Generalized average shadow prices and bottlenecks

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Abstract

We present a generalization of the average shadow price in 0-1-Mixed Integer Linear Programming problems and its relation with bottlenecks including the analysis relative to the coefficients matrix of resource constraints. A mathematical programming approach to find the strategy for investment in resources is presented.

Keywords Mixed Integer programming, Average shadow price, Bottlenecks, Resource Constraints

1 Introduction.

Usually some of the constraints of a 0-1-Mixed Integer Linear Programming (0-1-MILP) problem correspond to resources and in this paper we suppose that they may be redefined. For the availability of the resources the average shadow price (a.s.p) ([6],[14],[15],[18]) is the maximun price that the decision maker is willing to pay for an additional unit of the package (i.e. a combination) of resources defined by some direction predefined. An "unified" and "universal" concept of bottleneck has been suggested ([17]): "A set of constraints with strictly positive a.s.p is defined as a bottleneck".

A new definition of bottleneck has been presented recently as follows ([4],[5]): "A bottleneck is a modifiable specification of resources that by changing its value, the best achievable performance of the system can be improved". This new definition is more general because the authors consider the case in which the decision maker may redefine the system global configuration (including changes to the coefficients matrix). Also, the authors present "some limitations" to the use of the a.s.p for identifying bottlenecks, as follows:

1. "a.s.p is only applicable if objective and constraints are linear."
2. "a.s.p does not evaluate changes in the coefficients matrix and it is only limited to the right hand side vector."
3. "a.s.p does not provide information about the strategy for investment in resources, and the decision maker has to manually conduct analyses to find the best investment strategy."

We agree with the Limitation 1. In this paper we present a generalization of the a.s.p to include the analysis relative to the coefficients matrix (see Limitation 2). Also, we present a mathematical programming approach to find the strategy for investment in resources (see Limitation 3).

The Critical Slope (CS) of a function and an algorithm to find it are presented in section 2. The generalized a.s.p.(*gasp*) in 0-1-MILP problems is presented in section 3 by using the CS. Information about the strategy for investment in resources will be presented. Some examples are presented in section 4. Some extensions are presented is section 5 including a link to a generalization of our approach, developed in another work ([7]), following the ideas suggested in [4] and [5], in order to consider the global specification of the resource constraints. Finally the conclusions are presented in section 6.

A few words about our notation: If S is an optimization problem then $v(S)$ is its optimal value (if it exists) and $F(S)$ is its set of feasible solutions. A vector or matrix with zeros will be denoted 0. If we write $S(\theta, \cdots, \gamma)$ is a problem in $(x, \cdots, y)$ that means that $x, \cdots, y$ are the variable vectors and $\theta, \cdots, \gamma$ are data vectors that may change from one problem to another. The rest of the data for S are fixed and that must be clear in the context.

2 The critical slope of a function

Definition 1 *Let $f : [0,1] \longrightarrow \mathbb{R}$ be a bounded function with $f(0) = 0 \leq f(\theta) \leq f(1)$ for all $\theta \in [0,1]$. Let $Q = \{\frac{f(\theta)}{\theta} : \theta > 0\}$. Let us suppose that Q is a bounded set. Let $q = sup(Q)$. We say that q is the CS of f.*

Let us suppose that we can solve the following problems in θ for all $p \geq 0$:

$$(E(p)) \;\; max \;\; f(\theta) - p\theta \;\; s.t.$$
$$\theta \in [0,1]$$

$$(E^{+}(p)) \;\; max \;\; f^{+}(\theta) - p\theta \;\; s.t.$$

$$\theta \in [0,1]$$

where f^+ is the concave envelope of f in $[0,1]$.

Some elementary properties of the CS are listed in the Lemma 1. The proofs are very simple and may be seen in the Appendix. Later will be clear that the shadow price (s.p) for Linear Programming (LP) problems ([1],[2],[9],[13] and many others); the a.s.p for 0-1-MILP problems ([6],[14])and the generalization of the a.s.p. for 0-1-MILP problems, to be presented in this paper, have these properties.

Lemma 1

1. *$v(E(p)) \geq 0$ for all $p \geq 0$.*
2. *$v(E(q)) = 0$.*
3. *If $p > q$ then $v(E(p)) = 0$.*
4. *$q = 0$ if and only if $f(\theta) = 0 \ \ \forall \theta \in [0,1]$.*
5. *$v(E(p)) > 0$ if and only if $0 \leq p < q$.*
6. *If $0 \leq p \leq q$ and $v(E(p)) = 0$ then $p = q$.*
7. *θ^* is an optimal solution for $E(p)$ if and only if θ^* is an optimal solution for $E^+(p)$ and $v(E(p)) = v(E^+(p))$ for all $p \geq 0$.*

The CS algorithm to be presented below is the same presented by Crema ([6]) and suggested by Kim and Cho ([14]) to obtain the a.s.p relative to the right hand side in a 0-1-MILP problem. We present the algorithm here in order to point out that its validity is more general. We need to know $f(1)$ to apply the algorithm.

The CS algorithm

Let $p_0 = f(1)$, $\theta_0 = 1$ and $r = 0$.

1. Solve $E(p_r)$ to find $v(E(p_r))$. Let θ_{r+1} be an optimal solution of $E(p_r)$.
2. If $v(E(p_r)) = 0$ STOP.
3. $p_{r+1} = \frac{f(\theta_{r+1})}{\theta_{r+1}}$, $r = r + 1$ and return to step 1.

Note that if $v(E(p_r)) > 0$ then $\theta_{r+1} > 0$, therefore the algorithm is well defined.

Some elementary properties of the sequences generated by the algorithm are listed in the Lemmas 2 and 3. The proofs are very simple and may be seen in the Appendix. If the algorithm does not stop we have convergent sequences to appropriate values: $v(E(p_r))$ converges to 0 and p_r converges to q (this is proved in Lemma 4). The very important case when f^+ is a piecewise linear and nondecreasing function with a finite number of pieces is presented in Lemma 5.

Lemma 2 *Let $r \geq 0$. If $v(E(p_r)) > 0$ then:*

1. $p_{r+1} > p_r$.
2. $v(E(p_{r+1})) < v(E(p_r))$.
3. $\theta_{r+1} < \theta_r$.

Lemma 3 *If $v(E(p_r)) = 0$ (in step 2) then $q = p_r$.*

Lemma 4 *If the algorithm does not stop:*

1. $lim_{r \longrightarrow \infty} v(E(p_r)) = 0$.
2. $lim_{r \longrightarrow \infty} p_r = q$.

Proof:

(1) We have that $0 \leq v(E(p_{r+1})) < v(E(p_r))$ for all $r \geq 0$ and then there exists $s \geq 0$ such that $lim_{r \longrightarrow \infty} v(E(p_r)) = s$.

If $s > 0$ then $0 < s \leq v(E(p_r)) = f(\theta_{r+1}) - p_r \theta_{r+1}$ for all $r \geq 0$ and then $p_{r+1} \geq p_r + \frac{s}{\theta_{r+1}} \geq p_r + s$ for all $r \geq 0$. Hence we have that $lim_{r \longrightarrow \infty} p_r = \infty$ and then $lim_{r \longrightarrow \infty} v(E(p_r)) = 0$ and we have a contradiction. Therefore $s = 0$.

(2) We have that $0 \leq p_{r+1} < p_r$ for all $r \geq 0$ and then there exists $\hat{p}$ such that $\hat{p} \leq q$, $p_r \leq \hat{p}$ for all $r \geq 0$ and $lim_{r \longrightarrow \infty} p_r = \hat{p}$. Let $\hat{\theta}$ an optimal solution for $E(\hat{p})$ then $v(E(\hat{p})) \leq v(E(\hat{p})) + (\hat{p} - p_r)\hat{\theta} = f(\hat{\theta}) - p_r \hat{\theta} \leq f(\theta_{r+1}) - p_r \theta_{r+1} = (p_{r+1} - p_r)\theta_{r+1} \leq (p_{r+1} - p_r)$ for all $r \geq 0$ and then $v(E(\hat{p})) = 0$ because of $lim_{r \longrightarrow \infty}(p_{r+1} - p_r) = 0$. Therefore $q = \hat{p}$. •

Lemma 5 *If f^+ is a piecewise linear and nondecreasing function with a finite number of pieces then:*

1. *The CS algorithm is finite.*

2. *We can find f^+ by using the CS algorithm a finite number of times. With f^+ we can construct $v(E(p))$ for all $p \geq 0$*

Proof:

(1) Let f^+ the concave envelope of f. Let $l \geq 1$ and let $C^+(f) = \{(\overline{\theta}_i, q_i) : i = 1, \cdots, l\}$ with

(i) $0 = \overline{\theta_1} < \cdots < \overline{\theta_{l-1}} < \overline{\theta_l} = 1$,

(ii) $q = q_1 > \cdots > q_{l-1} > q_l = 0$,

(iii) $f^+(\theta) = q_1\theta$ if $0 = \overline{\theta_1} \leq \theta \leq \overline{\theta_2}$ and

(iv) $f^+(\theta) = f^+(\overline{\theta_{i-1}}) + q_{i-1}(\theta - \overline{\theta_{i-1}})$ if $\overline{\theta_{i-1}} \leq \theta \leq \overline{\theta_i}$ $(i = 2, \cdots, l)$

If $0 = q_l \leq p_r < q_{l-1}$ then $v(E(p_r)) = f^+(\overline{\theta_l}) - p_r\overline{\theta_l}$ with $\overline{\theta_l} = 1$ the unique optimal solution for $E(p_r)$.

If $p_r = q_1 = q$ then $v(E(p_r)) = 0$ and the algorithm stops.

Let us suppose that $q_i < p_r < q_{i-1}$ for some $i \leq l-1$ then $v(E(p_r)) = f^+(\overline{\theta_i}) - p_r\overline{\theta_i}$ with $\overline{\theta_i}$ the unique optimal solution for $E(p_r)$.

If $p_r = q_i$ for some $1 < i < l$ then any θ in $[\overline{\theta_i}, \overline{\theta_{i+1}}]$ is an optimal solution for $E(p_r)$ and $v(E(p_r)) > 0$. Let $\theta_{r+1} \in (\overline{\theta_i}, \overline{\theta_{i+1}})$. In this case $\theta_{r+2} \leq \overline{\theta_i}$.

It follows that each piece is visited at most two times. Because of the number of pieces is finite then the CS algorithm is finite.

(2) Let $f^i : [0,1] \longrightarrow \Re$ $(i = 1, \cdots, l-1)$ be functions defined as follows:

$f^i(\theta) = f(\overline{\theta_i} + (1 - \overline{\theta_i})\theta)$ for all $\theta \in [0,1]$.

Note that q_i is the critical slope of f^i $(i = 1, \cdots, l-1)$ and then we can find $C^+(f)$ by using the CS algorithm again and again. However in order to save time we may use the information previously generated to start the CS algorithm from an appropriate p_0.

In order to find $v(E(p))$ we proceed as follows:

$v(E(p)) = f^+(\overline{\theta_i}) - p\overline{\theta_i}$ for all $q_{i+1} \leq p \leq q_i$ for $i = 1, \cdots, l.\bullet$

In figure 1 we illustrate the execution of the CS algorithm with f^+ a piecewise linear and nondecreasing function with a finite number of pieces. In this example the CS algorithm finds q in a finite number of steps. In figure 2 we have f with $f = f^+$ and f^+ is not a piecewise linear function. In this example the CS algorithm generates convergent sequences. In figure 3 we illustrate the case in which $p_r = q_i$ for some $1 < i < l$. In this example $p_0 = q_2$ and any θ in $[\overline{\theta_2}, \overline{\theta_3}]$ is an optimal solution for $E(p_0)$ and $v(E(p_0)) > 0$. For example $\theta = \frac{3}{8}$ is optimal. Also $\theta_2 \leq \overline{\theta_2}$ and the piece is visited two times. In figure 4 we illustrate how to restart the algorithm in order to find f^+. First we use the CS algorithm to find $q_1 = q$. The next p_0 defined to find q_2 is not $\frac{f(1)-f(\frac{1}{4})}{1-\frac{1}{4}}$. We use $\hat{p}_0 = \frac{f(\frac{1}{2})-f(\frac{1}{4})}{\frac{1}{2}-\frac{1}{4}} = 1$

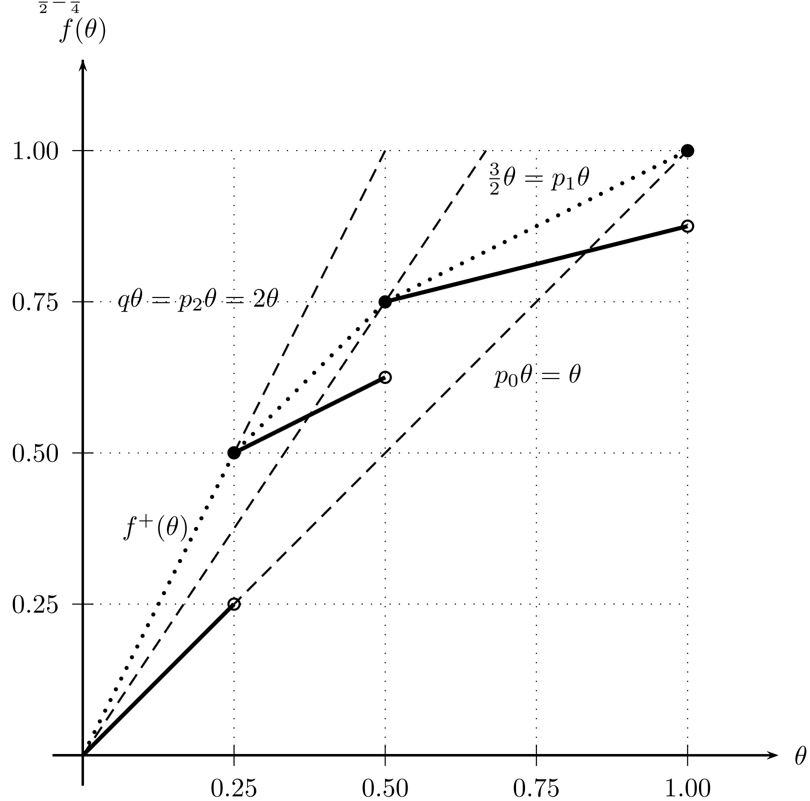

Figure 1: $p_0 = 1, \theta_1 = \frac{1}{2}, p_1 = \frac{\frac{3}{4}}{\frac{1}{2}} = \frac{3}{2}, \theta_2 = \frac{1}{4}, p_2 = \frac{\frac{1}{2}}{\frac{1}{4}} = 2, v(E(2)) = 0, q = 2$

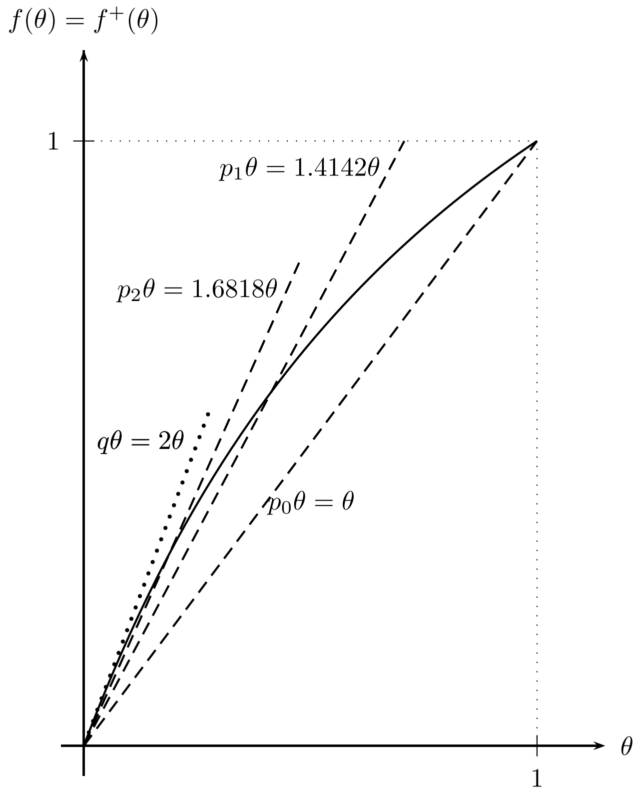

Figure 2: $p_0 = 1, \theta_1 = 0.4142, f(\theta_1) = 0.5858, p_1 = 1.4142, \theta_2 = 0.1892, f(\theta_2) = 0.3182, p_2 = 1.6818, \cdots, lim_{r \longrightarrow \infty} p_r = 2 = q$

3 A generalization of the average shadow price in 0-1-Mixed Integer Linear Programming problems

Let P be a 0-1-Mixed Integer Linear Progframming (0-1-MILP) problem in (x, y) defined as follows:

$$(P) \;\; max \;\; c^t x + d^t y \;\; s.t.$$

$$\hat{A}x + \hat{B}y \leq \hat{b}, \;\; Ax + By \leq b, \;\; x \geq 0$$

$$x \in \mathbb{R}^n, y \in \{0, 1\}^k$$

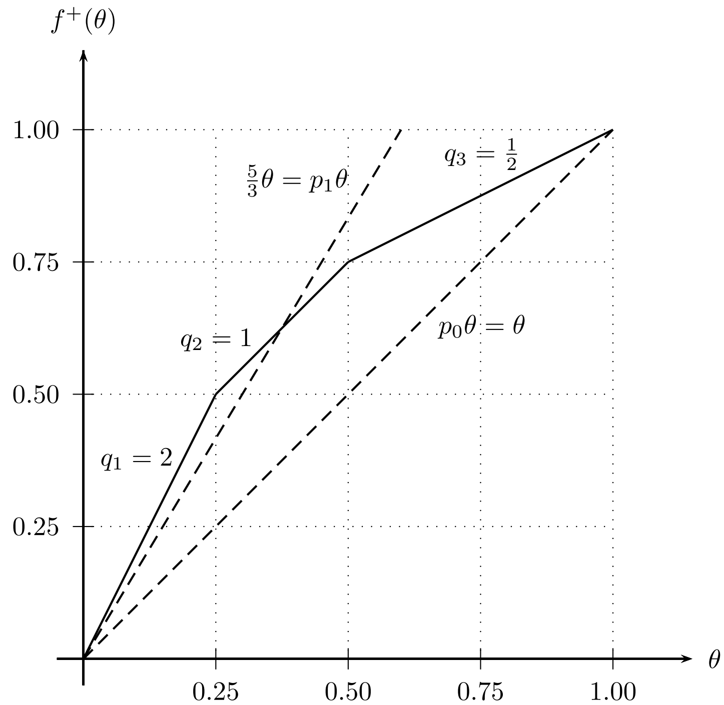

Figure 3: $C^+ = \{(0,2), (\frac{2}{8}, 1), (\frac{4}{8}, \frac{1}{2}), (1, 0)\}$. If $p_0 = 1$ any $\theta \in [\frac{2}{8}, \frac{4}{8}]$ is optimal. If $\theta_1 = \frac{3}{8}$ then $p_1 = \frac{5}{3}$ and $\theta_2 = \frac{2}{8}$

where $c \in \mathbb{R}^n,\ \ d \in \mathbb{R}^k,\ \ b \in \mathbb{R}^m,\ \ A \in \mathbb{R}^{m\times n},\ \ B \in \mathbb{R}^{m\times k},\ \ \hat{A} \in \mathbb{R}^{s\times n},\ \ \hat{B} \in \mathbb{R}^{s\times k}$ and $\hat{b} \in \mathbb{R}^s$.

Let $\Omega = \{(x,y) : \hat{A}x + \hat{B}y \leq \hat{b},\ x \in \mathbb{R}^n,\ x \geq 0,\ y \in \{0,1\}^k\}$. Let us suppose that $F(P) \neq \emptyset$ and Ω is a compact set.

Let us suppose that $c^t x + d^t y$ is a profit function and the constraints $Ax + By \leq b$ are resources constraints.

We will assume that: (i) the availability of some resources may be increased (for example the capacity of a knapsack, the capacity of a plant, the number of available machines, the available time to use a machine, etc...) and (ii) some of the coefficients of the matrix in the resources constraints may be reduced (for example the time that we need to process a job in a machine, the units of a resource to do a job, etc...).

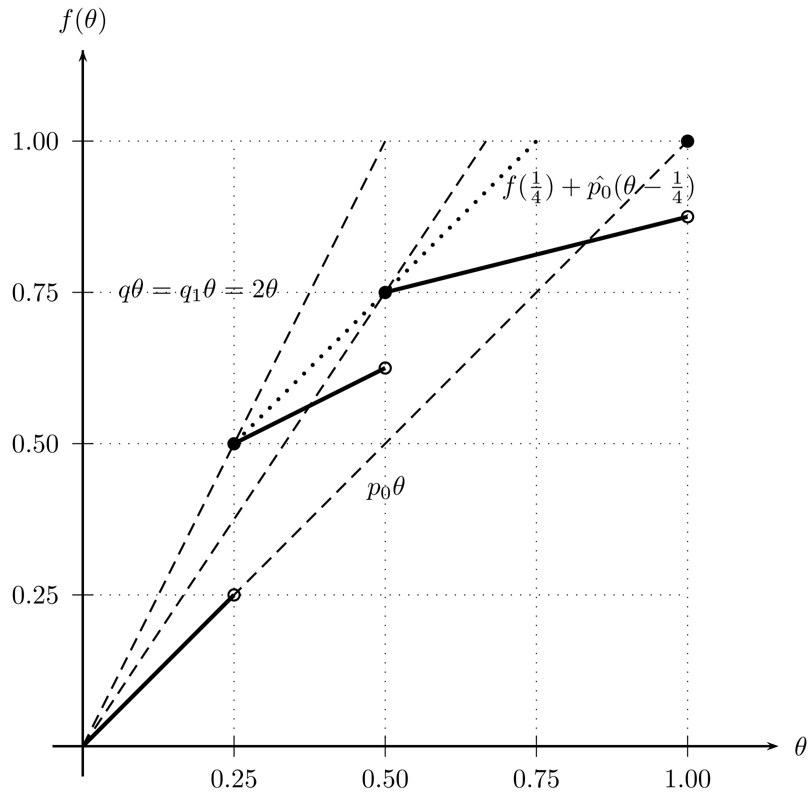

Figure 4: $p_0 = 1, \theta_1 = \frac{1}{2}, p_1 = \frac{\frac{3}{4}}{\frac{1}{2}} = \frac{3}{2}, \theta_2 = \frac{1}{4}, p_2 = \frac{\frac{1}{2}}{\frac{1}{4}} = 2, v(E(2)) = 0, q = 2 = q_1$. The *next* p_0 is defined as : $\hat{p_0} = \frac{f(\frac{1}{2}) - f(\frac{1}{4})}{\frac{1}{2} - \frac{1}{4}} = 1$

Let $(0,0,0) \le (\Delta b, \Delta A, \Delta B) \in \mathbb{R}^m \times \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times k}$.

Let $\theta \in [0,1]$. Let $P(\theta)$ be a problem in (x,y) defined as follows:

$$(P(\theta)) \;\; max \;\; c^t x + d^t y \;\; s.t.$$

$$(A - \theta\Delta A)x + (B - \theta\Delta B)y \le b + \theta\Delta b$$

$$(x,y) \in \Omega$$

Note that if $0 \le \theta_1 \le \theta_2 \le 1$ then $F(P(0)) \subseteq F(P(\theta_1)) \subseteq F(P(\theta_2)) \subseteq F(P(1))$ and $v(P) = v(P(0)) \le v(P(\theta_1)) \le v(P(\theta_2)) \le v(P(1))$.

Note that we use $\theta \in [0,1]$ without loss of generality. If $\theta \in [0,\infty)$ then again $F(P(0)) \subseteq F(P(\theta))$ for all $\theta \geq 0$ and because of Ω is a bounded set then we can find θ_0 such that $v(P(\theta)) = v(P(\theta_0))$ for all $\theta \geq \theta_0$. Also, if negative values are not valid for some coefficients of the matrix then it makes no sense that θ takes an arbitrary value. Therefore we may redefine $(\Delta b, \Delta A, \Delta B)$ to use $\theta \in [0,1]$ without loss of generality.

If the coefficients of the matrix are not mutually independent then ΔA and ΔB must be selected appropriately (an example will be present later). Let $H \subseteq \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times k}$ be the set of valid directions. In the rest of the paper we suppose that $(\Delta A, \Delta B) \in H$.

Let f be a function defined as:

$f(\theta) = v(P(\theta)) - v(P)$ for all $\theta \in [0,1]$.

Note that if $0 \leq \theta_1 \leq \theta_2 \leq 1$ then $0 \leq f(\theta_1) \leq f(\theta_2) \leq f(1)$.

The next Lemma is the generalization to the case $(0,0,0) \leq (\Delta b, \Delta A, \Delta B)$ of the same result restricted to $\Delta b \geq 0, \Delta A = 0$ and $\Delta B = 0$ presented previously in [6].

Lemma 6 *Let* $Q = \{\frac{f(\theta)}{\theta} : 0 < \theta \leq 1\}$. *$Q$ is a bounded set.*

Proof:

(1) Let $Y(\theta) = \{y \in \{0,1\}^k : \exists x \text{ such that } (x,y) \in F(P(\theta))\} \; \forall \theta \in [0,1]$.

Because of Ω is a compact set then there exists $\theta(y)$ such that $\theta(y) = min\{\theta : y \in Y(\theta), \;\; \theta \in [0,1]\} \; \forall y \in Y(1)$.

If $Y(0) = Y(1)$ let $\delta_1 = 1$, otherwise let $\delta_1 = min\{\theta(y) : y \notin Y(0)\}$ then $Y(0) = Y(\theta) \; \forall \theta \in [0, \delta_1)$.

(2) Let $y \in Y(0)$. Let $P(y,\theta)$ be a problem in x defined as follows:

$$(P(y,\theta)) \;\; d^t y + max \;\; c^t x \; s.t. \; (x,y) \in F(P(\theta))$$

$P(y,\theta)$ is a Linear Programming (LP) problem. Let $g_y : [0,1] \longrightarrow \Re$ be a function such that $g_y(\theta) = v(P(y,\theta))$. Note that $v(P) = v(P(0)) = max\{g_y(0) : y \in Y(0)\}$. It has been observed that g_y is locally rational and upper semicontinuous ([8],[10],[16],[21]). It follows that g_y is locally derivable at zero from the right.

Let $p(y) = \frac{dg_y(0)}{d\theta^+}$. Let $\epsilon > 0$ then there exists $\delta(y)$ such that $\frac{g_y(\theta)-g_y(0)}{\theta} \leq p(y) + \epsilon \; \forall \theta \in (0, \delta(y)) \forall y \in Y(0)$.

(3) Let $\delta^* = min\{\delta_1, min\{\delta(y) : y \in Y(0)\}\}$. Let $p^* = max\{p(y) : y \in Y(0)\}$ and let $\theta \in (0, \delta^*)$. We have that:

$$\frac{f(\theta)}{\theta} = \frac{v(P(\theta)) - v(P)}{\theta} = max\{\frac{v(P(y,\theta)) - v(P)}{\theta} : y \in Y(0)\} =$$

$$max\{\frac{g_y(\theta)) - v(P)}{\theta} : y \in Y(0)\}\} \leq max\{\frac{g_y(\theta)) - g_y(0)}{\theta} : y \in Y(0)\}\} \leq$$

$$max\{p(y) : y \in Y(0)\} \; + \epsilon = p^* + \epsilon$$

Therefore $\frac{f(\theta)}{\theta} \leq p^* + \epsilon$ for all $\theta \in (0, \delta^*)$. Let $p^{**} = \frac{f(1)}{\delta^*}$ then $\frac{f(\theta)}{\theta} \leq p^{**}$ for all $\theta \in [\delta^*, 1]$ and finally $max\{\frac{f(\theta)}{\theta} : \theta \in (0,1]\} \leq max\{p^{**}, p^* + \epsilon\}$ and Q is a bounded set.•

Now we can define a generalization of the a.s.p. for 0-1-MILP problems:

Definition 2 *The generalized average shadow price (gasp) relative to* $(\Delta b, \Delta A, \Delta B)$ *is the critical slope of* f.

If d, B, $\hat{B}$ and y does not appear then P is a Linear Programming (LP) problem and the *gasp* relative to $(\Delta b, 0, 0)$ is the usual shadow price (s.p.) relative to Δb and $q = \frac{df}{d\theta^+}(0)$.

If P is a 0-1-MILP problem the *gasp* relative to $(\Delta b, 0, 0)$ is the usual a.s.p relative to Δb.

For the resources the *gasp* is the maximun price that the decision maker is willing to pay for an additional unit of the package defined by $(\Delta b, 0, 0)$. For the coefficients matrix the *gasp* is the maximun price that the decision maker is willing to pay for a unit of reduction in the direction defined by $(0, \Delta A, \Delta B)$.

The interpretation of the *gasp* relative to $(\Delta b, \Delta A, \Delta B)$ is analogous.

In the general case the problem $E(p)$ relative to $(\Delta b, \Delta A, \Delta B)$ may be rewritten as a 0-1-Mixed Integer Bilinear Programming (0-1-MIBLP) problem ([12]) in (x, y, θ, δ) as follows :

$$(E(p)) \;\; max \;\; c^t x + d^t y - p\theta \;\; s.t.$$

$$(A - \theta \Delta A)x + By - \Delta B \delta \leq b + \Delta b$$

$$0 \leq \delta_j \leq y_j, \;\; 0 \leq \theta - \delta_j \leq 1 - y_j \;\; (j = 1, \cdots, k)$$

$$(x,y) \in \Omega, \quad \theta \in [0,1], \quad \delta \in \mathbb{R}^k$$

The problem $E(p)$ relative to $(\Delta b, 0, 0))$ may be rewritten as a 0-1-MILP in (x, y, θ) as follows:

$$(E(p)) \ \ max \ \ c^t x + d^t y - p\theta \ \ s.t.$$
$$Ax + By \leq b + \theta \Delta b$$
$$(x,y) \in \Omega, \quad \theta \in [0,1]$$

The problem $E(p)$ relative to $(\Delta b, 0, \Delta B)$ may be rewritten as a 0-1-MILP in (x, y, δ, θ) as follows:

$$(E(p)) \ \ max \ \ c^t x + d^t y - p\theta \ \ s.t.$$
$$Ax + By - \Delta B \delta \leq b + \theta \Delta b$$
$$0 \leq \delta_j \leq y_j, \quad 0 \leq \theta - \delta_j \leq 1 - y_j \quad (j = 1, \cdots, k)$$
$$(x,y) \in \Omega, \quad \theta \in [0,1], \quad \delta \in \mathbb{R}^k$$

If d, B, $\hat{B}$ and y does not appear and P is an LP problem then the problem $E(p)$ relative to $(\Delta b, \Delta A, 0)$ may be rewritten as a bilinear problem ([12]) in (x, θ) as follows:

$$(E(p)) \ \ max \ \ c^t x - p\theta \ \ s.t.$$
$$(A - \theta \Delta A)x \leq b + \Delta b$$
$$x \in \Omega, \quad \theta \in [0,1]$$

with Ω rewritten according the case.

For the cases $(\Delta b, 0, 0)$ and $(\Delta b, 0, \Delta B)$ f is a piecewise linear upper semi-continuous and nondecreasing function and then f^+ is a piecewise linear and nondecreasing function and the algorithm to obtain the *gasp* is finite.

The optimal value of $E(p)$ is the maximun additional profit which can be generated when purchasing (reducing) the resources (the coefficients matrix) at unit market price p. For the cases $(\Delta b, 0, 0)$ and $(\Delta b, 0, \Delta B)$ we can use f^+ to construct $v(E(p))$ for all $0 \leq p \leq q$ in order to find information about the strategy for investment in resources. In general, if f^+ is a piecewise linear and nondecreasing function with a finite number of pieces we can use it to construct $v(E(p))$ for all $0 \leq p \leq q$.

The properties from 1 until 7 from Lemma 1 have the usual economical interpretations (the same that we know for the s.p. in a LP problem).

With the *gasp* we can find bottlenecks as usual. If the *gasp* relative to $(\Delta b, \Delta A, \Delta B)$ is equal to zero then there is not chance to improved the performance of the system in that direction. Otherwise there is a modifiable specification of the resource constraints in that direction to improved the performance of the system.

Note that the modification of the system is restricted to the predefined direction $(\Delta b, \Delta A, \Delta B)$. In the extensions we deal with a generalization in order to consider any feasible modification.

4 Examples

Now we present some examples. This is not an exhaustive set of examples. We do not present examples where the *gasp* is equal either to the usual s.p. or to the usual a.s.p.. We present some of the new cases: a 0-1-ILP problem with a coefficient in the matrix that is (is not) a bottleneck, a 0-1-MILP problem with a coefficient in the matrix that is a bottleneck, a LP problem with a coefficient in the matrix that is a bottleneck and p_r converges to the *gasp* when we use the CS algorithm and a LP problem with a coefficient in the matrix that is a bottleneck and the CS algorithm finds the *gasp*.

Example 1 *P is a multidimensional Knapsack (KPm) ([19]) problem. The weight of item 1 is not a bottleneck.*

Let P a KPm problem ([19]) in y defined as follows:

$$(P) \;\; max \;\; y_1 + 2y_2 \;\; s.t.$$
$$2y_1 + y_2 \leq 2$$
$$y \in \Omega = \{y : y_1 + y_2 \leq 1, \;\; y \in \{0,1\}^2\}$$

The optimal solution of P is y^* with $y^{*t} = (0,1)$. The optimal value of P is $v(P) = 2$. Let $\Delta B = (2,0)$. The parametric problem $P(\theta)$ relative to $(0,0,\Delta B)$ with $\theta \in [0,1]$ is defined as follows:

$$(P(\theta)) \;\; max \;\; y_1 + 2y_2 \;\; s.t.$$
$$(2 - 2\theta)y_1 + y_2 \leq 2$$
$$y \in \Omega$$

We have that $v(P(\theta)) = v(P(0)) = v(P) = 2$ for all $\theta \in [0,1]$ and then $f(\theta) = 0$ for all $\theta \in [0,1]$. Therefore the *gasp* is $q = 0$ and the weight of item 1 is not a bottleneck.

In this example the CS algorithm finds q in one step.

Example 2 *P is a Fixed-Charge Multiple Knapsack (FCHMKP) problem ([20]). The weight of item 1 is a bottleneck.*

Let P a FCHMKP problem in (y, z) ([20]) defined as follows:

$$(P)\ \ max\ \ 7(y_{11} + y_{12}) + 5(y_{21} + y_{22}) + 12(y_{31} + y_{32}) - 10z_1 - 5z_2\ \ s.t.$$

$$3y_{11} + 2y_{21} + 2y_{31} \leq 4z_1$$

$$3y_{12} + 2y_{22} + 2y_{32} \leq 2z_2$$

$$y_{11} + y_{12} \leq 1,\ \ y_{21} + y_{22} \leq 1,\ \ y_{31} + y_{32} \leq 1$$

$$y_{ij} \in \{0, 1\}\ (i = 1, 2, 3)(j = 1, 2),\ \ z_j \in \{0, 1\}\ (j = 1, 2)$$

Let $\Omega = \{(y, z) :\ \ y_{11} + y_{12} \leq 1,\ \ y_{21} + y_{22} \leq 1,\ \ y_{31} + y_{32} \leq 1,\ \ y_{ij} \in \{0, 1\}\ (i = 1, 2, 3)(j = 1, 2),\ \ z_j \in \{0, 1\}\ (j = 1, 2)\}$

The optimal solution of P is (y^*, z^*) with:

${y^*}^t = (y^*_{11}, y^*_{12}, y^*_{21}, y^*_{22}, y^*_{31}, y^*_{32}) = (0, 0, 0, 0, 0, 1)$, ${z^*}^t = (z^*_1, z^*_2) = (0, 1)$ and $v(P) = 12 - 5 = 7$

We are interested in the weight of item 1. Let ΔB defined as:

$$\Delta B = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that the changes defined with ΔB are not mutually independent because the weight of item 1 appears in two constraints.

The parametric problem $P(\theta)$ relative to $(0, 0, \Delta B)$ with $\theta \in [0, 1]$ is defined as follows:

$$(P(\theta))\ \ max\ \ 7(y_{11} + y_{12}) + 5(y_{21} + y_{22}) + 12(y_{31} + y_{32}) - 10z_1 - 5z_2\ \ s.t.$$

$$(3 - 3\theta)y_{11} + 2y_{21} + 2y_{31} \leq 4z_1$$

$$(3 - 3\theta)y_{12} + 2y_{22} + 2y_{32} \leq 2z_2$$

$$(y, z) \in \Omega$$

$f(\theta)$ is defined as follows:

$$f(\theta) = \begin{cases} 0 & if \quad 0 \leq 3\theta < 1 \quad with \quad y^{*t} = (0,0,0,0,0,1), z^{*t} = (0,1) \\ 2 & if \quad 1 \leq 3\theta \leq 3 \quad with \quad y^{*t} = (0,1,1,0,1,0), z^{*t} = (1,1) \end{cases}$$

The *gasp* is $q = \frac{2}{\frac{1}{3}} = 6$ and the weight of item 1 is a bottleneck. In this example the CS algorithm finds q in two steps ($p_0 = 2, p_1 = 6, v(E(p_1) = 0)$. Also, we have f^+ in one more step (see figure 5).

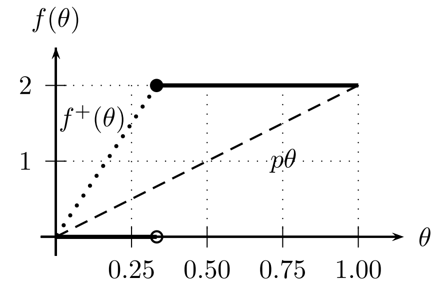

Figure 5: $q = 6$. The weight of item 1 is a bottleneck. If $p = 2$ then $v(E(p)) = \frac{4}{3}$ with $\theta^* = \frac{1}{3}$ the optimal decision.

Example 3 *P is a Capacitated Plant Location (CPL) ([3]) problem. The capacity of Plant 1 (a coefficient in the matrix) is a bottleneck.*

Let P a CPL problem in (x, y) (maximization case) defined as:

$$(P)\ max\ x_{11} + x_{12} + x_{21} + x_{22} + 1.1x_{31} + 1.1x_{32} - y_1 - 2y_2\ s.t$$
$$x_{11} + x_{12} - 10y_1 \leq 0$$
$$x_{21} + x_{22} - 10y_2 \leq 0$$
$$x_{31} + x_{32} - 4y_3 \leq 0$$
$$(x, y) \in \Omega$$

$$\Omega = \{(x, y) : \sum_{i=1}^{3} x_{i1} = 12,\ \sum_{i=1}^{3} x_{i2} = 12,\ x \geq 0,\ \ y_j \in \{0, 1\}\ (j = 1, 2)$$

The optimal solution of P is (x^*, y^*) with:

$x^{*t} = (x^*_{11}, x^*_{12}, x^*_{21}, x^*_{22}, x^*_{31}, x^*_{32}) = (10, 0, 0, 10, 2, 2),\ y^{*t} = (y^*_1, y^*_2, y^*_3) = (1, 1, 1)$ and $v(P) = 21.4$

We are interested in the capacity of plant 3. Let ΔB defined as:

$$\Delta B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 20 \end{bmatrix}$$

The parametric problem $P(\theta)$ relative to $(0,0,\Delta B)$ with $\theta \in [0,1]$ is defined as follows:

$$(P)\ max\ x_{11}+x_{12}+x_{21}+x_{22}+1.1x_{31}+1.1x_{32}-y_1-2y_2\ s.t$$
$$x_{11}+x_{12}-10y_1 \leq 0$$
$$x_{21}+x_{22}-10y_2 \leq 0$$
$$x_{31}+x_{32}-(4+20\theta)y_3 \leq 0$$
$$(x,y) \in \Omega$$

$f(\theta)$ is defined as follows:

$$f(\theta) = \begin{cases} 2\theta & if \quad 0 \leq 20\theta < 10 \quad with \quad y^{*t} = (1,1,1) \\ 3+2(\theta-\frac{1}{2}) & if \quad 10 \leq 20\theta < 20 \quad with \quad y^{*t} = (1,0,1) \\ 5 & if \quad 20\theta = 20 \quad with \quad y^{*t} = (0,0,1) \end{cases}$$

The *gasp* is $q = \frac{3}{\frac{1}{2}} = 6$ (see figure 6) and the capacity of plant 3 is a bottleneck. The CS algorithm finds q in two steps and f^+ in one more step.

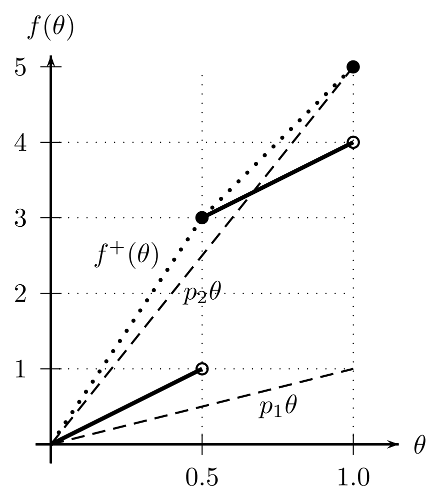

Figure 6: $q = 6$. The capacity of plant 3 is a bottleneck. If $p_1 = 1(p_2 = 5)$ then $v(E(p_1)) = 4(v(E(p_2)) = 0.5)$ with $\theta^* = 1(\theta^* = 0.5)$ the optimal decision.

Example 4 *P is a LP problem. The coefficient A_{11} is a bottleneck. The CS algorithm generates p_r that converges to q.*

Let P a LP problem in x defined as follows:

$$(P)\ \ max\ \ x_1+x_2\ \ s.t.$$

$$x_1 + x_2 \leq 2, \quad x \in \Omega$$

$$\Omega = \{x : \quad 2x_1 + x_2 \leq 3, \quad x \geq 0\}$$

The optimal solution of P is $x^{*t} = (1, 1)$ with $v(P) = 2$.

Let ΔA defined as $\Delta A = (1, 0)$. The parametric problem $P(\theta)$ in x relative to $(0, \Delta A, 0)$ with $\theta \in [0, 1]$ is defined as:

$$P(\theta) \;\; max \;\; x_1 + x_2 \;\; s.t.$$

$$(1 - \theta)x_1 + x_2 \leq 2, \quad x \in \Omega$$

$f(\theta)$ is defined as follows:

$f(\theta) = 1 - \frac{1}{1+\theta}$ if $0 \leq \theta \leq 1$ with $x_1^*(\theta) = \frac{1}{1+\theta}, x_2^*(\theta) = 3 - \frac{2}{1+\theta}$

The *gasp* is $q = 1$ (see figure 7) and the coefficient A_{11} is a bottleneck. In this case the CS algorithm generates a sequence p_r that converges to q. With a tolerance $\epsilon > 0$ the CS algorithm stops the first time that $v(E(p_r)) < \epsilon$. In that case $v(E(p)) < \epsilon$ for all $p \geq p_r$.

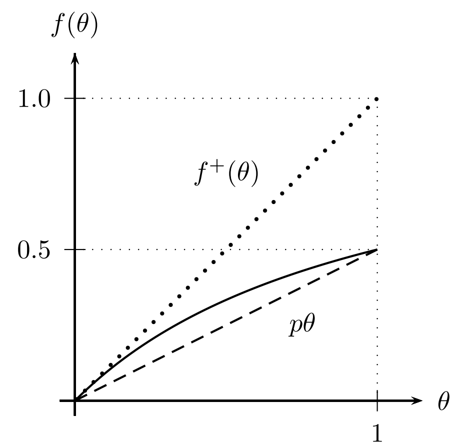

Figure 7: $q = 1$. The coefficient A_{11} is a bottleneck. If $p = \frac{1}{2}$ then $v(E(p)) = 2 - \sqrt{2}$ with $\theta^* = \sqrt{2} - 1$ the optimal decision.

Example 5 *P is a LP problem. The coefficient A_{11} is a bottleneck. The CS algorithm stops with $v(E(p_0)) = 0$.*

Let P a LP problem in x defined as follows:

$$(P)\ \ max\ \ x_1+x_2, 2x_1+x_2 \leq 2,\ \ x \in \Omega$$

$$\Omega = \{x:\ \ 2x_1+x_2 \leq 3,\ \ x \geq 0\}$$

The optimal solution of P is $x^{*t} =$ with $v(P) =$.

Let ΔA defined as $\Delta A = (2, 0)$. The parametric problem $P(\theta)$ in x relative to $(0, \Delta A, 0)$ with $\theta \in [0, 1]$ is defined as:

$$P(\theta)\ \ max\ \ x_1+x_2,\ \ (2-2\theta)x_1+x_2 \leq 2,\ \ x \in \Omega$$

$f(\theta)$ is defined as follows:

$$f(\theta) = \begin{cases} 0 & 0 \leq 2\theta \leq 1 \quad x_1^*(\theta) = 0, x_2^*(\theta) = 2 \\ 1 - \frac{1}{2\theta} & 1 \leq 2\theta \leq 2 \quad x_1^*(\theta) = \frac{1}{\theta}, x_2^*(\theta) = 3 - \frac{2}{\theta} \end{cases}$$

The *gasp* is $q = \frac{1}{2}$ (see figure 8) and the coefficient A_{11} is a bottleneck. In this case the CS algorithm stops with $v(E(p_0)) = 0$ to conclude that $q = p_0 = \frac{1}{2}$. Also we have f^+ in one step. Note that f^+ is a piecewise linear and nondecreasing function with one piece and the CS algorithms finds q.

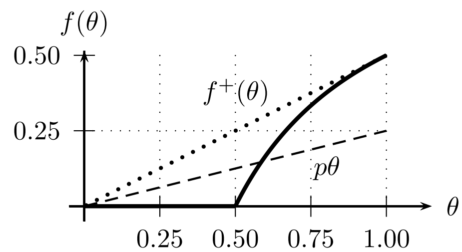

Figure 8: $q = \frac{1}{2}$. The coefficient A_{11} is a bottleneck. If $p = \frac{1}{4}$ then $v(E(p)) = \frac{1}{4}$ with $\theta^* = 1$ the optimal decision.

5 Extensions

5.1 The discrete case

In [18] the authors consider the discrete case ($\theta \in \{0, \theta_1, \cdots, \theta_l\}$) (with $\theta_i > 0$ for all i) instead of $\theta \in [0, 1]$ to define a generalization of the a.s.p..

The paper may be rewritten for the discrete case. In this case: (i) we have $Q = \{\frac{f(\theta)}{\theta} : \theta \in \{\theta_1, \cdots, l\}\}$ and again the *gasp* is defined as follows: $q = sup(Q)$ and (ii) The problem $E(p)$ relative to $(\Delta b, \Delta A, \Delta B)$ may be rewritten as a 0-1-MILP problem because of the nonlinear terms θx and θy may be linearized with standard procedures by using auxiliary binary variables ([11]).

5.2 A mathematical programming approach to redefine the global specification of the resource constraints

In [4] and [5] the authors consider the case in which the decision maker may redefine the system global configuration and then define a problem with two objectives: maximization of the original objective and minimization of the price of the modification. In order to consider nonlinear systems an evolutionary algorithm is proposed to obtain an approximation of the pareto set.

Let $r \geq 1$. Let $\underline{\theta} \in \Re^r$ and let $\overline{\theta} \in \Re^r$ with $\underline{\theta} \leq 0 \leq \overline{\theta}$. Let $\theta \in \Re^r$ the vector of parameters used to redefine the global configuration of the system then we consider the following problem:

$$(P(\theta)) \;\; max \;\; c(\theta)^t x + d(\theta)^t y \; s.t.$$
$$A(\theta)x + B(\theta)y \leq b(\theta)$$
$$(x, y) \in \Omega$$

Let $p \in \mathbb{R}^r$. Let us suppose that the price of the configuration redefined by θ is $p^t\theta$.

The biobjective mathematical programming problem in (x, θ) suggested in [4] and [5] is defined in this case as follows:

$$(BI)$$
$$max \; c(\theta)^t x + d(\theta)^t y, \;\; min \;\; p^t\theta \; s.t.$$
$$(x, y) \in F(P(\theta)), \;\; \underline{\theta} \leq \theta \leq \overline{\theta}$$

In this new contex the generalization of problem $E(p)$ is:

$$E(p) \;\; max \; c(\theta)^t x + d(\theta)^t y - p^t\theta \; s.t.$$
$$(x, y) \in F(P(\theta)), \;\; \underline{\theta} \leq \theta \leq \overline{\theta}$$

and its optimal solution is a specific non-dominated solution for BI.

In [7] we present a parametric approach to find an approximation of the pareto set of BI with linear changes defined by θ as follows:

Let $q^j \in \mathbb{R}^r$ $(j = 1, \cdots, n), d^i \in \mathbb{R}^r$ $(i = 1, \cdots, m)$ and $Q^{ij} \in \mathbb{R}^r$ $(i = 1, \cdots, m)(j = 1, \cdots, n)$ and let:

$$c(\theta)_j = c_j + {q^j}^t\theta \ \ \forall j$$

$$b(\theta)_i = b_i + {d^i}^t\theta \ \ \forall i$$

$$A(\theta)_{ij} = A_{ij} + {Q^{ij}}^t\theta \ \ \forall (i,j)$$

6 Conclusions

We present the *gasp*, a generalization of the a.s.p, in 0-1-Mixed Integer Linear Programming problems, and its relation with bottlenecks including the analysis relative to the coefficients matrix of resource constraints. An algorithm to find either the *gasp* or a convergent sequence to the *gasp* is presented. Also, a mathematical programming approach to find the strategy for investment in resources is presented by using the CS algorithm again and again to find the concave envelope of f and an optimal solution of $E(p)$ for all $p \geq 0$.

Now must be clear that the *gasp* may be used to overcome the limitations 2 and 3 presented in [4] and [5] at least from the theoretical point of view.

We must pointed out that we have an important limitation: the *gasp* is restricted to consider changes in the direction defined by $(\Delta b, \Delta A, \Delta B)$ (the same is true for the s.p and the a.s.p). In the extensions presented above we present an approach to overcome this limitation.

References

[1] Akgull, MA (1984) A note on shadow prices in Linear Programming.*J. Oper. Res. Soc.* 35:425-431.

[2] Aucamp DC, Steinberg DC (1982) The computation of shadow prices in Linear Programming. *J. Oper. Res. Soc.* 33:557-565.

[3] Barahona F, Chudak FA (2005) Near-optimal solutions to large-scale facility location problems. *Discrete Optimization.* 2:35-50.

[4] Bonyadi MR, Michalewicz Z (2014) Evolutionary Computation for Real-World Problems. Challenges in Computational Statistics and Data Mining. Stan Matwin-Jan Mielnikzuk Editors, Studies in Computational Intelligence, Springer Verlag. 605:1-21.

[5] Bonyadi MR, Michalewicz Z, Wagner M (2014) Beyond the Edge of Feasibility: Analysis of Bottlenecks. Simulated Evolution and Learning *Lect. Notes Comput. Sc.* 8886:431-442.

[6] Crema A (1995) Average shadow price in a mixed integer linear programming problem. *Eur. J. Oper. Res.* 85(3)625-635.

[7] Crema A (2016) A parametric programming approach to redefine the global configuration of resource constraints of 0-1-Integer Linear Programming problems. Optimization online, Applications OR and Management Sciences. Escuela de Computación, Facultad de Ciencias, Universidad Central de Venezuela.

[8] Freund RM (1985) Postoptimal Analysis of a Linear Program Under Simultaneous Changes in Matrix Coefficients. *Math. Program. Stud.* 24:1-13.

[9] Gal T (1986) Shadow prices and sensitivity analysis in Linear Programming under degeneracy. State-of-the-art-survey. *Operations Research Spectrum.*8:59-71.

[10] Gal T (1984) Linear Parametric Programming - a brief survey. *Math. Program. Stud.* 21:43-68.

[11] Fred Glover: Improved linear integer programming formulations of nonlinear integer problems *Management Science* Vol. 22, No. 4, December, pp. 455-460, 1975

[12] Gupte A, Ahmed S, Cheon MS and Dey S (2013) Solving mixed integer bilinear problems using milp formulations. *SIAM J. Optim.* 23(2)721744

[13] Jansen B, Roos C, de Jong JJ, Terlaky T (1997) Sensitivity analysis in linear programming: Just be careful! *Eur. J. Oper. Res.* 101(1):15-28.

[14] Kim S, Cho S (1988) A shadow price in integer programming for management decision. *Eur. J. Oper. Res.* 37(3)328-335.

[15] Kim S, Cho S (1992) Average shadow prices in mathematical programming. *J. Optimiz. Theory App.* 74(1)57-74.

[16] Martin DH (1975) On the Continuity of the Maximum in Parametric Linear Programming. *J. Optimiz. Theory App.* 17(3/4)205-210.

[17] Mukherjee S, Chatterjeeb AK (2006) Unified Concept of Bottleneck. Indian Institute of Management Ahmedabad 380015,INDIA. W.P. No. 2006-05-01.

[18] Mukherjee S, Chatterjeeb AK (2006) The average shadow price for MILPs with integral resource availability and its relationship to the marginal unit shadow price. *Eur. J. Oper. Res.* 169(1)5364.

[19] Puchinger J, Raidl G, Pferschy U (2010) The Multidimensional Knapsack Problem: Structure and Algorithms. *INFORMS J. Comput.* 22(2)250-265.

[20] Yamada T, Takeoka T (2009) An exact algorithm for the fixed-charge multiple knapsack problem. *Eur. J. Oper. Res.* 192:700705.

[21] Zuidwijk RA (2005) Linear Parametric Sensitivity Analysis of the Constraint Coefficient Matrix in Linear Programs. Erasmus Research Institute of Management. Report Series Research in Management.

Appendix

Proof of Lemma 1:

(1) Note that $0 \in F(E(p))$ and then $v(E(p)) \geq 0$ for all $p \geq 0$.

(2) If $v(E(q)) > 0$ there exists $\theta \in (0,1]$ with $f(\theta) - q\theta > 0$ and $\frac{f(\theta)}{\theta} > q$, therefore $v(E(q)) = 0$.

(3) If $p > q$ then $f(\theta) - p\theta \leq 0$ for all $\theta \in (0,1]$ and then $v(E(p)) = 0$.

(4) If $q = 0$ then $\frac{f(\theta)}{\theta} \leq 0 \ \ \forall \theta \in (0,1]$ and then $f(\theta) = 0 \ \ \forall \theta \in [0,1]$. If $f(\theta) = 0 \ \ \forall \theta \in [0,1]$ then $\frac{f(\theta)}{\theta} = 0 \ \ \forall \theta \in (0,1]$ and then $q = 0$.

(5) If $v(E(p)) > 0$ then there exists $\theta \in (0,1]$ with $f(\theta) - p\theta > 0$ and then $0 \leq p < \frac{f(\theta)}{\theta} \leq q$. If $0 \leq p < q$ and $v(E(p)) = 0$ then $f(\theta) - p\theta \leq 0 \ \ \forall \theta \in (0,1]$. Therefore $\frac{f(\theta)}{\theta} \leq p \ \ \forall \theta \in (0,1]$ and $q \leq p$. It follows that $v(E(p) > 0$.

(6) If $v(E(p)) = 0$ then $f(\theta) - p\theta \leq 0 \ \ \forall \theta \in (0,1]$ and $\frac{f(\theta)}{\theta} \leq p \ \ \forall \theta \in (0,1]$, therefore $q \leq p$ and $p = q$.•

(7) Let θ^* be an optimal solution for $E(p)$ then $f(\theta) - p\theta \leq f(\theta^*) - p\theta^*$ for all $\theta \in [0,1]$. It follows that $f^+(\theta) \leq f(\theta^*) + p(\theta - \theta^*)$ for all $\theta \in [0,1]$, therefore:

$$max \ \{f^+(\theta) - p\theta : \theta \in [0,1]\} \leq max \ \{f(\theta^*) + p(\theta - \theta^*) - p\theta : \theta \in [0,1]\} =$$

$$f(\theta^*) - p\theta^* = v(E(p)) = max \ \{f(\theta) - p\theta : \theta \in [0,1]\} \leq max \ \{f^+(\theta) - p\theta : \theta \in [0,1]\}.\bullet$$

Proof of Lemma 2:

(1) $0 < v(E(p_r)) = f(\theta_{r+1}) - p_r\theta_{r+1}$, therefore $p_{r+1} = \frac{f(\theta_{r+1})}{\theta_{r+1}} > p_r$.

(2) $v(E(p_{r+1})) = \ max\{f(\theta) - p_{r+1}\theta : \theta \in [0,1]\} < max\{f(\theta) - p_r\theta : \theta \in [0,1]\} = v(E(p_r))$.

(3) If $v(E(p_0)) > 0$ then $\theta_1 < 1 = \theta_0$. If $v(E(p_1)) > 0$ and $\theta_2 \geq \theta_1$ then:

$$f(\theta_1) - p_0\theta_1 = f(\theta_1) + p_0(\theta_2 - \theta_1) - p_0\theta_2 < f(\theta_1) + p_1(\theta_2 - \theta_1) - p_0\theta_2 =$$
$$(p_1 - p_0)\theta_2 < (p_1 - p_0)\theta_2 + f(\theta_2) - p_1\theta_2 = f(\theta_2) - p_0\theta_2$$

and we have a contradiction, therefore $\theta_2 < \theta_1$ and so on. •

Proof of Lemma 3:

Note that $p_0 = \frac{f(1)}{1} \leq q$. Let $p_r : \quad 0 < v(E(p_r)) = f(\theta_{r+1}) - p_r\theta_{r+1}$ then $q \geq p_{r+1} = \frac{f(\theta_{r+1})}{\theta_{r+1}} > p_r$. Therefore $p_r \leq q$ for all r and if $v(E(p_r)) = 0$ then $q = p_r \;\; \bullet$.