\sum_{i=1}^{I_k} \mu_i^k \exp\left\{\sum_{j=1}^{M} a_{ij}^k r_j\right\} + \sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \exp\left\{\sum_{j=1}^{M}(a_{ij}^k r_j + a_{pj}^k r_j) + 2\widetilde{\omega}_k^L\right\}} \\ \qquad \le \beta_k,\ k = 1,\dots,K, \\ l_{s,k} z_k^- + q_{s,k} \le \widetilde{\omega}_k^L,\ s = 1,\dots,S,\ k = 1,\dots,K. \end{cases} \tag{23}$$

As $\log(\Phi_{\varphi_k}^{-1}(z_k^-))$ is convex, (23) provides an inner approximation.

Moreover, we use the segments

$$G_{s,k}^U(z_k^-) = \tilde{l}_{s,k} z_k^- + \tilde{q}_{s,k},\ s = 1,\dots,S,$$

between $\tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_{S+1}$ to form a piecewise linear function

$$G_k^U(z_k^-) = \max_{s=1,\dots,S} G_{s,k}^U(z_k^-).$$

Here,

$$\tilde{l}_{s,k} = \frac{\log(\Phi_{\varphi_k}^{-1}(\tilde{\xi}_{s+1})) - \log(\Phi_{\varphi_k}^{-1}(\tilde{\xi}_s))}{\tilde{\xi}_{s+1} - \tilde{\xi}_s},$$

and

$$\tilde{q}_{s,k} = -\tilde{l}_{s,k}\tilde{\xi}_s + \log(\Phi_{\varphi_k}^{-1}(\tilde{\xi}_s)),\ s = 1,\dots,S.$$

Using the piecewise linear function $G_k^U(z_k^-)$ to replace $\log(\Phi_{\varphi_k}^{-1}(z_k^-))$ in (22) gives the following convex approximation of the constraint (22):

$$\begin{cases} \displaystyle\sum_{i=1}^{I_k} \mu_i^k \exp\left\{\sum_{j=1}^{M} a_{ij}^k r_j\right\} + \sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \exp\left\{\sum_{j=1}^{M}(a_{ij}^k r_j + a_{pj}^k r_j) + 2\widetilde{\omega}_k^U\right\}} \\ \qquad \le \beta_k,\ k = 1,\dots,K, \\ \tilde{l}_{s,k} z_k^- + \tilde{q}_{s,k} \le \widetilde{\omega}_k^U,\ s = 1,\dots,S,\ k = 1,\dots,K. \end{cases} \tag{24}$$

As $\log(\Phi_{\varphi_k}^{-1}(z_k^-))$ is convex, (24) provides an outer approximation.

3.3. Main result

Theorem 2. *Given Assumptions 1 and 2, we have the following convex approximations for the joint rectangular geometric chance constrained programs (SGP):*

$$
\begin{aligned}
&(SGP_L)\\
&\min_{r,z^+,z^-,x,\omega^L,\tilde{\omega}^L} \quad \sum_{i=1}^{I_0} \mu_i^0 \exp\left\{\sum_{j=1}^{M} a_{ij}^0 r_j\right\}\\
&\text{s.t.} \quad 2\alpha_k \sum_{i=1}^{I_k} \mu_i^k \exp\left\{\sum_{j=1}^{M} a_{ij}^k r_j\right\} + \sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \exp\left\{\sum_{j=1}^{M}(a_{ij}^k r_j + a_{pj}^k r_j)\right.\\
&\qquad\qquad \left. + \omega_{i,p,k}^L\right\} \le \alpha_k^2,\ k = 1,\dots,K,\\
&\qquad d_{s,i,p,k} z_k^+ + b_{s,i,p,k} \le \omega_{i,p,k},\ s = 1,\dots,S,\ i,p = 1,\dots,I_k,\ k = 1,\dots,K,\\
&\qquad \sum_{i=1}^{I_k} \mu_i^k \exp\left\{\sum_{j=1}^{M} a_{ij}^k r_j\right\} + \sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \exp\left\{\sum_{j=1}^{M}(a_{ij}^k r_j + a_{pj}^k r_j) + 2\widetilde{\omega}_k^L\right\}}\\
&\qquad\qquad \le \beta_k,\ k = 1,\dots,K,\\
&\qquad l_{s,k} z_k^- + q_{s,k} \le \widetilde{\omega}_k^L,\ s = 1,\dots,S,\ k = 1,\dots,K,\\
&\qquad z_k^+ + z_k^- - 1 \ge e^{x_k},\ 0 \le z_k^+, z_k^- \le 1,\ k = 1,\dots,K,\\
&\qquad \sum_{k=1}^{K} x_k \ge \log(1-\epsilon),\ x_k \le 0,\ k = 1,\dots,K.
\end{aligned}
$$

(SGP_U)

$$\min_{r,z^+,z^-,x,\omega^U,\tilde{\omega}^U} \quad \sum_{i=1}^{I_0} \mu_i^0 \exp\left\{\sum_{j=1}^{M} a_{ij}^0 r_j\right\}$$

$$\text{s.t.} \quad 2\alpha_k \sum_{i=1}^{I_k} \mu_i^k \exp\left\{\sum_{j=1}^{M} a_{ij}^k r_j\right\} + \sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \exp\left\{\sum_{j=1}^{M}(a_{ij}^k r_j + a_{pj}^k r_j) + \omega_{i,p,k}^U\right\} \le \alpha_k^2, \ k = 1,\ldots,K,$$

$$\tilde{d}_{s,i,p,k} z_k^+ + \tilde{b}_{s,i,p,k} \le \omega_{i,p,k}^U, \ s = 1,\ldots,S, \ i,p = 1,\ldots,I_k, \ k = 1,\ldots,K,$$

$$\sum_{i=1}^{I_k} \mu_i^k \exp\left\{\sum_{j=1}^{M} a_{ij}^k r_j\right\} + \sqrt{\sum_{i=1}^{I_k}\sum_{p=1}^{I_k} \exp\left\{\sum_{j=1}^{M}(a_{ij}^k r_j + a_{pj}^k r_j) + 2\widetilde{\omega}_k^U\right\}} \le \beta_k, \ k = 1,\ldots,K,$$

$$\tilde{l}_{s,k} z_k^- + \tilde{q}_{s,k} \le \widetilde{\omega}_k^U, \ s = 1,\ldots,S, \ k = 1,\ldots,K,$$

$$z_k^+ + z_k^- - 1 \ge e^{x_k}, \ 0 \le z_k^+, z_k^- \le 1, \ k = 1,\ldots,K,$$

$$\sum_{k=1}^{K} x_k \ge \log(1-\epsilon), \ x_k \le 0, \ k = 1,\ldots,K.$$

The optimal value of the approximation problem (SGP_L) is a lower bound of problem (SGP). The optimal value of the approximation problem (SGP_U) is an upper bound of problem (SGP). Moreover, when S goes to infinity, both (SGP_L) and (SGP_U) are reformulations of (SGP).

Proof. (SGP_L) is obtained from the reformulation (SGP_r) of (SGP) and the two outer approximations (20) and (23). Besides, we transform the variable y into $x_k = \log(y_k), \ k = 1,\ldots,K$. The outer approximations guarantee that the feasible region of (SGP_L) contains the feasible region of (SGP). Hence the optimal value of (SGP_L) is a lower bound of problem (SGP).

Meanwhile, (SGP_I) is obtained from the reformulation (SGP_r) of (SGP), the two inner approximations (21) and (24) and the variable transformation $x_k = \log(y_k), \ k = 1,\ldots,K$. The inner approximations guarantee that the feasible region of (SGP_U) is contained in the feasible region of (SGP). Hence the optimal value of (SGP_U) is an upper bound of problem (SGP).

Moreover, when S goes to infinity, $F^L(z_k^+)$ and $F^U(z_k^+)$ are close enough to $\log((\Phi_{\varphi_k}^{-1}(z_k^+))^2\sigma_{i,p}^k - \mu_i^k\mu_p^k)$, and $G^L(z_k^-)$ and $G^U(z_k^-)$ are close enough to $\log(\Phi_{\varphi_k}^{-1}(z_k^-))$. From the convexity of the six terms, the distance between the feasible sets of (SGP_L) and (SGP) (of (SGP_U) and (SGP)) is small enough when S goes to infinity. This means (SGP_L) and (SGP_U) are both reformulations of (SGP) when S goes to infinity.

□

Both (SGP_L) and (SGP_U) are convex programming problems. Interior point methods can be used to solve them efficiently.

4. Numerical experiments

We test the performances of our approximation methods by considering a stochastic rectangular shape optimization problem with a joint chance constraints.

We maximize $hw\zeta$, the volume of a box-shaped structure with height h, width w and depth ζ. We have a joint rectangular chance constraint on the total wall area $2(hw + h\zeta)$, and the floor area $w\zeta$.

$$\mathbb{P}\begin{pmatrix} \alpha_{wall}A_{wall} \le 2hw + 2h\zeta \le \beta_{wall}A_{wall} \\ \alpha_{flr}A_{flr} \le w\zeta \le \beta_{flr}A_{flr} \end{pmatrix} \ge 1 - \epsilon.$$

Here the upper and lower limits on the total wall area and the floor area are considered as random variables. Meanwhile, we suppose the upper limits are proportional to the lower limits. Moreover, there are some lower and upper bounds on the aspect ratios h/w and w/ζ. This example is a generalization of the shape optimization problem with random parameters in [13]. It can be formulated as a standard rectangular geometric stochastic program as follows:

$$\begin{aligned} (SCP) \quad \min_{h,w,\zeta} \quad & h^{-1}w^{-1}\zeta^{-1} \\ \text{s.t.} \quad & \mathbb{P}\begin{pmatrix} \alpha_{wall} \le (2/A_{wall})hw + (2/A_{wall})h\zeta \le \beta_{wall} \\ \alpha_{flr} \le (1/A_{flr})w\zeta \le \beta_{flr} \end{pmatrix} \ge 1 - \epsilon, \\ & \gamma_{wh}h^{-1}w \le 1, \ (1/\gamma_{hw})hw^{-1} \le 1, \\ & \gamma_{w\zeta}w\zeta^{-1} \le 1, \ (1/\gamma_{\zeta w})w^{-1}\zeta \le 1. \end{aligned}$$

In our experiments, we set $\gamma_{wh} = \gamma_{w\zeta} = 0.5$, $\gamma_{hw} = \gamma_{\zeta w} = 2$, $\epsilon = 5\%$, $\alpha_{wall} = 1$, $\beta_{wall} = 2$, $\alpha_{flr} = 1$ and $\beta_{flr} = 2$.

We first test normal distribution in the elliptical distribution group. We assume $2/A_{wall} \sim N(0.01, 0.01)$, $1/A_{flr} \sim N(0.01, 0.01)$, and $2/A_{wall}$ and $1/A_{flr}$ are independent.

For this simple example, we can easily verity that Assumption 2 holds. We solve six groups of piecewise linear approximation problems, (SGP_U) and (SGP_L), to compute six groups of lower and upper bounds for problem (SCP).

The first column in Table 1 gives the number of segments S used in (SGP_U) and (SGP_L). The second and third columns give the numbers of variables and the numbers of constraints of (SGP_U), respectively. The sixth and seventh columns give the numbers of variables and the numbers of constraints of (SGP_L), respectively. The forth and the fifth columns give the upper bounds and the CPU times of (SGP_U), respectively. The eighth and the ninth columns give the lower bounds and the CPU times of (SGP_L), respectively. We use Sedumi solver from CVX package [9] to solve the approximation problems with Matlab R2012b, on a PC with a 2.6 Ghz Intel Core i7-5600U CPU and 12.0 GB RAM. For better illustration, we compute the gaps of the two piecewise linear approximation bounds, which are the percentage differences between these lower bounds and the upper bound, and show them in the last column.

[Table 1 near here]

We then assume $2/A_{wall}$ follows a Student's t distribution with the location parameter $\mu_{2/A_{wall}} = 0.01$, the scale parameter $\Gamma_{2/A_{wall}} = 0.01$ and the degree of freedom $v_{2/A_{wall}} = 4$. $1/A_{flr}$ follows a Student's t distribution with the location parameter $\mu_{1/A_{flr}} = 0.01$, the scale parameter $\Gamma_{1/A_{flr}} = 0.01$ and the degree of freedom $v_{1/A_{flr}} = 4$. We further assume that $2/A_{wall}$ and $1/A_{flr}$ are pairwise independent.

We solve seven groups of piecewise linear approximation problems, (SGP_U) and (SGP_L), to compute seven groups of lower and upper bounds for problem (SCP). Similarly to Table 1, we show the number of segments, the bounds, the CPU times, the problem scales and the gaps in Table 2.

[Table 2 near here]

From Table 1, we can see that as the number of segments S increases, the gap of the corresponding piecewise linear approximation bounds for the problem with normal distribution becomes smaller. When the number of segments is equal to 10, the gap is tight. For the Student's t distribution in Table 2, the gap decreases when the number of segments increases. When $S = 500$, the gap is very small. As the Student's t distribution has heavier tail than the normal distribution, the convergence rate of the piecewise linear approximations in the tail part is not as fast as for the normal distribution.

Notice that the CPU time does not increase proportionally with the increase of S. When $S = 500$, the CPU time is less than 6 second.

Acknowledgement

This work was supported by the Programme Cai Yuanpei under grant number 34593YE; National Natural Science Foundation of China under Grant Numbers 71371152 and 11571270.

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Table 1: Computational results of approximations for normal distribution

S	Var.	Con.	UB	CPU(s)	Var.	Con.	LB	CPU(s)	Gap(%)
1	16	18	5.9119	1.2795	19	19	5.7587	1.2102	2.66
2	16	25	5.8188	0.9406	19	26	5.7587	1.0237	1.04
5	16	46	5.7644	0.9360	19	47	5.7639	1.0676	0.01
10	16	81	5.7645	1.1247	19	82	5.7643	0.9494	0.00
20	16	151	5.7644	1.2374	19	152	5.7643	1.2228	0.00
100	16	711	5.7644	2.0650	19	712	5.7643	1.9789	0.00

Table 2: Computational results of approximations for Student's t distribution

S	Var.	Con.	UB	CPU(s)	Var.	Con.	LB	CPU(s)	Gap(%)
1	16	18	13.8794	1.1772	19	19	5.8498	1.2815	137.26
2	16	25	8.8903	1.0984	19	26	5.8498	1.1373	51.98
5	16	46	6.0468	0.9796	19	47	5.8699	0.8857	3.01
10	16	81	5.9111	1.0510	19	82	5.8716	1.0800	0.67
20	16	152	5.8915	1.2446	19	152	5.8717	1.0739	0.34
100	16	711	5.8760	2.1234	19	712	5.8717	1.8112	0.07
500	16	3511	5.8725	5.9124	19	3512	5.8717	5.7727	0.01