To appear in *Engineering Optimization*
Vol. 00, No. 00, Month 20XX, 1–23

Sequential Linear Programming and Particle Swarm Optimization for the Optimization of Energy Districts

E. Riccietti [a*] and S. Bellavia[b] and S. Sello[c]

[a]*Dipartimento di Matematica e Informatica "Ulisse Dini", Università di Firenze, viale G.B. Morgagni 67a, 50134 Firenze, Italia.*; [b]*Dipartimento di Ingegneria Industriale, Università di Firenze, viale G.B. Morgagni 40, 50134 Firenze, Italia, stefania.bellavia@unifi.it*; [c]*Enel Ingegneria e Ricerca, Via Andrea Pisano 120, 56122 Pisa, Italia, stefano.sello@enel.com*

(*Received 00 Month 20XX; final version received 00 Month 20XX*)

This article deals with the optimization of energy resources management of industrial districts, with the aim of minimizing the customer energy expenses. In a district the number of possible energy system combinations is really large, and a manual design approach might lead to a suboptimal solution. A modelling of the district is employed, whose optimization gives rise to a nonlinear constrained optimization problem, here the focus is on its numerical solution. Two different methods are considered for its solution: a Sequential Linear Programming (SLP) and a Particle Swarm Optimization (PSO) method. Efficient implementations of both approaches are devised and the results of the tests performed on several energetic districts are reported.

Keywords: Nonlinear Optimization Problems; Sequential Linear Programming; Particle Swarm Optimization; Energy Districts; Penalty Functions

1. Introduction

In this article the numerical solution of optimization problems arising in energy districts is investigated. The considered energy districts comprise machines able to generate, absorb or accumulate energy, like generators, accumulators, thermal loads.

The electric system has been experiencing a dramatic evolution in the last years, due to the spread of unpredictable renewable energy source use and of distributed generation. On top of that, customers are increasing their attention towards a smarter approach to energy consumption, and paying more and more attention to convenient energy tariff schemes, choosing among different retailers. In a district the number of possible energy system combinations is too large to be handled by a manual design approach, that might lead to a suboptimal solution. This kind of problems have been addressed for example in (Ascione et al. 2016; Schiefelbein et al. 2015; Wang, Martinac, and Magny 2015). The research centre 'Enel Ingegneria e Ricerca' in Pisa in collaboration with the Department of Civil and Industrial Engineering of the University of Pisa developed a software package aimed at the optimized management of energy resources for energy districts, (Pannocchia and Mancuso 2014). This software provides a model of the energy district and taking into account all the available information (energy market prices, weather actual information and forecast) it builds the objective function measuring the costs related to the district

*Corresponding author. Email: elisa.riccietti@unifi.it management and functions modelling the physical constraints on the machines. The problem under consideration is then that of providing the optimal dispatching of local energy resources on day ahead in order to minimize the cost of energy at customer site, (Ferrara, Riccardi, and Sello 2014). The modelling gives rise to a nonlinear minimization problem with nonlinear constraints and here the focus is on its solution.

A wide range of methods have been developed for the solution of such problems, both exact optimization algorithms, (Newton, Quasi-Newton, Gauss-Newton, (Nocedal and Wright 2006; Bertsekas 1999)) and metaheuristic optimization algorithms (Particle Swarm Optimization, Cultural Algorithms, Musical Composition, Differential Evolution (Hu and Eberhart 2002; Lu and Chen 2008; Mezura-Montes, Miranda-Varela, and del Carmen Gómez-Ramón 2010; Aragón, Esquivel, and Coello 2010; Tessema and Yen 2009)). Over the last years this least category has been widened by employing hybrid methods that do not purely follow the concepts of one single traditional metaheuristic, but they combine various algorithmic ideas, borrowing strategies from different fields of optimization, sometimes also from outside of the traditional metaheuristics field, (de-los Cobos-Silva et al. 2016; Raidl 2006; Robles et al. 2006; Zhang and Xie 2003). These methods have over time also come to include any procedure for problem solving that employs a strategy for overcoming the trap of local optimality in complex solution spaces, (Glover and Kochenberger 2006).

The use of a method belonging to the first class does not guarantee convergence to a global minimum, actually the generated sequence usually converges to a local minimum close to the starting guess. Metaheuristic procedures on the other hand are specifically designed to find a global minimum. In solving the problem under consideration a compromise between two needs should be found. On one hand one aims at finding the machine asset corresponding to the lowest possible value of the objective function. On the other hand the optimization process should be quick, as it needs to be performed several times in a day, when the needs of the district or the weather conditions affecting the renewable resources change. Moreover the objective function does not have a simple analytical expression, it is expensive to evaluate, and the analytical form of just some components of the gradient is available. The remaining components have to be approximated, for example via finite differences. Thus, it is possible to rely only on derivative free methods or methods using just first order derivatives, as approximating the second derivatives with finite differences would be computationally heavy. Two methods belonging to the aforementioned categories were selected: a Sequential Linear Programming (SLP) and a Particle Swarm Optimization (PSO) method. SLP is an iterative procedure that approaches a solution of the original nonlinear problem by generating a sequence of solutions of linear subproblems approximating it, (Fletcher and de la Maza 1989; Byrd et al. 2003). PSO is an heuristic research procedure, (Hu and Eberhart 2002), it is derivative free and requires just function values. For this reason it usually shows a slower convergence rate than SLP, that is a first order method. PSO solver is then expected to find a better solution than SLP, but the optimization process is expected to be longer. To the authors' knowledge the performance of methods belonging to the above mentioned classes on this kind of problem has never been investigated and compared. Then, the aim of this work is to analyze and compare SLP and PSO performance when applied to optimization problems arising in energy districts, to understand if it is worth the use of a slower method, but able to find an approximation of the global minimum. Efficient and reliable implementations of the two procedures are also provided, specifically designed for the problem at hand. The adopted SLP method is close to the one presented in (Robinson 1972), but is equipped with a trust-region approach to promote the global convergence, (Nocedal and Wright 2006). The literature amount on PSO methods is huge, and many variants of the original approach have been proposed, see (Liu, Cai, and Wang 2010; Mazhoud et al. 2013; Mezura-Montes, Miranda-Varela, and del Carmen Gómez-Ramón 2010; Cagnina, Esquivel, and Coello 2011) just for few examples. The implemented method is based on the constraint handling strategy introduced in (Michalewicz and Attia 1994) and on a new strategy to prevent stagnation in local minima, proposed here.

The two solvers were inserted in the software package developed by Enel centre and University of Pisa, and tested on many different realistic examples of energy districts, and on a real energy district provided by 'Enel Ingegneria e Ricerca'. Preliminary analysis on this topic has been conducted by the authors in (Riccietti, Bellavia, and Sello 2017), where the problem under study is introduced and preliminary numerical results are given. In this article the numerical solution of the arising optimization problem is deeply analysed. Particularly, the new variant of PSO method is introduced, a convergence analysis for the proposed SLP is provided and a wide numerical experimentation is performed. A comparison of the proposed methods with the state-of-the-art of metaheuristic methods is also provided.

The article is organized as follows: in Section 2 the district modelling is introduced, the machines that are part of the district are presented, focusing on the description of the optimization variables. In Section 3 the arising optimization problem is stated. In Section 4 the SLP procedure that was implemented for the specific problem is described, the convergence analysis is reported in Appendix A. In Section 5 PSO method is described, with focus on the implemented version. The new strategy to prevent stagnation in local minima is introduced. Finally, in Section 6 numerical results are presented. First the two methods are applied to a famous set of benchmark functions, usually employed to test performance of metaheuristic methods (Liang et al. 2006), to provide a comparison of the implemented methods with the state-of-the-art of metaheuristic methods. Then the two solvers are applied to examples of energetic districts and the results of the tests performed are shown.

2. The District Model

The optimization problem to be solved arises form the district modelling described below, that has been designed in (Pannocchia and Mancuso 2014). Specifically, the user has to specify which devices the model is compound of, how they are connected to each other, choosing among different predefined thermal configurations for the heat/cold management, and specifying the characteristic parameters of the devices to simulate their real behaviour. The variables that need to be optimized are the physical parameters of the machine that affect their functioning. These parameters are scaled so that the quantities actually under control are dimensionless variables, called *set-points*. The aim of the optimization process is to individuate the optimal generation, load and energy storage profiles for each device. A plan of each machine set-points for the following day is built to minimize the expenses related to the district management, taking into account real time informations such as the price of the electricity (to sell or to buy), wind speed, solar radiation and ambient temperature. The time unit is set to be $\tau = 15$ minutes, so the day is divided into $N = \frac{24\times 60}{15} = 96$ quarters of an hour and the solver has to find the optimal set-points for each device and for each time unit. Let i be the time index, $i = 1 \dots N = 96$.

Particularly there are four different types of machines that can be included as part of an energy district: electrical generators, accumulators, electrical loads, thermal configurations, (Ferrara, Riccardi, and Sello 2014).

2.1 *Electrical generators and thermal configurations*

Thermal configurations are predefined configurations that can be found in an energy district for the heat/cold management, that might include CHPs (Combined Heat and Power), boilers, chillers. Let denote with N_{gen} the total number of generators and thermal configurations. For the k-th device $\alpha_k \in \mathbb{R}^N$ denotes the set-points vector, $k = 1\ldots, N_{gen}$. Its components $\alpha_k(i)$ are the fraction of power at i-th time unit $P_k(i)$, with respect to the rated output PN_k:

$$\alpha_k(i) = \frac{P_k(i)}{PN_k}, \qquad 0 \leq \alpha_k(i) \leq 1, \qquad i = 1, \ldots, N, \tag{1}$$

where $\alpha_k(i) = 0$ means that the device is off, $\alpha_k(i) = 1$ means that it works on maximum power. For fuel burning generators there is also an upper bound on the number of ignitions NI, that is a nonlinear function of α_k, so that the following nonlinear constraint arises:

$$NI(\alpha_k) \leq NI_{\max}. \tag{2}$$

2.2 *Electrical accumulators*

Let $\beta_b \in \mathbb{R}^N$ be the set-points vector for the b-th accumulator, $b = 1, \ldots, N_{acc}$ with N_{acc} the total number of accumulators in the district. Its components $\beta_b(i)$ are the set-points at the i-th time unit, $i = 1, \ldots, N$:

$$\beta_b(i) = \frac{\sum_{k=1}^{N_{dev}} \delta(k,b,i) PN_k}{PB_b}, \qquad -1 \leq \beta_b(i) \leq 1, \qquad i = 1, \ldots, N, \tag{3}$$

where $\delta(k, b, i)$ is a function of k, b, i, PN_k is the rated output of the k-th device, N_{dev} is the total number of devices, PB_b is the rated output of the b-th battery, (Ferrara, Riccardi, and Sello 2014). Negative values of $\beta_b(i)$ means that the accumulator is providing power, positive values of $\beta_b(i)$ means that the accumulator is gaining power. For each accumulator and for each time unit there are also two physical restrictions on the state of charge SOC_b, that is a nonlinear function of β_k:

$$SOC_{b\min} \leq SOC_b(i) \leq SOC_{b\max}, \qquad i = 1, \ldots, N. \tag{4}$$

2.3 *Electrical loads*

Three different types of electrical loads are considered: *L1 loads*, that are mandatory electrical consumptions, *L2 loads*, that are electrical cycles that need to be completed one or more times at no specific time in the day, *L3 loads*, that are normally on, and can be shut down for a limited amount of time without compromising the related process operation. L1 loads are given parameters for each time unit, so these loads do not have set-points associated and are not included in the count of total number of loads that is $N_{loads} = N_{L2} + N_{L3}$ where N_{L2} and N_{L3} are total numbers of L2 and L3 loads respectively. Let denote with γ_m the vector of set-points for m-th (L2 or L3) load. In this case set-points are not time dependent. What has to be under control is, for L2 loads, the starting times of cycles and, for L3 loads, switch-off and switch-on times, that are all integer variables. This would give rise to a mixed-integer nonlinear programming problem (MINLP). In order to avoid working with a MINLP, what is usually done in these applications, and also in Enel package (Pannocchia and Mancuso 2014), is to consider as set-points scalar continuous variables in $(0, 1]$ and then relate them to the integer quantities one actually wants to control. For L2 loads $\gamma_m \in \mathbb{R}^{N_m}$, where N_m is the number of cycles that need to be completed by m-th L2 load, and those set-points are then used to compute the starting time of l-th cycle $i_l = \lceil \gamma_m(l)N \rceil$, $l = 1, \dots, N_m$, so that $i_l \in \{1, 2, \dots, N\}$. For m-th L3 load the vector of set-points $\gamma_m \in \mathbb{R}^{2NI_m}$, with NI_m maximum number of ignitions. The odd components of γ_m are related to switch-off times s_l and the even ones to switch-on times a_l, for $l = 1, 2, \dots, NI_m$:

$$s_l = \lceil \gamma_m(2l-1)N \rceil, \qquad a_l = \lceil \gamma_m(2l)N \rceil.$$

On the loads set-points there are the following bound constraints:

$$\frac{1}{N} \leq \gamma_m(l) \leq 1,\ l = 1, \dots, N_m, \qquad \frac{1}{N} \leq \gamma_m(j) \leq 1,\ j = 1, \dots, 2NI_m, \tag{5}$$

and also some physical nonlinear constraints. For L2 loads they ensure that the time between the starting points of two successive cycles is enough to complete the first of them and that the time between the beginning of the last cycle and the end of the day is enough to complete the cycle. For L3 loads they guarantee that the shut down of the load precedes the turn on, that the load is not off for more than a given amount of time, and that if a load is shut down and then turned on, a suitable amount of time passes until it is turned down again, (Ferrara, Riccardi, and Sello 2014).

3. Arising Optimization Problem

The objective function f represents the overall daily cost of energy obtained as a result of the difference between purchase costs (fuel and electric energy) and incomings (incentives and sales revenues), (Pannocchia and Mancuso 2014). It is calculated as the sum of the overall district cost of each time instant $i = 1, \dots, N$:

$$f(x) = \sum_{i=1}^{N} \bar{f}_i(x)$$

where $\bar{f}_i(x)$ is the sum of all the partial cost functions of district devices for the i-th time unit:

$$\bar{f}_i(x) = \sum_{k=1}^{N_{gen}} f_{i,k}(x) + \sum_{b=1}^{N_{acc}} f_{i,b}(x) + \sum_{m=1}^{N_{loads}} f_{i,m}(x).$$

If n is the problem dimension, with

$$n = N[N_{gen} + N_{acc}] + \sum_{m=1}^{N_{L2}} N_m + \sum_{m=1}^{N_{L3}} NI_m,$$

then $f : \mathbb{R}^n \to \mathbb{R}$ is a nonlinear, non-convex function. As discussed in the previous section, a number of devices have physical constraints on their set-points (see equations (2), (4)). These process constraint can be stacked together obtaining a vector of constraints

$g : \mathbb{R}^n \to \mathbb{R}^p$, with p total number of constraints:

$$g(x) = \left[g_1(x), \dots, g_{N_{gen}}(x), \dots, g_{N_{gen}+N_{acc}}(x), , \dots, g_{N_{gen}+N_{acc}+N_{loads}}(x)\right]^T,$$

where $g_j(x)$ for $j = 1, \dots, N_{gen} + N_{acc} + N_{loads}$ is a vector containing the constraints on the j-th device. The resulting optimization problem is the following nonlinear constrained problem:

$$\min_x \ f(x), \tag{6a}$$

$$x_{\min} \le x \le x_{\max}, \tag{6b}$$

$$g(x) \le 0, \tag{6c}$$

where $x \in \mathbb{R}^n$ is the stacked vector of all devices set-points, $x_{\min} \in \mathbb{R}^n$ and $x_{\max} \in \mathbb{R}^n$ denote the bound constraints from (1), (3), (5), i.e.

$$\begin{aligned} x &= \left[\alpha_1, \dots, \alpha_{N_{gen}}, \beta_1, \dots, \beta_{N_{acc}}, \gamma_1, \dots, \gamma_{N_{loads}}\right]^T, \\ x_{\min} &= \left[0, \dots, 0, -1 \cdots -1, \tfrac{1}{N}, \dots, \tfrac{1}{N}\right]^T, \\ x_{\max} &= \left[1, \dots, 1, \dots 1,\right]^T. \end{aligned}$$

4. Sequential Linear Programming

Sequential Linear Programming (SLP) is an iterative method to find local minima of nonlinear constrained optimization problems, by solving a sequence of linear programming problems, (Robinson 1972; Byrd et al. 2003). At each iteration k, f and g are approximated in a neighbourhood of the current solution approximation x_k with first order Taylor series:

$$m_k(d) = f(x_k) + \nabla f(x_k)^T d \tag{7}$$

$$g_i(x_k) + \nabla g_i(x_k)^T d \le 0 \qquad i = 1, \dots, p, \tag{8}$$

where $d = x - x_k$. Moreover, to obtain a globally convergent method, a trust region strategy is employed (Conn, Gould, and Toint 2000). Then, a new constraint is added, that consists of a bound on the step-length of this form: $\|d\|_\infty \le \Delta_k$, where Δ_k is called the trust-region radius. The new constraint is added to the bound constraints, so that at each iteration the following problem is solved:

$$\min_d \ m_k(d), \tag{9a}$$

$$g_i(x_k) + \nabla g_i(x_k)^T d \le 0; \ \ i = 1, \dots, p, \tag{9b}$$

$$\max((x_{\min} - x_k)_j, -\Delta_k) \le d_j \le \min((x_{\max} - x_k)_j, \Delta_k), \ \ j = 1, \dots, n. \tag{9c}$$

At each iteration the computed solution d_k of (9) is used as a step to define the new solution approximation: $x_{k+1} = x_k + d_k$. Anyway, it is not possible to work directly with problem (9), as the linearized constraints could be inconsistent, i.e. it could be impossible to find a d for which (9b) holds, or the solution found d_k could be such that the new approximation $x_k + d_k$ does not satisfy the nonlinear constraints:

$$x_k + d_k \notin \Omega = \{x : \; x_{\min} \leq x \leq x_{\max}, \;\; g(x) \leq 0\}.$$

To deal with this, the constraints are usually added in the objective function as a penalty parameter, i.e. a new term is added to function f that is positive if the constraints are not satisfied, and is zero otherwise. The resulting new objective function is called *a penalty function.* This term can be chosen in many different ways, so that different penalty functions are obtained. Following (Fletcher and de la Maza 1989; Byrd et al. 2003) the l_1 penalty function was chosen and the following penalized objective function is obtained:

$$\Phi(x; \nu) = f(x) + \nu \sum_{i=1}^{p} \max(0, -g_i(x)) \tag{10}$$

where $\nu > 0$ is the penalty parameter. If ν is sufficiently large, i.e.

$$\nu \geq \nu^* = \max\{\lambda_i^*, i = 1, \dots, p\}, \tag{11}$$

where λ_i^*, $i = 1, \dots, p$ are the Lagrange multiplier of the inequality constraints $g(x) \leq 0$, it is possible to show that l_1 is an exact penalty function, (Nocedal and Wright 2006):

Definition 4.1 Exact Penalty Function *A penalty function $\Phi(x, \nu)$ is said to be exact if there exists a positive scalar ν^* such that for any $\nu \geq \nu^*$, any local solution of the nonlinear programming problem* (6) *is a local minimizer of $\Phi(x, \nu)$.*

Since it is necessary to take into account the bound constraints, the following problem is actually to be dealt with:

$$\min_x \Phi(x; \nu) = f(x) + \nu \sum_{i=1}^{p} \max(0, -g_i(x)) \tag{12a}$$

$$x_{\min} \leq x \leq x_{\max}. \tag{12b}$$

If (11) holds, local minimizers of (12) are equivalent to local solutions of (6), to a large extent (see (Fletcher 1987; Han and Mangasarian 1979; Janesch and Santos 1997) for details). The general scheme of the algorithm is the same as before, with the difference that instead of linearising both f and the constraints, function Φ is linearised and at each iteration a problem with just bound constraints is solved, that surely has a solution. As choosing the right value of ν a priori is difficult, a sequence of linearized penalized problems is actually solved, adjusting the penalty parameter during the course of the computation. Then, at iteration k given the current iterate x_k and the current penalty parameter ν_k, the following linear programming problem has to be solved:

$$\min_d l_k(d) := f(x_k) + \nabla f(x_k)^T d + \nu_k \sum_{i=1}^{p} \max(0, -g_i(x_k) - \nabla g_i(x_k)^T d); \tag{13a}$$

$$\max((x_{\min} - x_k)_j, -\Delta_k) \leq d_j \leq \min((x_{\max} - x_k)_j, \Delta_k), \; j = 1, \dots, n. \tag{13b}$$

After the solution d_k is found, it is necessary to decide whether to accept the step or not. The step acceptance is based on the agreement between the model function l_k and the objective function Φ, which is measured by the ratio between the *actual reduction* and the *predicted reduction* (Conn, Gould, and Toint 2000):

$$\rho_k = \frac{\Phi(x_k;\nu_k) - \Phi(x_k + d_k;\nu_k)}{l_k(0) - l_k(d_k)} = \frac{\Delta\Phi_k}{\Delta l_k}. \tag{14}$$

If $\rho_k > \rho_{bad}$, where ρ_{bad} is a tolerance to be fixed, typically $\rho_{bad} \in \left(0, \frac{1}{4}\right)$, the step is accepted. In this case the trust-region radius is left unchanged or it is possibly enlarged. Otherwise the step is rejected, the trust-region is shrink and (13) is solved again.

In Algorithm 1 it is sketched the k-th iteration of SLP method described above.

Algorithm 1 k-th iteration of SLP algorithm

```
(1) Given $x_k$, $\Delta_k$, $\Delta_{\max}$, $\nu_k$, $0 < \rho_{bad} < \rho_{good} < 1$.
(2) Evaluate $\nabla f(x_k)$ and $\nabla g_i(x_k)$ for $i = 1, \dots, p$.
(3) Solve the linear programming problem (13) obtaining a candidate step $d_k$.
(4) Let $\Phi(x;\nu) = f(x) + \nu \sum_{i=1}^{p} \max(0, -g_i(x))$, compute the step evaluation parameter $\rho_k$
    in (14).
    (a) If $\rho_k \le \rho_{bad}$, reduce the trust-region radius: $\Delta_{k+1} = \frac{1}{2}\Delta_k$, and go to step (5).
    (b) ElseIf $\rho_k \ge \rho_{good}$, and in addition $\|d_k\|_\infty \ge 0.8\Delta_k$, increase the trust-region
        radius
        $\Delta_{k+1} = \min(2\Delta_k, \Delta_{\max})$. Go to step (5).
    (c) Else set $\Delta_{k+1} = \Delta_k$ and go to step (5).
(5) If $\rho_k > \rho_{bad}$ accept the step, set: $x_{k+1} = x_k + d_k$ and choose $\nu_{k+1} > 0$.
    Otherwise reject the step: $x_{k+1} = x_k$. Set $\nu_{k+1} = \nu_k$.
```

It is possible to prove the global convergence of the sequence $\{x_k\}$ generated by Algorithm 1 to a stationary point of problem (12), as stated in the following theorem. See Appendix A for the proof.

Theorem 4.2 Global convergence of Algorithm 1
Let f and g be $\mathbb{C}^1$ functions and let $\{x_k\}$ be the sequence generated by Algorithm 1. Then, either there exists an iteration index $\bar{k}$ such that $x_{\bar{k}}$ is a stationary point for problem (12)*, or there exists a subsequence S of indexes such that $\{x_k\}_{k\in S}$ has an accumulation point x^* which is a stationary point for problem* (12)*.*

Theorem 4.2 states the existence of an accumulation point x^* of the sequence $\{x_k\}$ generated by Algorithm 1 that is a stationary point of problem (12), regardless of the starting point. In Section 4.1 it will be described the chosen penalty parameter update strategy, which ensures that the penalty parameter ν is large enough to let x^* be a solution of the original problem.

4.1 *Implementation issues*

In this section three important features of the algorithm are discussed: the solution of subproblems (13), the stopping criterion and the updating strategy for the penalty parameter. Function l_k in (13a) is non-differentiable, but problem (13) can be written as the following equivalent smooth linear programming problem, introducing the vector of slack variables t, (Byrd et al. 2003):

$$\min_{d,t} \nabla f(x_k)^T d + \nu_k \sum_{i\in\mathcal{I}} t_i \tag{15a}$$

$$g_i(x_k) + \nabla g_i(x_k)^T d \le t_i, \quad i = 1, \dots, p \tag{15b}$$

$$\max((x_{\min} - x_k)_j, -\Delta_k) \le d \le \min((x_{\max} - x_k)_j, \Delta_k)_j, \quad j = 1, \dots, n, \tag{15c}$$

$$t \ge 0. \tag{15d}$$

Then, the solution d_k of the above problem is sought, along with the Lagrange multipliers vectors λ_k, π_k and $\bar{\lambda}_k$ of constraints (15b), (15c) and (15d) respectively. These multipliers are employed to implement a reliable stopping criterion for Algorithm 1. Algorithm 1 is stopped whenever a pair (x_k, λ_k) satisfies the following conditions:

$$\max\{\|\nabla f(x_k) + \nabla g(x_k)^T \lambda_k\|_\infty, \|g(x_k)^T\lambda_k\|_\infty\} < \epsilon(1 + \|\lambda_k\|_2), \tag{16}$$

$$\max\{\max_{i\in\mathcal{I}}(0, g_i(x_k)), \max(0, x_{\min} - x_k), \max(0, x_k - x_{\max})\} < \epsilon(1 + \|x_k\|_2), \tag{17}$$

with ϵ a tolerance to be fixed. This stopping criterion provides us a measure of the closeness of the computed solution to a point satisfying first order optimality conditions for problem (6). See Theorem A.5 in Appendix A for a theoretical support for the employment of such criterion.

As far as the penalty parameter is concerned, it is desirable to have penalty function (10) to be exact, according to equation (11) and Definition 4.1. Then, at Step 5 of Algorithm 1, in case of successful iteration, the following updating strategy is adopted:

$$\text{If } \nu_k < \max\{\|\lambda_k\|_\infty, \|\bar{\lambda}_k\|_\infty\} \text{ then } \nu_{k+1} = \max\{\|\lambda_k\|_\infty, \|\bar{\lambda}_k\|_\infty\} \tag{18}$$

is set. Both in (16) and (18) current Lagrange multiplier estimates of the LP subproblems (15), provided by the function used to solve the LPs, are used as an approximation to those of the original problem.

5. Particle Swarm Optimization

Particle Swarm Optimization (PSO) is a stochastic evolutionary method designed to converge to a global minimum of a function f, (Kennedy 2011). It is inspired to the behaviour of bird swarms. Following the natural metaphor, PSO evolves a population of individuals, referred to as particles, within the search space, that behave according to simple rules and interact to produce a collective behaviour to pursuit a common aim, in this case the localization of a global minimum. The swarm is composed of s particles, each of them represents an approximation of the global minimum of the optimization problem and it is represented by a vector $x \in \mathbb{R}^n$. To each particle it is associated a velocity vector v too. The method is an iterative procedure where at each iteration k vectors x_k and v_k are updated as follows:

$$v^i_{k+1} = wv^i_k + c_1 r_1(p^i_{best,k} - x^i_k) + c_2 r_2(p^g_{best,k} - x^i_k) \tag{19}$$

$$x^i_{k+1} = x^i_k + v^i_{k+1} \tag{20}$$

where x^i_k and v^i_k are the position and velocity vector of the i-th particle, $i = 1, \dots, s$, at the k-th iteration, $p^i_{best,k}$ and $p^g_{best,k}$ are respectively the best position reached by the i-th particle so far and the best position reached by the whole swarm (the best position is the one that corresponds to the lowest value of the objective function), c_1, c_2 and w are positive weights, r_1 and r_2 are random variables with uniform distribution in $[0, 1]$, $r_1, r_2 \sim U(0, 1)$. At each iteration vectors $p^i_{best,k}$ and $p^g_{best,k}$ are updated too:

$$p^i_{best,k+1} = \begin{cases} x^i_k & \text{if } f(x^i_k) < f(p^i_{best,k}) \\ p^i_{best,k} & \text{otherwise} \end{cases} \qquad p^g_{best,k+1} = \begin{cases} p & \text{if } f(p) < f(p^g_{best,k}) \\ p^g_{best,k} & \text{otherwise} \end{cases} \tag{21}$$

where $p = \arg\min_{i=1\dots s} f(p^i_{best,k+1})$.

The solution approximation provided by the procedure is $p^g_{best,k*}$ where k^* is the last iteration index. Bound constraints are handled bringing back on the nearest boundary a particle x that has left the search space. It is also necessary to change the particle velocity, otherwise at the next iteration it is likely to have a new violation: $(v^i_k)_j = -r(v^i_k)_j$, where $(v^i_k)_j$ is the j-th component of vector v^i_k and r is a random variable uniformly distributed in $(0, 1)$. Originally PSO methods were developed to deal with problems with just bound constraints, and later they were employed to solve also constrained problems (Aziz et al. 2011; Parsopoulos, Vrahatis et al. 2002). See also (Mezura-Montes and Coello 2011; Jordehi 2015) for a more recent review on constraints handling strategies for PSO methods. Penalty function approaches are widely employed to make the method suitable to the solution of that kind of problems. Following (Michalewicz and Attia 1994; Michalewicz and Schoenauer 1996), the following quadratic penalty function is employed:

$$\Psi(x; \tau) = f(x) + \frac{1}{2\tau} \sum_{i=1}^{p} g_i(x)^2, \tag{22}$$

with τ penalty parameter that is decreased at each iteration to penalize with increasing severity constraints violations. Then, given a penalty parameter τ_k, function $\Psi(x; \tau_k)$ is used in (21) in place of the objective function $f(x)$. The k-th iteration of PSO procedure is sketched in Algorithm 2.

Algorithm 2 k-th iteration of PSO algorithm

```
(1) Given $c_1, c_2, w_{\min}, w_{\max}$, $k_{\max}$, $\tau_k$, $x_{\max}$, $x_{\min}$, $x^i_k$, $v^i_k$, $p^i_{best,k}$, $p^g_{best,k}$ for $i = 1, \dots, s$,
    perform the following steps.
(2) Compute $w_k$ in (24), $r_1, r_2 \sim U(0, 1)$ and evolve the swarm according to (19), (20).
(3) Handle the bound constraints: for $i = 1, \dots, s$ and $j = 1, \dots, n$
    • If $(x^i_{k+1})_j < (x_{\min})_j$ compute $r \sim U(0, 1)$ and set
          $(x^i_{k+1})_j = (x_{\min})_j,$
          $(v^i_{k+1})_j = -r(v^i_{k+1})_j,$
    • Elseif $(x^i_{k+1})_j > (x_{\max})_j$ compute $r \sim U(0, 1)$ and set
          $(x^i_{k+1})_j = (x_{\max})_j,$
          $(v^i_{k+1})_j = -r(v^i_{k+1})_j.$
```

(4) Evaluate the objective function (22) of each particle of the swarm and update vectors

$$p^i_{best,k+1} = \begin{cases} x^i_k & \text{if } \Psi(x^i_k;\tau_k) < \Psi(p^i_{best,k};\tau_k) \\ p^i_{best,k} & \text{otherwise} \end{cases}$$

$$p^g_{best,k+1} = \begin{cases} p & \text{if } \Psi(p;\tau_k) < \Psi(p^g_{best,k};\tau_k) \\ p^g_{best,k} & \text{otherwise} \end{cases}$$

where $p = \arg\min_{i=1\dots s} \Psi(p^i_{best,k+1};\tau_k)$ and choose $\tau_{k+1} < \tau_k$.

In the implemented PSO method a new strategy to help the swarm escape from local minima is proposed. When the swarm appears to be stuck, i.e. when after a certain number of iterations the value of the objective function is not decreased, the velocity updating scheme (19) is modified adding a new term in the equation, that is proportional to the distance of the particle best position from the global best position:

$$v^i_{k+1} = wv^i_k + c_1r_1(p^i_{best,k} - x^i_k) + c_2r_2(p^g_{best,k} - x^i_k) + c_3r_3(p^g_{best} - p^i_{best}), \tag{23}$$

where c_3 is a weighting parameter to be set, and $r_3 \sim U(0,1)$. When a new improvement in the objective function is obtained, coefficient c_3 is set back to zero and the standard update is employed. Numerical experiments have shown that in some tests the addition of this term helps the swarm escape from a stalemate, allowing it to find a better solution. In the following we will address this PSO variant as PSOc_3.

The main advantage of PSO methods is that they do not require neither regularity assumptions on the objective function nor to compute the first derivatives and are so suitable when few information on the objective function are available. Clearly the fact that few information on f are used, leads to a slow method that requires many iterations to converge. However, the method could be efficiently implemented on a parallel architecture.

These methods are heuristic and standard convergence results, like those proved for exact optimization methods, are not usually provided. However, a different kind on analysis of the algorithm can be performed. Using results from the dynamic system theory, it is possible to provide an understanding about how the swarm searches the problem space through the analysis of a single particle trajectory or of the the swarm seen as a stochastic system, (Trelea 2003; Clerc and Kennedy 2002). The analysis provides useful guidelines for the choice of the free parameters, to control the system's convergence tendencies.

5.1 *Implementation issues*

The choices regarding the method implementation are made taking into account that the problems under consideration have hundreds of variables, an objective function that has not a simple analytic form and that is expensive to evaluate, and that the algorithm must return results quickly to be useful in practise.

Regarding the number of particles, it is impossible to use wide swarms because each particle of the swarm requires two function evaluations, the objective and the constraints functions, at each iteration. On the other hand using few particles means low exploration capability. After several numerical tests, it was found that a swarm of 20 particles represents a good compromise between solution quality and execution time. For parameter w in (19), instead of using a fixed value, a linearly decreasing scheme for is adopted, (Shi and Eberhart 1998):

$$w_k = w_{\max} - (w_{\max} - w_{\min})\frac{k}{k_{\max}}, \tag{24}$$

where $k_{\max}$ is the maximum number of allowed iterations. With this choice convergence velocity is slowed down ensuring a more accurate exploration of the search space. The process is stopped when a maximum number of iterations $k_{\max}$ is performed or when there are no improvements over a fixed number of iterations κ. An improvement is measured in terms of a decrease in the objective function, and it is judged not to be sufficient when the following condition is satisfied for κ consecutive iterations and for ϵ a tolerance to be fixed:

$$\frac{|f(x_{k-2}) - f(x_k)|}{|f(x_{k-2})|} < \epsilon \tag{25}$$

6. Numerical Tests

Both solvers were implemented in MATLAB and in this section results of numerical tests are shown.

First in Section 6.1 a set of benchmark functions commonly used to test performance of metaheuristic methods is considered. Then in Section 6.2 13 different problems arising from the described industrial application and corresponding to models of synthetic energy districts are taken into account, plus one arising from the modelling of an existing district in Pisa, provided by Enel centre. To solve these problems, the two solvers were inserted in the software tool previously developed by Enel centre and University of Pisa.

The numerical experimentation in Section 6.1 is performed on an Intel(R) Core(TM) i7-4510U 2.00GHz, 16 GB RAM; the one in Section 6.2 on an Intel(R)Xeon(R) CPU E5430, 2.66 GHz, 8.00 GB RAM. In both cases MATLAB R2016a is used, the machine precision is $\epsilon_m \sim 2 \cdot 10^{-16}$.

Here the values of the free coefficients used in the procedures are specified. For PSOc_3 in (19) $c_1 = 1.3$ and $c_2 = 2.8$, in (23) $c_3 = 1$, in (24) $w_{\max} = 0.6$, $w_{\min} = 0.1$, the tolerance for the stopping criterion (25) is $\epsilon = 10^{-3}$, $\kappa = 20$ and $k_{\max} = 1700$. The swarm size is 100 in Section 6.1 and 20 in Section 6.2. This choice is motivated by the fact that in energy districts test cases the objective function is expensive to evaluate and the use of wider swarms would lead to prohibitive computational costs. On the other hand in test problems taken from the literature this is not the case, so a higher number of particle is employed to allow a better exploration of the search space. Moreover for the test problems in Section 6.1 it was found harder to satisfy the constraints than in the tests of energy districts. Then for both PSO and SLP algorithm constraints violations are penalized with higher severity in Section 6.1 than in Section 6.2. Then, in the PSO constraints handling strategy (22) $\tau_{k+1} = (1 - 0.01)\tau_k$ and $\tau_0 = 0{,}0000001$ in Section 6.1 while $\tau_0 = 0.1$ in Section 6.2. For SLP in (10) it was set $\nu_0 = 5$ in Section 6.1 and $\nu_0 = 1$ in Section 6.2. In Algorithm 1 $\rho_{bad} = 0.10$ and $\rho_{good} = 0.75$, and the linear subproblems were solved using the Matlab function `linprog`(Interior Point Algorithm) with default choice parameters. Notice that the structure of the subproblems is not taken into consideration and a special purpose code could lead to a more efficient solution of the subproblems, but this is out of the scope of this article.

Results shown in the following tables are the average of those obtained over 25 runs in Section 6.1 and 100 runs in Section 6.2, varying the starting point for SLP solver. In the tables for each one of the test cases the following statistics are reported: $\boldsymbol{f}$ and $\boldsymbol{k}$

arithmetic mean of the function values and the number of iterations, $\boldsymbol{\sigma_f}$ the standard deviation on function values, **min *f*** and **max *f*** minimum and maximum function values obtained, **time(·)** total time in seconds or minutes.

6.1 *Comparison with the state-of-the-art*

In this section the performance of the proposed methods is evaluated on the set of CNOP (constrained nonlinear optimization problems) called G-suite, proposed in (Liang et al. 2006) for the competition on constrained optimization of the Congress on Evolutionary Computation in 2006 (CEC'2006). This set was chosen since it provides a challenging set of functions and it makes possible to compare the proposed methods to the state-of-the-art. Indeed, a large part of papers dealing with evolutionary methods refers to this set of functions, see for example (Das and Suganthan 2011; de-los Cobos-Silva et al. 2016; Liu et al. 2016; Mezura-Montes and Lopez-Ramirez 2007). The G-suite is composed of 24 CNOP. Among them, those subject just to inequality constraints were selected, as the proposed methods are not designed to handle equality constraints. Table 1 sums up the relevant features of each test function. In the heading, n is the number of variables, $\rho = |F|/|S|$ is the estimated ratio between the feasible region and the search space, LI is the number of linear inequality constraints and NI is the number of nonlinear inequality constraints.

Table 1.: Details of the test problems.

Prob.	n	**Type**	ρ	**LI**	**NI**
g01	13	quadratic	0.0111%	9	0
g02	20	nonlinear	99.9971%	0	2
g04	5	quadratic	52.1230%	0	6
g06	2	cubic	0.0066%	0	2
g07	10	quadratic	0.0003%	3	5
g08	2	nonlinear	0.8560%	0	2
g09	7	polynomial	0.5121%	0	4
g10	8	linear	0.0010%	3	3
g12	3	quadratic	4.7713%	0	1
g16	5	nonlinear	0.0204%	4	34
g18	9	quadratic	0.0000%	0	13
g19	15	nonlinear	33.4761%	0	5
g24	2	linear	76.6556%	0	2

For each test function 25 runs were performed, as required by the CEC'2006 Special Session. The results are reported in Table 2, in the last column **feas** is the percentage of runs in which a feasible solution is found. Noticeably, PSOc_3 always finds feasible solutions in all runs, while SLP method is not so robust from this point of view, in 4 out of 13 tests (g07, g08, g09, g10) a feasible solution is found in 75%-90% of the runs and especially in instance g06 a feasible solution is never found. The results reported refers just to the feasible runs. In 9 out of 13 instances PSOc_3 manages to find the best known solution (g01, g04, g06, g07, g08, g09, g12, g16, g24). In two instances (g18, g19) it finds the best values reported in (Liang et al. 2006), but recently even better values have been found in (de-los Cobos-Silva et al. 2016). In the remaining two instances values close to the minimum are found, but the global minimum is not reached. Regarding SLP method, in 6 instances it finds the global minimum (g01, g04, g06, g07, g09, g24). In instances g08 and g12 the best value found is close to it. In instances g02 and g10 its performance is worst than that of PSOc_3 and as PSOc_3 in instances g18, g19 SLP finds the best values reported in (Liang et al. 2006), but not the ones found in (de-los Cobos-Silva et al. 2016). Noticeably, in many runs the standard deviation is really low.

Table 2.: Tests on 13 test functions taken from the G-suit of CEC'2006. Comparison of PSOc_3 and SLP solvers.

Test	**Solver**	**f**	$\sigma_{\mathbf{f}}$	max **f**	min **f**	**k**	**time(s)**	**feas**
g01	PSOc_3	-14.54	0.94	-12.00	-15.00	130	3.31	100%
	SLP	-14.28	0.76	-12.66	-15.00	68.75	1.62	100%
g02	PSOc_3	-0.60	0.05	-0.49	-0.69	130	3.36	100%
	SLP	-0.19	0.06	-0.076	-0.27	17.65	2.10	100%
g04	PSOc_3	-30665	0.10	-30665	-30666	130	2.80	100%
	SLP	-30666	1.18e-08	-30666	-30666	300	6.67	100%
g06	PSOc_3	-6951.6	7.32	-6935	-6961.1	130	2.72	100%
	SLP	-6961.8	1.70e-05	-6961.8	-6961.8	200	3.90	100%
g07	PSOc_3	25.20	0.70	26.56	24.35	1322.9	33.50	100%
	SLP	24.55	0.9784	28.22	24.31	47.75	1.40	80%
g08	PSOc_3	-0.0958	3.18e-17	-0.0958	-0.0958	180	5.08	100%
	SLP	-0.0120	0.03	0.0460	-0.0860	28.20	0.68	90%
g09	PSOc_3	681.28	0.39	682.63	681.03	636.30	17.26	100%
	SLP	680.63	8.86e-09	680.63	680.63	62.11	1.80	90%
g10	PSOc_3	7760.8	775.93	10607	7241.6	1520.6	33.53	100%
	SLP	14237	40400	22300	8617	15.6	1.34	75%
g12	PSOc_3	-1	0	-1	-1	180	7.32	100%
	SLP	-0.799	0.11	-0.5540	-0.9864	20.65	0.62	100%
g16	PSOc_3	-1.90	1.5e-3	-1.8997	-1.9041	239.95	8.82	100%
	SLP	-	-	-	-	-	-	0%
g18	PSOc_3	-0.73	0.13	-0.49	-0.8577	180	4.52	100%
	SLP	-0.84	0.07	-0.6750	-0.8660	43.45	1.25	100%
g19	PSOc_3	40.71	5.89	57.64	34.44	1229.1	35.08	100%
	SLP	32.66	3.42e-09	32.66	32.66	52.35	1.46	100%
g24	PSOc_3	-5.5080	5.24e-09	-5.5080	-5.5080	180	4.81	100%
	SLP	-4.28	1.08	-2.23	-5.5080	4.35	0.07	100%

As suggested for example in (Derrac et al. 2011; García et al. 2009) a nonparametric analysis of the results was performed. First, the Wilcoxon test was used to compare the methods performance, based on the mean values found, through the Matlab function `signrank`. Wilcoxon test is a nonparametric test that is used to determine whether two independent samples were selected from populations having the same distribution, therefore it can be employed to detect significant differences between the behaviour of two algorithms. The null hypothesis is that the difference between two sample means is zero. It emerged that the null hypothesis of equivalence of the two algorithm cannot be rejected at significance level $\alpha = 0.05$. Denoting with R^+ the sum of ranks for the problems in which SLP outperforms PSOc_3, and R^- the sum of ranks for the opposite, $R^+ = 33$ and $R^- = 58$ are obtained. Anyway it is worth remembering that the test is performed without taking into account that SLP solver doesn't find a feasible solution in all runs.

We can conclude that both solvers find good results, both in terms of mean values and in terms of runtime. PSO method manages to find a solution approximation requiring a really low number of function evaluations, that is about 150000 for three test cases (g07, g10, g19) but is much lower for the others: less than 50000 for g09 and g16 and less than 20000 for all the others test cases. SLP requires considerably less function evaluations despite n extra function evaluations for iteration are needed to approximate the gradient. Indeed, the maximum number of f-evaluations required is 1800 for g04. As problems dimensions are really small and a quite large number of particles is used for PSO method, the execution time for SLP is lower than for PSO.

We compared the proposed methods with seven hybrid procedures introduced in de-los Cobos-Silva et al. (2016). These procedures, namely MMC-DE, MMC-DE-SC, MMC-DE-interleaved, MMC-DE-SC-interleaved, MMC-DE-batch, MMC-DE-SC-batch and PSO-3P-SC, are based on metaheuristic approaches. The comparison was carried out through pairwise comparison by the Wilcoxon test, using the statistics provided in (de-los Cobos-Silva et al. 2016). It emerges that the null hypothesis of equivalence should be rejected for the comparisons of PSOc_3 with MMC-DE-SC-interleaved, PSO-3P-SC at significance level $\alpha = 0.01$, and with MMC-DE at significance level $\alpha = 0.03$. PSOc_3 indeed shows an improvement over these methods, getting respectively $R^+ = 91, 78, 67$, where in this case R^+ is the sum of ranks for the problems in which PSOc_3 outperforms the method it is compared with. On the other hand SLP method results to be equivalent to all the hybrids, except for MMC-DE-SC-interleaved for which SLP shows an improvement at significance level $\alpha = 0.01$ and $R^+ = 77$, with R^+ the sum of ranks for the problems in which SLP outperforms MMC-DE-SC-interleaved. Then a Friedman test was performed to compare PSOc_3, PSO-3P-SC, MMC-DE, MMC-DE-SC-interleaved and SLP, followed by a post-hoc analysis through the Matlab function `multcompare`. This function, using the statistics provided by the Friedman test, performs $N \times N$ multiple comparisons, where N is the number of algorithms to be tested, and estimates the difference of average ranks (MathWorks 2017; King and Mody 2010). The result of the test is a matrix of multiple comparison results, returned as an p-by-6 matrix of scalar values, where p is the number of pairs. Each row of the matrix contains the result of one paired comparison test. The matrix obtained in the test is shown in Table 3. **Method 1** and **Method 2** denote the methods being compared, **lb**, **diff** and **ub** denotes respectively the lower confidence interval, the estimate, and the upper confidence interval, **p-value** is the p-value for the hypothesis test that the corresponding mean difference is not equal to 0. Then, in each row the numbers indicate that the mean of Method 1 minus the mean of Method 2 is estimated to be **diff**, and a 95% confidence interval for the true difference of the means is [**lb**, **ub**]. If the confidence interval contains 0, the difference is significant at the 5% significance level, otherwise it is not. The post-hoc analysis confirms all the differences detected by the Wilcoxon test, except the one between PSOc_3 and MMC-DE. It emerged indeed that PSOc_3 and PSO-3P-SC (p=0.0178), PSOc_3 and MMC-DE-SC-interleaved (p=0.001) and MMC-DE-SC-interleaved and SLP (p=0.0221) have mean ranks significantly different, as it is shown in Figure 1. In the figure, estimates and comparison intervals are shown for each method. Each method mean is represented by the symbol ”o”, and the interval is represented by a line extending out from the symbol. Two methods means are significantly different if their intervals are disjoint, while they are not significantly different if their intervals overlap, (MathWorks 2017; King and Mody 2010).

Table 3.: Matrix of multiple comparisons from Matlab function `multcompare`.

Method 1	Method 2	lb	diff	ub	p-value
PSOc_3	PSO-3P-SC	-3.5301	-1.8846	-0.2391	**0.0154**
PSOc_3	MMC-DE	-2.8378	-1.1923	0.4532	0.2775
PSOc_3	MMC-DE-SC-inter	-4.2994	-2.6538	-1.0083	**0.0001**
PSOc_3	SLP	-2.4532	-0.8077	0.8378	0.6668
PSO-3P-SC	MMC-DE	-0.9532	0.6923	2.3378	0.7810
PSO-3P-SC	MMC-DE-SC-inter	-2.4147	-0.7692	0.8763	0.7066
PSO-3P-SC	5.0000	-0.5686	1.0769	2.7224	0.3822
MMC-DE	MMC-DE-SC-inter	-3.1071	-1.4615	0.1840	0.1092
MMC-DE	5.0000	-1.2609	0.3846	2.0301	0.9689
MMC-DE-SC-inter	5.0000	0.2006	1.8462	3.4917	**0.0188**

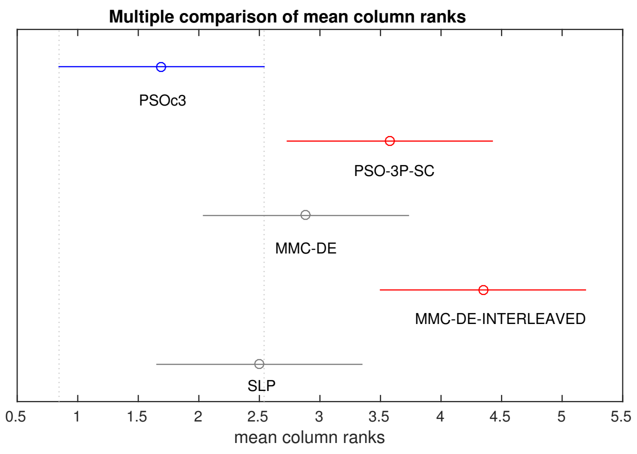


Figure 1.: Post-hoc analysis through Matlab function `multcompare`.

6.2 *Examples of synthetic energy districts*

In this section the results gained by the two methods when applied to 13 synthetic test examples of energy districts are presented. The arising optimization problems have different dimensions, the number of variables is comprised between 294 and 494, the number of bound constraints between 588 and 796 and the number of process constraints is 10 for the first test and 213 for the others. The results of the tests performed are reported in Table 4.

At first glance, from these results it is possible to deduce the following remarks. As a consequence of the fact that SLP solver is a first order method, its convergence rate is much higher then that of PSO method, that requires an high number of iteration to allow the swarm to carefully explore the search space. In term of computational time an iteration of PSO method is cheaper than one of SLP, that requires the solution of a Linear Programming problem and the computation of some derivatives by finite differences, which is computationally expensive. Despite this, the total time required by SLP is really lower than that required by PSO method, as SLP performs far less iterations. As expected SLP is more suitable for real time optimization.

Regarding function values it can be observed that on many tests PSO presents higher means compared to SLP, but generally the values on different runs are close to the mean value, as the standard deviation is really low. On the other hand SLP provides lower means but high standard deviation, which means both higher worst values and lower best values. Then, the probability of finding a worst result on a single run using SLP

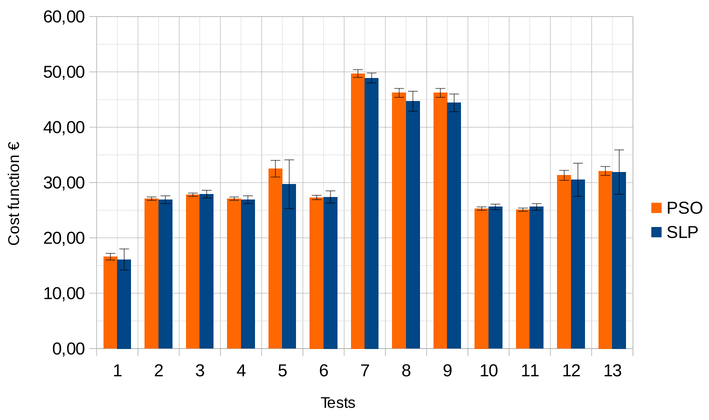
Table 4.: Tests on 13 synthetic examples of energy districts and on Pisa district (last row), comparison of PSOc_3 and SLP solvers.

Problem	**Solver**	f	σ_f	$\max f$	$\min f$	k	**time(m)**
Test 1	PSOc_3	16.6	0.6	18.3	15.4	1207	1.8
	SLP	16.1	1.9	25.8	15.4	25.4	0.06
Test 2	PSOc_3	27.1	0.3	27.7	26.4	1585	13
	SLP	26.9	0.7	29.3	25.8	78.9	1.0
Test 3	PSOc_3	27.8	0.3	28.5	27.1	1554	14
	SLP	27.9	0.7	30.7	26.8	84	1.3
Test 4	PSOc_3	27.1	0.3	27.8	26.5	1583	11
	SLP	26.9	0.7	29.2	26.1	82.5	0.5
Test 5	PSOc_3	32.5	1.5	36.6	29.2	1493	14
	SLP	29.7	4.4	48.7	26.7	78.8	1.0
Test 6	PSOc_3	27.3	0.4	28.6	26.7	1588	11
	SLP	27.4	1.1	31.2	26.2	83.8	0.5
Test 7	PSOc_3	49.7	0.7	52.2	48.2	1509	10
	SLP	48.9	0.9	53.1	46.9	76.3	0.4
Test 8	PSOc_3	46.2	0.8	49.2	44.3	1531	10
	SLP	44.7	1.8	51.9	42.9	78.9	0.4
Test 9	PSOc_3	46.2	0.8	48.6	44.5	1543	15
	SLP	44.4	1.6	51.3	42.9	79.6	0.4
Test 10	PSOc_3	25.3	0.3	26.1	24.7	1484	14
	SLP	25.6	0.5	28.2	24.9	73.0	1.0
Test 11	PSOc_3	25.1	0.3	26.3	24.5	1406	14
	SLP	25.6	0.6	28.4	24.8	70.3	1.0
Test 12	PSOc_3	31.3	0.9	34.7	29.3	1379	12
	SLP	30.5	3.0	45.4	29.0	64.4	0.6
Test 13	PSOc_3	32.1	0.8	35.1	30.4	1385	11
	SLP	31.9	4.0	53.5	30.0	64.8	0.7
Pisa	PSOc_3	73.4	0.03	73.6	73.4	377	2.7
	SLP	74.0	0.7	80.6	73.4	23	0.08

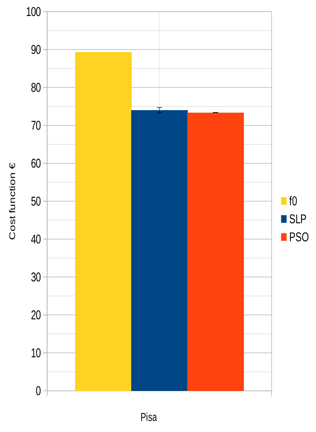
is high, while for PSO is more likely to have a result close to the mean value. This is highlighted in Figure 2 (a), where for each test case the mean values together with the standard deviation for the two solvers are compared. The left bars refer to PSOc_3 method and the right ones to SLP method, the standard deviation is highlighted by black vertical lines. Note that overall PSO provides good results, even if a small number of particles is employed. Its performance could be improved increasing the swarm size, to better explore the search space. The difference between PSO and SLP is indeed especially evident in Test 5, which is the one in which the search space has highest dimension. However, from the numerical experience it emerged that increasing the number of particles would be prohibitive from a computational point of view.

Then, a deeper statistical analysis of the results was performed. First it was checked if the results obtained in the tests satisfy the conditions for the use of a parametric test, (García et al. 2009; Derrac et al. 2011). They were tested with the Kolmogorov-Smirnov test, through the Matlab function `kstest`, and it emerged that the data are not normally distributed, meaning that a non-parametric analysis would be more meaningful. Therefore the data were analysed trough the Wilcoxon signed ranks test (Derrac et al. 2011). From the test it emerges that the data produce an evidence that is sufficient to reject the null hypothesis, according to significance level $\alpha = 0.05$. If we denote with R^+

the sum of ranks for the problems in which SLP outperforms PSOc_3, and R^- the sum of ranks for the opposite, we got $R^+ = 84, R^- = 21$. Then, the statistical tests confirm that SLP shows an improvement over PSO on these problems, if the mean values are considered.



(a)



(b)

Figure 2.: (a) Comparison of objective function values obtained by PSOc_3 (left bars) and SLP (right bars) on the 13 tests cases; (b) comparison of actual management ($f_0 = 89.3$, left bar) and optimized management provided by SLP (central bar) and PSOc_3 (right bar), on the test case of Pisa district.

In the next section we show results of the tests performed on a real energy district.

6.2.1 Pisa District

This is a real district in Pisa provided by the research centre 'Enel Ingegneria e Ricerca', that comprises: a CHP characterized by rated power 25 kWe and rated thermal power 75 kWt, a gas boiler with rated thermal power 35 kWt, a tank for the storage of hot water with capacity 9400 kj/ °C, a photovoltaic generator with rated power 14 kWe, a wind farm with rated power 3 kWe, 2 loads. The arising optimization problem has 288 variables, 576 bound constraints and 1 physical constraint. The detailed results of the optimization process are reported in the last row of Table 4.

This test is much less constrained than those presented in the previous section, it has just one mild nonlinear constraint, and the dimensionality of the search space is also lower. In this case PSO algorithm performs better than SLP algorithm, providing a lower mean value of the objective function and also a really smaller standard deviation. Notice that in this case also for PSOc_3 algorithm the execution time is quite reasonable.

This test case is particularly interesting, because data referring to the cost of unoptimized management of local resources, i.e. the management that is actually running the district, are available. Then, it is possible to evaluate savings arising from the optimized management provided by the proposed procedures. The value of the unoptimized objective function is $f_0 = 89.3$, as it is depicted in Figure 2 (b), left bar. Comparing this cost to that provided by the optimized management (right bars, referring to SLP and PSO respectively), it is possible to see that the proposed approach provides considerable saving, about 18% daily saving.

7. Conclusions

This article deals with the numerical solution of optimization problems arising in energy districts. A Particle Swarm Optimization solver and a Sequential Linear Programming solver were implemented, to investigate if the gain in the objective function given by the evolutionary method is worth the longer computational time. The solvers were tested both on constrained tests taken from the literature, and on many different examples of synthetic energy districts and on a real one. From this study it emerges that the two solvers can compare with other methods proposed in the literature, and that in districts test cases the use of PSO gains values less distributed around the mean, but lower means are generally provided by SLP method. Noticeably, from the test performed on a real energetic district it arises that the optimized management of resources gained by the optimization package provides considerable savings in the energy bill.

Acknowledgements

We thank G. Pannocchia and G.M. Mancuso, Dipartimento di Ingegneria Civile e Industriale dell' Università di Pisa, for providing us the MATLAB code of the software tool for the test problems.
Work partially supported by INdAM-GNCS.

References

Aragón, V.S., S.C. Esquivel, and C.A.C. Coello. 2010. "A Modified Version of a T-Cell Algorithm for Constrained Optimization Problems." *International Journal for Numerical Methods in Engineering* 84 (3): 351–378.

Ascione, F., N. Bianco, C. De Stasio, G. M. Mauro, and G.P. Vanoli. 2016. "Simulation-Based Model Predictive Control by the Multi-Objective Optimization of Building Energy Performance and Thermal Comfort." *Energy and Buildings* 111: 131–144.

Aziz, N.A.A., M.Y. Alias, A.W. Mohemmed, and K.A. Aziz. 2011. "Particle Swarm Optimization for Constrained and Multiobjective Problems: a Brief Review." In *Proc IPEDR: Int Conf Manage Artif Intell,* Vol. 6146–2.

Bertsekas, D.P. 1999. *Nonlinear Programming.* Athena scientific Belmont.

Byrd, R.H., N.I.M. Gould, J. Nocedal, and R.A. Waltz. 2003. "An Algorithm for Nonlinear Optimization Using Linear Programming and Equality Constrained Subproblems." *Mathematical Programming* 100 (1): 27–48.

Cagnina, L.C., S.C. Esquivel, and C.A.C. Coello. 2011. "Solving Constrained Optimization Problems with a Hybrid Particle Swarm Optimization Algorithm." *Engineering Optimization* 43 (8): 843–866.

Clerc, M., and J. Kennedy. 2002. "The Particle Swarm - Explosion, Stability, and Convergence in a Multidimensional Complex Space." *IEEE Transactions on Evolutionary Computation* 6 (1): 58–73.

Conn, A.R., N.I.M. Gould, and Ph.L. Toint. 2000. *Trust Region Methods.* Vol. 1. Siam.

Das, S., and P.N. Suganthan. 2011. "Differential Evolution: A Survey of the State-of-the-Art." *IEEE Transactions on Evolutionary Computation* 15 (1): 4–31.

de-los Cobos-Silva, Sergio Gerardo, Roman Anselmo Mora-Gutiérrez, Miguel Angel Gutiérrez-Andrade, Eric Alfredo Rincón-García, Antonin Ponsich, and Pedro Lara-Velázquez. 2016. "Development of Seven Hybrid Methods Based on Collective Intelligence for Solving Nonlinear Constrained Optimization Problems." *Artificial Intelligence Review* 1–35.

Derrac, J., S. García, D. Molina, and F. Herrera. 2011. "A Practical Tutorial on the Use of Nonparametric Statistical Tests as a Methodology for Comparing Evolutionary and Swarm Intelligence Algorithms." *Swarm and Evolutionary Computation* 1 (1): 3–18.

Ferrara, W., J. Riccardi, and S. Sello. 2014. "Enel Customer Optimization Service Expert System Development." In *Technical Report,* Enel Ingegneria e Ricerca, Pisa.

Fletcher, R. 1987. "Practical Methods of Optimization." *Wiley-Interscience* .

Fletcher, R., and E.S. de la Maza. 1989. "Nonlinear Programming and Nonsmooth Optimization by Successive Linear Programming." *Mathematical Programming* 43 (1-3): 235–256.

García, S., D. Molina, M. Lozano, and F. Herrera. 2009. "A Study on the Use of Non-Parametric Tests for Analyzing the Evolutionary Algorithms Behaviour: a Case Study on the CEC2005 Special Session on Real Parameter Optimization." *Journal of Heuristics* 15 (6): 617–644.

Glover, F.W., and G.A. Kochenberger. 2006. *Handbook of Metaheuristics.* Vol. 57. International Series in Operations Research and Management Science.

Han, S.P., and O. . Mangasarian. 1979. "Exact Penalty Functions in Nonlinear Programming." *Mathematical programming* 17 (1): 251–269.

Hu, X., and R. Eberhart. 2002. "Solving Constrained Nonlinear Optimization Problems with Particle Swarm Optimization." In *6th world multiconference on systemics, cybernetics and informatics,* 203–206.

Janesch, S.M.H., and L.T. Santos. 1997. "Exact Penalty Methods with Constrained Subproblems." *Investigacion Operativa* 7 (1-2): 55–65.

Jordehi, A.R. 2015. "A Review on Constraint Handling Strategies in Particle Swarm Optimisation." *Neural Computing and Applications* 26 (6): 1265–1275.

Kennedy, J. 2011. "Particle Swarm Optimization." In *Encyclopedia of Machine Learning,* 760–766. Springer.

King, Michael R, and Nipa A Mody. 2010. *Numerical and statistical methods for bioengineering: applications in MATLAB.* Cambridge University Press.

Liang, J.J., T. P. Runarsson, E. Mezura-Montes, M. Clerc, P.N. Suganthan, C.A. Coello, and K. Deb. 2006. "Problem Definitions and Evaluation Criteria for the CEC 2006 Special Session on Constrained Real-Parameter Optimization." *Journal of Applied Mechanics* 41 (8).

Liu, H., Z. Cai, and Y. Wang. 2010. "Hybridizing Particle Swarm Optimization with Differential Evolution for Constrained Numerical and Engineering Optimization." *Applied Soft Computing* 10 (2): 629–640.

Liu, J., K. L. Teo, X. Wang, and C. Wu. 2016. "An Exact Penalty Function-Based Differential Search Algorithm for Constrained Global Optimization." *Soft Computing* 20 (4): 1305–1313.

Lu, H., and W. Chen. 2008. "Self-Adaptive Velocity Particle Swarm Optimization for Solving Constrained Optimization Problems." *Journal of Global Optimization* 41 (3): 427–445.

MathWorks. 2017. "Multiple Comparison Test." https://it.mathworks.com/help/stats/multcompare.html.

Mazhoud, I., K. Hadj-Hamou, J. Bigeon, and P. Joyeux. 2013. "Particle Swarm Optimization for Solving Engineering Problems: A New Constraint-Handling Mechanism." *Engineering Applications of Artificial Intelligence* 26 (4): 1263 – 1273.

Mezura-Montes, E., and C.A.C. Coello. 2011. "Constraint-Handling in Nature-Inspired Numerical Optimization: Past, Present and Future." *Swarm and Evolutionary Computation* 1 (4): 173–194.

Mezura-Montes, E., and B.C. Lopez-Ramirez. 2007. "Comparing Bio-Inspired Algorithms in Constrained Optimization Problems." In *2007 IEEE Congress on Evolutionary Computation,* 662–669. Sept.

Mezura-Montes, Efrén, Mariana Edith Miranda-Varela, and Rubí del Carmen Gómez-Ramón. 2010. "Differential Evolution in Constrained Numerical Optimization: An Empirical Study." *Inf. Sci.* 180 (22): 4223–4262.

Michalewicz, Z., and N. Attia. 1994. "Evolutionary Optimization of Constrained Problems." In *Proceedings of the 3rd annual conference on evolutionary programming,* 98–108.

Michalewicz, Z., and M. Schoenauer. 1996. "Evolutionary Algorithms for Constrained Parameter Optimization Problems." *Evolutionary Computation* 4 (1): 1–32.

Nocedal, J., and S. Wright. 2006. *Numerical Optimization.* Springer Science & Business Media.

Pannocchia, G., and G.M. Mancuso. 2014. "Steady State Optimizer of Energy Districts." In *Technical Report,* Dipartimento di Ingegneria Civile e Industriale, Pisa.

Parsopoulos, K.E., M.N. Vrahatis, et al. 2002. "Particle Swarm Optimization Method for Constrained Optimization Problems." *Intelligent Technologies–Theory and Application: New Trends in Intelligent Technologies* 76 (1): 214–220.

Raidl, G.R. 2006. *A Unified View on Hybrid Metaheuristics.* 1–12. Hybrid Metaheuristics: Third International Workshop, HM 2006 Gran Canaria, Spain, October 13-14, 2006 Proceedings: Springer Berlin Heidelberg.

Riccietti, E., S. Bellavia, and S. Sello. 2017. "Numerical Methods for Optimization Problems Arising in Energetic Districts." In *Proceedings of ECMI 2016, Mathematics in Industry,* edited by Springer.

Robinson, S.M. 1972. "A Quadratically-Convergent Algorithm for General Nonlinear Programming Problems." *Mathematical Programming* 3 (1): 145–156.

Robles, V., J.M. Peña, P. Larrañaga, M.S. Pérez, and V. Herves. 2006. *GA-EDA: A New Hybrid Cooperative Search Evolutionary Algorithm.* 187–219. Towards a New Evolutionary Computation: Advances in the Estimation of Distribution Algorithms: Springer Berlin Heidelberg.

Schiefelbein, J., J. Tesfaegzi, R. Streblow, and D. Müller. 2015. "Design of an Optimization Algorithm for the Distribution of Thermal Energy Systems and Local Heating Networks within a City District." In *Proceedings of ECOS 2015,* 28th International Conference on Efficiency, Cost, Optimization, Simulation and Environmental Impact of Energy Systems.

Shi, Y., and R. Eberhart. 1998. "A Modified Particle Swarm Optimizer." In *Evolutionary Computation Proceedings, 1998. IEEE World Congress on Computational Intelligence,* 69–73. IEEE.

Tessema, B., and G.G. Yen. 2009. "An Adaptive Penalty Formulation for Constrained Evolutionary Optimization." *IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans* 39 (3): 565–578.

Trelea, Ioan Cristian. 2003. "The Particle Swarm Optimization Algorithm: Convergence Analysis and Parameter Selection." *Information Processing Letters* 85 (6): 317 – 325. http://www.sciencedirect.com/science/article/pii/S0020019002004477.

Wang, C., I. Martinac, and A. Magny. 2015. "Multi-Objective Robust Optimization of Energy Systems for a Sustainable District in Stockholm." In *Proceedings of BS2015,* 14th Conference of the International Building Performance Simulation Association.

Zhang, Wen-Jun, and Xiao-Feng Xie. 2003. "DEPSO: hybrid particle swarm with differential evolution operator." In *Systems, Man and Cybernetics, 2003. IEEE International Conference on,* Vol. 43816–3821. IEEE.

Appendix A. Convergence analysis

In this Appendix Theorem 4.2 is proved, which states the global convergence of the sequence $\{x_k\}$ generated by Algorithm 1 to a stationary point of problem (12). Here it is restated in a more general form, as it is valid not only when l_1 penalty function is chosen, but also for all polyhedral penalty convex functions H (Fletcher and de la Maza 1989). Then, Φ is going to be expressed in the following general compact form: $\Phi(x;\nu) = f(x) + H(g(x;\nu))$, where $H(g(x;\nu))$ is the penalty term and $g : \mathbb{R}^n \to \mathbb{R}^p$ and, for sake of simplicity, in the following analysis $g(x;\nu)$ and $\Phi(x;\nu)$ will be denoted as $g(x)$ and $\Phi(x)$ respectively. Moreover for the step a generic norm $\|d_k\|$ can be considered, while, for seek of simplicity, bound constraints are not considered.

So, as a consequence of the above assumptions, theoretical results are referred to the following unconstrained problem:

$$\min_x \Phi(x) = f(x) + H(g(x)), \tag{A1}$$

and subproblem (13) becomes:

$$\min_d l_k(d) = f(x_k) + \nabla f(x_k)^T d + H(g(x_k) + \nabla g(x_k)^T d) \tag{A2a}$$

$$\|d\| \leq \Delta_k, \tag{A2b}$$

where $\nabla g(x) \in \mathbb{R}^{p\times n}$ is the Jacobian matrix of $g(x)$.

To prove the theorem it is necessary to take into account that objective functions of problems (A1) and (A2) are non differentiable, so to characterize their stationary points it is necessary to introduce KKT conditions for problems with non-smooth objective function, (Fletcher 1987).

Definition A.1 *Let f be a convex function defined in $\mathcal{D} \subseteq \mathbb{R}^n$. A vector v is a subgradient of f at $x \in \mathcal{D}$, if $f(y) \geq f(x) + v^T(y-x)$ for all $y \in \mathcal{D}$.*

Definition A.2 *Let f be a convex function defined in $\mathcal{D} \subseteq \mathbb{R}^n$. The subdifferential $\partial f(x)$ of f at x is the set of all subgradients:*

$$\partial f(x) = \{v : v^T(y-x) \leq f(y) - f(x) \quad \text{for all} \quad y \in \mathcal{D}\}.$$

First order necessary KKT conditions for x^* to solve (A1) are that there exists vectors of multipliers $\lambda^* \in \partial H(g^*)$ such that, (see Theorem 14.2.1 in (Fletcher 1987)): $\nabla f(x^*) + \nabla g(x^*)\lambda^* = 0$. First order conditions for subproblem (A2) are that there exist multipliers $\lambda_k \in \partial H(g(x_k) + \nabla g(x_k)^T d_k)$, $w_k \in \partial\|d_k\|$ and $\pi_k \geq 0$, (see Theorem 14.6.1 in (Fletcher 1987)), such that:

$$\nabla f(x_k) + \nabla g(x_k)^T \lambda_k + \pi_k w_k = 0, \tag{A3}$$

$$\pi_k(\|d_k\| - \Delta_k) = 0. \tag{A4}$$

The following Lemma is proved in (Fletcher 1987) and is useful for the convergence theorems.

Lemma A.3 *[Lemma 14.2.1 of (Fletcher 1987)] Let $f : \mathcal{K} \to \mathbb{R}$ be a convex function, $\mathcal{K} \subset \mathbb{R}^n$ a convex set. Then $\partial f(x)$ is a closed convex set and it is bounded for all $x \in B \subset \overset{o}{\mathcal{K}}$ where B is compact and $\overset{o}{\mathcal{K}}$ denotes the interior of $\mathcal{K}$.*

The following theorem is the reformulation of Theorem 4.2 in a slightly more general form. It is proved following the lines of the proof of Theorem 2.1 in (Fletcher and de la Maza 1989). Note that in (Fletcher and de la Maza 1989) it is assumed to have at disposal an approximation of the Hessian matrix of the Lagrangian function, while the method considered in this article exploits just first order informations.

Theorem A.4 *[Global convergence of Algorithm 1]*
Let f and g be $\mathbb{C}^1$ functions and let $H(g)$ be a convex function. Let $\{x_k\}$ be the sequence generated by Algorithm 1. Either there exists an iteration index $\bar{k}$ such that $x_{\bar{k}}$ is a KKT point for $\Phi(x)$, or $\Phi(x_k) \to -\infty$ $k \to \infty$, or if the sequence $\{x_k\}$ is bounded, then there exists a subsequence S of indexes such that $\{x_k\}_{k\in S}$ has an accumulation point x^* which satisfies the KKT conditions for $\Phi(x)$, that is it exists a vector of multipliers λ^* such that:

$$\nabla f(x^*) + \nabla g(x^*)\lambda^* = 0. \tag{A5}$$

Proof. To prove the theorem it is sufficient to consider the case in which $\{\Phi_k\}$ is bounded below and $\{x_k\}$ is bounded. Because $\{x_k\}$ is bounded, there exists a subsequence S of iterations such that $\{x_k\}_{k\in S} \to x^*$. Suppose that:
a) d_k does not satisfy $\rho_k > \rho_{bad}$ for any $k \in S$ and $\{\Delta_k\}_{k\in S} \to 0$ and hence $\{\|d_k\|\}_{k\in S} \to 0$.
Let define $\Delta\Phi_k = \Phi(x_k) - \Phi(x_k + d_k)$ and $\Delta l_k = l_k(0) - l_k(d_k) = \Phi(x_k) - l_k(d_k)$. A consequence of $\mathbb{C}^1$ continuity of f and g, convexity of $H(g)$ and boundedness of $\partial H(g)$, which follows from Lemma A.3, and of the use of the first order Taylor expansion, is

that: $\Delta\Phi_k = \Delta l_k + o(\|d_k\|)$ and hence $\Delta\Phi_k/\Delta l_k \to 1$ as $k \to \infty$, which contradicts the fact that $\rho_k = \frac{\Delta\Phi_k}{\Delta l_k} > \rho_{bad}$ fails for all $k \in S$. Therefore this case is inconsistent and it certainly exists a subsequence of indexes S such that:
b) d_k satisfies $\rho_k > \rho_{bad}$ and $\liminf\limits_{k\in S} \Delta_k > 0$.
In fact, let S' be a sequence of indexes of unsuccessful iterations. If S' is finite, then clearly $\rho_k > \rho_{bad}$ for k sufficiently large. Otherwise suppose $k_0 \in S'$. Since $\{\Delta_k\}_{k\in S'} \nrightarrow 0$, otherwise case (a) is obtained again, for each $k_0 \in S'$ it exists i such that $k_0 + i$ is a successful iteration with $k_0 + i \notin S'$. Therefore $x_{k_0+i} = x_{k_0}$ and the subsequence $\{x_{k_0+i}\}_{k_0\in S'} \to x^*$. Let $S = \{k_0 + i, k_0 \in S'\}$, then $\{x_k\}_{k\in S}$ is the subsequence of case (b). In case (b) it can be assumed that $\liminf_{k\in S} \Delta_k > \bar{\Delta} > 0$, as if $\liminf_{k\in S} \Delta_k = 0$ thus imply $\liminf \Delta_k = 0$ and this yields case (a) that has be proved to be inconsistent. Because $\Phi_1 - \Phi^* \geq \sum_{k\in S} \Delta\Phi_k$, it follows that $\sum_{k\in S} \Delta\Phi_k$ converges. Then, $\rho_k \geq \rho_{bad}$, i.e. $\Delta\Phi_k \geq \Delta l_k \rho_{bad}$, yields the convergence of the series $\sum_{k\in S} \Delta l_k$, and hence $\{\Delta l_k\} \to 0$.

Define $l^*(d) = f(x^*) + \nabla f(x^*)d + H(g(x^*) + \nabla g(x^*)^T d)$. Let

$$\bar{d} = \arg\min l^*(d), \quad s.t. \;\; \|d\| \leq \bar{\Delta}$$

and denote $\bar{x} = x^* + \bar{d}$. Then

$$\|\bar{x} - x_k\| \leq \|\bar{x} - x^*\| + \|x^* - x_k\| = \|\bar{d}\| + \|x^* - x_k\| \leq \bar{\Delta} + \|x^* - x_k\| \leq \Delta_k$$

for all k sufficiently large, $k \in S$. Thus $\bar{x}$ is feasible for problem (A2), so
$l_k(\bar{x} - x_k) \geq l_k(d_k) = \Phi(x_k) - \Delta l_k$. In the limit, for $k \in S$, $\nabla f(x_k) \to \nabla f(x^*)$, $g(x_k) \to g(x^*), \nabla g(x_k) \to \nabla g(x^*), \bar{x} - x_k \to \bar{d}$, and $\Delta l_k \to 0$, so it follows that $l^*(\bar{d}) \geq \Phi(x^*) = l^*(0)$. Thus $d = 0$ also minimizes $l^*(d)$ subject to $\|d\| \leq \bar{\Delta}$, and since the latter constraint is not active it follows from (A4) that $\pi^* = 0$ and from (A3) that it exists λ^* such that $\nabla f(x^*) + \nabla g(x^*)\lambda^* = 0$, then x^* is a KKT point. □

Theorem A.4 states that in case the objective function is not unbounded, which is ensured in the particular case considered in Section 4, due to the presence of the bound constraints, it exists an accumulation point x^* of the sequence $\{x_k\}$ generated by Algorithm 1 that satisfies KKT conditions (A5) for $\Phi(x)$, regardless of the starting point.

It is also possible to prove the convergence of multipliers, i.e. if the subsequence S of Theorem A.4 exists, than the subsequence of multipliers of subproblems (A2) approximates multipliers of problem (A1). In fact, the following theorem holds:

Theorem A.5 Convergence of multipliers, Theorem 2.2 in (Fletcher and de la Maza 1989) *Let $f, g \in \mathbb{C}^1$, $H(g)$ a convex function, π_k and λ_k multipliers of subproblems* (A2)*, defined in* (A3) *and* (A4)*. If the subsequence S in the statement of Theorem 4.2 exists, then $\{\pi_k\}_{k\in S} \to 0$. Moreover any accumulation point λ^* of the multiplier vectors λ_k, $k \in S$, satisfies $\lambda^* \in \Lambda^*$, where $\Lambda^* = \{\lambda : \lambda$ satisfies KKT conditions* (A5) *at $x^*\}$, and such an accumulation point exists.*

The stopping criterion employed for Algorithm 1, that is described in Section 4.1, relies on this important theoretical result, that allows to use the multipliers of subproblems (A2), provided by the function used to solve the LPs, as an approximation of those of problem (A1) .