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PATH COVER AND PATH PACK INEQUALITIES FOR THE CAPACITATED FIXED-CHARGE NETWORK FLOW PROBLEM

ALPER ATAMTÜRK, BIRCE TEZEL AND SIMGE KÜÇÜKYAVUZ

Abstract. Capacitated fixed-charge network flows are used to model a variety of problems in telecommunication, facility location, production planning and supply chain management. In this paper, we investigate capacitated path substructures and derive strong and easy-to-compute *path cover and path pack inequalities*. These inequalities are based on an explicit characterization of the submodular inequalities through a fast computation of parametric minimum cuts on a path, and they generalize the well-known flow cover and flow pack inequalities for the single-node relaxations of fixed-charge flow models. We provide necessary and sufficient facet conditions. Computational results demonstrate the effectiveness of the inequalities when used as cuts in a branch-and-cut algorithm.

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1. Introduction

Given a directed graph with demand or supply on the nodes, and capacity, fixed and variable cost of flow on the arcs, the capacitated fixed-charge network flow (CFNF) problem is to choose a subset of the arcs and route the flow on the chosen arcs while satisfying the supply, demand and capacity constraints, so that the sum of fixed and variable costs is minimized.

There are numerous polyhedral studies on the fixed-charge network flow problem. In a seminal paper Wolsey (1989) introduces the so-called submodular inequalities, which subsume almost all valid inequalities known for capacitated fixed-charge networks. Although the submodular inequalities are very general, their coefficients are defined implicitly through value functions. In this paper, we give explicit valid inequalities that simultaneously make use of the path substructures of the network as well as the arc capacities.

A. Atamtürk: Department of Industrial Engineering & Operations Research, University of California, Berkeley, CA 94720. atamturk@berkeley.edu

B. Tezel: Department of Industrial Engineering & Operations Research, University of California, Berkeley, CA 94720. btezel@berkeley.edu

S. Küçükyavuz: Department of Industrial and Systems Engineering, University of Washington, Seattle, WA 98195. simge@uw.edu .

For the *uncapacitated* fixed-charge network flow problem, van Roy and Wolsey (1985) give flow path inequalities that are based on path substructures. Rardin and Wolsey (1993) introduce a new family of dicut inequalities and show that they describe the projection of extended multicommodity formulation onto the original variables of fixed-charge network flow problem. Ortega and Wolsey (2003) present a computational study on the performance of path and cut-set (dicut) inequalities.

For the *capacitated* fixed-charge network flow problem, almost all known valid inequalities are based on single-node relaxations. Padberg et al. (1985), van Roy and Wolsey (1986) and Gu et al. (1999) give flow cover, generalized flow cover and lifted flow cover inequalities. Stallaert (1997) introduces a complementary class of flow cover inequalities; Atamtürk (2001) describes lifted flow pack inequalities. Both uncapacitated path inequalities and capacitated flow cover inequalities are highly valuable in solving a host of practical problems and are part of the suite of cutting planes implemented in modern mixed-integer programming solvers.

The *path* structure arises naturally in network models of the lot-sizing problem. Atamtürk and Muñoz (2004) introduce valid inequalities for the capacitated lot-sizing problems with infinite inventory capacities. Atamtürk and Küçükyavuz (2005) give valid inequalities for the lot-sizing problems with finite inventory and infinite production capacities. Van Vyve (2013) introduces valid inequalities for the uncapacitated fixed charge transportation problems. Van Vyve and Ortega (2004) and Gade and Küçükyavuz (2011) give valid inequalities and extended formulations for uncapacitated lot-sizing with fixed charges on stocks. For uncapacitated lot-sizing with backlogging, Pochet and Wolsey (1988) and Pochet and Wolsey (1994) provide valid inequalities and Küçükyavuz and Pochet (2009) provide an explicit description of the convex hull.

Contributions. In this paper we consider a generic path relaxation, with supply and/or demand nodes and capacities on incoming and outgoing arcs. By exploiting the path substructure of the network and introducing notions of *path cover* and *path pack* we provide two explicitly-described subclasses of the submodular inequalities of Wolsey (1989). The most important consequence of the explicit derivation is that the coefficients of the submodular inequalities on a path can be computed efficiently. In particular, we show that *all* coefficients of such an inequality can be computed by solving max-flow/min-cut problems parametrically over the path in linear time. For a path with a single node, the inequalities reduce to the well-known flow cover and flow pack inequalities. Moreover, we show that the path cover and path pack inequalities dominate flow cover and flow pack inequalities for the corresponding single node relaxation of a path obtained by merging the path into a single node. We give necessary and sufficient facet-defining conditions. Finally, we report on computational experiments demonstrating the effectiveness of the proposed inequalities when used as cuts in a branch-and-cut algorithm.

Outline. The remainder of this paper is organized as follows: In Section 2, we describe the capacitated fixed-charge flow problem on a path, its formulation and the assumptions we make. In Section 3, we review the submodular inequalities, discuss their computation on a path, and introduce two explicit subclasses: path cover inequalities and path pack inequalities. In Section 4, we analyze the sufficient and necessary facet-defining conditions.

In Section 5, we present computational experiments showing the effectiveness of the path cover and path pack inequalities compared to other network inequalities.

2. Capacitated fixed-charge network flow on a path

Let $G = (N', A)$ be a directed graph with nodes N' and arcs A. Let s_N and t_N be the source and the sink nodes of G. Let $N := N' \setminus \{s_N, t_N\}$ and without loss of generality we label $N := \{1, \ldots, n\}$ such that a directed *forward path* arc exists from node i to node $i+1$ and a directed *backward path* arc exists from node $i+1$ to node i for each node $i = 1, \ldots, n-1$ (see Figure 1 for an illustration). In Remarks 1 and 2, we discuss how to obtain a "path" graph G from a more general directed graph.

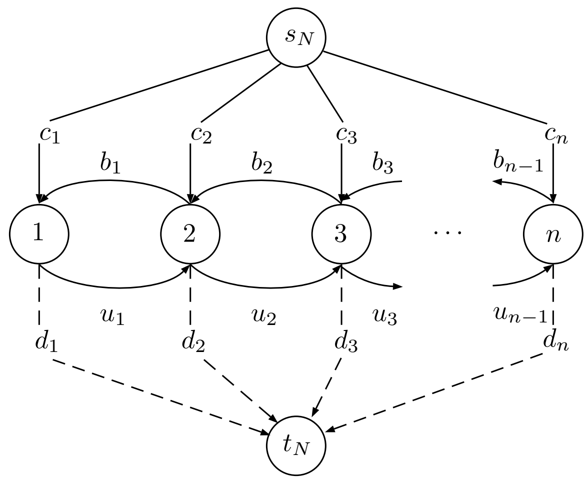

Figure 1. Fixed-charge network representation of a path.

Let $E^+ = \{(i,j) \in A : i \notin N, j \in N\}$ and $E^- = \{(i,j) \in A : i \in N, j \notin N\}$. Moreover, let us partition the sets E^+ and E^- such that $E^+_k = \{(i,j) \in A : i \notin N, j = k\}$ and $E^-_k = \{(i,j) \in A : i = k, j \notin N\}$ for $k \in N$. We refer to the arcs in E^+ and E^- as *non-path arcs*. Finally, let $E := E^+ \cup E^-$ be the set of all non-path arcs. For convenience, we generalize this set notation scheme. Given an arbitrary subset of non-path arcs $Y \subseteq E$, let $Y^+_j = Y \cap E^+_j$ and $Y^-_j = Y \cap E^-_j$.

Remark 1. If there is an arc $t = (i,j)$ from node $i \in N$ to $j \in N$, where $|i - j| > 1$, then we construct a relaxation by removing arc t, and replacing it with two arcs $t^- \in E^-_i$ and $t^+ \in E^+_j$.

Remark 2. Given a graph $\tilde{G} = (\tilde{N}, \tilde{A})$ with nodes $\tilde{N}$, arcs $\tilde{A}$ and a path that passes through nodes N, we can construct G as described above by letting $E^+ = \{(i,j) \in \tilde{A} : i \in \tilde{N} \setminus N, j \in N\}$ and $E^- = \{(i,j) \in \tilde{A} : i \in N, j \in \tilde{N} \setminus N\}$ and letting all the arcs in E^+ be the outgoing arcs from a dummy source s_N and all the arcs in E^- to be incoming to a dummy sink t_N.

Throughout the paper, we use the following notation: Let $[k,j] := \{k, k+1, \ldots, j\}$, $c(S) = \sum_{t \in S} c_t$, $y(S) = \sum_{t \in S} y_t$, $(a)^+ := \max\{0, a\}$ and $d_{kj} = \sum_{t=k}^{j} d_t$ if $j \geq k$ and 0 otherwise. Moreover, let $\dim(A)$ denote the dimension of a polyhedron A and $\text{conv}(S)$ be the convex hull of a set S.

The capacitated fixed-charge network flow problem on a path can be formulated as a mixed-integer optimization problem. Let d_j be the demand at node $j \in N$. Let the flow on forward path arc $(j, j+1)$ be represented by i_j with an upper bound u_j for $j \in N \setminus \{n\}$. Similarly, let the flow on backward path arc $(j+1, j)$ be represented by r_j with an upper bound b_j for $j \in N \setminus \{n\}$. Let y_t be the amount of flow on arc $t \in E$ with an upper bound c_t. Define binary variable x_t to be 1 if $y_t > 0$, and zero otherwise for all $t \in E$. An arc t is *closed* if $x_t = 0$ and *open* if $x_t = 1$. Moreover, let f_t be the fixed cost and p_t be the unit flow cost of arc t. Similarly, let h_j and g_j be the costs of unit flow, on forward and backward arcs $(j, j+1)$ and $(j+1, j)$ respectively for $j \in N \setminus \{n\}$. Then, the problem is formulated as

$$
\text{(F1)} \quad
\begin{aligned}
\min \quad & \sum_{t \in E} (f_t x_t + p_t y_t) + \sum_{j \in N} (h_j i_j + g_j r_j) && \text{(1a)}\\
\text{s. t.} \quad & i_{j-1} - r_{j-1} + y(E_j^+) - y(E_j^-) - i_j + r_j = d_j, \quad j \in N, && \text{(1b)}\\
& 0 \le y_t \le c_t x_t, \quad t \in E, && \text{(1c)}\\
& 0 \le i_j \le u_j, \quad j \in N, && \text{(1d)}\\
& 0 \le r_j \le b_j, \quad j \in N, && \text{(1e)}\\
& x_t \in \{0,1\}, \quad t \in E, && \text{(1f)}\\
& i_0 = i_n = r_0 = r_n = 0. && \text{(1g)}
\end{aligned}
$$

Let $\mathcal{P}$ be the set of feasible solutions of (F1). Figure 1 shows an example network representation of (F1). In this representation, the dummy source node s_N has a supply of d_{1n} and the dummy sink node t_N has demand d_{1n}.

Throughout we make the following assumptions on (F1):

(A.1) The set $\mathcal{P}_t = \{(x, y, i, r) \in \mathcal{P} : x_t = 0\} \neq \emptyset$ for all $t \in E$,
(A.2) $c_t > 0$, $u_j > 0$ and $b_j > 0$ for all $t \in E$ and $j \in N$,
(A.3) $c_t \le d_{1n} + c(E^-)$ for all $t \in E^+$,
(A.4) $c_t \le b_{j-1} + u_j + (d_j)^+ + c(E_j^-)$, for all $j \in N, t \in E_j^+$,
(A.5) $c_t \le b_j + u_{j-1} + (-d_j)^+ + c(E_j^+)$ for all $j \in N, t \in E_j^-$.

Assumptions (A.1)–(A.2) ensure that $\dim\big(\text{conv}(\mathcal{P})\big) = 2|E| + |N| - 2$. Notice that, if (A.1) does not hold for some $t \in E$, then $x_t = 1$ for all points in $\mathcal{P}$. Similarly, if (A.2) does not hold, the flow on such an arc can be fixed to zero. Finally, assumptions (A.3)–(A.5) are without loss of generality. The flow values on arcs $t \in E$ cannot exceed the capacities implied in (A.3)–(A.5).

Next, we review the submodular inequalities introduced by Wolsey (1989) that are valid for any capacitated fixed-charge network flow problem. Then, using the path structure, we obtain the explicit submodular inequality coefficients in $O(|E| + |N|)$ time.

3. Submodular inequalities on paths

Let $S^+ \subseteq E^+$ and $L^- \subseteq E^-$. Wolsey (1989) shows that the value function of the following optimization problem is submodular:

$$
\begin{aligned}
v(S^+, L^-) = \max \quad & \sum_{t \in E} a_t y_t && \text{(2a)}\\
\text{s. t.} \quad & i_{j-1} - r_{j-1} + y(E_j^+) - y(E_j^-) - i_j + r_j \leq d_j, \quad j \in N, && \text{(2b)}\\
& 0 \leq i_j \leq u_j, \quad j \in N, && \text{(2c)}\\
& 0 \leq r_j \leq b_j, \quad j \in N, && \text{(2d)}\\
\text{(F2)} \qquad & 0 \leq y_t \leq c_t, \quad t \in E, && \text{(2e)}\\
& i_0 = i_n = r_0 = r_n = 0, && \text{(2f)}\\
& y_t = 0, \quad t \in (E^+ \setminus S^+) \cup L^-, && \text{(2g)}
\end{aligned}
$$

where $a_t \in \{0, 1\}$ for $t \in E^+$ and $a_t \in \{0, -1\}$ for $t \in E^-$. The set of feasible solutions of (F2) is represented by $\mathcal{Q}$.

We call the sets S^+ and L^- that are used in the definition of $v(S^+, L^-)$ the *objective sets*. For ease of notation, we also represent the objective sets as $C := S^+ \cup L^-$. Following this notation, let $v(C) := v(S^+, L^-)$, $v(C \setminus \{t\}) = v(S^+ \setminus \{t\}, L^-)$ for $t \in S^+$ and $v(C \setminus \{t\}) = v(S^+, L^- \setminus \{t\})$ for $t \in L^-$. Moreover, let

$$\rho_t(C) = v(C \cup \{t\}) - v(C)$$

be the marginal contribution of adding an arc t to C with respect to the value function v. Wolsey (1989) shows that the following inequalities are valid for $\mathcal{P}$:

$$\sum_{t \in E} a_t y_t + \sum_{t \in C} \rho_t(C \setminus \{t\})(1 - \bar{x}_t) \leq v(C) + \sum_{t \in E \setminus C} \rho_t(\emptyset)\bar{x}_t, \tag{3}$$

$$\sum_{t \in E} a_t y_t + \sum_{t \in C} \rho_t(E \setminus \{t\})(1 - \bar{x}_t) \leq v(C) + \sum_{t \in E \setminus C} \rho_t(C)\bar{x}_t, \tag{4}$$

where the variable $\bar{x}_t$ is defined as

$$\bar{x}_t = \begin{cases} x_t, & t \in S^+ \\ 1 - x_t, & t \in L^-. \end{cases}$$

We refer to submodular inequalities (3) and (4) derived for path structures as *path inequalities*. In this paper, we consider sets S^+ and L^- such that (F2) is feasible for all objective sets C and $C \setminus \{t\}$ for all $t \in C$.

3.1. Equivalence to the maximum flow problem.

Equivalence to the maximum flow problem. Define sets K^+ and K^- such that the coefficients of the objective function (2a) are:

$$a_t = \begin{cases} 1, & t \in K^+ \\ -1, & t \in K^- \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

where $S^+ \subseteq K^+ \subseteq E^+$ and $K^- \subseteq E^- \setminus L^-$. We refer to the sets K^+ and K^- as *coefficient sets*. Let the set of arcs with zero coefficients in (2a) be represented by $\bar{K}^+ = E^+ \setminus K^+$ and

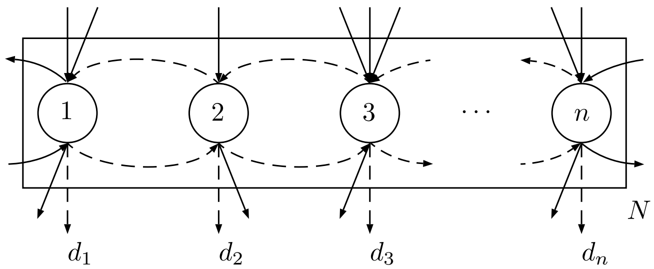

FIGURE 2. Set of arcs E, represented by solid directed lines.

$\bar{K}^- = E^- \setminus K^-$. Given a selection of coefficients as described in (5), we claim that (F2) can be transformed to a maximum flow problem. We first show this result assuming $d_j \geq 0$ for all $j \in N$; however, we show in Appendix A that for the derivation of the inequalities the nonnegativity of demand is without loss of generality through an appropriate transformation.

Proposition 1. *Let $S^+ \subseteq E^+$ and $L^- \subseteq E^-$ be the objective set in (F2) and let $\mathcal{Y}$ be the nonempty set of optimal solutions of (F2). If $d_j \geq 0$ for all $j \in N$, then there exists at least one optimal solution $(\boldsymbol{y}^*, \boldsymbol{r}^*, \boldsymbol{i}^*) \in \mathcal{Y}$ such that $y^*_t = 0$ for $t \in \bar{K}^+ \cup K^-$.*

Proof. Observe that $y^*_t = 0$ for all $t \in E^+ \setminus S^+$, due to constraints (2g). Since $\bar{K}^+ \subseteq E^+ \setminus S^+$, $y^*_t = 0$, for $t \in \bar{K}^+$ from feasibility of (F2). Similarly, $y^*_t = 0$ for all $t \in L^-$ by constraints (2g).

Now suppose that, $y^*_t = \epsilon > 0$ for some $t \in K^-_j$ (i.e., $a_t = -1$ for arc t in (F2)). Let the slack value at constraint (2b) for node j be

$$s_j = d_j - \left[i^*_{j-1} - r^*_{j-1} + y^*(E^+_j) - y^*(E^-_j \setminus \{t\}) - y^*_t - i^*_j + r^*_j\right].$$

If $s_j \geq \epsilon$, then decreasing y^*_t by ϵ both improves the objective function value and conserves feasibility since $s_j - \epsilon \geq 0$ at the flow balance inequalities (2b).

If $s_j < \epsilon$, then decreasing y^*_t by ϵ violates flow balance inequality since $s_j - \epsilon < 0$. In this case, there must exist a simple directed path P from either the source node s_N or a node $k \in N \setminus \{j\}$ to node j where all arcs have at least a flow of $(\epsilon - s_j)$. This is guaranteed because, $s_j < \epsilon$ implies that, without the outgoing arc t, there is more incoming flow to node j than outgoing. Then, notice that decreasing the flow on arc t and all arcs in path P by $\epsilon - s_j$ conserves feasibility. At the end of this transformation, the slack value s_j does not change, however; the flow at arc t is now $y^*_t = s_j$ which is the first case that is discussed above. As a result, we obtain a new solution to (F2) where $y^*_t = 0$ and the objective value is at least as large. □

Proposition 2. *If $d_j \geq 0$ for all $j \in N$, then (F2) is equivalent to a maximum flow problem from source s_N to sink t_N.*

Proof. By Proposition 1, the decision variables y_t, for $t \in \bar{K}^+ \cup K^-$ can be assumed to be zero in the optimization problem 2 with objective set S^+, L^-. Then, notice that the problem defined by (F2) reduces to:

$$\max\left\{y(K^+) : \; i_{j-1} - r_{j-1} + y(K^+_j) - y(\bar{K}^-_j) - i_j + r_j \leq d_j, \; j \in N, \; (2c) - (2g)\right\}.$$

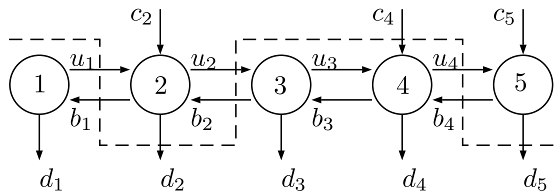

FIGURE 3. A representation of minimum $s_N - t_N$ cut.

Since, constraint (2g) pushes the flow on arcs in $E^+ \setminus S^+$ and arcs in L^- to zero, we drop the flow decisions of these arcs and the constraint set (2g):

$$\max\left\{y(S^+):\ i_{j-1} - r_{j-1} + y(S_j^+) - y(\bar{K}_j^-) - i_j + r_j \le d_j,\ j \in N,\ (2c) - (2f)\right\}. \tag{6}$$

Now, we reformulate (6) by representing the left hand side of the flow balance constraint by a new nonnegative decision variable z_j that has an upper bound of d_j for each $j \in N$:

$$\max\left\{y(S^+): i_{j-1} - r_{j-1} + y(S_j^+) - y(\bar{K}_j^-) - i_j + r_j = z_j,\ j \in N,\right.$$
$$\left. 0 \le z_j \le d_j,\ j \in N, \quad (2c) - (2f)\right\}.$$

Notice that, the formulation above is equivalent the maximum flow formulation from the source node s_N to the sink node t_N for the path structures we are considering in this paper. □

Under the assumption that $d_j \ge 0$ for all $j \in N$, Proposition 1 and Proposition 2 together show that the optimal objective function value $v(S^+, L^-)$ can be computed by solving a maximum flow problem from source s_N to sink t_N. We generalize this result in Appendix A for node sets N such that $d_j < 0$ for some $j \in N$. As a result, obtaining the explicit coefficients of submodular inequalities (3) and (4) reduces to solving $|E| + 1$ maximum flow problems. For a general underlying graph, solving $|E| + 1$ maximum flow problems would take $O(|E|^2|N|)$ time (e.g., see King et al. (1994)), where $|E|$ and $|N|$ are the number of arcs and nodes, respectively. In the following subsection, by utilizing the equivalence of maximum flow and minimum cuts and the path structure, we show that all coefficients of (3) and (3) can be obtained in $O(|E| + |N|)$ time using dynamic programming.

3.2. **Computing the coefficients of the submodular inequalities.** Throughout the paper, we use minimum cut arguments to find the explicit coefficients of inequalities (3) and (4). Figure 3 illustrates an example where $N = [1, 5]$, $S^+ = \{2, 4, 5\}$, $E^- = \emptyset$ and the cut represented by the dashed line corresponds to the partition $\{s_N, 2, 5\}$ and $\{t_N, 1, 3, 4\}$. Therefore, the value of this cut is $b_1 + d_2 + u_2 + c_4 + b_4 + d_5$. Moreover, we say that a cut *passes above node* j if j is in the source partition and *passes below node* j if j is in the sink partition.

Let α_j^u and α_j^d be the minimum value of a cut on nodes $[1, j]$ that passes above and below node j, respectively. Similarly, let β_j^u and β_j^d be the minimum values of cuts on nodes $[j, n]$ that passes above and below node j respectively. Finally, let

$$S^- = E^- \setminus (K^- \cup L^-),$$

where K^- is defined in (5). Recall that S^+ and L^- are the given objective sets. Given the notation introduced above, note that all of the arcs in sets S^- and L^- have a coefficient zero in (F2). Therefore, dropping an arc from L^- is equivalent to adding that arc to S^-. We compute $\alpha_j^{\{u,d\}}$ by a forward recursion and $\beta_j^{\{u,d\}}$ by a backward recursion:

$$\alpha_j^u = \min\{\alpha_{j-1}^d + u_{j-1}, \alpha_{j-1}^u\} + c(S_j^+) \tag{7}$$

$$\alpha_j^d = \min\{\alpha_{j-1}^d, \alpha_{j-1}^u + b_{j-1}\} + d_j + c(S_j^-), \tag{8}$$

where $\alpha_0^u = \alpha_0^d = 0$ and

$$\beta_j^u = \min\{\beta_{j+1}^u, \beta_{j+1}^d + b_j\} + c(S_j^+) \tag{9}$$

$$\beta_j^d = \min\{\beta_{j+1}^u + u_j, \beta_{j+1}^d\} + d_j + c(S_j^-), \tag{10}$$

where $\beta_{n+1}^u = \beta_{n+1}^d = 0$.

Let m_j^u and m_j^d be the values of minimum cuts for nodes $[1,n]$ that pass above and below node j, respectively. Notice that

$$m_j^u = \alpha_j^u + \beta_j^u - c(S_j^+) \tag{11}$$

and

$$m_j^d = \alpha_j^d + \beta_j^d - d_j - c(S_j^-). \tag{12}$$

For convenience, let

$$m_j := \min\{m_j^u, m_j^d\}.$$

Notice that m_j is the minimum of the minimum cut values that passes above and below node j. Since the minimum cut corresponding to $v(C)$ has to pass either above or below node j, m_j is equal to $v(C)$ for any $j \in N$. As a result, the value of minimum cut (or maximum flow) for the objective set $C = S^+ \cup L^-$ is

$$v(C) = m_1 = \cdots = m_n. \tag{13}$$

Proposition 3. *All values m_j, for $j \in N$, can be computed in $O(|E| + |N|)$ time.*

Obtaining the explicit coefficients of inequalities (3) and (4) also requires finding $v(C\setminus\{t\})$ for $t \in C$ and $v(C \cup \{t\})$ for $t \notin C$ in addition to $v(C)$. It is important to note that we do not need to solve the recursions above repeatedly. Once the values m_j^u and m_j^d are obtained for the set C, the marginals $\rho_t(C \setminus \{t\})$ and $\rho_t(C)$ can be found in $O(1)$ time for each $t \in E$.

We use the following observation while providing the marginal values $\rho_t(C)$ and $\rho_t(C\setminus\{t\})$ in closed form.

Observation 1. Let $c \geq 0$ and $d := (b-a)^+$, then,

1. $\min\{a + c, b\} - \min\{a, b\} = \min\{c, d\}$,
2. $\min\{a, b\} - \min\{a, b - c\} = (c - d)^+$.

In the remainder of this section, we give the coefficients ρ_t for inequalities (3) and (4) explicitly.

Coefficients of inequality (3)**: Path cover inequalities.** Let S^+ and L^- be the objective sets in (F2) and $S^- \subseteq E^- \setminus L^-$. We select the coefficient sets in (5) as $K^+ = S^+$ and

$K^- = E^- \setminus (L^- \cup S^-)$ to obtain the explicit form of inequality (3). Note that, as a result, the set definition of $S^- = E^- \setminus (K^- \cup L^-)$ is conserved.

Definition 1. Let the coefficient sets in (5) be selected as above and (S^+, L^-) be the objective set. The set (S^+, S^-) is called a *path cover* for the node set N if

$$v(S^+, L^-) = d_{1n} + c(S^-).$$

For inequality (3), we assume that the set (S^+, S^-) is a path cover for N. Then, by definition,

$$v(C) = m_1 = \cdots = m_n = d_{1n} + c(S^-).$$

After obtaining the values m_j^u and m_j^d for a node $j \in N$ using recursions in (7)-(10), it is trivial to find the minimum cut value after dropping an arc t from S_j^+:

$$v(C \setminus \{t\}) = \min\{m_j^u - c_t, m_j^d\}, \quad t \in S_j^+, \quad j \in N.$$

Similarly, dropping an arc $t \in L_j^-$ results in the minimum cut value:

$$v(C \setminus \{t\}) = \min\{m_j^u, m_j^d + c_t\}, \quad t \in L_j^-, \quad j \in N.$$

Using Observation 1, we obtain the marginal values

$$\rho_t(C \setminus \{t\}) = (c_t - \lambda_j)^+, \quad t \in S^+$$

and

$$\rho_t(C \setminus \{t\}) = \min\{\lambda_j, c_t\}, \quad t \in L^-$$

where

$$\lambda_j = (m_j^u - m_j^d)^+, \quad j \in N.$$

On the other hand, all the coefficients $\rho_t(\emptyset) = 0$ for arcs $t \in E \setminus C$. First, notice that, for $t \in E^+ \setminus S^+$, $v(\{t\}) = 0$, because the coefficient $a_t = 0$ for $t \in E^+ \setminus S^+$. Notice that, $v(\{t\}) = 0$ for $t \in E^- \setminus L^-$, since all incoming arcs would be closed for an objective set $(\emptyset, \{t\})$. As a result, inequality (3) for the objective set (S^+, L^-) can be written as

$$y(S^+) + \sum_{t \in S^+} (c_t - \lambda_j)^+ (1 - x_t) \le d_{1n} + c(S^-) + \sum_{t \in L^-} \min\{c_t, \lambda_j\} x_t + y(E^- \setminus (L^- \cup S^-)). \tag{14}$$

We refer to inequalities (14) as the *path cover inequalities*.

Remark 3. Observe that for a path consisting of a single node $N = \{j\}$ with demand $d := d_j > 0$, the path cover inequalities (14) reduce to the flow cover inequalities (Padberg et al., 1985, van Roy and Wolsey, 1986). Suppose that the path consists of a single node $N = \{j\}$ with demand $d := d_j > 0$. Let $S^+ \subseteq E^+$ and $S^- \subseteq E^-$. The set (S^+, S^-) is a flow cover if $\lambda := c(S^+) - d - c(S^-) > 0$ and the resulting path cover inequality

$$y(S^+) + \sum_{t \in S^+} (c_t - \lambda)^+ (1 - x_t) \le d + c(S^-) + \lambda x(L^-) + y(E^- \setminus L^-) \tag{15}$$

is a flow cover inequality.

Proposition 4. *Let (S^+, S^-) be a path cover for the node set N. The path cover inequality for node set N is at least as strong as the flow cover inequality for the single node relaxation obtained by merging the nodes in N.*

Proof. Flow cover and path cover inequalities differ in the coefficients of variables x_t for $t \in S^+$ and $t \in L^-$. Therefore, we compare the values λ_j, $j \in N$ of path cover inequalities (14) and value λ of flow cover inequalities (15). First, notice that the merging of node set N in graph G is equivalent to relaxing the values u_j and b_j to be infinite for $j \in [1, n-1]$. As a result, the value of the minimum cut that goes above a the merged node is $\bar{m}^u = c(S^+)$ and the value of the minimum cut that goes below the merged node is $\bar{m}^d = d_{1n} + c(S^-)$. Now, observe that the recursions in (7)-(10) imply that the minimum cut values for the original graph G are smaller:

$$m_j^u = \alpha_j^u + \beta_j^u - c(S_j^+) \leq c(S^+) = \bar{m}^u$$

and

$$m_j^d = \alpha_j^d + \beta_j^d - d_j - c(S_j^-) \leq d_{1n} + c(S^-) = \bar{m}^d$$

for all $j \in N$. Recall that the coefficient for the flow cover inequality is $\lambda = (\bar{m}^u - \bar{m}^d)^+$ and the coefficients for path cover inequality are $\lambda_j = (m_j^u - m_j^d)^+$ for $j \in N$. The fact that (S^+, S^-) is a path cover implies that $m_j^d = d_{1n} + c(S^-)$ for all $j \in N$. Since $\bar{m}^d = m_j^d$ and $m_j^u \leq \bar{m}^u$ for all $j \in N$, we observe that $\lambda_j \leq \lambda$ for all $j \in N$. Consequently, the path cover inequality (14) is at least as strong as the flow cover inequality (15). $\square$

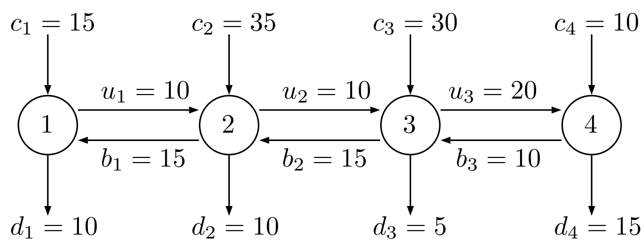

FIGURE 4. A lot-sizing instance with backlogging.

Example 1. Consider the lot-sizing instance in Figure 4 where $N = [1,4]$, $S^+ = \{2,3\}$, $L^- = \emptyset$. Observe that $m_1^u = 45$, $m_1^d = 40$, $m_2^u = 65$, $m_2^d = 40$, $m_3^u = 60$, $m_3^d = 40$, and $m_4^u = 45$, $m_4^d = 40$. Then, $\lambda_1 = 5$, $\lambda_2 = 25$, $\lambda_3 = 20$, and $\lambda_4 = 5$ leading to coefficients 10 and 10 for $(1 - x_2)$ and $(1 - x_3)$, respectively. Also, notice that the maximum flow values are $v(C) = 40$, $v(C \setminus \{2\}) = 30$, and $v(C \setminus \{3\}) = 30$. Then, the resulting path cover inequality (14) is

$$y_2 + y_3 + 10(1 - x_2) + 10(1 - x_3) \leq 40, \tag{16}$$

and it is facet-defining for $\text{conv}(\mathcal{P})$ as it will be shown in Section 4. Now, consider the relaxation obtained by merging the nodes in $[1,4]$ into a single node with incoming arcs $\{1,2,3,4\}$ and demand $d = 40$. As a result, the flow cover inequalities can be applied to the merged node set. The excess value for the set $S^+ = \{2,3\}$ is $\lambda = c(S^+) - d = 25$. Then, the resulting flow cover inequality (15) is

$$y_2 + y_3 + 10(1 - x_2) + 5(1 - x_3) \leq 40,$$

and it is weaker than the path cover inequality (16). $\square$

Coefficients of inequality (4)**: Path pack inequalities.** Let S^+ and L^- be the objective sets in (F2) and let $S^- \subseteq E^- \setminus L^-$. We select the coefficient sets in (5) as $K^+ = E^+$ and $K^- = E^- \setminus (S^- \cup L^-)$ to obtain the explicit form of inequality (4). Note that, as a result, the set definition of $S^- = E^- \setminus (K^- \cup L^-)$ is conserved.

Definition 2. Let the coefficients in (5) be selected as above and (S^+, L^-) be the objective set. The set (S^+, S^-) is called a *path pack* for node set N if

$$v(S^+, L^-) = c(S^+).$$

For inequality (4), we assume that the set (S^+, S^-) is a path pack for N and $L^- = \emptyset$. Now, we need to compute the values of $v(C)$, $v(E)$, $v(E \setminus \{t\})$ for $t \in C$ and $v(C \cup \{t\})$ for $t \in E \setminus C$. The value of $v(C \cup \{t\})$ can be obtained using the values m_j^u and m_j^d that are given by recursions (7)-(10). Then,

$$v(C \cup \{t\}) = \min\{m_j^u + c_t, m_j^d\}, \quad t \in E_j^+ \setminus S_j^+, \quad j \in N$$

and

$$v(C \cup \{t\}) = \min\{m_j^u, m_j^d + c_t\}, \quad t \in S_j^-, \quad j \in N.$$

Then, using Observation 1, we compute the marginal values

$$\rho_t(C) = \min\{c_t, \mu_j\}, \quad t \in E_j^+ \setminus S_j^+$$

and

$$\rho_t(C) = (c_t - \mu_j)^+, \quad t \in S_j^-,$$

where

$$\mu_j = (m_j^d - m_j^u)^+, \quad j \in N.$$

Next, we compute the values $v(E)$ and $v(E \setminus \{t\})$ for $t \in C$. Note that feasibility of (F1) implies that $(E^+, \emptyset)$ is a path cover for N. By Assumption (A.1), $(E^+ \setminus \{t\}, \emptyset)$ is also a path cover for N for each $t \in S^+$. Then $v(E) = v(E \setminus \{t\}) = d_{1n}$ and

$$\rho_t(E \setminus \{t\}) = 0, \quad t \in S^+ \cup L^-.$$

Then, inequality (4) can be explicitly written as

$$y(S^+) + \sum_{t \in E^+ \setminus S^+} (y_t - \min\{c_t, \mu_j\} x_t) + \sum_{t \in S^-} (c_t - \mu_j)^+ (1 - x_t) \leq c(S^+) + y(E^- \setminus S^-). \quad (17)$$

We refer to inequalities (17) as the *path pack inequalities*.

Remark 4. Observe that for a path consisting of a single node $N = \{j\}$ with demand $d := d_j > 0$, the path pack inequalities (17), reduce to the flow pack inequalities ((Atamtürk, 2001)). Let (S^+, S^-) be a flow pack and $\mu := d - c(S^+) + c(S^-) > 0$. Also, notice that the maximum flow that can be sent through S^+ for demand d and arcs in S^- is $c(S^+)$. Then, the value function $v(S^+) = c(S^+)$ and the resulting path pack inequality

$$y(S^+) + \sum_{t \in E^+ \setminus S^+} (y_t - \min\{c_t, \mu\} x_t) + \sum_{t \in S^-} (c_t - \mu)^+ (1 - x_t) \leq c(S^+) + y(E^- \setminus S^-) \quad (18)$$

is equivalent to the flow pack inequality.

Proposition 5. *Let (S^+, S^-) be a path pack for the node set N. The path pack inequality for N is at least as strong as the flow pack inequality for the single node relaxation obtained by merging the nodes in N.*

Proof. The proof is similar to that of Proposition 4. Notice that, flow pack and path pack inequalities only differ in the coefficients of variables x_t for $t \in E^+ \setminus S^+$ and $t \in S^-$. Therefore, we compare the values μ_j, $j \in N$ of path pack inequalities (17) and value μ of flow pack inequalities (18). For the single node relaxation, the values of the minimum cuts that pass above and below the merged node are $\bar{m}^u = c(S^+)$ and $\bar{m}^d = d_{1n} + c(S^-)$, respectively. The recursions in (7)-(10) imply that

$$m_j^u = \alpha_j^u + \beta_j^u - c(S_j^+) \leq c(S^+) = \bar{m}^u$$

and

$$m_j^d = \alpha_j^d + \beta_j^d - d_j - c(S_j^-) \leq d_{1n} + c(S^-) = \bar{m}^d.$$

The coefficient for flow pack inequality is $\mu = (\bar{m}^d - \bar{m}^u)^+$ and for path pack inequality $\mu_j = (m_j^d - m_j^u)^+$. Since (S^+, S^-) is a path pack, the minimum cut passes above all nodes in N and $m_j^u = c(S^+)$ for all $j \in N$. As a result, $m_j^u = \bar{m}^u$ for all $j \in N$ and $m_j^d \leq \bar{m}^d$. Then, observe that the values

$$\mu_j \leq \mu, \quad j \in N.$$

□

Example 1 (continued). Recall the lot-sizing instance with backlogging given in Figure 4. Let the node set $N = [1, 4]$ with $E^- = \emptyset$ and $S^+ = \{3\}$. Then, $m_1^u = 30$, $m_1^d = 40$, $m_2^u = 30$, $m_2^d = 40$, $m_3^u = 30$, $m_3^d = 30$, $m_4^u = 30$, $m_4^d = 30$, leading to $\mu_1 = 10$, $\mu_2 = 10$, $\mu_3 = 0$ and $\mu_4 = 0$. Also notice that the maximum flow values are $v(C) = 30$, $v(C \cup \{1\}) = 40$, $v(C \cup \{2\}) = 40$, $v(C \cup \{4\}) = 30$, $v(E) = 40$, and $v(E \setminus \{3\}) = 40$. Then the resulting path pack inequality (17) is

$$y_1 + y_2 + y_3 + y_4 \leq 30 + 10x_1 + 10x_2 \tag{19}$$

and it is facet-defining for $\text{conv}(\mathcal{P})$ as it will be shown in Section 4. Now, suppose that the nodes in $[1, 4]$ are merged into a single node with incoming arcs $\{1, 2, 3, 4\}$ and demand $d = 40$. For the same set S^+, we get $\mu = 40 - 30 = 10$. Then, the corresponding flow pack inequality (18) is

$$y_1 + y_2 + y_3 + y_4 \leq 30 + 10x_1 + 10x_2 + 10x_4,$$

which is weaker than the path pack inequality (19). □

Proposition 6. *If $|E^+ \setminus S^+| \leq 1$ and $S^- = \emptyset$, then inequalities* (14) *and* (17) *are equivalent.*

Proof. If $E^+ \setminus S^+ = \emptyset$ and $S^- = \emptyset$, then it is easy to see that the coefficients of inequality (17) are the same as (14). Moreover, if $|E^+ \setminus S^+| = 1$ (and wlog $E^+ \setminus S^+ = \{j\}$), then the resulting inequality (17) is

$$\begin{aligned} y(E^+) - y(E^-) &\leq v(C) + \rho_j(C)x_j \\ &= v(C) + \big(v(C \cup \{j\}) - v(C)\big)x_j \\ &= v(C \cup \{j\}) - \rho_j(C)(1 - x_j), \end{aligned}$$

which is equivalent to path cover inequality (14) with the objective set $(E^+, \emptyset)$. □

4. The strength of the path cover and pack inequalities

The capacities of the forward and the backward path arcs play an important role in finding the coefficients of the path cover and pack inequalities (14) and (17). Recall that K^+ and K^- are the coefficient sets in (5), (S^+, L^-) is the objective set for (F2) and $S^- = E^- \setminus (K^- \cup L^-)$.

Definition 3. A node $j \in N$ is called *backward independent* for set (S^+, S^-) if

$$\alpha_j^u = \alpha_{j-1}^d + u_{j-1} + c(S_j^+),$$

or

$$\alpha_j^d = \alpha_{j-1}^u + b_{j-1} + d_j + c(S_j^-).$$

Definition 4. A node $j \in N$ is called *forward independent* for set (S^+, S^-) if

$$\beta_j^u = \beta_{j+1}^d + b_j + c(S_j^+),$$

or

$$\beta_j^d = \beta_{j+1}^u + u_j + d_j + c(S_j^-).$$

In Lemmas 7 and 8 below, we further explain how forward and backward independency affect the coefficients of path cover and pack inequalities. First, let $S_{jk}^+ = \cup_{i=j}^k S_i^+$, $S_{jk}^- = \cup_{i=j}^k S_i^-$ and $L_{jk}^- = \cup_{i=j}^k L_i^-$ if $j \leq k$, and $\emptyset$ otherwise.

Lemma 7. *If a node $j \in N$ is backward independent for set (S^+, S^-), then the values λ_j and μ_j do not depend on the sets S_{1j-1}^+, S_{1j-1}^- and the value d_{1j-1}.*

Proof. If a node j is backward independent, then either $\alpha_j^u = \alpha_{j-1}^d + u_{j-1} + c(S_j^+)$ or $\alpha_j^d = \alpha_{j-1}^u + b_{j-1} + d_j + c(S_j^-)$. If $\alpha_j^u = \alpha_{j-1}^d + u_{j-1} + c(S_j^+)$, then the equality in (7) implies $\alpha_{j-1}^d + u_{j-1} \leq \alpha_{j-1}^u$. As a result, the equality in (8) gives $\alpha_j^d = \alpha_{j-1}^d + d_j + c(S_j^-)$. Following the definitions in (11)–(12), the difference $w_j := m_j^u - m_j^d$ is $\beta_j^u - \beta_j^d + u_{j-1}$ which only depends on sets S_k^+ and S_k^- for $k \in [j, n]$, the value d_{jn} and the capacity of the forward path arc $(j-1, j)$.

If $\alpha_j^d = \alpha_{j-1}^u + b_{j-1} + d_j + c(S_j^-)$, then the equality in (8) implies $\alpha_{j-1}^u + b_{j-1} \leq \alpha_{j-1}^d$. As a result, the equality in (7) gives $\alpha_j^u = \alpha_{j-1}^u + c(S_j^+)$. Then, the difference $w_j = \beta_j^u - \beta_j^d - b_{j-1}$ which only depends on sets S_k^+ and S_k^- for $k \in [j, n]$, the value d_{jn} and the capacity of the backward path arc $(j, j-1)$.

Since the values λ_j and μ_j are defined as $(w_j)^+$ and $(-w_j)^+$ respectively, the result follows. □

Remark 5. Let $w_j := m_j^u - m_j^d$. If a node $j \in N$ is backward independent for a set (S^+, S^-), then we observe the following: (1) If $\alpha_j^u = \alpha_{j-1}^d + u_{j-1} + c(S_j^+)$, then

$$w_j = \beta_j^u - \beta_j^d + u_{j-1},$$

and (2) if $\alpha_j^d = \alpha_{j-1}^u + b_{j-1} + d_j + c(S_j^-)$, then

$$w_j = \beta_j^u - \beta_j^d - b_{j-1}.$$

Lemma 8. *If a node $j \in N$ is forward independent for set (S^+, S^-), then the values λ_j and μ_j do not depend on the sets S^+_{j+1n}, S^-_{j+1n} and the value d_{j+1n}.*

Proof. Notice that the backward independency implies either $\beta^u_j = \beta^d_{j+1} + b_j + c(S^+_j)$ and $\beta^d_j = \beta^d_{j+1} + d_j + c(S^-_j)$ or $\beta^u_j = \beta^u_{j+1} + c(S^+_j)$ and $\beta^d_j = \beta^u_{j+1} + u_j + d_j + c(S^-_j)$. Then, the difference $w_j = m^u_j - m^d_j$ is either $\alpha^u_j + \alpha^d_j + b_j$ or $\alpha^u_j + \alpha^d_j - u_j$ and in both cases, it is independent of the sets S^+_k, S^-_k for $k \in [1, j-1]$ and the value d_{1j-1}. □

Remark 6. Let $w_j := m^u_j - m^d_j$. If a node $j \in N$ is forward independent for a set (S^+, S^-), then we observe the following: (1) If $\beta^u_j = \beta^d_{j+1} + b_j + c(S^+_j)$, then

$$w_j = \alpha^u_j - \alpha^d_j + b_j,$$

and (2) if $\beta^d_j = \beta^u_{j+1} + u_j + d_j + c(S^-_j)$, then

$$w_j = \alpha^u_j - \alpha^d_j - u_j.$$

Corollary 9. *If a node $j \in N$ is backward independent for set (S^+, S^-), then the values λ_k and μ_k for $k \in [j, n]$ are also independent of the sets S^+_{1j-1}, S^-_{1j-1} and the value d_{1j-1}. Similarly, if a node $j \in N$ is forward independent for set (S^+, S^-), then the values λ_k and μ_k for $k \in [1, j]$ are also independent of the sets S^+_{j+1n}, S^-_{j+1n} and the value d_{j+1n}.*

Proof. The proof follows from recursions in (7)–(10). If a node j is backward independent, we write α^u_{j+1} and α^d_{j+1} in terms of α^u_{j-1} and α^d_{j-1} and observe that the difference $w_{j+1} = m^u_{j+1} - m^d_{j+1}$ does not depend on α^u_{j-1} nor α^d_{j-1} which implies independency of sets S^+_{1j-1}, S^-_{1j-1} and the value d_{1n}. We can repeat the same argument for w_j, $j \in [j+2, n]$ to show independency.

We show the same result for forward independency by writing β^u_{j-1} and β^d_{j-1} in terms of β^u_{j+1} and β^d_{j+1}, we observe that w_j does not depend on β^u_{j+1} nor β^d_{j+1}. Then, it is clear that w_{j-1} is also independent of the sets S^+_{j+1n}, S^-_{j+1n} and the value d_{j+1n}. We can repeat the same argument for w_j, $j \in [1, j-1]$ to show independency. □

Proving the necessary facet conditions frequently requires a partition of the node set N into two disjoint sets. Suppose, N is partitioned into $N_1 = [1, j-1]$ and $N_2 = [j, n]$ for some $j \in N$. Let E_{N1} and E_{N2} be the set of non-path arcs associated with node sets N_1 and N_2. We consider the forward and backward path arcs $(j-1, j)$ and $(j, j-1)$ to be in the set of non-path arcs E_{N1} and E_{N2} since the node $j-1 \in N_i$ and $j \notin N_i$ for $i = 1, 2$. In particular, $E^+_{N1} := (j, j-1) \cup E^+_{1j-1}$, $E^-_{N1} := (j-1, j) \cup E^-_{1j-1}$ and $E^+_{N2} := (j-1, j) \cup E^+_{jn}$, $E^-_{N2} := (j, j-1) \cup E^-_{jn}$, where $E^+_{k\ell}$ and $E^-_{k\ell}$ are defined as $\cup^{\ell}_{i=k} E^+_i$ and $\cup^{\ell}_{i=k} E^-_i$ if $k \leq \ell$ respectively, and empty set otherwise. Since the path arcs for N do not have associated fixed-charge variables, one can assume that there exists auxiliary binary variables $\tilde{x}_k = 1$ for $k \in \{(j-1, j), (j, j-1)\}$. Moreover, we partition the sets S^+, S^- and L^- into $S^+_{N1} \supseteq S^+_{1j-1}$, $S^-_{N1} \supseteq S^-_{1j-1}$, $L^-_{N1} := L^-_{1j-1}$ and $S^+_{N2} \supseteq S^+_{jn}$, $S^-_{N2} \supseteq S^-_{jn}$, $L^-_{N2} := L^-_{jn}$. Then, let v_1 and v_2 be the value functions defined in (F2) for the node sets N_1 and N_2 and the objective sets (S^+_{N1}, L^-_{N1}) and (S^+_{N2}, L^-_{N2}). Moreover, let α^u_j, α^d_j, β^u_j and β^d_j be defined for $j \in N$ in recursions (7)-(10) for the set (S^+, S^-) and recall that $S^- = E^- \setminus (K^- \cup L^-)$.

Lemma 10. *Let (S^+, L^-) be the objective set for the node set for $N = [1,n]$. If $\alpha^u_j = \alpha^d_{j-1} + u_{j-1} + c(S^+_j)$ or $\beta^u_{j-1} = \beta^d_j + b_{j-1} + c(S^+_{j-1})$, then*

$$v(S^+, L^-) = v_1(S^+_{N1}, L^-_{N1}) + v_2(S^+_{N2}, L^-_{N2}),$$

where $N_1 = [1, j-1]$, $N_2 = [j,n]$ and the arc sets are $S^+_{N1} = (j, j-1) \cup S^+_{1j-1}$, $S^+_{N2} = (j-1, j) \cup S^+_{jn}$, $S^-_{N1} = S^-_{1j-1}$, $S^-_{N2} = S^-_{jn}$.

Proof. Recall that $C = S^+ \cup L^-$ and let $C_1 = S^+_{N1} \cup L^-_{N1}$ and $C_2 = S^+_{N2} \cup L^-_{N2}$. In (13), we showed that the value of the minimum cut is

$$v(C) = m_i = \min\{\alpha^u_i + \beta^u_i - c(S^+_i), \alpha^d_i + \beta^d_i - d_i - c(S^-_i)\}$$

for all $i \in N$. For node set N_1 and the arc set C_1, notice that the value of the minimum cut is

$$v_1(C_1) = \min\{\alpha^u_{j-1} + b_{j-1}, \alpha^d_{j-1}\}.$$

This is because of three observations: (1) the values $\alpha^{\{u,d\}}_i$ for $i \in [1, j-2]$ are the same for the node sets N_1 and N, (2) for the arc set C_1 the set S^+_{j-1} now includes the backward path arc $(j, j-1)$ and (3) node $j-1$ is the last node of the first path. Similarly, for node set N_2 and the arc set C_2, the value of the minimum cut is

$$v_2(C_2) = \min\{\beta^u_j + u_{j-1}, \beta^d_j\}.$$

For nodes N_2 and the arc set C_2, notice that (1) the values $\beta^{\{u,d\}}_i$ for $i \in [j+1, n]$ are the same for the node sets N_2 and N, (2) for the arc set C_2 the set S^+_j now includes the forward path arc $(j-1, j)$ and (3) node j is the first node of the second path.

Now, if $\alpha^u_j = \alpha^d_{j-1} + u_{j-1} + c(S^+_j)$, then $\alpha^d_j = \alpha^d_{j-1} + d_j + c(S^-_j)$ from equations in (7)–(8). Then, rewriting $v(C) = m_j$ and $v_1(C_1)$ in terms of α^d_{j-1}:

$$v(C) = \alpha^d_{j-1} + \min\{\beta^u_j + u_{j-1}, \beta^d_j\}$$

and

$$v_1(C_1) = \alpha^d_{j-1}.$$

As a result, the values $v_1(C_1)$ and $v_2(C_2)$ summed gives the value $v(C)$ under the assumption for the value of α^u_j.

Similarly, if $\beta^u_{j-1} = \beta^d_j + b_{j-1} + c(S^+_{j-1})$, then $\beta^d_{j-1} = \beta^d_j + d_{j-1} + c(S^-_{j-1})$ from equations in (9)–(10). Then, rewriting $v(C) = m_{j-1}$ and $v_2(C_2)$ in terms of β^d_j:

$$v(C) = \beta^d_j + \min\{\alpha^u_{j-1} + b_{j-1}, \alpha^d_{j-1}\}$$

and

$$v_2(C_2) = \beta^d_j.$$

As a result, the values $v_1(C_1)$ and $v_2(C_2)$ summed gives the value $v(C)$ under the assumption for the value of β^u_{j-1}. □

Lemma 11. *Let (S^+, L^-) be the objective set for the node set for $N = [1,n]$. If $\alpha^d_j = \alpha^u_{j-1} + b_{j-1} + d_{j-1} + c(S^-_j)$ or $\beta^d_{j-1} = \beta^u_j + u_{j-1} + d_{j-1} + c(S^-_{j-1})$, then*

$$v(S^+, L^-) = v_1(S^+_{N1}, L^-_{N1}) + v_2(S^+_{N2}, L^-_{N2}),$$

where $N_1 = [1, j-1]$*,* $N_2 = [j, n]$ *and the arc sets are* $S^+_{N1} = S^+_{1j-1}$*,* $S^+_{N2} = S^+_{jn}$*,* $S^-_{N1} = (j-1, j) \cup S^-_{1j-1}$*,* $S^-_{N2} = (j, j-1) \cup S^-_{jn}$*.*

Proof. The proof follows very close to that of Lemma 10. Let $C = S^+ \cup L^-$, $C_1 = S^+_{N1} \cup L^-_{N1}$ and $C_2 = S^+_{N2} \cup L^-_{N2}$. For node set N_1 and the arc set C_1, notice that the value of the minimum cut is

$$v_1(C_1) = \min\{\alpha^u_{j-1}, \alpha^d_{j-1} + u_{j-1}\},$$

where u_{j-1} is added because $c(S^-_{N1}) = c(S^-_{1j-1}) + u_{j-1}$. Similarly, for node set N_2 and the arc set C_2, the value of the minimum cut is

$$v_2(C_2) = \min\{\beta^u_j, \beta^d_j + b_{j-1}\},$$

where b_{j-1} is added because $c(S^-_{N2}) = c(S^-_{jn}) + b_{j-1}$.

Now, if $\alpha^d_j = \alpha^u_{j-1} + b_{j-1} + d_{j-1} + c(S^-_j)$, then $\alpha^u_j = \alpha^u_{j-1} + c(S^+_j)$ from equations in (7)–(8). Then, rewriting $v(C) = m_j$ and $v_1(C_1)$ in terms of α^u_{j-1}:

$$v(C) = \alpha^u_{j-1} + \min\{\beta^u_j, \beta^d_j + b_{j-1}\}$$

and

$$v_1(C_1) = \alpha^u_{j-1}.$$

As a result, the values $v_1(C_1)$ and $v_2(C_2)$ summed gives the value $v(C)$ under the assumption for the value of α^d_j.

Similarly, if $\beta^d_{j-1} = \beta^u_j + u_{j-1} + d_{j-1} + c(S^-_{j-1})$, then $\beta^u_{j-1} = \beta^u_j + c(S^+_{j-1})$ from equations in (9)–(10). Then, rewriting $v(C) = m_{j-1}$ and $v_2(C_2)$ in terms of β^u_j:

$$v(C) = \beta^u_j + \min\{\alpha^u_{j-1}, \alpha^d_{j-1} + u_{j-1}\}$$

and

$$v_2(C_2) = \beta^u_j.$$

As a result, the values $v_1(C_1)$ and $v_2(C_2)$ summed gives the value $v(C)$ under the assumption for the value of β^d_{j-1}. ☐

Lemma 12. *Let* (S^+, L^-) *be the objective set for the node set for* $N = [1, n]$*. If* $\alpha^u_j = \alpha^d_{j-1} + u_{j-1} + c(S^+_j)$ *and* $\beta^d_{j-1} = \beta^u_j + u_{j-1} + d_{j-1} + c(S^-_{j-1})$*, then*

$$v(S^+, L^-) = v_1(S^+_{N1}, L^-_{N1}) + v_2(S^+_{N2}, L^-_{N2}),$$

where $N_1 = [1, j-1]$*,* $N_2 = [j, n]$ *and the arc sets are* $S^+_{N1} = S^+_{1j-1}$*,* $S^+_{N2} = (j-1, j) \cup S^+_{jn}$*,* $S^-_{N1} = S^-_{1j-1}$*,* $S^-_{N2} = (j, j-1) \cup S^-_{jn}$*.*

Proof. The proof follows very close to that of Lemmas 10 and 11. Let $C = S^+ \cup L^-$, $C_1 = S^+_{N1} \cup L^-_{N1}$ and $C_2 = S^+_{N2} \cup L^-_{N2}$. For node set N_1 and the arc set C_1, notice that the value of the minimum cut is

$$v_1(C_1) = \min\{\alpha^u_{j-1}, \alpha^d_{j-1}\}$$

and for node set N_2 and the arc set C_2, the value of the minimum cut is

$$v_2(C_2) = \min\{\beta^u_j + u_{j-1}, \beta^d_j + b_{j-1}\}.$$

Now, if $\alpha_j^u = \alpha_{j-1}^d + u_{j-1} + c(S_j^+)$ and $\beta_{j-1}^d = \beta_j^u + u_{j-1} + d_{j-1} + c(S_{j-1}^-)$, then $\alpha_j^d = \alpha_{j-1}^d + d_j + c(S_j^-)$ and $\beta_{j-1}^u = \beta_j^u + c(S_j^+)$. Then, rewriting $v(C) = m_j$, $v_1(C_1)$ and $v_2(C_2)$:

$$v(C) = \alpha_{j-1}^d + \min\{u_{j-1} + \beta_j^u, \beta_j^d\} = \alpha_{j-1}^d + u_{j-1} + \beta_j^u,$$

$$v_1(C_1) = \alpha_{j-1}^d \text{ and } v_2(C_2) = \beta_j^u + u_{j-1}.$$

As a result, the values $v_1(C_1)$ and $v_2(C_2)$ summed gives the value $v(C)$ under the assumption for the values of α_j^u and β_{j-1}^d. □

Lemma 13. *Let (S^+, L^-) be the objective set for the node set for $N = [1, n]$. If $\alpha_j^d = \alpha_{j-1}^u + b_{j-1} + d_j + c(S_j^-)$ and $\beta_{j-1}^u = \beta_j^d + b_{j-1} + c(S_{j-1}^+)$, then*

$$v(S^+, L^-) = v_1(S_{N1}^+, L_{N1}^-) + v_2(S_{N2}^+, L_{N2}^-),$$

where $N_1 = [1, j-1]$, $N_2 = [j, n]$ and the arc sets are $S_{N1}^+ = (j, j-1) \cup S_{1j-1}^+$, $S_{N2}^+ = S_{jn}^+$, $S_{N1}^- = (j-1, j) \cup S_{1j-1}^-$, $S_{N2}^- = S_{jn}^-$.

Proof. The proof follows very close to that of Lemmas 10 and 11. Let $C = S^+ \cup L^-$, $C_1 = S_{N1}^+ \cup L_{N1}^-$ and $C_2 = S_{N2}^+ \cup L_{N2}^-$. For node set N_1 and the arc set C_1, notice that the value of the minimum cut is

$$v_1(C_1) = \min\{\alpha_{j-1}^u + b_{j-1}, \alpha_{j-1}^d + u_{j-1}\}$$

and for node set N_2 and the arc set C_2, the value of the minimum cut is

$$v_2(C_2) = \min\{\beta_j^u, \beta_j^d\}.$$

Now, if $\alpha_j^d = \alpha_{j-1}^u + b_{j-1} + d_j + c(S_j^-)$ and $\beta_{j-1}^u = \beta_j^d + b_{j-1} + c(S_{j-1}^+)$, then $\alpha_j^u = \alpha_{j-1}^u + c(S_j^+)$ and $\beta_{j-1}^d = \beta_j^d + d_j + c(S_j^-)$. Then, rewriting $v(C) = m_j$, $v_1(C_1)$ and $v_2(C_2)$:

$$v(C) = \alpha_{j-1}^u + \min\{\beta_j^u, \beta_j^d + b_{j-1}\} = \alpha_{j-1}^u + \beta_j^d + b_{j-1},$$

$$v_1(C_1) = \alpha_{j-1}^u + b_{j-1} \text{ and } v_2(C_2) = \beta_j^d.$$

As a result, the values $v_1(C_1)$ and $v_2(C_2)$ summed gives the value $v(C)$ under the assumption for the values of α_j^d and β_{j-1}^u. □

In the remainder of this section, we give necessary and sufficient conditions for path cover and pack inequalities (14) and (17) to be facet-defining for the convex hull of $\mathcal{P}$.

Theorem 14. *Let $N = [1, n]$, and $d_j \geq 0$ for all $j \in N$. If $L^- = \emptyset$ and the set (S^+, S^-) is a path cover for N, then the following conditions are necessary for path cover inequality (14) to be facet-defining for conv($\mathcal{P}$):*

(i) $\rho_t(C \setminus \{t\}) < c_t$*, for all* $t \in C$*,*
(ii) $\max_{t \in S^+} \rho_t(C \setminus \{t\}) > 0$*,*
(iii) if a node $j \in [2, n]$ is forward independent for set (S^+, S^-), then node $j-1$ is not backward independent for set (S^+, S^-),
(iv) if a node $j \in [1, n-1]$ is backward independent for set (S^+, S^-), then node $j+1$ is not forward independent for set (S^+, S^-),
(v) if $\max_{t \in S_i^+}(c_t - \lambda_i)^+ = 0$ for $i = p, \ldots, n$ for some $p \in [2, n]$, then the node $p-1$ is not forward independent for (S^+, S^-),

(vi) if $\max_{t\in S_i^+}(c_t-\lambda_i)^+=0$ *for* $i=1,\dots,q$ *for some* $q\in[1,n-1]$*, then the node* $q+1$ *is not backward independent for* (S^+,S^-)*.*

Proof. (i) If for some $t'\in S^+$, $\rho_{t'}(C\setminus\{t'\})\geq c_{t'}$, then the path cover inequality with the objective set $S^+\setminus\{t'\}$ summed with $y_{t'}\leq c_{t'}x_{t'}$ results in an inequality at least as strong. Rewriting the path cover inequality using the objective set S^+, we obtain

$$\sum_{t\in S^+\setminus\{t'\}}(y_t+\rho_t(S^+\setminus\{t\})(1-x_t))+y_{t'}\leq v(S^+)-\rho_{t'}(S^+\setminus\{t'\})(1-x_{t'})+y(E^-\setminus S^-)$$
$$=v(S^+\setminus\{t'\})+\rho_{t'}(S^+\setminus\{t'\})x_{t'}+y(E^-\setminus S^-).$$

Now, consider summing the path cover inequality for the objective set $S\setminus\{t'\}$

$$\sum_{t\in S\setminus\{t'\}}(y_t+\rho_t(S\setminus\{t,t'\})(1-x_t))\leq v(S\setminus\{t'\})+y(E^-\setminus S^-),$$

and $y_{t'}\leq c_{t'}x_{t'}$. The resulting inequality dominates inequality (3) because $\rho_t(S^+\setminus\{t\})\leq\rho_t(S^+\setminus\{t,t'\})$, from the submodularity of the set function v.

(ii) If $L^-=\emptyset$ and $\max_{t\in S^+}\rho_t(C\setminus\{t\})=0$, then summing flow balance inequalities (1b) for all nodes $j\in N$ gives an inequality at least as strong.

(iii) Suppose a node j is forward independent for (S^+,S^-) and the node $j-1$ is backward independent for (S^+,S^-) for some $j\in[2,n]$. Lemmas 10–13 show that the nodes N and the arcs $C=S^+\cup L^-$ can be partitioned into $N_1=[1,j-1]$, $N_2=[j,n]$ and C_1, C_2 such that the sum of the minimum cut values for N_1, N_2 is equal to the minimum cut for N. From Remarks 5 and 6 and Corollary 9, it is easy to see that λ_i for $i\in N$ will not change by the partition procedures described in Lemmas 10–13. We examine the four cases for node $j-1$ to be forward independent and node j to be backward independent for the set (S^+,S^-).

(a) Suppose $\alpha_j^u=\alpha_{j-1}^d+u_{j-1}+c(S_j^+)$ and $\beta_{j-1}^u=\beta_j^d+b_{j-1}+c(S_{j-1}^+)$. Consider the partition procedure described in Lemma 10, where $S_{N1}^+=(j,j-1)\cup S_{1j-1}^+$, $S_{N2}^+=(j-1,j)\cup S_{jn}^+$, $S_{N1}^-=S_{1j-1}^-$, $S_{N2}^-=S_{jn}^-$. Then, the path cover inequalities for nodes N_1 and N_2

$$r_{j-1}+\sum_{i=1}^{j-1}\sum_{t\in S_i^+}\big(y_t+(c_t-\lambda_i)^+(1-x_t)\big)\leq v_1(S_{N1}^+)+\sum_{i=1}^{j-1}y(E_i^-\setminus S_i^-)+i_{j-1}$$

and

$$i_{j-1}+\sum_{i=j}^{n}\sum_{t\in S_i^+}\big(y_t+(c_t-\lambda_i)^+(1-x_t)\big)\leq v_2(S_{N2}^+)+\sum_{i=j}^{n}y(E_i^-\setminus S_i^-)+r_{j-1}$$

summed gives

$$\sum_{i=1}^{n}\sum_{t\in S_i^+}\big(y_t+(c_t-\lambda_i)^+(1-x_t)\big)\leq v(S^+)+y(E^-\setminus S^-),$$

which is the path cover inequality for N with the objective set S^+.

(b) Suppose $\alpha_j^d=\alpha_{j-1}^u+b_{j-1}+d_{j-1}+c(S_j^-)$ and $\beta_{j-1}^d=\beta_j^u+u_{j-1}+d_{j-1}+c(S_{j-1}^-)$. Consider the partition described in Lemma 11, where $S_{N1}^+=S_{1j-1}^+$, $S_{N2}^+=S_{jn}^+$,

$S^-_{N1} = (j-1,j) \cup S^-_{1j-1}$, $S^-_{N2} = (j,j-1) \cup S^-_{jn}$. The path cover inequalities for nodes N_1 and N_2

$$\sum_{i=1}^{j-1} \sum_{t \in S_i^+} \big(y_t + (c_t - \lambda_i)^+(1-x_t)\big) \le v_1(S^+_{N1}) + \sum_{i=1}^{j-1} y(E_i^- \setminus S_i^-)$$

and

$$\sum_{i=j}^{n} \sum_{t \in S_i^+} \big(y_t + (c_t - \lambda_i)^+(1-x_t)\big) \le v_2(S^+_{N2}) + \sum_{i=j}^{n} y(E_i^- \setminus S_i^-).$$

summed gives the path cover inequality for nodes N and arcs C.

(c) Suppose $\alpha^u_j = \alpha^d_{j-1} + u_{j-1} + c(S^+_j)$ and $\beta^d_{j-1} = \beta^u_j + u_{j-1} + d_{j-1} + c(S^-_{j-1})$. Consider the partition described in Lemma 12, where $S^+_{N1} = S^+_{1j-1}$, $S^+_{N2} = (j-1,j) \cup S^+_{jn}$, $S^-_{N1} = S^-_{1j-1}$, $S^-_{N2} = (j,j-1) \cup S^-_{jn}$. The path cover inequalities for nodes N_1 and N_2

$$\sum_{i=1}^{j-1} \sum_{t \in S_i^+} \big(y_t + (c_t - \lambda_i)^+(1-x_t)\big) \le v_1(S^+_{N1}) + \sum_{i=1}^{j-1} y(E_i^- \setminus S_i^-) + i_{j-1}$$

and

$$i_{j-1} + \sum_{i=j}^{n} \sum_{t \in S_i^+} \big(y_t + (c_t - \lambda_i)^+(1-x_t)\big) \le v_2(S^+_{N2}) + \sum_{i=j}^{n} y(E_i^- \setminus S_i^-).$$

summed gives the path cover inequality for nodes N and arcs C.

(d) Suppose $\alpha^d_j = \alpha^u_{j-1} + b_{j-1} + d_j + c(S^-_j)$ and $\beta^u_{j-1} = \beta^d_j + b_{j-1} + c(S^+_{j-1})$. Consider the partition described in Lemma 13, where $S^+_{N1} = (j,j-1) \cup S^+_{1j-1}$, $S^+_{N2} = S^+_{jn}$, $S^-_{N1} = (j-1,j) \cup S^-_{1j-1}$, $S^-_{N2} = S^-_{jn}$. The path cover inequalities for nodes N_1 and N_2

$$r_{j-1} + \sum_{i=1}^{j-1} \sum_{t \in S_i^+} \big(y_t + (c_t - \lambda_i)^+(1-x_t)\big) \le v_1(S^+_{N1}) + \sum_{i=1}^{j-1} y(E_i^- \setminus S_i^-)$$

and

$$\sum_{i=j}^{n} \sum_{t \in S_i^+} \big(y_t + (c_t - \lambda_i)^+(1-x_t)\big) \le v_2(S^+_{N2}) + \sum_{i=j}^{n} y(E_i^- \setminus S_i^-) + r_{j-1}.$$

summed gives the path cover inequality for nodes N and arcs C.

(iv) The same argument for condition (iii) above also proves the desired result here.

(v) Suppose $(c_t - \lambda_i)^+ = 0$ for all $t \in S_i^+$ and $i \in [p,n]$ and the node $p-1$ is forward independent for some $p \in [2,n]$. Then, we partition the node set $N = [1,n]$ into $N_1 = [1,p-1]$ and $N_2 = [p,n]$. We follow Lemma 10 if $\beta^u_{p-1} = \beta^d_p + b_{p-1} + c(S^+_{p-1})$ and follow Lemma 11 if $\beta^d_{p-1} = \beta^u_p + u_{p-1} + d_{p-1} + c(S^-_{p-1})$ to define S^+_{N1}, S^-_{N1}, S^+_{N2} and S^-_{N2}. Remark 6 along with the partition procedure described in Lemma 10 or 11 implies that λ_i will remain unchanged for $i \in N_1$. Notice that the path cover inequality for nodes N and arcs C is

$$y(S^+) + \sum_{i=1}^{p-1} \sum_{t \in S_i^+} (c_t - \lambda_i)^+ (1 - x_t) \le v(S^+) + y(E^- \setminus S^-).$$

If $\beta_{p-1}^u = \beta_j^d + b_{p-1} + c(S_{p-1}^+)$, then the path cover inequality for nodes N_1 and arcs S_{N1}^+, S_{N1}^- described in Lemma 10 is

$$r_{p-1} + \sum_{i=1}^{p-1} \sum_{t \in S_i^+} \big(y_t + (c_t - \lambda_i)^+ (1 - x_t)\big) \le v(S_{N1}^+) + \sum_{i=1}^{p-1} y(E_i^- \setminus S_i^-) + i_{p-1}.$$

Moreover, let $\bar{m}_p^u$ and $\bar{m}_p^d$ be the values of minimum cut that goes above and below node p for the node set N_2 and arcs S_{N2}^+, S_{N2}^- and observe that

$$\bar{m}_p^u = \beta_p^u + u_{p-1} \text{ and } \bar{m}_p^d = \beta_p^d.$$

Then, comparing the difference $\bar{\lambda}_p := (\bar{m}_p^u - \bar{m}_p^d)^+ = (\beta_p^u - \beta_p^d + u_{p-1})^+$ to $\lambda_p = (m_p^u - m_p^d)^+ = (\beta_p^u - \beta_p^d + \alpha_p^u - \alpha_p^d + c(S_p^+) - d_p - c(S_p^-))^+$, we observe that $\bar{\lambda}_p \ge \lambda_p$ since $\alpha_p^u - \alpha_p^d + c(S_p^+) - d_p - c(S_p^-) \le u_{p-1}$ from (7)–(8). Since $(c_t - \lambda_p)^+ = 0$, then $(c_t - \bar{\lambda}_p)^+ = 0$ as well. Using the same technique, it is easy to observe that $\bar{\lambda}_i \ge \lambda_i$ for $i \in [p+1, n]$ as well. As a result, the path cover inequality for N_2 with sets S_{N2}^+, S_{N2}^- is

$$i_{p-1} + \sum_{i=p}^{n} y(S_i^+) \le v(S_{N2}^+) + \sum_{i=p}^{n} y(E_i^- \setminus S_i^-) + r_{p-1}.$$

Notice that the path cover inequalities for N_1, S_{N1}^+, S_{N1}^- and for N_2, S_{N2}^+, S_{N2}^- summed gives the path cover inequality for N, S^+, S^-.

Similarly, if $\beta_{p-1}^d = \beta_j^u + u_{p-1} + d_{p-1} + c(S_{p-1}^-)$, the proof follows very similar to the previous argument using Lemma 11. Letting $\bar{m}_p^u$ and $\bar{m}_p^d$ be the values of minimum cut that goes above and below node p for the node set N_2 and arcs S_{N2}^+, S_{N2}^-, we get

$$\bar{m}_p^u = \beta_p^u \text{ and } \bar{m}_p^d + b_{p-1} = \beta_p^d$$

under this case. Now, notice that that $\alpha_p^u - \alpha_p^d + c(S_p^+) - d_p - c(S_p^-) \ge -b_{p-1}$ from (7)–(8), which leads to $\bar{\lambda}_p \ge \lambda_p$. Then the proof follows same as above.

(vi) The proof is similar to that of the necessary condition (v). We use Lemmas 12 and 13 and Remark 6 to partition the node set N and arcs S^+, S^- into node sets $N_1 = [1, q]$ and $N_2 = [q+1, n]$ for $q \in [2, n]$ and arcs S_{N1}^+, S_{N1}^- and S_{N2}^+, S_{N2}^-. Next, we check the values of minimum cut that goes above and below node q for the node set N_1 and arcs S_{N1}^+, S_{N1}^-. Then, observing $-u_q \le \beta_q^u - \alpha_q^d + c(S_q^+) - d_q - c(S_q^-) \le b_q$ from (9)–(10), it is easy to show that the coefficients x_t for $t \in S_{N1}^+$ are equal to zero in the path cover inequality for node set N_1. As a result, the path cover inequalities for N_1, S_{N1}^+, S_{N1}^- and for N_2, S_{N2}^+, S_{N2}^- summed gives the path cover inequality for N, S^+, S^-.

□

Theorem 15. *Let $N = [1, n]$, $E^- = \emptyset$, $d_j \ge 0$ and $|E_j^+| = 1$, for all $j \in N$ and let the set S^+ be a path cover. The necessary conditions in Theorem 14 along with*

(i) $(c_t - \lambda_j)^+ > 0$ *for all* $t \in S_j^+$, $j \in N$,

(ii) $(c_t - \lambda_j)^+ < c(E^+ \setminus S^+)$ *for all* $t \in S_j^+$, $j \in N$

are sufficient for path cover inequality (14) *to be facet-defining for conv*($\mathcal{P}$)*.*

Proof. Recall that $\dim\big(\text{conv}(\mathcal{P})\big) = 2|E| + n - 2$. In this proof, we provide $2|E| + n - 2$ affinely independent points that lie on the face F

$$F = \left\{ (\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{r}) \in \mathcal{P} \ : \ y(S^+) + \sum_{t \in S^+} (c_t - \lambda_j)^+ (1 - x_t) = d_{1n} \right\}.$$

First, we provide Algorithm 1 which outputs an initial feasible solution $(\bar{\mathbf{y}}, \bar{\mathbf{x}}, \bar{\mathbf{i}}, \bar{\mathbf{r}})$, where all the arcs in S^+ have non-zero flow. Let $\bar{d}_j$ be the effective demand on node j, that is, the sum of d_j and the minimal amount of flow that needs to be sent from the arcs in S_j^+ to ensure $v(S^+) = d_{1n}$. In Algorithm 1, we perform a backward pass and a forward pass on the nodes in N. This procedure is carried out to obtain the minimal amounts of flow on the forward and backward path arcs to satisfy the demands. For each node $j \in N$, these minimal outgoing flow values added to the demand d_j give the effective demand $\bar{d}_j$.

Algorithm 1

Initialization: Let $\bar{d}_j = d_j$ for $j \in N$
for $j = (n-1)$ **to** 1 **do**
  Let $\Delta = \min\left\{ u_j, \big(\bar{d}_{j+1} - c(S_{j+1}^+)\big)^+ \right\}$,
  $\bar{d}_j = \bar{d}_j + \Delta$, $\bar{d}_{j+1} = \bar{d}_{j+1} - \Delta$,
  $\bar{i}_j = \Delta$.
end for
for $j = 2$ **to** n **do**
  Let $\Delta = \big(\bar{d}_{j-1} - c(S_{j-1}^+)\big)^+$,
  $\bar{d}_j = \bar{d}_j + \Delta$, $\bar{d}_{j-1} - \Delta$
  $\bar{r}_{j-1} = \Delta - \min\{\Delta, \bar{i}_{j-1}\}$
  $\bar{i}_{j-1} = \bar{i}_{j-1} - \min\{\Delta, \bar{i}_{j-1}\}$
end for
$\bar{y}_j = \bar{d}_j$, for all $j \in S^+$.
$\bar{x}_j = 1$ if $j \in S^+$, 0 otherwise.
$\bar{y}_j = \bar{x}_j = 0$, for all $j \in E^-$.

Algorithm 1 ensures that at most one of the path arcs $(j-1, j)$ and $(j, j-1)$ have non-zero flow for all $j \in [2, n]$. Also, note that sufficient condition (i) ensures that all the arcs in S^+ have nonzero flow. Moreover, for at least one node $i \in N$, it is guaranteed that $c(S_i^+) > \bar{d}_i$. Otherwise, $\rho_t(C) = c_t$ for all $t \in S^+$ which contradicts with the necessary condition (i). Necessary conditions (iii) and (iv) ensure that $\bar{i}_j < u_j$ and $\bar{r}_j < b_j$ for all $j = 1, \ldots, n-1$. Let

$$e := \arg\max_{i \in N} \{c(S_i^+) - \bar{d}_i\}$$

be the node with the largest excess capacity. Also let $\mathbf{1}_j$ be the unit vector with 1 at position j.

Next, we give $2|S^+|$ affinely independent points represented by $\bar{w}^t = (\bar{\mathbf{y}}^t, \bar{\mathbf{x}}^t, \bar{\mathbf{i}}^t, \bar{\mathbf{r}}^t)$ and $\tilde{w}^t = (\tilde{\mathbf{y}}^t, \tilde{\mathbf{x}}^t, \tilde{\mathbf{i}}^t, \tilde{\mathbf{r}}^t)$ for $t \in S^+$:

(i) Select $\bar{w}^e = (\bar{\mathbf{y}}, \bar{\mathbf{x}}, \bar{\mathbf{i}}, \bar{\mathbf{r}})$ given by Algorithm 1. Let $\varepsilon > 0$ be a sufficiently small value. We define $\bar{w}^t$ for $e \neq t \in S^+$ as $\bar{\mathbf{y}}^t = \bar{\mathbf{y}}^e + \varepsilon \mathbf{1}_e - \varepsilon \mathbf{1}_t$, $\bar{\mathbf{x}}^t = \bar{\mathbf{x}}^e$. If $t < e$, then $\bar{\mathbf{i}}^t = \bar{\mathbf{i}}^e$ and $\bar{r}^t_j = \bar{r}^e_j$ for $j < t$ and for $t \geq e$, $\bar{r}^t_j = \bar{r}^e_j + \varepsilon$ for $t \leq j < e$.

(ii) In this class of affinely independent solutions, we close the arcs in S^+ one at a time and open all the arcs in $E^+ \setminus S^+$: $\tilde{\mathbf{x}}^t = \bar{\mathbf{x}} - \mathbf{1}_t + \sum_{j \in E^+ \setminus S^+} \mathbf{1}_j$. Next, we send an additional $\bar{y}_t - (c_t - \lambda_j)^+$ amount of flow from the arcs in $S^+ \setminus \{t\}$. This is a feasible operation because $v(C \setminus \{t\}) = d_{1n} - (c_t - \lambda_j)^+$. Let $(\mathbf{y}^*, \mathbf{i}^*, \mathbf{r}^*)$ be the optimal solution of (F2) corresponding to $v(S^+ \setminus \{t\})$. Then let, $\tilde{y}^t_j = y^*_j$ for $j \in S^+ \setminus \{t\}$. Since $v(C \setminus \{t\}) < d_{1n}$, additional flow must be sent through nodes in $E^+ \setminus S^+$ to satisfy flow balance equations (1b). This is also a feasible operation, because of assumption (A.1). Then, the forward and backward path flows $\tilde{\mathbf{i}}^t$ and $\tilde{\mathbf{r}}^t$ are calculated using the flow balance equations.

In the next set of solutions, we give $2|E^+ \setminus S^+| - 1$ affinely independent points represented by $\hat{w}^t = (\hat{\mathbf{y}}^t, \hat{\mathbf{x}}^t, \hat{\mathbf{i}}^t, \hat{\mathbf{r}}^t)$ and $\check{w}^t = (\check{\mathbf{y}}^t, \check{\mathbf{x}}^t, \check{\mathbf{i}}^t, \check{\mathbf{r}}^t)$ for $t \in E^+ \setminus S^+$.

(iii) Starting with solution $\bar{w}^e$, we open arcs in $E^+ \setminus S^+$, one by one. $\hat{\mathbf{y}}^t = \bar{\mathbf{y}}^e$, $\hat{\mathbf{x}}^t = \bar{\mathbf{x}}^e + \mathbf{1}_t$, $\hat{\mathbf{i}}^t = \bar{\mathbf{i}}^e$, $\hat{\mathbf{r}}^t = \bar{\mathbf{r}}^e$.

(iv) If $|E^+ \setminus S^+| \geq 2$, then we can send a sufficiently small $\varepsilon > 0$ amount of flow from arc $t \in E^+ \setminus S^+$ to $t \neq k \in E^+ \setminus S^+$. Let this set of affinely independent points be represented by $\check{w}^t$ for $t \in E^+ \setminus S^+$. While generating $\check{w}^t$, we start with the solution $\tilde{w}^e$, where the non-path arc in S^+_e is closed. The feasibility of this operation is guaranteed by the sufficiency conditions (ii) and necessary conditions (iii) and (iv).
   (a) If $\tilde{y}^e_t = c_t$, then there exists at least one arc $t \neq m \in E^+ \setminus S^+$ such that $0 \leq \tilde{y}^e_m < c_m$ due to sufficiency assumption (ii), then for each $t \in E^+ \setminus S^+$ such that $\tilde{y}^e_t = c_t$, let $\check{\mathbf{y}}^t = \tilde{\mathbf{y}}^e - \varepsilon \mathbf{1}_t + \varepsilon \mathbf{1}_m$, $\check{\mathbf{x}}^t = \tilde{\mathbf{x}}^e$. If $t < m$, then $\check{\mathbf{i}}^t = \tilde{\mathbf{i}}^e$ and $\check{\mathbf{r}}^t = \tilde{\mathbf{r}}^e + \varepsilon \sum_{i=t}^{m-1} \mathbf{1}_i$. If $t > m$, then $\check{\mathbf{i}}^t = \tilde{\mathbf{i}}^e + \varepsilon \sum_{i=m}^{t-1} \mathbf{1}_i$ and $\check{\mathbf{r}}^t = \tilde{\mathbf{r}}^e$.
   (b) If $\tilde{y}^e_t < c_t$ and there exists at least one arc $t \neq m \in E^+ \setminus S^+$ such that $\tilde{y}^e_m = 0$, then the same point described in (a) is feasible.
   (c) If $\tilde{y}^e_t < c_t$ and there exists at least one arc $t \neq m \in E^+ \setminus S^+$ such that $\tilde{y}^e_m = c_m$, then, we send ε amount of flow from t to m, $\check{\mathbf{y}}^t = \tilde{\mathbf{y}}^e + \varepsilon \mathbf{1}_t - \varepsilon \mathbf{1}_m$, $\check{\mathbf{x}}^t = \tilde{\mathbf{x}}^e$. If $t < m$, then $\check{\mathbf{i}}^t = \tilde{\mathbf{i}}^e + \varepsilon \sum_{i=t}^{m-1} \mathbf{1}_i$ and $\check{\mathbf{r}}^t = \tilde{\mathbf{r}}^e$. If $t > m$, then $\check{\mathbf{i}}^t = \tilde{\mathbf{i}}^e$ and $\check{\mathbf{r}}^t = \tilde{\mathbf{r}}^e + \varepsilon \sum_{i=m}^{t-1} \mathbf{1}_i$.

Finally, we give $n-1$ points that perturb the flow on the forward path arcs $(j, j+1)$ for $j = 1, \ldots, n-1$ represented by $\breve{w}^j = (\breve{\mathbf{y}}^j, \breve{\mathbf{x}}^j, \breve{\mathbf{i}}^j, \breve{\mathbf{r}}^j)$. Let $k = \min\{i \in N : S^+_i \neq \emptyset\}$ and $\ell = \max\{i \in N : S^+_i \neq \emptyset\}$. The solution given by Algorithm 1 guarantees $\bar{i}_j < u_j$ and $\bar{r}_j < b_j$ for $j = 1, \ldots, n-1$ due to necessary conditions (iii) and (iv).

(v) For $j = 1, \ldots, n-1$, we send an additional ε amount of flow from the forward path arc $(j, j+1)$ and the backward path arc $(j+1, j)$. Formally, the solution $\breve{w}^j$ can be obtained by: $\breve{\mathbf{y}}^j = \bar{\mathbf{y}}^e$, $\breve{\mathbf{x}}^j = \bar{\mathbf{x}}^e$, $\breve{\mathbf{i}}^j = \bar{\mathbf{i}}^e + \varepsilon \mathbf{1}_j$ and $\breve{\mathbf{r}}^j = \bar{\mathbf{r}}^e + \varepsilon \mathbf{1}_j$.

□

Next, we identify the conditions under which path pack inequality (17) is facet-defining for conv($\mathcal{P}$).

Theorem 16. *Let $N = [1, n]$, $d_j \geq 0$ for all $j \in N$, let the set (S^+, S^-) be a path pack and $L^- = \emptyset$. The following conditions are necessary for path pack inequality (17) to be facet-defining for conv($\mathcal{P}$):*

(i) $\rho_j(S^+) < c_j$, *for all* $j \in E^+ \setminus S^+$,

(ii) $\max_{t \in S^-} \rho_t(C) > 0$,

(iii) if a node $j \in [2, n]$ *is forward independent for set* (S^+, S^-), *then node* $j-1$ *is not backward independent for set* (S^+, S^-),

(iv) if a node $j \in [1, n-1]$ *is backward independent for set* (S^+, S^-), *then node* $j+1$ *is not forward independent for set* (S^+, S^-),

(v) if $\max_{t \in E_i^+ \setminus S_i^+} \rho_t(C) = 0$ *and* $\max_{t \in S_i^-} \rho_t(C) = 0$ *for* $i = p, \ldots, n$ *for some* $p \in [2, n]$, *then the node* $p-1$ *is not forward independent for* (S^+, S^-),

(vi) if $\max_{t \in E_i^+ \setminus S_i^+} \rho_t(C) = 0$ *and* $\max_{t \in S_i^-} \rho_t(C) = 0$ *for* $i = 1, \ldots, q$ *for some* $q \in [1, n-1]$, *then the node* $q+1$ *is not backward independent for* (S^+, S^-).

Proof. (i) Suppose that for some $k \in E^+ \setminus S^+$, $\rho_k(S^+) = c_k$. Then, recall the implicit form of path pack inequality (17) is

$$y(E^+ \setminus \{k\}) + y_k + \sum_{t \in S^-} \rho_t(S^+)(1 - x_t) \leq v(S^+) + \sum_{k \neq t \in E^+ \setminus S^+} \rho_t(S^+) x_t + c_k x_k + y(E^- \setminus S^-).$$

Now notice that, if we select $a_k = 0$ in (F2), then the coefficients of x_k and y_k become zero and summing the path cover inequality

$$y(E^+ \setminus \{k\}) + \sum_{t \in S^-} \rho_t(S^+)(1 - x_t) \leq v(S^+) + \sum_{k \neq t \in E^+ \setminus S^+} \rho_t(S^+) x_t + y(E^- \setminus S^-).$$

with $y_k \leq c_k x_k$ gives the first path cover inequality.

(ii) Suppose that $\rho_j(S^+) = 0$ for all $j \in S^-$. Then the path pack inequality is

$$y(E^+) \leq v(S^+) + \sum_{t \in E^+ \setminus S^+} \rho_t(S^+) x_t + y\left(E^- \setminus (L^- \cup S^-)\right),$$

where $L^- = \emptyset$. If an arc j is dropped from S^- and added to L^-, notice that $v(S^+) = v(S^+ \cup \{j\})$ since $\rho_j(S^+) = 0$ for $j \in S^-$. Then, the path pack inequality with $S^- = S^- \setminus \{j\}$ and $L^- = \{j\}$

$$y(E^+) + \sum_{t \in S^-} \rho_t(S^+ \cup \{j\})(1 - x_t) \leq v(S^+) + \sum_{t \in E^+ \setminus S^+} \rho_t(S^+ \cup \{j\}) x_t + y\left(E^- \setminus (L^- \cup S^-)\right).$$

But since $0 \leq \rho_t(S^+ \cup \{j\}) \leq \rho_t(S^+)$ from submodularity of v and $\rho_t(S^+) = 0$ for all $t \in S^-$, we observe that the path pack inequality above reduces to

$$y(E^+) \leq v(S^+) + \sum_{t \in E^+ \setminus S^+} \rho_t(S^+ \cup \{j\}) x_t + y\left(E^- \setminus (L^- \cup S^-)\right)$$

and it is at least as strong as the first pack inequality for S^+, S^- and $L^- = \emptyset$.

(iii)–(iv) We repeat the same argument of the proof of condition (iii) of Theorem 14. Suppose a node j is forward independent for (S^+, S^-) and the node $j-1$ is backward independent for (S^+, S^-) for some $j \in [2, n]$. Lemmas 10–13 show that the nodes N and the arcs $C = S^+ \cup L^-$ can be partitioned into $N_1 = [1, j-1]$, $N_2 = [j, n]$ and C_1, C_2 such that the sum of the minimum cut values for N_1, N_2 is equal to the minimum cut for N. From Remarks 5 and 6 and Corollary 9, it is easy to see that μ_i for $i \in N$ will not change by the partition procedures described in Lemmas 10–13. We examine the four cases for node $j-1$ to be forward independent and node j to be backward independent for the set (S^+, S^-). For ease of notation, let

$$Q_{jk}^+ := \sum_{i=j}^{k} \sum_{t \in E_i^+ \setminus S_i^+} (y_t - \min\{\mu_i, c_t\} x_t)$$

and

$$Q_{jk}^- := \sum_{i=j}^{k} \sum_{t \in S_i^-} (c_t - \mu_i)^+ (1 - x_t)$$

for $j \leq k$ and $j \in N$, $k \in N$ (and zero if $j > k$), where the values μ_i are the coefficients that appear in the path pack inequality (17). As a result, the path pack inequality can be written as

$$y(S^+) + Q_{1n}^+ \leq v(C) + Q_{1n}^- + y(E^- \setminus S^-) \tag{20}$$

(a) Suppose $\alpha_j^u = \alpha_{j-1}^d + u_{j-1} + c(S_j^+)$ and $\beta_{j-1}^u = \beta_j^d + b_{j-1} + c(S_{j-1}^+)$. Consider the partition procedure described in Lemma 10, where $S_{N1}^+ = (j, j-1) \cup S_{1j-1}^+$, $S_{N2}^+ = (j-1, j) \cup S_{jn}^+$, $S_{N1}^- = S_{1j-1}^-$, $S_{N2}^- = S_{jn}^-$. Then, the path pack inequalities for nodes N_1 is

$$r_{j-1} + y(S_{1j-1}^+) + Q_{1j-1}^+ \leq v_1(C_1) + Q_{1j-1}^- + y(E_{1j-1}^- \setminus S_{1j-1}^-) + i_{j-1}. \tag{21}$$

Similarly, the path pack inequality for N_2 is

$$i_{j-1} + y(S_{jn}^+) + Q_{jn}^+ \leq v_2(C_2) + Q_{jn}^- + y(E_{jn}^- \setminus S_{jn}^-) + r_{j-1}. \tag{22}$$

Inequalities (21)–(22) summed gives the path pack inequality (20).

(b) Suppose $\alpha_j^d = \alpha_{j-1}^u + b_{j-1} + d_j + c(S_j^-)$ and $\beta_{j-1}^d = \beta_j^u + u_{j-1} + d_{j-1} + c(S_{j-1}^-)$. Consider the partition described in Lemma 11, where $S_{N1}^+ = S_{1j-1}^+$, $S_{N2}^+ = S_{jn}^+$, $S_{N1}^- = (j-1, j) \cup S_{1j-1}^-$, $S_{N2}^- = (j, j-1) \cup S_{jn}^-$. The submodular inequality (4) for nodes N_1 where the objective coefficients of (F2) are selected as $a_t = 1$ for $t \in E_{1j-1}^+$, $a_t = 0$ for $t = (j, j-1)$, $a_t = -1$ for $t \in E_{N1}^- \setminus S_{N1}^-$ and $a_t = 0$ for $t \in S_{N1}^-$ is

$$y(S_{1j-1}^+) + \sum_{t \in S_{N1}^+} k_t (1 - x_t) + Q_{1j-1}^+ \leq v_1(C_1) - Q_{1j-1}^- + y(E_{1j-1}^- \setminus S_{1j-1}^-), \tag{23}$$

where k_t for $t \in S_{N1}^+$ are some nonnegative coefficients. Similarly, the submodular inequality (4) for nodes N_2, where the objective coefficients of (F2) are selected as $a_t = 1$ for $t \in E_{jn}^+$, $a_t = 0$ for $t = (j-1, j)$, $a_t = -1$ for $t \in E_{N2}^- \setminus S_{N2}^-$ and $a_t = 0$ for $t \in S_{N2}^-$ is

$$y(S_{jn}^+) + \sum_{t \in S_{N2}^+} k_t (1 - x_t) + Q_{jn}^+ \leq v_2(C_2) - Q_{jn}^- + y(E_{jn}^- \setminus S_{jn}^-), \tag{24}$$

where k_t for $t \in S_{N2}^+$ are some nonnegative coefficients. The sum of inequalities (23)–(24) is at least as strong as the path pack inequality (20).

(c) Suppose $\alpha^u_j = \alpha^d_{j-1} + u_{j-1} + c(S^+_j)$ and $\beta^d_{j-1} = \beta^u_j + u_{j-1} + d_{j-1} + c(S^-_{j-1})$. Consider the partition described in Lemma 12, where $S^+_{N1} = S^+_{1j-1}$, $S^+_{N2} = (j-1,j) \cup S^+_{jn}$, $S^-_{N1} = S^-_{1j-1}$, $S^-_{N2} = (j,j-1) \cup S^-_{jn}$. The submodular inequality (4) for nodes N_1 where the objective coefficients of (F2) are selected as $a_t = 1$ for $t \in E^+_{1j-1}$, $a_t = 0$ for $t = (j,j-1)$, $a_t = -1$ for $t \in E^-_{N1} \setminus S^-_{N1}$ and $a_t = 0$ for $t \in S^-_{N1}$ is

$$y(S^+_{1j-1}) + \sum_{t \in S^+_{N1}} k_t(1-x_t) + Q^+_{1j-1} \leq v_1(C_1) - Q^-_{1j-1} + y(E^-_{1j-1} \setminus S^-_{1j-1}) + i_{j-1}, \quad (25)$$

where k_t for $t \in S^+_{N1}$ are some nonnegative coefficients. The path pack inequality for N_2 is

$$i_{j-1} + y(S^+_{jn}) + Q^+_{jn} \leq v_2(C_2) + Q^-_{jn} + y(E^-_{jn} \setminus S^-_{jn}). \quad (26)$$

The sum of inequalities (25)–(26) is at least as strong as inequality (20).

(d) Suppose $\alpha^d_j = \alpha^u_{j-1} + b_{j-1} + d_j + c(S^-_j)$ and $\beta^u_{j-1} = \beta^d_j + b_{j-1} + c(S^+_{j-1})$. Consider the partition described in Lemma 13, where $S^+_{N1} = (j,j-1) \cup S^+_{1j-1}$, $S^+_{N2} = S^+_{jn}$, $S^-_{N1} = (j-1,j) \cup S^-_{1j-1}$, $S^-_{N2} = S^-_{jn}$. The path pack inequalities for nodes N_1 is

$$r_{j-1} + y(S^+_{1j-1}) + Q^+_{1j-1} \leq v_1(C_1) + Q^-_{1j-1} + y(E^-_{1j-1} \setminus S^-_{1j-1}). \quad (27)$$

The submodular inequality (4) for nodes N_2 where the objective coefficients of (F2) are selected as $a_t = 1$ for $t \in E^+_{jn}$, $a_t = 0$ for $t = (j-1,j)$, $a_t = -1$ for $t \in E^-_{N2} \setminus S^-_{N2}$ and $a_t = 0$ for $t \in S^-_{N2}$ is

$$y(S^+_{jn}) + \sum_{t \in S^+_{N2}} k_t(1-x_t) + Q^+_{jn} \leq v_2(C_2) - Q^-_{jn} + y(E^-_{jn} \setminus S^-_{jn}) + r_{j-1}, \quad (28)$$

where k_t for $t \in S^+_{N2}$ are some nonnegative coefficients. The sum of inequalities (27)–(28) is at least as strong as the path pack inequality (20).

(v) Suppose $(c_t - \mu_i)^+ = 0$ for all $t \in S^-_i$ and $i \in [p,n]$ and the node $p-1$ is forward independent. Then, we partition the node set $N = [1,n]$ into $N_1 = [1,p-1]$ and $N_2 = [p,n]$. We follow Lemma 10 if $\beta^u_{p-1} = \beta^d_p + b_{p-1} + c(S^+_{p-1})$ and follow Lemma 11 if $\beta^d_{p-1} = \beta^u_p + u_{p-1} + d_{p-1} + c(S^-_{p-1})$ to define S^+_{N1}, S^-_{N1}, S^+_{N2} and S^-_{N2}. Remark 6 along with the partition procedure described in Lemma 10 or 11 implies that μ_i will remain unchanged for $i \in N_1$.

If $\beta^u_{p-1} = \beta^d_j + b_{p-1} + c(S^+_{p-1})$, then the coefficients μ_i of the path pack inequality for nodes N_1 and arcs S^+_{N1}, S^-_{N1} described in Lemma 10 is the same as the coefficients of the path pack inequality for nodes N and arcs S^+, S^-. Moreover, let $\bar{m}^u_p$ and $\bar{m}^d_p$ be the values of minimum cut that goes above and below node p for the node set N_2 and arcs S^+_{N2}, S^-_{N2} and observe that

$$\bar{m}^u_p = \beta^u_p + u_{p-1} \text{ and } \bar{m}^d_p = \beta^d_p.$$

Then, comparing the difference $\bar{\mu}_p := (\bar{m}^d_p - \bar{m}^u_p)^+ = (\beta^d_p - \beta^u_p - u_{p-1})^+$ to $\mu_p = (m^d_p - m^u_p)^+ = (\beta^d_p - \beta^u_p + \alpha^d_p - \alpha^u_p - c(S^+_p) + d_p + c(S^-_p))^+$, we observe that $\bar{\mu}_p \geq \mu_p$ since $\alpha^d_p - \alpha^u_p - c(S^+_p) + d_p + c(S^-_p) \geq -u_{p-1}$ from (7)–(8). Since $(c_t - \mu_p)^+ = 0$, then $(c_t - \bar{\mu}_p)^+ = 0$ as well. Using the same technique, it is easy to observe that $\bar{\mu}_i \geq \mu_i$ for $i \in [p+1, n]$ as well. As a result, the path pack inequality for N_2 with sets S^+_{N2}, S^-_{N2}, summed with the path pack inequality for nodes N_1 and arcs S^+_{N1}, S^-_{N1} give the path pack inequality for nodes N and arc S^+, S^-.

Similarly, if $\beta^d_{p-1} = \beta^u_j + u_{p-1} + d_{p-1} + c(S^-_{p-1})$, the proof follows very similar to the previous argument using Lemma 11. Letting $\bar{m}^u_p$ and $\bar{m}^d_p$ be the values of minimum cut that goes above and below node p for the node set N_2 and arcs S^+_{N2}, S^-_{N2}, we get

$$\bar{m}^u_p = \beta^u_p \text{ and } \bar{m}^d_p + b_{p-1} = \beta^d_p$$

under this case. Now, notice that that $\alpha^d_p - \alpha^u_p - c(S^+_p) + d_p + c(S^-_p) \leq b_{p-1}$ from (7)–(8), which leads to $\bar{\mu}_p \geq \mu_p$. Then the proof follows same as above.

(vi) The proof is similar to that of the necessary condition (v) above. We use Lemmas 12–13 and Remark 6 to partition the node set N and arcs S^+, S^- into node sets $N_1 = [1, q]$ and $N_2 = [q+1, n]$ and arcs S^+_{N1}, S^-_{N1} and S^+_{N2}, S^-_{N2}. Next, we check the values of minimum cut that goes above and below node q for the node set N_1 and arcs S^+_{N1}, S^-_{N1}. Then, observing $-b_q \leq \beta^d_q - \alpha^u_q - c(S^+_q) + d_q + c(S^-_q) \leq u_q$ from (9)–(10), it is easy to see that the coefficients of x_t for $t \in S^-_{N1}$ and $t \in E^+_{N1} \setminus S^+_{N1}$ are equal to zero in the path pack inequality for node set N_1. As a result, the path pack inequalities for N_1, S^+_{N1}, S^-_{N1} and for N_2, S^+_{N2}, S^-_{N2} summed gives the path pack inequality for N, S^+, S^-.

□

Theorem 17. *Let $N = [1, n]$, $E^- = \emptyset$, $d_j \geq 0$ and $|E^+_j| = 1$, for all $j \in N$ and let the objective set S^+ be a path pack for N. The necessary conditions in Theorem 16 along with*

(i) for each $j \in E^+ \setminus S^+$, either $S^+ \cup \{j\}$ is a path cover for N or $\rho_j(S^+) = 0$,

(ii) for each $t \in S^+$, there exists $j_t \in E^+ \setminus S^+$ such that $S^+ \setminus \{t\} \cup \{j_t\}$ is a path cover for N,

(iii) for each $j \in [1, n-1]$, there exists $k_j \in E^+ \setminus S^+$ such that the set $S^+ \cup \{k_j\}$ is a path cover and neither node j is backward independent nor node $j+1$ is forward independent for the set $S^+ \cup \{k_j\}$

are sufficient for path pack inequality (17) to be facet-defining for conv($\mathcal{P}$).

Proof. We provide $2|E| + n - 2$ affinely independent points that lie on the face:

$$F = \left\{ (\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{r}) \in \mathcal{P} \ : \ y(S^+) + \sum_{t \in E^+ \setminus S^+} (y_t - \min\{\mu_j, c_t\} x_t) = c(S^+) \right\}.$$

Let $(\mathbf{y}^*, \mathbf{i}^*, \mathbf{r}^*) \in \mathcal{Q}$ be an optimal solution to (F2). Since S^+ is a path pack and $E^- = \emptyset$, $v(S^+) = c(S^+)$. Then, notice that $y^*_t = c_t$ for all $t \in S^+$. Moreover, let e be the arc with largest capacity in S^+, $\varepsilon > 0$ be a sufficiently small value and $\mathbf{1}_j$ be the unit vector with 1 at position j. First, we give $2|E^+ \setminus S^+|$ affinely independent points represented by $\bar{z}^t = (\bar{\mathbf{y}}^t, \bar{\mathbf{x}}^t, \bar{\mathbf{i}}^t, \bar{\mathbf{r}}^t)$ and $\tilde{z}^t = (\tilde{\mathbf{y}}^t, \tilde{\mathbf{x}}^t, \tilde{\mathbf{i}}^t, \tilde{\mathbf{r}}^t)$ for $t \in E^+ \setminus S^+$.

(i) Let $t \in E^+ \setminus S^+$, where $S^+ \cup \{t\}$ is a path cover for N. The solution $\bar{z}^t$ has arcs in $S^+ \cup \{t\}$ open, $\bar{x}^t_j = 1$ for $j \in S^+ \cup \{t\}$, 0 otherwise, $\bar{y}^t_j = y^*_j$ for $j \in S^+$ and

$\bar{y}_t^t = \rho_t(S^+)$, 0 otherwise. The forward and backward path arc flow values $\bar{i}_j^t$ and $\bar{r}_j^t$ can then be calculated using flow balance equalities (1b) where at most one of them can be nonzero for each $j \in N$. Sufficiency condition (i) guarantees the feasibility of $\bar{z}^t$.

(ii) Let $t \in E^+ \setminus S^+$, where $\rho_t(S^+) = 0$ and let $t \neq \ell \in E^+ \setminus S^+$, where $S^+ \cup \{\ell\}$ is a path cover for N. The solution $\bar{z}^t$ has arcs in $S^+ \cup \{t, \ell\}$ open, $\bar{x}_j^t = 1$ for $j \in S^+ \cup \{t, \ell\}$, and 0 otherwise, $\bar{y}_j^t = y_j^*$ for $j \in S^+$, $\bar{y}_t^t = 0$, $\bar{y}_\ell^t = \rho_\ell(S^+)$, and 0 otherwise. The forward and backward path arc flow values $\bar{i}_j^t$ and $\bar{r}_j^t$ can then be calculated using flow balance equalities (1b) where at most one of them can be nonzero for each $j \in N$. Sufficiency condition (i) guarantees the feasibility of $\bar{z}^t$.

(iii) The necessary condition (i) ensures that $\rho_t(S^+) < c_t$, therefore $\bar{y}_t^t < c_t$. In solution $\tilde{z}^t$, starting with $\bar{z}^t$, we send a flow of ε from arc $t \in E^+ \setminus S^+$ to $e \in S^+$. Let $\tilde{\mathbf{y}}^t = \bar{\mathbf{y}}^t + \varepsilon \mathbf{1}_t - \varepsilon \mathbf{1}_e$ and $\tilde{\mathbf{x}}^t = \bar{\mathbf{x}}^t$. If $e < t$, then $\tilde{\mathbf{r}}^t = \bar{\mathbf{r}}^t + \varepsilon \sum_{i=e}^{t-1} \mathbf{1}_i$, $\tilde{\mathbf{i}}^t = \bar{\mathbf{i}}^t$ and if $e > t$, then $\tilde{\mathbf{i}}^t = \bar{\mathbf{i}}^t + \varepsilon \sum_{i=t}^{e-1} \mathbf{1}_i$, $\tilde{\mathbf{r}}^t = \bar{\mathbf{r}}^t$.

Next, we give $2|S^+| - 1$ affinely independent feasible points $\hat{z}^t$ and $\check{z}^t$ corresponding to $t \in S^+$ that are on the face F. Let k be the arc in $E^+ \setminus S^+$ with largest capacity.

(iv) In the feasible solutions $\hat{z}^t$ for $e \neq t \in S^+$, we open arcs in $S^+ \cup \{k\}$ and send an ε flow from arc k to arc t. Let $\hat{\mathbf{y}}^t = \bar{\mathbf{y}}^k + \varepsilon \mathbf{1}_k - \varepsilon \mathbf{1}_t$ and $\hat{\mathbf{x}}^t = \bar{\mathbf{x}}^k$. If $t < k$, then $\hat{\mathbf{r}}^t = \bar{\mathbf{r}}^k + \varepsilon \sum_{i=t}^{k-1} \mathbf{1}_i$, $\hat{\mathbf{i}}^t = \bar{\mathbf{i}}^k$ and if $t > k$, then $\hat{\mathbf{i}}^t = \bar{\mathbf{i}}^k + \varepsilon \sum_{i=k}^{t-1} \mathbf{1}_i$, $\hat{\mathbf{r}}^t = \bar{\mathbf{r}}^k$.

(v) In the solutions $\check{z}^t$ for $t \in S^+$, we close arc t and open arc $j_t \in E^+ \setminus S^+$ that is introduced in the sufficient condition (ii). Then, $\check{x}_j^t = 1$ if $j \in S^+ \setminus \{t\}$ and if $j = j_t$ and $\check{x}_j^t = 0$ otherwise. From sufficient condition (ii), there exists $\check{y}_j^t$ values that satisfy the flow balance equalities (1b). Moreover, these $\check{y}_j^t$ values satisfy inequality (17) at equality since both $S^+ \cup \{j_t\}$ and $S^+ \setminus \{t\} \cup \{j_t\}$ are path covers for N. Then, the forward and backward path arc flows are found using flow balance equalities where at most one of $\check{i}_j^t$ and $\check{r}_j^t$ are nonzero for each $j \in N$.

Finally, we give $n-1$ points $\breve{z}^j$ corresponding to forward and backward path arcs connecting nodes j and $j+1$.

(vi) In the solution set $\breve{z}^j$ for $j = 1, \dots, n-1$, starting with solution $\bar{z}^{k_j}$, where k_j is introduced in the sufficient condition (iii), we send a flow of ε from both forward path arc $(j-1, j)$ and backward path arc $(j, j-1)$. Since the sufficiency condition (iii) ensures that $\bar{r}_j^{k_j} < b_j$ and $\bar{i}_j^{k_j} < u_j$, the operation is feasible. Let $\breve{\mathbf{y}}^j = \bar{\mathbf{y}}^{k_j}$, $\breve{\mathbf{x}}^j = \bar{\mathbf{x}}^{k_j}$, $\breve{\mathbf{i}}^j = \bar{\mathbf{i}}^{k_j} + \varepsilon \mathbf{1}_j$ and $\breve{\mathbf{r}}^j = \bar{\mathbf{r}}^{k_j} + \varepsilon \mathbf{1}_j$.

□

5. Computational study

We test the effectiveness of path cover and path pack inequalities (14) and (17) by embedding them in a branch-and-cut framework. The experiments are ran on a Linux workstation with 2.93 GHz Intel®Core™ i7 CPU and 8 GB of RAM with 1 hour time limit and 1 GB memory limit. The branch-and-cut algorithm is implemented in C++ using IBM's Concert Technology of CPLEX (version 12.5). The experiments are ran with one hour limit on elapsed time and 1 GB limit on memory usage. The number of threads is set one and the dynamic search is disabled. We also turn off heuristics and preprocessing as the purpose is to see the impact of the inequalities by themselves.

Instance generation. We use a capacitated lot-sizing model with backlogging, where constraints (1b) reduce to:

$$i_{j-1} - r_{j-1} + y_j - i_j + r_j = d_j, \quad j \in N.$$

Let n be the total number of time periods and f be the ratio of the fixed cost to the variable cost associated with a non-path arc. The parameter c controls how large the non-path arc capacities are with respect to average demand. All parameters are generated from a discrete uniform distribution. The demand for each node is drawn from the range $[0, 30]$ and non-path arc capacities are drawn from the range $[0.75 \times c \times \bar{d}, 1.25 \times c \times \bar{d}]$, where $\bar{d}$ is the average demand over all time periods. Forward and backward path arc capacities are drawn from $[1.0 \times \bar{d}, 2.0 \times \bar{d}]$ and $[0.3 \times \bar{d}, 0.8 \times \bar{d}]$, respectively. The variable costs p_t, h_t and g_t are drawn from the ranges $[1, 10]$, $[1, 10]$ and $[1, 20]$ respectively. Finally, fixed costs f_t are set equal to $f \times p_t$. Using these parameters, we generate five random instances for each combination of $n \in \{50, 100, 150\}$, $f \in \{100, 200, 500, 1000\}$ and $c \in \{2, 5, 10\}$.

Finding violated inequalities. Given a feasible solution $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{i}^*, \mathbf{r}^*)$ to a linear programming (LP) relaxation of (F1), the separation problem aims to find sets S^+ and L^- that maximize the difference

$$y^*(S^+) - y^*(E^- \setminus L^-) + \sum_{t \in S^+} (c_t - \lambda_j)^+(1 - x_t^*) - \sum_{t \in L^-} (\min\{\lambda_j, c_t\}) x_t^* - d_{1n} - c(S^-)$$

for path cover inequality (14) and sets S^+ and S^- that maximize

$$y^*(S^+) - y^*(E^- \setminus S^-) - \sum_{t \in E^+ \setminus S^+} \min\{c_t, \mu_j\} x_t^* + \sum_{t \in S^-} (c_t - \mu_j)^+(1 - x_t) - c(S^+)$$

for path pack inequality (17). We use the knapsack relaxation based heuristic separation strategy described in (Wolsey and Nemhauser, 1999, pg. 500) for flow cover inequalities to choose the objective set S^+ with a knapsack capacity d_{1n}. Using S^+, we obtain the values λ_j and μ_j for each $j \in N$ and let $S^- = \emptyset$ for path cover and path pack inequalities (14) and (17). For path cover inequalities (14), we add an arc $t \in E^-$ to L^- if $\lambda_j x_t^* < y_t^*$ and $\lambda_j < c_t$. We repeat the separation process for all subsets $[k, \ell] \subseteq [1, n]$.

Results. We report multiple performance measures. Let z_{INIT} be the objective function value of the initial LP relaxation and z_{ROOT} be the objective function value of the LP relaxation after all the valid inequalities added. Moreover, let z_{UB} be the objective function value of the best feasible solution found within time/memory limit among all experiments for an instance. Let `init gap`$= 100 \times \frac{z_{\text{UB}} - z_{\text{INIT}}}{z_{\text{UB}}}$, `root gap`$= 100 \times \frac{z_{\text{UB}} - z_{\text{ROOT}}}{z_{\text{UB}}}$. We compute the improvement of the relaxation due to adding valid inequalities as `gap imp`$= 100 \times \frac{\texttt{init gap} - \texttt{root gap}}{\texttt{init gap}}$. We also measure the optimality gap at termination as `end gap` $= \frac{z_{\text{UB}} - z_{\text{LB}}}{z_{\text{UB}}}$, where z_{LB} is the value of the best lower bound given by CPLEX. We report the average number of valid inequalities added at the root node under column `cuts`, average elapsed time in seconds under `time`, average number of branch-and-bound nodes explored under `nodes`. If there are instances that are not solved to optimality within the time/memory

Table 1. Effect of path size on the performance.

				$p=1$		$p \leq 5$				$p \leq 0.5 \times n$				$p \leq n$			
			init	gap imp	cuts	gap imp		cuts		gap imp		cuts		gap imp		cuts	
n	f	c	gap	(m)spi	(m)spi	spi	mspi	spi	mspi	spi	mspi	spi	mspi	spi	mspi	spi	mspi
50	100	2	14.8	34%	21	87%	52%	212	106	97%	52%	1164	158	97%	52%	1233	158
		5	44.3	56%	52	99%	69%	303	148	99%	69%	664	151	99%	69%	664	151
		10	58.3	60%	54	96%	70%	277	147	99%	70%	574	167	99%	70%	574	167
	200	2	14.5	32%	22	81%	57%	257	133	96%	61%	1965	241	96%	61%	2387	241
		5	49.8	43%	44	99%	57%	378	162	100%	57%	1264	171	100%	57%	1420	171
		10	66.3	38%	47	98%	50%	392	169	99%	51%	1235	197	99%	51%	1283	197
	500	2	19.1	23%	19	77%	48%	266	128	90%	49%	2286	222	90%	49%	3249	222
		5	54.4	35%	36	99%	49%	522	185	100%	49%	1981	205	100%	49%	2074	205
		10	73.0	34%	43	99%	40%	498	196	99%	40%	1336	236	99%	40%	1445	236
	1000	2	14.6	18%	15	73%	39%	264	99	83%	40%	2821	211	83%	40%	3918	212
		5	59.7	31%	36	98%	45%	487	201	100%	45%	2077	239	100%	45%	2329	239
		10	76.9	30%	41	100%	36%	529	215	100%	37%	1935	268	100%	37%	2149	268
Average:			**45.5**	**36%**	**36**	**92%**	**51%**	**365**	**157**	**97%**	**52%**	**1609**	**206**	**97%**	**52%**	**1894**	**206**

limit, we report the average end gap and the number of unsolved instances under `unslvd` next to `time` results. All numbers except initial gap, end gap and time are rounded to the nearest integers.

In Tables 1 and 2, we present the performance with the path cover (14) and path pack (17) inequalities under columns `spi`. To understand how the forward and backward path arc capacities affect the computational performance, we also apply them to the single node relaxations obtained by merging a path into a single node, where the capacities of forward and backward path arcs within a path are considered to be infinite. Note that, in this case, the path inequalities reduce to the flow cover and flow pack inequalities. These results are presented under columns `mspi`.

In Table 1, we focus on the impact of path size on the gap improvement of the path cover and path pack inequalities for instances with $n = 50$. In the columns under $p = 1$, we obtain the same results for both `mspi` and `spi` since the paths are singleton nodes.. We present these results under `(m)spi`. In columns $p \leq q$, we add valid inequalities for paths of size $1, \ldots, q$ and observe that as the path size increases, the gap improvement of the path inequalities increase rapidly. On average 97% of the initial gap is closed as longer paths are used. On the other hand, flow cover and pack inequalities from merged paths reduce about half of the initial gap. These results underline the importance of exploiting path arc capacities on strengthening the formulations. We also observe that the increase in gap improvement diminishes as path size grows. Therefore, we limit the maximum path size to $0.75 \times n$ for the experiments reported in Tables 2 and 3.

In Table 2, we present other performance measures as well for instances with 50,100, and 150 nodes. We observe that the forward and backward path arc capacities have a large impact on the performance level of the path cover and pack inequalities. Compared to flow cover and pack inequalities added from merged paths, path cover and path pack inequalities reduce the number of nodes and solution times by orders of magnitude. This is mainly due to better integrality gap improvement (50% vs 95% on average).

In Table 3, we examine the incremental effect of path cover and path pack inequalities over the fixed-charge network cuts of CPLEX, namely flow cover, flow path and multi-commodity flow cuts. Under `cpx`, we present the performance of flow cover, flow path and multi-commodity flow cuts added by CPLEX and under `cpx_spi`, we add path cover and path pack inequalities addition to these cuts. We observe that with the addition of path cover and pack inequalities, the gap improvement increases from 86% to 95%. The number of branch and bound nodes explored is reduced about 900 times. Moreover, with path cover and path pack inequalities the average elapsed time is reduced to almost half and the total number of unsolved instances reduces from 13 to 6 out of 180 instances.

Tables 1, 2 and 3 show that submodular path inequalities are quite effective in tackling lot-sizing problems with finite arc capacities. When added to the LP relaxation, they improve the optimality gap by 95% and the number of branch and bound nodes explored decreases by a factor of 1000. In conclusion, our computational experiments indicate that the use of path cover and path pack inequalities is beneficial in improving the performance of the branch-and-cut algorithms.

Table 2. Comparison of path inequalities applied to paths (`spi`) versus applied to merged paths (`mspi`).

				`gapimp`		`nodes`		`cuts`		`time (endgap:unslvd)`		
n	f	c	`init gap`	`spi`	`mspi`	`spi`	`mspi`	`spi`	`mspi`	`spi`	`mspi`	
50	100	2	14.8	96%	52%	7	430	1151	195	0.2	0.2	
		5	44.3	97%	69%	9	553	435	146	0.1	0.1	
		10	58.3	93%	70%	63	468	386	160	0.1	0.1	
	200	2	14.5	92%	59%	18	330	2469	226	0.6	0.2	
		5	49.8	100%	57%	3	1112	1022	176	0.1	0.3	
		10	66.3	97%	53%	14	615	739	173	0.1	0.2	
	500	2	19.1	92%	43%	19	2041	2577	238	1.0	0.7	
		5	54.4	99%	48%	1	705	1164	214	0.1	0.3	
		10	73.0	97%	48%	11	5659	698	248	0.1	1.4	
	1000	2	14.6	90%	45%	60	612	2094	301	0.9	0.4	
		5	59.7	100%	50%	2	2265	1792	241	0.1	0.7	
		10	76.9	99%	40%	5	9199	1032	314	0.1	2.3	
100	100	2	13.9	95%	65%	39	7073	3114	410	1.3	3.2	
		5	42.2	98%	70%	19	20897	1337	297	0.2	4.8	
		10	57.8	94%	59%	230	395277	1298	346	0.4	88.2	
	200	2	16.1	89%	56%	290	151860	6919	478	11.0	58.4	
		5	47.6	99%	55%	7	455192	2355	331	0.3	126.1	
		10	65.7	95%	54%	104	4130780	1872	399	0.5	962.3	(1.1:1)
	500	2	17.5	84%	36%	1047	956745	11743	475	47.7	390.9	
		5	53.9	99%	41%	4	332041	3874	444	0.4	115.5	
		10	72.9	96%	42%	34	1175647	1495	474	0.3	352.5	
	1000	2	17.9	91%	41%	173	57147	10919	570	21.3	23.0	
		5	58.5	100%	45%	1	284979	3261	501	0.3	92.8	
		10	75.7	97%	36%	88	3158262	2358	657	0.5	1047.0	(0.7:1)
150	100	2	13.2	94%	64%	336	163242	5159	704	11.3	107.6	
		5	44.8	99%	65%	17	3024118	2087	431	0.5	929.6	
		10	56.9	95%	65%	404	7254052	1492	476	0.9	2087.3	(0.7:1)
	200	2	14.7	92%	53%	519	2772494	12636	744	27.2	1390.6	(0.1:1)
		5	48.1	99%	55%	15	3802938	2462	508	0.6	1483.0	(1.2:2)
		10	65.2	95%	50%	330	9377122	2047	567	0.9	3585.9	(8.2:5)
	500	2	19.3	86%	33%	7927	7619674	22275	792	1087.3	3165.6	(4.0:4)
		5	54.4	100%	45%	7	7873043	4927	641	0.8	2813.6	(4.3:3)
		10	72.3	97%	41%	250	10219548	2678	713	1.2	3422.8	(11.0:5)
	1000	2	19.6	88%	34%	2824	7316675	33729	724	804.8	3260.3	(2.5:3)
		5	57.5	100%	39%	2	9661586	6710	709	0.8	3578.9	(9.7:5)
		10	75.8	96%	37%	99	9910056	3981	829	1.2	3412.3	(15.2:5)
Average:			**45.2**	**95%**	**50%**	**416**	**2504012**	**4619**	**440**	**56.3**	**903.0**	**(1.6:36)**

Table 3. Effectiveness of the path inequalities when used together with CPLEX's network cuts.

n	f	c	init gap	gapimp cpx_spi	gapimp cpx	nodes cpx_spi	nodes cpx	time (endgap:unslvd) cpx_spi	time (endgap:unslvd) cpx
100	100	2	13.9	96%	85%	35	1715	1.0	0.5
		5	42.2	99%	97%	5	75	0.2	0.1
		10	57.8	99%	93%	10	2970	0.3	0.6
	200	2	16.1	90%	79%	288	9039	6.6	2.1
		5	47.6	99%	95%	7	52	0.3	0.1
		10	65.7	97%	89%	61	3186	0.4	0.7
	500	2	17.5	85%	63%	1232	455068	57.3	95.2
		5	53.9	99%	94%	6	92	0.4	0.1
		10	72.9	98%	89%	11	4621	0.4	0.9
	1000	2	17.9	91%	76%	173	18109	22.2	3.6
		5	58.5	100%	93%	1	156	0.3	0.1
		10	75.7	97%	85%	117	5297	0.7	1.0
150	100	2	13.2	94%	86%	365	60956	9.7	19.0
		5	44.8	100%	97%	5	119	0.4	0.1
		10	56.9	99%	92%	16	15929	0.5	3.9
	200	2	14.7	92%	80%	954	216436	44.9	66.7
		5	48.1	99%	96%	11	284	0.5	0.2
		10	65.2	97%	91%	181	3992	0.9	1.2
	500	2	19.3	86%	69%	7647	4943603	1049.9	1215.1 (0.2:1)
		5	54.4	100%	94%	5	5434	0.8	1.6
		10	72.3	97%	88%	141	141211	1.4	35.9
	1000	2	19.6	88%	71%	3051	2788993	917.4 (0.2:1)	619.4 (0.4:1)
		5	57.5	100%	90%	3	4322	0.8	1.2
		10	75.8	96%	89%	196	10588	2.5	2.8
200	100	2	14.1	94%	82%	1623	864841	32.2	384.0
		5	42.7	100%	97%	8	213	0.5	0.1
		10	57.5	99%	93%	26	45263	0.7	13.8
	200	2	16.3	89%	78%	4279	5634851	259.9	1940.4 (0.1:1)
		5	48.0	99%	95%	13	1310	0.9	0.5
		10	65.0	98%	90%	128	163145	1.2	52.3
	500	2	16.3	88%	72%	8083	6805861	1226.3 (0.3:1)	2137.6 (0.7:3)
		5	54.5	99%	93%	7	6606	1.6	2.2
		10	72.0	96%	90%	376	900152	3.2	302.4
	1000	2	18.0	82%	63%	13906	9894589	3000.5 (1.2:4)	2835.9 (3.0:5)
		5	57.9	100%	94%	4	1977	3.4	0.8
		10	75.6	96%	84%	704	6127929	15.0	1785.0 (1.8:2)
Average:			**45.0**	**95%**	**86%**	**1213**	**1087194**	**185.1 (0.0:6)**	**320.2 (0.2:13)**

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Appendix A. Equivalency of (F2) to the maximum flow problem

In Section 3, we showed the maximum flow equivalency of $v(S^+, L^-)$ under the assumption that $d_j \geq 0$ for all $j \in N$. In this section, we generalize the equivalency for the paths where $d_j < 0$ for some $j \in N$.

Observation 2. If $d_j < 0$ for some $j \in N$, one can represent the supply amount as a dummy arc incoming to node j (i.e., added to E_j^+) with a fixed flow and capacity of $-d_j$ and set the modified demand of node j to be $d_j = 0$.

Given the node set N with at least one supply node, let $\mathcal{T}(N)$ be the transformed path using Observation 2. Transformation $\mathcal{T}$ ensures that the dummy supply arcs are always open. As a result, they are always in the set S^+. We refer to the additional constraints that fix the flow to the supply value on dummy supply arcs as *fixed-flow constraints*. Notice that, $v(S^+, L^-)$ computed for $\mathcal{T}(N)$ does not take fixed-flow constraints into account. In the next proposition, for a path structure, we show that there exists at least one optimal solution to (F2) such that the fixed-flow constraints are satisfied.

Proposition 18. *Suppose that $d_j < 0$ for some $j \in N$. If (F2) for the node set N is feasible, then it has at least one optimal solution that satisfies the fixed-flow constraints.*

Proof. We need to show that $v(S^+, L^-)$ has an optimal solution where the flow at the dummy supply arcs is equal to the supply values. Notice that, the transformation $\mathcal{T}$ makes Proposition 1 applicable to the modified path $\mathcal{T}(N)$. Let $\mathcal{Y}$ be the set of optimal solutions of (F2). Then, there exists a solution $(\mathbf{y}^*, \mathbf{i}^*, \mathbf{r}^*) \in \mathcal{Y}$ where $y_t^* = 0$ for $t \in E^- \setminus (S^- \cup L^-)$. Let $p \in S_j^+$ represent the index of the dummy supply arc with $c_p = -d_j$. If $y_p^* < c_p$, then satisfying the fixed-flow constraints require pushing flow through the arcs in $E^- \setminus L^-$. We use Algorithm 2 to construct an optimal solution with $y_p^* = c_p$. Note that each arc in $E_k^- \setminus L_k^-$ for $k \in N$ appear in (F2) with the same coefficients, therefore we merge these outgoing arcs into one in Algorithm 2. We represent the merged flow and capacity by $\bar{Y}_k^- = \sum_{t \in E_k^- \setminus (S_k^- \cup L_k^-)} y_t^*$ and $\bar{C}_k = c\big(E_k^- \setminus (S_k^- \cup L_k^-)\big)$ for $k \in N$.
□

Proposition 18 shows that, under the presence of supply nodes, transformation $\mathcal{T}$ both captures the graph's structure and does not affect (F1)'s validity. As a result, Propositions 1 and 2 become relevant to the transformed path and submodular path inequalities (14) and (17) are also valid for paths where $d_j < 0$ for some $j \in N$.

Algorithm 2

$\mathcal{J}$: Set of supply nodes in N where the nodes are sorted with respect to their order in N.

$(\mathbf{y}^*, \mathbf{i}^*, \mathbf{r}^*) \in \mathcal{Y}$: $y_t^* = 0$ for all $t \in E^-$.
for $q \in \mathcal{J}$ **do**
  Let p be the dummy supply arc in S_q^+
  $\Delta = c_p - y_p^*$
  for $j = q$ **to** n **do**
    $\bar{Y}_j^- = \bar{Y}_j^- + \min\{\bar{C}_j - \bar{Y}_j^-, \Delta\}$
    $\Delta = \Delta - \min\{\bar{C}_j - \bar{Y}_j^-, \Delta\}$
    $i_j^* = i_j^* + \Delta$
    if $i_j^* > u_j$ **then**
      $\Delta = i_j^* - u_j$
      $i_j^* = u_j$
      Let $k := j$
      break inner loop
    end if
  end for
  if $\Delta > 0$ **then**
    for $j = k$ **to** 1 **do**
      $\bar{Y}_j^- = \bar{Y}_j^- + \min\{\bar{C}_j - \bar{Y}_j^-, \Delta\}$
      $\Delta = \Delta - \min\{\bar{C}_j - \bar{Y}_j^-, \Delta\}$
      $r_j^* = r_j^* + \Delta$
      if $r_j^* > b_j$ **then**
        $\Delta = r_j^* - b_j$
        break inner loop
      end if
    end for
  end if
  if $\Delta > 0$ **then**
    (F2) is infeasible for the node set N.
  end if
end for