

An Extension of Chubanov's Polynomial-Time Linear Programming Algorithm to Second-Order Cone Programming

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Abstract

Recently, Chubanov proposed an interesting new polynomial-time algorithm for linear program. In this paper, we extend his algorithm to second-order cone programming.

Key words: Chubanov's algorithm, Linear programming, Second-order cone programming

1 Introduction

In linear programming, the ellipsoid method [8, 9] and the interior-point method [7, 13] were the only two algorithms which enjoy polynomiality for a long time. Recently, an interesting new polynomial-time algorithm was proposed by Chubanov [3, 4, 5]. Related studies include, for instance, [2, 10, 15, 16]. In this paper, we develop a word-by-word extension of Chubanov's algorithm to second-order cone programming [1, 6, 12, 13, 14, 17]. Among the related works, Peña and Soheili [15] developed a polynomial-time projection and rescaling algorithm for a symmetric cone feasibility problem. Their algorithm utilizes Chubanov's idea and is closely related to ours in its direction. We briefly compare the two approaches later to highlight the difference.

Consider the following homogeneous second-order cone programming feasibility problem (P):

$$(P) \quad \text{find}_{(x_1; \dots; x_n)} \sum A_i x_i = 0, \quad x_i \in \mathcal{K}_i, \quad i = 1, \dots, n,$$

where $A_i \in \mathbb{R}^{m \times d_i}$ for each i and $\mathcal{K}_i \subseteq \mathbb{R}^{d_i}$ is either a half-line or a second-order cone, i.e.,

$$\begin{aligned} \mathcal{K}_i &= \{x_i \in \mathbb{R} \mid x_i \geq 0\} \quad (\text{if } d_i = 1), \\ \mathcal{K}_i &= \{x_i = (x_{i0}; x_{i1}) \in \mathbb{R} \times \mathbb{R}^{d_i-1} \mid \|x_{i1}\| \leq x_{i0}\} \quad (\text{if } d_i \geq 2). \end{aligned}$$

We assume that vectors are in column form by default and the vertical concatenation of two vectors a and b is written as $(a; b)$. We denote by SOC and \mathcal{LI} the set of indices i where \mathcal{K}_i is a second-order cone and a half-line, respectively. The dual problem (D) is

$$(D) \quad \text{find}_{(s_1; \dots; s_n)} s_i = - \sum A_i^T u, \quad s_i \in \mathcal{K}_i, \quad i = 1, \dots, n, \quad u \in \mathbb{R}^m.$$

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Throughout this paper we use a notation analogous to $(x_{i0}; x_{i1}) \in \mathcal{K}_i$ concerning a second-order cone. When we deal with a vector in a space where a second-order cone is defined, the first element x_{i0} with “the index 0” always represents the center axis of a second-order cone, and the second element x_{i1} with the “index 1” represents the rotational part, unless otherwise stated. In the following, for a cone $\tilde{\mathcal{K}}$, say, we use the notations $x \succeq y$ and $x \succ y$ to mean that $x - y \in \tilde{\mathcal{K}}$ and $x - y \in \text{int}(\tilde{\mathcal{K}})$, respectively. Letting $A = (A_1, \dots, A_n)$, $\mathcal{K} = \mathcal{K}_1 \times \dots \times \mathcal{K}_n$, (P) and (D) are written as

$$(P) \text{ find}_x Ax = 0, \quad x \succeq 0, \quad (D) \text{ find}_{(s,u)} s = -A^T u, \quad s \succeq 0,$$

where $x = (x_1; \dots; x_n) \in \mathbb{R}^{\bar{n}}$, $s = (s_1; \dots; s_n) \in \mathbb{R}^{\bar{n}}$, $y \in \mathbb{R}^m$ and $\bar{n} = \sum_{i=1}^n d_i$. For simplicity, we assume that $A \in \mathbb{R}^{m \times \bar{n}}$ is row independent.

By generalized Gordan’s theorem, (P) has an interior feasible solution if and only if (D) does not have a nonzero solution (i.e., zero is the only solution to (D)), and (D) has an interior feasible solution if and only if (P) does not have a nonzero solution. If we let

$$(GP) \text{ find}_x Ax = 0, \quad x \succ 0, \quad (GD) \text{ find}_s s = -A^T u, \quad s \succeq 0, \quad s \neq 0, \quad u \in \mathbb{R}^m,$$

the Generalized Gordan’s Theorem says that (GP) has a solution if and only if (GD) does not have a solution and (GD) has a solution if and only if (GP) does not have a solution.

Given a matrix B , say, let P_B be an orthogonal projection matrix to $\text{Ker}(B)$. If B is a row independent matrix, $P_B = I - B^T(BB^T)^{-1}B$. The problem (P) is written as

$$\text{find}_x x = P_A y, \quad x \succeq 0, \quad y \in \mathbb{R}^{\bar{n}}.$$

and the problem (D) is written as

$$\text{find}_y P_A y = 0, \quad y \succeq 0,$$

(where “the free variable” u is eliminated).

We will develop a polynomial-time algorithm for finding a solution to (GP) or (GD) or detecting no ε -interior feasible solution exists to (P) (the definition of ε -interior feasible solution is given below). In Appendix we describe how we can solve approximately a general second-order cone program with a primal-dual interior feasible solution by the algorithm developed in this paper.

The problem (P) is equivalent to finding an interior feasible solution to the following system.

$$Ax = 0, \quad \|x\|_\infty \leq 1, \quad x \succeq 0.$$

We denote by \mathcal{F} the set of feasible solutions to this system. We define the projection \mathcal{F}_i of \mathcal{F} onto the block i as follows:

$$\mathcal{F}_i = \{x_i \in \mathcal{K}_i \subset \mathbb{R}^{d_i} \mid x \in \mathcal{F}\}. \quad (1)$$

For a point $x \in \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_n}$, its minimum eigenvalue $\lambda_{\min}(x)$ is defined as

$$\lambda_{\min}(x) = \min\left(\min_{i \in \mathcal{L}\mathcal{I}} x_i, \min_{i \in \mathcal{S}\mathcal{O}\mathcal{C}} x_{i0} - \|x_{i1}\|\right).$$

A point $x \in \mathcal{K}$ is called an ε -interior point of \mathcal{K} if $\lambda_{\min}(x) \geq \varepsilon$. We define the maximum eigenvalue $\lambda_{\max}(x)$ as

$$\lambda_{\max}(x) = \max(\max_{i \in \mathcal{LI}} x_i, \max_{i \in \mathcal{SOC}} x_{i0} + \|x_{i1}\|).$$

If $x \in \mathcal{K}$, then the following equivalence relation holds between $\|\cdot\|_{\infty}$ and $\lambda_{\max}(\cdot)$:

$$\|x\|_{\infty} \leq \lambda_{\max}(x) \leq 2\|x\|_{\infty} \quad (2)$$

For $x_i \in \mathcal{K}_i$, the following quantity is called the determinant of x_i :

$$\det(x_i) \equiv x_i \quad \text{if } i \in \mathcal{LI} \quad \text{and} \quad \det(x_i) \equiv x_{j0}^2 - \|x_{j1}\|^2 \quad \text{if } i \in \mathcal{SOC}.$$

The determinant of $x \in \mathcal{K}$ is defined as:

$$\det(x) \equiv \prod_{i \in \mathcal{LI}} x_i \prod_{j \in \mathcal{SOC}} (x_{j0}^2 - \|x_{j1}\|^2).$$

Let $e = (e_1; \dots; e_n)$, where

$$e_i = 1 \text{ if } i \in \mathcal{LI}, \quad e_i = (1; 0_{d_i-1}) \in \mathbb{R} \times \mathbb{R}^{d_i-1} \text{ if } i \in \mathcal{SOC}.$$

Here 0_{d_i-1} denotes the d_i-1 dimensional zero vector and we use analogous notation onwards. We have the following proposition.

Proposition 1.1 *Let $x \in \mathbb{R}^{\bar{n}}$. The following relations hold:*

1. $\lambda_{\min}(x) \geq \varepsilon \Leftrightarrow x \succeq \varepsilon e$.
2. $M \geq \lambda_{\max}(x) \Leftrightarrow Me \succeq x$.

Let $\varepsilon > 0$. A point $x \in \mathbb{R}^{\bar{n}}$ is called an ε -interior-feasible solution to (P) if the following condition is satisfied:

$$Ax = 0, \quad \lambda_{\min}(x) \geq \varepsilon, \quad \text{or equivalently,} \quad Ax = 0, \quad x \succeq \varepsilon e.$$

We develop a polynomial-time algorithm to find an interior feasible solution to (P) or a nonzero feasible solution to (D), or conclude that no ε -interior feasible solution exists to (P). The algorithm terminates in $O(n \log \varepsilon^{-1})$ iterations of a procedure called a basic procedure. The basic procedure requires $O(n^3 \bar{n} \max_i d_i + m \bar{n}^2)$ arithmetic operations (assuming that the standard linear algebraic procedures are employed). Therefore, the algorithm terminates in $O((n^4 \bar{n} \max_i d_i + nm \bar{n}^2) \log \varepsilon^{-1})$ arithmetic operations. The basic procedure is a heart of Chubanov's algorithm.

In the following, we explain our algorithm in comparison with Chubanov's original algorithm, and discuss the difference between our algorithm and Peña and Soheili's algorithm. Chubanov's algorithm is to find a point in the intersection of a linear space and a unit hypercube, i.e., the direct product of 0-1 segments. For simplicity, we assume that the system is interior feasible. The algorithm first performs the basic procedure. The basic procedure either (i) finds an interior feasible solution, or (ii) detects a variable whose value cannot be

greater than $1/2$ in the feasible region. Detection is done by finding a “cut”, a hyperplane to cut off the region where no feasible solution exists. Once such “cut” is found, then the associated coordinate is rescaled by a factor of two so that the hypercube is recovered, to continue the same procedure. In terms of the original coordinate, this process is regarded as generating a series of shrinking convex bodies of the same type, i.e., a hyper-rectangle, which enclose feasible solutions.

Now we illustrate our algorithm. For the ease of understanding, we assume that \mathcal{K} is just a single second-order cone and that there exists an interior feasible solution. We let $\varepsilon = 0$. Let \mathcal{F} be the intersection of the feasible solution to (P) and $\mathcal{C}_S = \{x \in \mathcal{K} \mid x_0 \leq 1\}$, which we call “the standard truncated second-order cone.” The algorithm is to find an interior feasible solution in \mathcal{F} . To this end, first we perform the basic procedure. The basic procedure either (i) finds an interior solution to (P), or (ii) finds a hyperplane called a “cut” $\{x \mid w^T x = w^T v\}$. The cut defines an obliquely truncated second-order cone $\mathcal{C}_O = \{x \mid w^T x \leq w^T v, x \in \mathcal{K}\}$. This cut is a natural extension of the one by Chubanov, and is one of the key ideas of this paper. In virtue of the basic procedure, the set \mathcal{C}_O contains \mathcal{F} and has smaller volume than \mathcal{C}_S at least by a constant factor. Thus, if a cut is found, we can shrink the region where the feasible solutions exist. Then, \mathcal{C}_O is transformed to \mathcal{C}_S with an automorphism transformation of the cone \mathcal{K} . We apply the same procedure to the transformed problem, and repeat it over and over. This way, the algorithm constructs a series of shrinking obliquely truncated second-order cone which contains a nonzero feasible solution to (P). It is shown that the volume of obliquely truncated second-order cone converges linearly to zero. Therefore, if there exists an interior feasible solution, then shrinkage cannot last forever and the algorithm and the basic procedure ends with (i) at a certain point. This is a rough sketch of the algorithm, and the idea will be generalized to the multiple cone case in the rest of this paper.

Interestingly, while the new algorithm has similarity to the ellipsoid method in the sense that it generates a series of shrinking convex bodies of the same type, it has some flavor of the interior-point method in that it utilizes the automorphism group of the cone. The idea of the cut and the basic procedure is two key concepts in Chubanov’s algorithm, and will be extended to the second-order cone programming case in this paper. As is readily seen from the above explanations of the two algorithms, our algorithm is a word-by-word generalization of Chubanov’s algorithm.

Peña and Soheili [15] developed a polynomial-time projection and rescaling algorithm for the symmetric cone feasibility problem. Their algorithm consists of rescaling step and the basic procedure to find a vector for rescaling, where rescaling procedure is inspired by Chubanov’s idea. They measure the progress of the algorithm with a condition number of the system which is essentially the determinant. The condition number is bounded above by one, and the system whose condition number is closer to one is better conditioned. In their algorithm, the condition number is increased by a constant factor at each iteration by rescaling, or the algorithm finds an interior feasible solution. The Chubanov’s cut vector is used as an algebraic machinery to rescale the system properly. Their algorithm plays with scaling (or metric), but does not change the shape of the region on focus. This makes a remarkable contrast with our approach as we argue below.

Our algorithm uses the cut to confine the region of existence of the feasible solutions and generates a series of shrinking convex bodies of the same type containing the feasible region.

In this regard, our algorithm is geometrically intuitive and can be considered as a direct and word-by-word extension of Chubanov’s algorithm. In our algorithm, we measure the progress of the algorithm with the volume of the shrinking area of existence of the feasible solutions, which is essentially the determinant. Thus, the determinant plays crucial roles in the both algorithms and they share some features in common.

The paper is organized as follows. In Section 2, we introduce the second-order cone and its automorphism group, and study some basic properties of the truncated second-order cones. In Section 3, we discuss an extension of Chubanov’s fundamental relation in the context of second-order cone programming. In Section 4, we extend and analyze the basic procedure. In Sections 5, we develop the main algorithm. In Section 6, we make some remarks. Section 7 is a conclusion.

Note added at the Second Revision (January 2017):

We removed “Section 6: Full Version” of the paper, because we found a flaw in its complexity analysis. The intension of that section was to reduce the complexity by a factor of n from the algorithm in Section 5 by adapting Chubanov’s elegant idea [5] of initiating a basic procedure using the second last iterate of the preceding basic procedure. We realized that the argument we made in the previous version does not work. We feel very sorry to the readers about this mistake, but we consider that the main part of the paper, development of an extension of Chubanov’s algorithm to second-order cone program and its polynomial-time complexity analysis, yet survives.

Note added at the First Revision (December 2016):

1. We corrected a nontrivial error in evaluating complexity of the basic procedure. In the first version released in November 2016, we conducted analysis assuming that one iteration of the basic procedure can be done in $O(\bar{n})$ arithmetic operations like in the case of linear program. But later we realized that the argument in the first version was not correct and that one iteration of the basic procedure requires $O(\bar{n}d_i)$ arithmetic operations. At Step 5 of the basic procedure, we compute $P_A\eta$ and this requires $O(\bar{n}d_i)$ arithmetic operations. This affects overall complexity estimate of the entire algorithms. We corrected them accordingly. We feel very sorry for the confusion caused by this flaw.
2. We refer the reference [15] and added related considerations in this introduction. We also updated references and corrected some misleading statements related to [16]. A few (easily fixable) mathematical errors are also corrected.

2 Preliminary Observations

We assume that \mathcal{K} is a d -dimensional second-order cone ($d \geq 2$). For $w, v \in \mathbb{R}^d$, we define

$$H(w, v) = \{x \in \mathbb{R}^d \mid w^T x \leq v^T x\},$$

i.e., $H(w, v)$ is the half space in \mathbb{R}^d whose boundary normal vector is w and v is on the boundary. The boundary hyperplane of $H(w, v)$ is written as $\partial H(w, v)$.

Let $e = (1; 0_{d-1}) \in \mathbb{R} \times \mathbb{R}^{d-1}$. The intersection of the second-order cone and the half space $H(e, e) = \{(x_0; x_1) \in \mathbb{R} \times \mathbb{R}^{d-1} \mid x_0 \leq 1\}$ is referred to as the standard truncated

second-order cone (S-TSOC). We denote by V_d the volume of k -dimensional S-TSOC. Its concrete formula is:

$$V_d = \frac{\pi^{(d-1)/2}}{(d-1)\Gamma(\frac{d-1}{2} + 1)},$$

which is obtained by integrating the volume of $(d-1)$ -dimensional hypersphere from the radius 0 to 1.

Let $w \in \text{int}(\mathcal{K})$ and $v \in \text{int}(\mathcal{K})$. Then $\mathcal{K} \cap H(w, v)$ is a non-empty bounded domain which is obtained by cutting \mathcal{K} with a tilted hyperplane. This set is referred to as an obliquely truncated second-order cone (O-TSOC). We let

$$\mathcal{C}(w, v) = \{x | x \in \mathcal{K} \cap H(w, v)\}.$$

With this notation, S-TSOC is written as $\mathcal{C}(e, e)$.

The automorphism group of a cone $\tilde{\mathcal{K}}$ is the set of linear transformations \tilde{G} such that

$$\tilde{\mathcal{K}} = \tilde{G}\tilde{\mathcal{K}}.$$

We denote by $\text{Aut}(\tilde{\mathcal{K}})$ the automorphism group of $\tilde{\mathcal{K}}$.

Let $w \in \text{int}(\mathcal{K})$ and $v \in \mathbb{R}^d$ be such that $\mathcal{C}(w, v) \neq \emptyset$. In the following, we show that there exists an element G of $\text{Aut}(\mathcal{K})$ which maps S-TSOC $\mathcal{C}(e, e)$ to $\mathcal{C}(w, v)$. This G plays an important role throughout our algorithm development and analysis.

We start with the following statement.

Proposition 2.1 *If $G \in \mathbb{R}^{d \times d}$ satisfies the following conditions:*

1.

$$G^T \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix} G = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix}, \lambda > 0 \quad (3)$$

2. *There exists a point $\eta \in \text{int}(\mathcal{K})$ such that $G\eta \in \text{int}(\mathcal{K})$,*

then, $\mathcal{K} = G\mathcal{K}$, $\mathcal{K} = G^T\mathcal{K}$ and hence G and G^T are elements of $\text{Aut}(\mathcal{K})$.

Proof. We fix $\lambda = 1$ and show that if the condition 1 with $\lambda = 1$ and the condition 2 are satisfied, then $G \in \text{Aut}(\mathcal{K})$ holds. This is enough to prove the proposition with general $\lambda > 0$.

The main part of the proof is to show that G is invertible and $G\partial\mathcal{K} = \partial\mathcal{K}$, where $\partial\mathcal{K}$ is the boundary of \mathcal{K} . Once this is shown, $G\mathcal{K} = \mathcal{K}$ readily follows since G is a linear transformation. After this, we will proceed to demonstrate that $G^T \in \text{Aut}(\mathcal{K})$.

We prove that G is invertible and $G\partial\mathcal{K} = \partial\mathcal{K}$. The condition 1 immediately implies that G is an invertible matrix. Consider the image $G\partial\mathcal{K}$ where

$$\partial\mathcal{K} = \{(x_0; x_1) \in \mathbb{R} \times \mathbb{R}^{d-1} | x_0 = \|x_1\|, x_0 \geq 0\}.$$

Let $y(x) = Gx$ and let $x \in \partial\mathcal{K}$. Since $\lambda = 1$ and hence $y_0(x)^2 - \|y_1(x)\|^2 = x_0^2 - \|x_1\|^2$ holds, we have $y_0(x) = \|y_1(x)\|$ or $y_0(x) = -\|y_1(x)\|$. We show that the second case never occurs. Suppose that there exists a point $\hat{x} \in \partial\mathcal{K}$ such that $y_0(\hat{x}) = -\|y_1(\hat{x})\| < 0$. Consider the line $x(t) = (1-t)\eta + t\hat{x}$, and let $y(t) = Gx(t)$. Then, since $y(0) = G\eta \in \text{int}(\mathcal{K})$, we

have $y_0(0) > 0$ but $y_0(1) < 0$, yielding that $y_0(\hat{t}) = 0$ for some $0 \leq \hat{t} < 1$. This implies that $y_1(\hat{t}) = 0$ as well. However, since $x(\hat{t})$ is in the interior of \mathcal{K} , we have $x_0(\hat{t})^2 - \|x_1(\hat{t})\|^2 > 0$ whereas $y_0(\hat{t}) = \|y_0(\hat{t})\|$ and hence $y_0(\hat{t}) - \|y_0(\hat{t})\| = 0$, which is a clear contradiction to the condition 1. Thus, whenever $x_0 = \|x_1\|$, we have $y_0(x) = \|y_1(x)\|$. This shows that $G\partial\mathcal{K} \subseteq \partial\mathcal{K}$. If we take $G^{-1} := G$, G^{-1} satisfies the conditions 1 and 2. Therefore, we have $G^{-1}\partial\mathcal{K} \subseteq \partial\mathcal{K}$ and hence $\partial\mathcal{K} \subseteq G\partial\mathcal{K}$. Thus, we have shown $\partial\mathcal{K} = G\partial\mathcal{K}$. Since G is a linear transformation, $\mathcal{K} = G\mathcal{K}$ follows immediately.

Now we show that $G^T \in \text{Aut}(\mathcal{K})$. Let

$$E = \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix}.$$

Multiplying GE from the left on the both sides of (3) and by using $E^2 = I$ and that G is invertible, we obtain $GE G^T = E$. In order to show $G^T e \in \text{int}(\mathcal{K})$, we use the fact that $\eta \in \text{int}(\mathcal{K})$ if and only if $\eta^T e / \|\eta\| \|e\| > 1/\sqrt{2}$. We apply this by choosing $\eta = G^T e$. Since $Ge \in \text{int}(\mathcal{K})$, we have $e^T Ge / \|Ge\| \|e\| > 1/\sqrt{2}$. Then it follows that $e^T (G^T e) / \|G^T e\| \|e\| > 1/\sqrt{2}$. It remains to show that $\|Ge\| = \|G^T e\|$. Since $G^T EG = GE G^T = E$, we have

$$(Ge)_0^2 - \|(Ge)_1\|^2 = (G^T e)_0^2 - \|(G^T e)_1\|^2 = 1,$$

Since $(Ge)_0 = (G^T e)_0$, we have $\|Ge\| = \|G^T e\|$, and we are done. \blacksquare

In the following, we will find $G \in \text{Aut}(\mathcal{K})$ such that

$$\mathcal{C}(w, v) = G\mathcal{C}(e, e).$$

Since

$$\mathcal{C}(w, v) = H(w, v) \cap \mathcal{K} = G\mathcal{C}(e, e) = G(H(e, e) \cap \mathcal{K}) = (GH(e, e)) \cap G\mathcal{K} = (GH(e, e)) \cap \mathcal{K},$$

it is enough to find an element G of the automorphism group such that $H(w, v) = GH(e, e)$, and since $w^T v > 0$, this amounts to finding G such that $\partial H(w, v) = G\partial H(e, e)$ where $\partial H(w, v) = \{x | w^T x = w^T v\}$ and $\partial H(e, e) = \{u | e^T u = e^T e\} = \{u | u_0 = 1\}$.

Since $\partial H(e, e) = \{u | u \in (1; u_1), u_1 \in \mathbb{R}^{d-1}\}$, We have $G\partial H(e, e) = \{x = Gu | u \in (1; u_1), u_1 \in \mathbb{R}^{d-1}\}$. The tangent space of $G\partial H(e, e)$ is $T_1 = \{\Delta x = G(0, u_1) | u_1 \in \mathbb{R}^{d-1}\}$, and this should be equal to the tangent space $T_2 = \{\Delta x | w^T \Delta x = 0, \Delta x \in \mathbb{R}^d\}$ of $\partial H(w, v)$. Since $T_1 = T_2$ should hold,

$$w^T G \begin{pmatrix} 0 \\ u_1 \end{pmatrix} = 0 \quad \forall u_1 \in \mathbb{R}^{d-1}.$$

Therefore, we have $w^T G = (\lambda; 0)^T = \lambda e^T$ with $\lambda \neq 0$ (c.f. $0 \neq w \in \text{int}(\mathcal{K})$). This implies that $w = \lambda G^{-T} e$ and equivalently $\lambda e = G^T w$. Since $G \in \text{Aut}(\mathcal{G})$, so is G^T , then we have $\lambda e \in \text{int}(\mathcal{K})$ and hence $\lambda > 0$. Note that λG^{-T} is an element of $\text{Aut}(\mathcal{K})$ which maps e to w . Since $Ge \in \partial H(w, v) (= \{x | w^T x = w^T v\})$, we have $w^T Ge = w^T v$. Substituting $w = \lambda G^{-T} e$ into this formula, we obtain that $\lambda = w^T v$. In summary, if $G\mathcal{C}(e, e) = \mathcal{C}(w, v)$ and $G \in \text{Aut}(\mathcal{K})$, G should satisfy $w = w^T v G^{-T} e$. On the other hand, if we can find $G \in \text{Aut}(\mathcal{K})$ satisfying this condition, we have $G\mathcal{C}(e, e) = \mathcal{C}(w, v)$. In the following, we find $G \in \text{Aut}(\mathcal{K})$ satisfying the condition $w = w^T v G^{-T} e$.

Let $\alpha = w_0/\gamma$, $\beta = w_1/\gamma$, where $\gamma = \sqrt{w_0^2 - \|w_1\|^2}$, and let

$$\tilde{G} = \begin{pmatrix} \alpha & \beta^T \\ \beta & I + \frac{\beta\beta^T}{1+\alpha} \end{pmatrix}.$$

It is not difficult to check that \tilde{G} satisfies the conditions 1 and 2 in Proposition 2.1, being a member of $\text{Aut}(\mathcal{K})$. In particular, we see that $\det(\tilde{G}) = 1$ and $\tilde{G}e = (\alpha; \beta) = w/\gamma$. Hence we have $\gamma\tilde{G}e = w$. Since $w^T v G^{-T} e = w$, we let $w^T v G^{-T} = \gamma\tilde{G}$ and obtain

$$G = w^T v \gamma^{-1} \tilde{G}^{-1}.$$

By direct calculation it is easy to confirm that

$$\tilde{G}^{-1} = \begin{pmatrix} \alpha & -\beta^T \\ -\beta & I + \frac{\beta\beta^T}{1+\alpha} \end{pmatrix}.$$

Therefore, we have

$$G = \gamma^{-1} w^T v \tilde{G}^{-1} = (\alpha v_0 + \beta^T v_1) \begin{pmatrix} \alpha & -\beta^T \\ -\beta & I + \frac{\beta\beta^T}{1+\alpha} \end{pmatrix} \quad (4)$$

and

$$\text{vol}(\mathcal{C}(w, v)) = (\alpha v_0 + \beta^T v_1)^d V_d. \quad (5)$$

Thus, we obtain the following proposition.

Proposition 2.2 *Let $w = (w_0; w_1) \in \text{int}(\mathcal{K})$, $v \in \text{int}(\mathcal{K})$, and consider O-TSOC $\mathcal{C}(w, v) = \{x | w^T x \leq w^T v, x \in \mathcal{K}\}$. Then the matrix*

$$G = (\alpha; \beta)^T v \begin{pmatrix} \alpha & -\beta^T \\ -\beta & I + \frac{\beta\beta^T}{1+\alpha} \end{pmatrix},$$

where

$$\alpha = \frac{w_0}{\sqrt{w_0^2 - \|w_1\|^2}} \quad \text{and} \quad \beta = \frac{w_1}{\sqrt{w_0^2 - \|w_1\|^2}},$$

maps S-TSOC $\mathcal{C}(e, e)$ to $\mathcal{C}(w, v)$, i.e.,

$$\mathcal{C}(w, v) = G\mathcal{C}(e, e)$$

and

$$\text{vol}(\mathcal{C}(w, v)) = \left(\frac{w^T v}{\sqrt{w_0^2 - \|w_1\|^2}} \right)^d V_d.$$

Suppose that $v \in \text{int}(\mathcal{K})$ is given, and that we want to find $w = (w_0; w_1) \in \mathbb{R} \times \mathbb{R}^{d-1}$ which minimizes the volume of $\mathcal{C}(w, v)$. Since $w \in \text{int}(\mathcal{K})$, without loss of generality, we may assume that $w = (\alpha; \beta) \in \mathbb{R} \times \mathbb{R}^{d-1}$ satisfies $\alpha^2 - \|\beta\|^2 = 1$. Furthermore, due to

rotational symmetry with respect to the 0th axis, we assume that, without loss of generality, $v = (\xi_0; \xi_1; 0_{d-2}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-2}$. In order to minimize $\text{vol}(\mathcal{C}(w, v))$, we just minimize

$$\min \alpha \xi_0 + \beta_1 \xi_1, \quad \text{s.t. } \alpha^2 - \beta^T \beta = 1.$$

Solving this problem, we obtain that

$$\alpha = \frac{1}{\sqrt{1 - \eta^2}}, \quad \beta = -\frac{\eta}{\sqrt{1 - \eta^2}},$$

where $\eta = \xi_1/\xi_0$. and the optimal value is:

$$\sqrt{\xi_0^2 - \xi_1^2}.$$

This consideration is summarized as the following proposition:

Proposition 2.3 *Let $v \in \text{int}(\mathcal{K})$. A normal vector $w \in \text{int}(\mathcal{K})$ which minimizes the volume $\text{vol}(\mathcal{C}(w, v))$ is given as*

$$w = \begin{pmatrix} v_0 \\ -v_1 \end{pmatrix}$$

and

$$\text{vol}(\mathcal{C}(w, v)) = (v_0^2 - \|v_1\|^2)^{d/2} V_d.$$

Corollary 2.4 *Let $\hat{x} \in \text{int}(\mathcal{K})$. Then, the minimum volume O-TSOC containing \hat{x} is given as*

$$\mathcal{C}((\hat{x}_0; -\hat{x}_1), \hat{x}),$$

and hence the minimum volume is given as

$$\text{vol}(\mathcal{C}((\hat{x}_0; -\hat{x}_1), \hat{x})) = (\hat{x}_0^2 - \|\hat{x}_1\|^2)^{d/2} V_d.$$

Proof. Let us denote by $\mathcal{C}(w, v)$ an O-TSOC satisfying the condition. Then we can take $v = \hat{x}$, since, otherwise, we can make a parallel shift of the boundary hyperplane $\partial H(w, v)$ until it touches \hat{x} after the shift. Now we can apply the previous lemma to obtain the result. ■

Proposition 2.5 *Let $(x_0; x_1) \in \text{int}(\mathcal{K})$. If $\sqrt{x_0^2 - \|x_1\|^2} \leq \varepsilon$, then*

$$0 \leq x_0 - \|x_1\| \leq \varepsilon.$$

The strict inequality version of this relation also holds.

Proof. The relation obviously holds because

$$(x_0 - \|x_1\|)^2 \leq (x_0 + \|x_1\|)(x_0 - \|x_1\|) \leq x_0^2 - \|x_1\|^2 = \varepsilon^2.$$

■

Proposition 2.6 *Let \mathcal{F} and \mathcal{F}_i be as defined in Section 1. Let $w \in \text{int}(\mathcal{K}_i)$ and $v \in \text{int}(\mathcal{K}_i)$, and suppose that $\mathcal{F}_i \subseteq \mathcal{C}(w, v)$ and $\text{vol}(\mathcal{C}(w, v)) < \varepsilon^{d_i} V_{d_i}$. Then, there does not exist an ε -interior solution in \mathcal{F} .*

Proof. By contradiction, we assume that there exists an ε -interior solution \hat{x} , say, in \mathcal{F} . The i -th block $(\hat{x}_{i0}; \hat{x}_{i1})$ of this solution satisfies

$$\sqrt{\hat{x}_{i0}^2 - \|\hat{x}_{i1}\|^2} \geq \varepsilon$$

because $\hat{x}_{i0} + \|\hat{x}_{i1}\| \geq \hat{x}_{i0} - \|\hat{x}_{i1}\| \geq \varepsilon$ (the contraposition of Proposition 2.5). Since $\hat{x}_i \in \mathcal{F}_i \subseteq \mathcal{C}(w, v)$, $\mathcal{C}(w, v)$ is an O-TSOC containing \hat{x} and therefore, in view of Corollary 2.4,

$$\text{vol}(\mathcal{C}(w, v)) \geq (\hat{x}_{i0}^2 - \|\hat{x}_{i1}\|^2)^{d_i/2} V_{d_i} \geq \varepsilon^{d_i} V_{d_i}$$

holds, which is a contradiction to the initial assumption that $\text{vol}(\mathcal{C}(w, v)) < \varepsilon^{d_i} V_{d_i}$. \blacksquare

In the end of this section, we introduce a scaling operation of (P) and (D). Let $G_i \in \text{Aut}(\mathcal{K}_i)$ for $i = 1, \dots, n$, and consider the dual pair of the problems (SP) and (SD):

$$\text{(SP)} \quad \text{find}_{(\tilde{x}_1; \dots; \tilde{x}_n)} \sum \tilde{A}_i \tilde{x}_i = 0, \quad \tilde{x}_i \in \mathcal{K}_i, \quad i = 1, \dots, n,$$

where $\tilde{A}_i = A_i G_i$ for $i = 1, \dots, n$ and

$$\text{(SD)} \quad \text{find}_{(y, (\tilde{s}_1; \dots; \tilde{s}_n))} s_i = - \sum \tilde{A}_i^T y, \quad s_i \in \mathcal{K}_i, \quad i = 1, \dots, n.$$

(SP) and (SD) are mutually dual and they are obviously equivalent to (P) and (D). Following interior-point terminology, we call (SP) and (SD) “scaled problems.” In the algorithm developed in this paper, we mostly work with scaled problems (SP) and (SD). The original problems (P) and (D) appear only in the beginning and in the end of the algorithm.

3 Basic Lemma and its Consequences

We extend a fundamental relation established by Chubanov (Formula (2), Section 2.1, [5]) and its consequences to the second-order cone case. For notational convenience, we develop the results in terms of (P) and (D) in Section 1. Later we will apply the results in this section to scaled problems (SP) and (SD). It is easy to translate the results written in terms of (P) and (D) into the corresponding results in terms of (SP) and (SD). In the rest of the paper, we denote the S-TSOC of the k -th block by \mathcal{C}_k , i.e., $\mathcal{C}_k = \mathcal{C}(e_k, e_k) \subset \mathcal{K}_k \in \mathbb{R}^{d_k}$. The extension of Chubanov’s fundamental relation to the second-order cone case is described as follows.

Lemma 3.1 (*Basic Lemma*) *Suppose $x \in \mathcal{F}$. Suppose that $y \in \mathcal{K}$ satisfies the homogeneous inequality*

$$2\sqrt{n}\|P_A y\| \leq y_{k0}$$

for some index k . Then, if \mathcal{K}_k is a half-line, we have

$$\mathcal{F}_k \subseteq \left[0, \frac{1}{\sqrt{2}}\right],$$

and if \mathcal{K}_k is a second-order cone, then,

$$\mathcal{F}_k \subseteq H\left(y_k, \frac{1}{\sqrt{2}}e_k\right) \cap \mathcal{K}_k = \mathcal{C}\left(y_k, \frac{1}{\sqrt{2}}e_k\right).$$

In other words, any feasible solution $x \in \mathcal{F}$ to (P) satisfies $x_k \in [0, 1/\sqrt{2}]$ if \mathcal{K}_k is a half-line, and $x_k \in \mathcal{C}(y_k, e_k/\sqrt{2})$ if \mathcal{K}_k is a second order cone.

Proof. We give a proof for the case where \mathcal{K}_k is a second-order cone. The half-line case is analogous and easy. We have $\|x\| \leq \sqrt{2n}$ if $\|x\|_\infty = 1$ and $x \in \mathcal{F}$. Therefore,

$$y_k^T x_k \leq y^T x = y^T P_A x \leq \|x\| \|P_A y\| \leq \|x\| \frac{y_{k0}}{2\sqrt{n}} \leq \frac{1}{\sqrt{2}} y_{k0}.$$

Thus, we see that any $x_k \in \mathcal{F}_k$ is contained in the half space

$$\left\{ x_k \in \mathbb{R}^{d_k} \mid y_k^T \left(x_k - \frac{1}{\sqrt{2}} e_k \right) \leq 0 \right\} = H\left(y_k, \frac{1}{\sqrt{2}} e_k\right).$$

■

We call y satisfying the condition of lemma as “a cut generating vector,” and k and y_k are referred to as “generating index” and “generating block,” respectively.

Suppose that $\mathcal{F}_k \in \mathcal{C}_k$, and a cut generating vector y with generating index k is found. In the rest of this section, we construct an O-TSOC $\mathcal{C}(w, v)$ which encloses \mathcal{F}_k and with smaller volume than \mathcal{C}_k by choosing appropriate $w \in \mathbb{R}^{d_k}$ and $v \in \mathbb{R}^{d_k}$. Specifically, we find w, v satisfying the following two conditions:

$$\mathcal{F}_k \subseteq H\left(y_k, \frac{1}{\sqrt{2}}e_k\right) \cap \mathcal{C}_k \subseteq H(w, v) \cap \mathcal{K}_k = \mathcal{C}(w, v), \quad (6)$$

$$\begin{aligned} \text{vol}\left(H\left(y_k, \frac{1}{\sqrt{2}}e_k\right) \cap \mathcal{C}_k\right) &\leq \text{vol}(H(w, v) \cap \mathcal{K}_k) = \text{vol}(\mathcal{C}(w, v)) \\ &\leq 0.96^{d_k} \text{vol}(\mathcal{C}_k) = 0.96^{d_k} V_{d_k}. \end{aligned} \quad (7)$$

If these conditions are satisfied, the pair (w, v) is called “a cut.” We also use the term “cut” for the hyperplane $\partial H(w, v)$.

We illustrate the situation in Figure 1 for the case where the dimension of \mathcal{K}_k is two. In the beginning, we only know that \mathcal{F}_k is enclosed in the triangle AOB ($= \mathcal{C}_k$). Let a cut generating vector y is given with generating index k . The quadrangle COBD, which is the intersection of the two triangles AOB and COE ($= \mathcal{C}_k \cap H(y_k, e_k/\sqrt{2})$), is the reduced area where \mathcal{F}_k is still enclosed. (The triangle ACD is the area which was “cut off.”) Here, the line CE can be taken as a cut. Figure 1 intuitively shows that the triangle OCE satisfies the conditions (6) and (7), since it encloses the quadrangle COBD and the area of the triangle COE is smaller than the triangle AOB

Now we generalize this intuitive observation in a more quantitative manner to construct an O-TSOC we are aiming at. We branch into two cases: (Case 1) The angle between y_k and the center axis e_k is small; (Case 2) The angle between y_k and e_k is large.

(Analysis of Case 1)

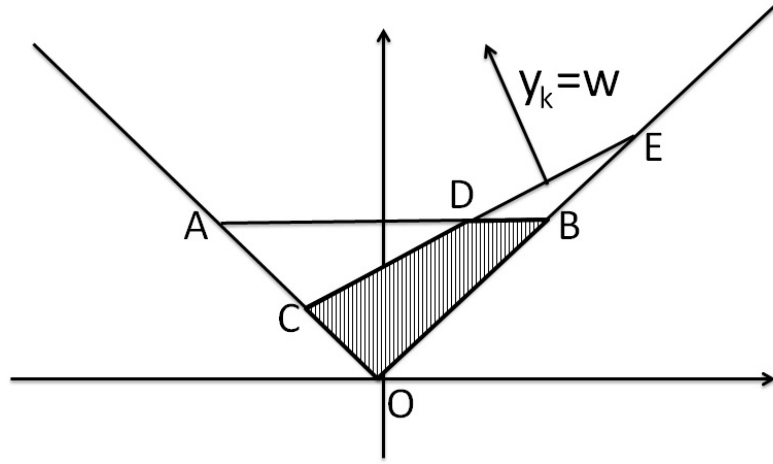


Figure 1: Case 1

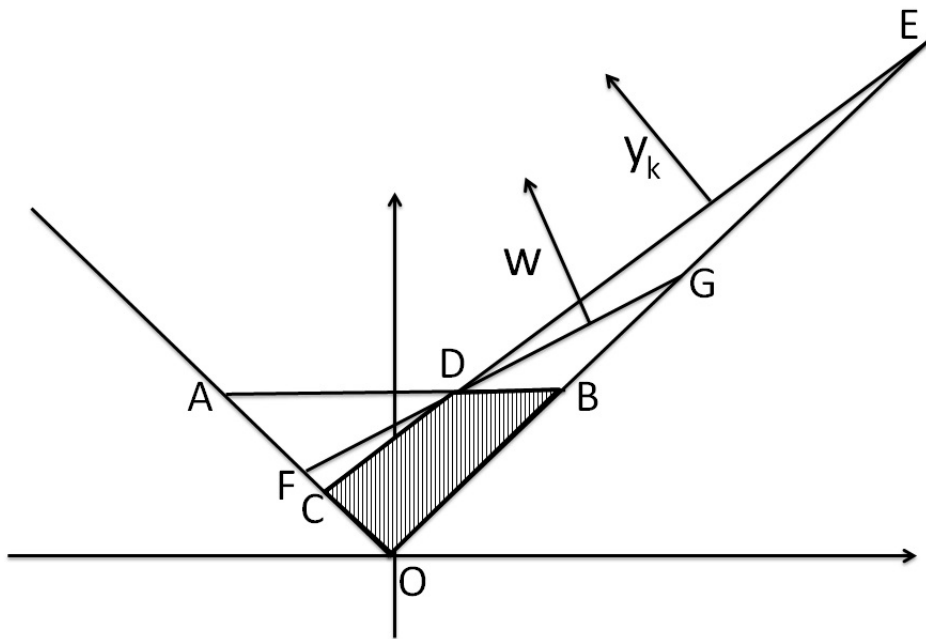


Figure 2: Case 2

In general, since $\mathcal{C}_k \subset \mathcal{K}_k$, we can take, as was suggested in the above, $w = y_k$ and $v = e_k/\sqrt{2}$. This means that we just use

$$H\left(y_k, \frac{1}{\sqrt{2}}e_k\right) \cap \mathcal{K}_k = \mathcal{C}\left(y_k, \frac{1}{\sqrt{2}}e_k\right)$$

as a confined enclosing area for \mathcal{F}_k satisfying (6) and (7). The volume of $\mathcal{C}(y_k, e_k/\sqrt{2})$ is, by letting $w = y_k, v = e_k/\sqrt{2}$ in Proposition 2.3, given as

$$\text{vol}\left(\mathcal{C}\left(y_k, \frac{1}{\sqrt{2}}e_k\right)\right) = \left(\frac{1}{\sqrt{2}} \frac{y_{k0}}{\sqrt{y_{k0}^2 - \|y_{k1}\|^2}}\right)^k V_{d_k}.$$

If $y_{k0}^2/(y_{k0}^2 - \|y_{k1}\|^2) < 2$, the volume is ensured to decrease. For later convenience, let $\hat{y}_k = y_k/y_{k0}$. We have $\hat{y}_{k0} = 1$, and let $\eta = \|\hat{y}_{k1}\|$. Since $y_k \in \mathcal{K}_k$ and so is \hat{y}_k , the range of η is $[0, 1]$. In terms of η , the condition $y_{k0}^2/(y_{k0}^2 - \|y_{k1}\|^2) < 2$ is written equivalently as $\eta < 1/\sqrt{2}$. Under this condition, the reduction ratio of the volume is written as:

$$g_1(\eta) = \frac{1}{\sqrt{2}(1 - \eta^2)^k}.$$

Observe that this is a monotonically increasing function.

The above idea does not work for $\eta \geq 1/\sqrt{2}$. See Figure 2. Observe that y_k is almost parallel to the edge of the cone. The vertex E is seen further than before. The quadrangle COBD contains \mathcal{F}_k . The triangle COE is larger than the original triangle AOB though it contains the quadrangle COBD. The triangle COE cannot be used to confine the existing area of \mathcal{F}_k this time.

We consider the following approach to deal with this case. We continue explanation with Figure 2. This time, we generate a supporting line (segment) FG which touches the quadrangle COBD at the vertex D, and enclose the quadrangle COBD with the triangle FOG. The line FG is chosen so that the triangle FOG contains the quadrangle COBD and its volume gets smaller than the original triangle AOB. We already developed a formula to find a line which goes through D and minimizes the area of the triangle FOG, see Proposition 2.4. We also need to check that the resulting line *does not* intersect the quadrangle COBD.

(Analysis of Case 2)

Consider a half-space $H(w, v)$ which contains $H(y_k, \frac{1}{\sqrt{2}}e_k) \cap H(e_k, e_k)$ such that its boundary $\partial H(w, v) = \{x | w^T x = v\}$ is a supporting hyperplane to $H(y_k, \frac{1}{\sqrt{2}}e_k) \cap H(e_k, e_k)$. Obviously, we have

$$H\left(y_k, \frac{1}{\sqrt{2}}e_k\right) \cap \mathcal{C}_k = H\left(y_k, \frac{1}{\sqrt{2}}e_k\right) \cap H(e_k, e_k) \cap \mathcal{K}_k \subseteq H(w, v) \cap \mathcal{K}_k = \mathcal{C}(w, v).$$

Since $\partial H(w, v)$ is a supporting hyperplane to $H(y_k, \frac{1}{\sqrt{2}}e_k) \cap H(e_k, e_k)$, without loss of generality, we assume that

$$v \in \partial H(y_k, \frac{1}{\sqrt{2}}e_k) \cap \partial H(e_k, e_k). \quad (8)$$

Let $\hat{y}_k = y_k/y_{k0}$, then $\hat{y}_{k0} = 1$ and $\|\hat{y}_{k1}\| < 1$. We let $\eta = \|\hat{y}_{k1}\|$ as before. We assume that v is written as follows

$$v = -\tilde{\alpha}\hat{y}_k + \tilde{\beta}e_k, \quad (9)$$

and try to find w and v satisfying the condition such that

$$\text{vol}(\mathcal{C}(w, v)) < 0.96^{d_k} V_{d_k}, \quad H\left(y_k, \frac{1}{\sqrt{2}}e_k\right) \cap H(e_k, e_k) \subseteq H(w, v)$$

under a certain condition on y_k .

Due to (8), we have

$$\hat{y}_k^T \left(v - \frac{e_k}{\sqrt{2}}\right) = 0, \quad e^T(v - e) = 0.$$

Therefore, taking $\hat{y}_{k0} = 1$ into account, we obtain

$$-\tilde{\alpha}(1 + \|\hat{y}_{k1}\|^2) + \tilde{\beta} = \frac{1}{\sqrt{2}}, \quad -\tilde{\alpha} + \tilde{\beta} = 1.$$

Solving this with respect to $\tilde{\alpha}$, we have

$$\tilde{\alpha} = \frac{1}{\|\hat{y}_{k1}\|^2} \left(1 - \frac{1}{\sqrt{2}}\right), \quad \tilde{\beta} = 1 + \tilde{\alpha}. \quad (10)$$

Note that $\tilde{\alpha} > 0$. Plugging (10) into (8), we obtain

$$v(y_k) = \left(1; -\frac{1}{\|\hat{y}_{k1}\|^2} \left(1 - \frac{1}{\sqrt{2}}\right) \hat{y}_{k1}\right) = (1; -\tilde{\alpha}\hat{y}_{k1}).$$

Then Proposition 2.3 yields that the minimum volume O-TSOC $\mathcal{C}(w, v)$ is obtained by taking

$$w(y_k) = \left(1; \frac{1}{\|\hat{y}_{k1}\|^2} \left(1 - \frac{1}{\sqrt{2}}\right) \hat{y}_{k1}\right) = (1; \tilde{\alpha}\hat{y}_{k1})$$

and

$$\text{vol}(\mathcal{C}(w(y_k), v(y_k))) = g_2(\eta(\hat{y}_k)), \quad \text{where } g_2(\eta) \equiv \left(1 - \left(\frac{1}{\sqrt{2}} - 1\right)^2 \frac{1}{\eta^2}\right)^{d_k/2} V_{d_k}.$$

Observe that g_2 is a monotonically increasing function of η whose value is positive in the interval $1 - 1/\sqrt{2} < \eta \leq 1$. It is easy to see that $g_2(1) \leq 0.96^{d_k} V_{d_k}$.

Now we examine the condition that $\partial H(w, v)$ defines a supporting hyperplane of $H(e_k, e_k) \cap H(y_k, e_k/\sqrt{2})$. Since $v \in \partial H(e_k, e_k)$ and $v \in \partial H(y_k, e_k/\sqrt{2})$, a necessary and sufficient condition for $\partial H(w, v)$ to be the supporting hyperplane is that w is written as a nonnegative combination of y_k and e_k . Since $(0; \hat{y}_{k1}) = \hat{y}_k - e_k$,

$$w = \tilde{\alpha}(\hat{y}_k - e_k) + e_k = \tilde{\alpha}\hat{y}_k + (1 - \tilde{\alpha})e_k.$$

Thus, w is on the line connecting \hat{y}_k and e_k , and can be represented as a conic combination of \hat{y}_k (or equivalently y_k) and e_k if and only if $0 \leq \tilde{\alpha} \leq 1$, i.e.,

$$0 \leq \frac{1}{\eta^2} \left(1 - \frac{1}{\sqrt{2}}\right) \leq 1.$$

The analysis so far is summarized as follows:

1. Suppose that $\eta \leq 1/\sqrt{2}$. Then, by letting $w = y_k$ and $v = \frac{1}{\sqrt{2}}e_k$, O-TSOC $\mathcal{C}(w, v)$ encloses \mathcal{F}_k and its volume is bounded by

$$g_1(\eta) = \left(\frac{1}{\sqrt{2(1-\eta^2)}} \right)^{d_k} V_{d_k}$$

The function $g_1(\eta)$ is well-defined in the interval $[0, 1)$ and is monotonically increasing. In particular, if $\eta = 0.6 \leq 1/\sqrt{2}$, we have $g_1(0.6) = \frac{V_{d_k}}{\sqrt{1.28^k}} < 0.96^{d_k} V_{d_k}$.

2. Suppose that $\sqrt{1 - \sqrt{1/2}} \leq \eta \leq 1$. Then by letting

$$w = \left(1; \frac{1}{\eta^2} \left(1 - \frac{1}{\sqrt{2}} \right) \hat{y}_{k1} \right) \quad \text{and} \quad v = \left(1; -\frac{1}{\eta^2} \left(1 - \frac{1}{\sqrt{2}} \right) \hat{y}_{k1} \right),$$

O-TSOC $\mathcal{C}(w, v)$ encloses \mathcal{F}_k and its volume is bounded by

$$g_2(\eta) \equiv \left(1 - \frac{1.5 - \sqrt{2}}{\eta^2} \right)^{d_k/2} V_{d_k}.$$

as long as $\sqrt{1 - \sqrt{1/2}} \leq \eta \leq 1$. In particular, $\eta = 0.6$ is in the interval. The function is monotone increasing, and we have $g_2(1) \leq 0.96^{d_k} V_{d_k}$.

Therefore, given the cut generating vector y with generating index k , if we determine w and v according to the rule that

1. if $0 \leq \eta \leq 0.6$, then take w and v as in the item 1 above,
2. if $0.6 < \eta \leq 1$, then take w and v as in the item 2 above.

Then, O-TSOC $\mathcal{C}(w, v)$ encloses \mathcal{F}_k and the bound

$$\text{vol}(\mathcal{C}(w, v)) \leq 0.96^{d_k} V_{d_k},$$

is ensured. Finally, Proposition 2.2 yields that an element of the automorphism group which maps \mathcal{C}_k to $\mathcal{C}(w, v)$ is:

$$G = (\alpha; \beta)^T v \begin{pmatrix} \alpha & -\beta^T \\ -\beta & I + \frac{\beta\beta^T}{1+\alpha} \end{pmatrix}$$

where $\alpha = w_0/\sqrt{w_0^2 - \|w_1\|^2}$ and $\beta = w_1/\sqrt{w_0^2 - \|w_1\|^2}$.

4 Basic Procedure and its Analysis

In this section, we explain and analyze the basic procedure which is a direct extension of Chubanov's. The basic procedure deals with a pair of the dual problems $Ax = 0$, $x \succeq 0$ and $y = -A^T u$, $y \succeq 0$, and finds either a primal interior solution, or dual nonzero solution, or a cut generating vector. In the procedure, the iterate y satisfying $e^T y = 1$, $y \succ 0$ is updated

every iteration. As will be discussed later, the iteration complexity estimate is based on the fact that the quantity $1/\|P_A y\|^2$ increases at least by 1/2 at each iteration. On the other hand, we can show that y is a cut generating vector if $1/\|P_A y\|^2 \geq 4n^3$. Then, the basic procedure is ensured to terminate in $O(n^3)$ iterations regardless of the choice of initial value of y . Before we proceed, we make two important comments:

1. The basic procedure is mainly applied to a scaled system (SP) and (SD). But we describe the procedure and conduct analysis just for (P) and (D) to avoid that the notation gets too heavy.
2. In Chubanov's algorithm, one iteration of his basic procedure requires just $O(\bar{n})$ arithmetic operations though it computes projection of a vector to $\text{Ker}(A)$ and appears to require $O(\bar{n}^2)$ arithmetic operations. In our case, the complexity of one iteration of the basic procedure is a bit higher and $O(\bar{n} \max_i d_i)$ arithmetic operations, because the second-order cone is a bit more complicated than linear inequalities. We mention that Peña and Soheili [15] extends the basic procedure to general symmetric cone programming.

The Basic Procedure

Input Matrix A and vector y such that $e^T y = 1$, $y \succ 0$,

Output One of the followings : (i) A cut generating vector y and its generating index k , (ii) Solution x to $Ax = 0$, $x \succ 0$; (iii) Solution y to $y = A^T u \succeq 0$, $y \neq 0$,

Procedure

1. Compute P_A and $z = P_A y$.
2. Check termination conditions (based on z):
 - (a) If $z = 0$, then y is dual interior feasible. Return “(iii)” and y .
 - (b) If $z \succ 0$, then, z is primal interior feasible. Return “(ii)” and z .
 - (c) If $2\sqrt{\bar{n}}\|P_A y\| \leq y_{k_0}$ holds for some k , return “(i)”, and, y and k as a cut generating vector and k a generating index, respectively.
 - (d) If (a)–(c) does not hold, then, go to Step 3.
3. Since the conditions (a) and (b) do not hold, $z \neq 0$ and $z \notin \text{int}(\mathcal{K})$. Therefore, there exists an index i , say, such that $z_i \neq 0$ and $z_i \notin \text{int}(\mathcal{K}_i)$ hold. In the following, we construct $\eta_i \in \mathcal{K}_i$ such that $\eta_{i0} = 1$, $\eta_i^T z_i \leq 0$.
 - If \mathcal{K}_i is a half-line, then, we set $\eta_i = e_i$. (η_i, z_i are scalars and $z_i \leq 0$.)
 - If \mathcal{K}_k is a second-order cone, such η_i is computed as follows.
 - If $z_{i0} \leq 0$, then we let $\eta_i = e_i$;
 - If $z_{i0} > 0$, then, we let

$$\eta_i = e_i - \frac{\hat{z}_i - e_i}{\|\hat{z}_i - e_i\|},$$

where $\hat{z}_i = z_i/z_{i0}$ ($\hat{z}_{i0} = 1$). In this case, $\|\hat{z}_i - e_i\| \geq 1$ holds because $\hat{z}_i \notin \text{int}(\mathcal{K}_i)$. Therefore, we have

$$\hat{z}_i^T \eta_i = \hat{z}_i^T e_i - \frac{\|\hat{z}_i - e_i\|^2 + e_i^T(\hat{z}_i - e_i)}{\|\hat{z}_i - e_i\|} = 1 - \|\hat{z}_i - e_i\| \leq 0.$$

4. Let $\eta = (0, \dots, 0, \eta_i, 0, \dots, 0)$. Then we have $e^T \eta = 1$ and $\eta^T z \leq 0$.
5. Let $p = P_A \eta$. Computation of p requires $O(\bar{n}d_i)$ arithmetic operations, since P_A is already computed and η contains d_i nonzero elements.
6. Check termination conditions (based on p):
 - (a) If $p = 0$, then η is dual feasible. Return “(iii)” and η .
 - (b) If $p \succ 0$, then, p is primal interior feasible. Return “(ii)” and p .
 - (c) If neither of (a) nor (b) holds, go to Step 7.
7. Construct a new iterate \tilde{y} as follows:

$$\tilde{y} = \alpha y + (1 - \alpha)\eta, \quad \tilde{z} = \alpha z + (1 - \alpha)p, \quad \alpha = \frac{p^T(p - z)}{\|z - p\|^2}.$$

Note that $p \neq 0$ and $z \neq 0$ are ensured, and that α is chosen so that $\tilde{z} = P_A \tilde{y}$ is closest to the origin. Then it follows that α is positive as is discussed below. Since $e^T y = 1$ and $e^T \eta = 1$, we have $e^T \tilde{y} = 1$. Since $\alpha > 0$, $\tilde{y} \succ 0$. So we continue iteration by letting $y := \tilde{y}$, $z := \tilde{z}$ and going to Step 2.

(Analysis of change of $1/\|z\|^2$)

We show that $1/\|z\|^2$ increases by at least 1/2 per iteration. Observe that

$$p^T z = (P_A \eta)^T z = \eta^T (P_A z) = \eta^T z = \eta_i^T z_i \leq 0.$$

Furthermore, since

$$\|\tilde{z} - p\|^2 = \|z\|^2 + \|p\|^2 - 2z^T p$$

and neither z nor p is zero, $\alpha \in (0, 1)$. Therefore, we have $\tilde{y} \succ 0$.

Letting $\tilde{z} = P_A \tilde{y}$, we have

$$\tilde{z} = p + \alpha(z - p).$$

Therefore,

$$\|\tilde{z}\|^2 = \alpha^2 \|z - p\|^2 + 2\alpha p^T(z - p) + \|p\|^2.$$

Substituting the concrete formula of α into the above and using $p^T z \leq 0$, we obtain

$$\|\tilde{z}\|^2 = \|p\|^2 - \frac{(p^T(z - p))^2 - (z^T p)^2}{\|z - p\|^2} = \frac{\|z\|^2 \|p\|^2 - (z^T p)^2}{\|z\|^2 + \|p\|^2 - 2z^T p} \leq \frac{\|p\|^2 \|z\|^2}{\|z\|^2 + \|p\|^2}.$$

Since P_A is a projection matrix, we have $\|p\|^2 \leq \|\eta\|^2 \leq 2$. This implies that

$$\frac{1}{\|\tilde{z}\|^2} \geq \frac{1}{\|z\|^2} + \frac{1}{\|p\|^2} \geq \frac{1}{\|z\|^2} + \frac{1}{2}.$$

Complexity Analysis of the Basic Procedure

Now we analyze overall complexity of the basic procedure. Prior to the iteration of the basic procedure, we compute P_A and $P_A y$. This requires $O(\bar{n}^2 m)$ arithmetic operations. In one iteration of the basic procedure, we need to compute $p = P_A \eta$. This can be done in $O(\bar{n} \max_i d_i)$ arithmetic operations as explained in the previous section.

We analyze that the number of iterations of the basic procedure is $O(n^3)$. Recall the condition that y is a cut generating vector with generating index i is $2\sqrt{\bar{n}}\|z\| \leq y_{i0} \leq \|y\|_\infty$. since $e^T y = 1$ and $y \succ 0$, we obtain that $1/n \leq \|y\|_\infty$. Therefore, if

$$4n^3 \leq \frac{1}{\|z\|^2},$$

then y associated with $z = P_A y$ is a cut generating vector. As we already analyzed, at each step of the basic procedure $1/\|z\|^2$ increases by $1/2$. Therefore, in $O(n^3)$ iterations of the basic procedure, we find a cut generating vector or a primal interior feasible solution or dual nonzero feasible solution. Since one iteration of the basic procedure requires $O(\bar{n} \max_i d_i)$ arithmetic operations, the basic procedure terminates in $O(n^3 \bar{n} \max_i d_i + m \bar{n}^2)$ arithmetic operations.

5 Main Algorithm

We are ready to describe the main algorithm. This algorithm (i) finds an interior feasible solution to (P), (ii) finds a nonzero solution to (D), or (iii) concludes that there exists no ε -interior feasible solution in $O(n \log \varepsilon^{-1})$ iterations of the basic procedure, where the basic procedure requires $O(n^3 \bar{n} \max_i d_i + m \bar{n}^2)$ arithmetic operations. Thus, the overall arithmetic operations of the algorithm presented in this section is $O(n(n^3 \bar{n} \max_i d_i + m \bar{n}^2)) \log \varepsilon^{-1}$.

The Main Algorithm

Input A matrix A and a cone \mathcal{K} which is the direct product of second-order cones and half-lines,

Output One of the followings : (i) Solution x to $Ax = 0$, $x \succ 0$; (ii) Solution y to $y = -A^T u \succeq 0$, $y \neq 0$, (iii) Declare “No ε -interior solution to $Ax = 0$, $x \succeq 0$.”

Algorithm

1. Let $t := 0$, $v_i := 1$, $i = 1, \dots, n$, $A^{(0)} := A$, $y^{in} := e/n$, $M^{(0)} = I$, $A^{(0)} = A$;
2. Call the Basic Procedure (BP) by setting $A^{(t)}$ and y^{in} as Input (See Section 4).
 - (C1) If (BP) returns a cut generating vector y and generating index k , then proceed to Step 3.
 - (C2) If (BP) returns an interior solution \tilde{x} to $A^{(t)}\tilde{x} = 0$, $\tilde{x} \succeq 0$, then we let $x := M^{(t)}\tilde{x}$ and return x as an interior solution to (P).
 - (C3) If (BP) returns a nonzero solution y to $y = -(A^{(t)})^T u$, $y \succeq 0$, $y \neq 0$, then return y as a nonzero solution to (D).
3. In the case of (C1),

- if \mathcal{K}_k is a half space then set $G = 1/\sqrt{2}$.
- if \mathcal{K}_k is a second-order cone and $\eta \leq 0.6$ where $\eta = \|y_{k1}\|/y_{k0}$, then set $w = (y_{k0}; y_{k1})$, $v = e_k/\sqrt{2}$, and construct G according to Proposition 2.2 as an automorphism transformation of \mathcal{K}_k mapping \mathcal{C}_k to $\mathcal{C}(w, v)$.
- if \mathcal{K}_k is a second-order cone and $0.6 < \eta \leq 1$ where $\eta = \|y_{k1}\|/y_{k0}$, then set

$$w = \left(1; \frac{1}{\eta^2} \left(1 - \frac{1}{\sqrt{2}}\right) \frac{y_{k1}}{y_{k0}}\right) \quad \text{and} \quad v = \left(1; -\frac{1}{\eta^2} \left(1 - \frac{1}{\sqrt{2}}\right) \frac{y_{k1}}{y_{k0}}\right),$$

and construct G according to Proposition 2.2 as an automorphism transformation of \mathcal{K}_k mapping \mathcal{C}_k to $\mathcal{C}(w, v)$.

4. We set

$$A_k^{(t+1)} := A_k^{(t)}G, \quad v_k := \det(G)v_k, \quad M_k^{(t+1)} = M_k^{(t)}G.$$

Regarding other blocks than k , we let $A_i^{(t+1)} = A_i^{(t)}$.

5. If $v_k \leq \varepsilon^{d_k}$, then, we conclude that there is no ε interior feasible solution to (P). Otherwise, we set $t := t + 1$ and return to Step 2.

Overall Complexity Analysis

Now we analyze the complexity of the main algorithm. In the beginning of the algorithm, \mathcal{F}_k is enclosed in \mathcal{C}_k for all k . The algorithm terminates

- In the middle of the basic procedure by finding an interior solution to (P) or nonzero solution to (D);
- Detecting that there is no ε -interior feasible solution.

Let $p_1 < \dots < p_q$ be the iteration number in which the k -th block is transformed. Let $G_k^{p_1}, \dots, G_k^{p_q}$ be the matrix of the automorphism transformation associated with \mathcal{K}_k performed in the course of the algorithm. Then it follows that

$$\mathcal{F}_k \subseteq G_k^{(p_q)}G_k^{(p_1)} \dots G_k^{(p_1)}\mathcal{C}(e_k, e_k).$$

So, in view of Proposition 2.4, we conclude (ii) when the following relation holds:

$$\det(G_k^{(p_q)}) \dots \det(G_k^{(p_1)})V_{d_k} \leq 0.96^{qd_k}V_{d_k} < \varepsilon^{d_k}V_{d_k}.$$

This relation implies that q is bounded by $O(\log \varepsilon^{-1})$. The most time consuming case is that the number of occurrence of cutting process is almost even ($\sim \log \varepsilon^{-1}$) for all cones before termination of the algorithm. Then, the number of execution of the basic procedure is bounded by $O(n \log \varepsilon^{-1})$.

Since $O(n \log \varepsilon^{-1})$ executions of the basic procedure might be necessary before completion of the whole procedure, the complexity of the proposed algorithm is $O(n(n^3\bar{n} \max_i d_i + m\bar{n}^2) \log \varepsilon^{-1})$.

6 Remarks

Before concluding this paper, we make some remarks.

6.1 Condition Number

We define a condition number of (P) as follows:

$$\text{cond}(A, \mathcal{K}) = \min_{x \in \mathcal{F}} \left(\frac{\lambda_{\max}(x)}{\lambda_{\min}(x)} \right)$$

If (P) have an interior feasible solution, $\text{cond}(A, \mathcal{K})$ stays finite, but it becomes infinity if (P) is feasible but is not interior feasible. This quantity is useful in evaluating the complexity of the main algorithm developed in this paper. It is worth noting that $\text{cond}(A, \mathcal{K}) = \varepsilon_P^{-1}$, where ε_P is the optimal value of the following problem:

$$\max \varepsilon, \quad Ax = 0, \quad e \succeq x \succeq \varepsilon e.$$

6.2 Running Time to Find a Feasible Solution to an Interior-feasible System

Suppose that we set $\varepsilon = 0$ and run the main algorithm. The algorithm will never stop if (P) does not have an interior feasible solution. But if there exists an interior feasible solution, then the algorithm is ensured to terminate in $O(n \log \varepsilon_P^{-1})$ execution of of the basic procedure, where ε_P is the optimal value of the following optimization problem:

$$\max \varepsilon \quad \text{subject to } Ax = 0, \quad \|x\|_{\infty} \leq 1, \quad x \succeq \varepsilon e.$$

In view of (2), we have $(2\varepsilon_P)^{-1} \leq \text{cond}(A, \mathcal{K}) \leq \varepsilon_P^{-1}$. Therefore, the algorithm is capable of finding an interior solution to (P) in $O(n \log \text{cond}(A, \mathcal{K}))$ times execution of the basic procedure.

6.3 SOCP Feasibility Problem

Suppose that we deal with the problem of finding an interior-feasible solution x to

$$Ax = b, \quad x \in \tilde{\mathcal{K}}, \tag{11}$$

where $\tilde{\mathcal{K}}$ is a direct product of n second-order cones/half-lines. We assume the system is interior-feasible. To solve this problem, we consider the homogenized system

$$Ax - b\tau = 0, \quad x \in \tilde{\mathcal{K}}, \quad \tau \in \mathbb{R}_+,$$

where \mathbb{R}_+ is a half-line. We run the main algorithm with $\varepsilon = 0$. The algorithm stops in $O(n \log(\text{cond}((A \ -b); \tilde{\mathcal{K}} \times \mathbb{R}_+)))$ iterations.

For any feasible solution to \tilde{x} ,

$$\frac{\lambda_{\max}(\tilde{x}; 1)}{\lambda_{\min}(\tilde{x}; 1)}$$

is an upper bound for $\text{cond}((A \ -b); \tilde{\mathcal{K}} \times \mathbb{R}_+)$.

This implies that if the system (11) have an interior feasible solution whose components are more or less in the same magnitude, then less number of iterations is required to find an interior feasible solution.

7 Conclusion

We extended Chubanov's algorithm for linear programming to second-order cone programming. The extension to semidefinite programming and symmetric cone programming is an interesting topic for further research. In the case of linear program, Chubanov [5] developed a technique to reduce the complexity by a factor of n by initiating a basic procedure using the second last iterate of the preceding basic procedure. This idea does not directly carry over to the second-order cone program. Extending the technique to the second-order cone program is another interesting subject. As was mentioned in introduction, Peña and Soheili developed an extension to Chubanov's algorithm to symmetric cone programming. We hope that comparison of our extension and their algorithm will shed new insight into substance of Chubanov's idea in conic programming.

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Appendix

In this section, we describe how we can solve a standard SOCP problem with the algorithm developed in this paper. Consider the pair of primal and dual SOCP:

$$(P) \min c^T x \quad \text{subject to } Ax = b, \quad x \succeq 0$$

and

$$(D) \max b^T y \quad \text{subject to } s = c - A^T y, \quad s \succeq 0.$$

Suppose (P) and (D) have interior feasible solutions. Then (P) and (D) have optimal solutions with the same optimal value. Furthermore, the optimal set is bounded for the both problems. In this appendix, we explain, given any $\delta > 0$, how the algorithm developed in this paper can be used to find a feasible solutions satisfying $c^T x - b^T y \leq \delta$. If δ is sufficiently small, then x , y and s are approximate optimal solutions to (P) and (D).

It is well-known that (P) and (D) are equivalent to the following problem.

$$(PD) \text{ find } Ax = b, \quad c - A^T y = s, \quad c^T x - b^T y = 0, \quad x \succeq 0, \quad s \succeq 0.$$

We have the following proposition.

Proposition A.1 *Let \tilde{x} and (\tilde{y}, \tilde{s}) be an interior feasible solution to (P) and (D), respectively. Let*

$$\hat{\varepsilon} = \min(\lambda_{\min}(\tilde{x}), \lambda_{\min}(\tilde{s}), 1) \quad \text{and} \quad M = c^T \tilde{x} - b^T \tilde{y}.$$

If $0 \leq t \leq 1/2$,

$$(PD(t)) \text{ find } Ax = b, \quad c - A^T y = s, \quad 0 \leq c^T x - b^T y \leq 2tM, \quad x \succeq 0, \quad s \succeq 0,$$

has $t \min(\hat{\varepsilon}, M)$ -interior feasible solution.

Proof. $(\tilde{x}, (\tilde{y}, \tilde{s}))$ is a $\hat{\varepsilon}$ -interior feasible solution to (PD). Let x^* , $(y^*; s^*)$ be optimal solutions to (P) and (D), and define

$$x(t) = t\tilde{x} + (1-t)x^*, \quad y(t) = t\tilde{y} + (1-t)y^*, \quad s(t) = t\tilde{s} + (1-t)s^*.$$

Then, $(x(t), (y(t), s(t)))$ is a $t\hat{\varepsilon}$ -interior feasible solution to (PD) for any $t \in [0, 1]$. We also have

$$c^T x(t) - b^T y(t) = t(c^T \tilde{x} - b^T \tilde{y}) = tM.$$

It is easy to check that $x(t)$, $(y(t), s(t))$ is indeed $t \min(\hat{\varepsilon}, M)$ -interior feasible solution to (PD(t)) for $t \in [0, 1/2]$, and we are done. ■

We may consider \tilde{x} , \tilde{s} and \tilde{y} as a feasible solution obtained in Phase I. Now we are ready to describe an algorithm to solve (P) and (D) approximately. The algorithm works in two phases.

1. **(Phase I)** We apply the feasibility algorithm described in Section 6.3 to

$$(PD) \quad Ax = b, \quad s = c - A^T y, \quad x \succeq 0, \quad s \succeq 0$$

with $\varepsilon = 0$. (This problem contains y as a free variable, but we can apply the main algorithm after rewriting the condition $s = c - A^T y$ with $P_A(c - s) = 0$ to eliminate y . In the end of the algorithm, we can recover y from s .) Then, we will find an interior feasible solution $(x, s, y) = (\tilde{x}, \tilde{s}, \tilde{y})$.

The complexity of this step is estimated with the result in Section 6.3, in terms of the condition number. Let $\tilde{\varepsilon} = \min(\lambda_{\min}(\tilde{x}), \lambda_{\min}(\tilde{s}), 1)$. Then, $(\tilde{x}, \tilde{s}, \tilde{y})$ is an $\tilde{\varepsilon}$ -interior feasible solution to (PD).

2. **(Phase II)** If we want to reduce the objective value by a factor of $t (\leq 1/2)$ from $c^T \tilde{x} - b^T \tilde{y}$, we solve the interior-feasibility problem (PD(t)) above, which is ensured to have an $t \min(\tilde{\varepsilon}, c^T \tilde{x} - b^T \tilde{y})$ -interior feasible solution. The complexity is estimated with the result in Section 6.3, again, in terms of the condition number of (PD(t)).

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