The Min-up/Min-down Unit Commitment polytope

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Received: date / Accepted: date

Abstract The *Min-up/min-down Unit Commitment Problem* (MUCP) is to find a minimum-cost production plan on a discrete time horizon for a set of fossil-fuel units for electricity production. At each time period, the total production has to meet a forecasted demand. Each unit must satisfy minimum up-time and down-time constraints besides featuring production and start-up costs. A full polyhedral characterization of the MUCP with only one production unit is provided by Rajan and Takriti (2005). In this article, we analyze polyhedral aspects of the MUCP with n production units. We first translate the classical *extended cover inequalities* of the knapsack polytope to obtain the so-called *up-set inequalities* for the MUCP polytope. We introduce the *interval up-set inequalities* as a new class of valid inequalities, which generalizes both up-set inequalities and minimum up-time inequalities. We provide a characterization of the cases when interval up-set inequalities are valid and not dominated by other inequalities. We devise an efficient Branch & Cut algorithm, using up-set and interval up-set inequalities.

Keywords Unit Commitment Problem (UCP) · Min-up/min-down · Polytope · Facet · Branch & Cut

1 Introduction

Given a discrete time horizon $\mathcal{T} = \{1,...,T\}$, a demand for electric power D_t is to be met at each time period $t \in \mathcal{T}$. Power is provided by a set $\mathcal{N}$ of n production units. At each time period, unit $i \in \mathcal{N}$ is either down or up, and in the latter case, its production is within $[P^i_{min}, P^i_{max}]$. Each unit must satisfy minimum up-time (resp. down-time) constraints, *i.e.* each unit i must remain up (resp. down) during at least L^i (resp. ℓ^i) periods after start up (resp. shut down). Each unit i has *initial conditions* (e^i, τ^i), indicating unit i has commitment status e^i before time 1, and it should remain in the same status up to time τ^i (if $\tau^i = 0$, unit i is free to shut down or start up at time 1). Each unit i also features three different costs: a fixed cost c^i_f, incurred each time period the unit is up; a start-up cost c^i_0, incurred each time the unit starts up; and a cost

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c_p^i proportional to its production. The Min-up/min-down Unit Commitment Problem (MUCP) is to find a production plan minimizing the total cost while satisfying the demand and the minimum up and down time constraints. As for $T = 1$ and $P^i_{min} = P^i_{max}$, $i \in \mathcal{N}$, the 0-1 knapsack problem reduces to the MUCP, it follows that the MUCP is NP-hard (see Tseng (1996)).

Électricité de France (EDF) manages a mix of production units composed of nearly 60 nuclear power plants, 30 fossil-fuel power plants, and 500 hydropower plants dispatched in 50 valleys. Given the large number and variety of units, the daily production planning problem is solved at EDF (cf Renaud (1993)) using a Lagrangian relaxation – commonly referred as price decomposition – where coupling demand constraints are dualized and prices are updated using a subgradient method (see Cohen (1978); Frangioni (2005)). Each nuclear or fossil-fuel unit (resp. each valley) is treated as a subproblem and is solved using dynamic programming (resp. linear programming). As this Lagrangian relaxation does not systematically produce feasible solutions, an augmented Lagrangian relaxation is considered at a second stage in order to improve feasibility recovery. In the real-world fossil-fuel unit commitment problem, usually called *Unit Commitment Problem* (UCP), some more technical constraints have also to be taken into account, e.g. ramp constraints or reserve requirement constraints, and the start-up costs are an exponential function of the unit downtime. From a combinatorial point of view, the MUCP is the core structure of the fossil-fuel problem solved daily at EDF.

As shown by Tahanan et al (2015), several techniques to solve variants of the UCP have been used in the literature. In order to solve problems on a smaller scale, like subproblems in a decomposition schema, dynamic programming is often used. This method allows for non-linearity and non-concavity, and to alleviate the curse of dimensionality, several approximations have been considered (see Tahanan et al (2015) for references on the different techniques used).

Various *Mixed Integer Linear Programming* (MILP) formulations for the UCP have also been proposed (see references in the survey by Tahanan et al (2015)). Carrion and Arroyo (2006) formulate the problem with one set of binary variables, indicating whether a unit is up or down at each time period. However, using additional binary variables has been proven to be computationally more efficient (Arroyo and Conejo (2000); Ostrowski et al (2012)). In order to obtain a more realistic cost function, a stairwise formulation of the start-up cost, depending on how much time a unit has been down, and a linear piecewise approximation of non-convex and non-differentiable production costs, are proposed by Arroyo and Conejo (2000). In López et al (2015), a model for hot and cold start-ups is presented. Frangioni et al (2009) compare the use of perspective cuts to the classical linear piecewise approximation for quadratic production costs. Arroyo and Conejo (2004) propose to model the start-up and shut-down trajectories of production units, drawing a distinction between power and energy, and thus providing a more accurate description of the actual operating process of production units. In Morales-Espana et al (2013b), a compact model of these trajectories is presented, using up/down variables as well as online/offline variables. Morales-Espana et al (2015) generalize this model to include quick-start units. In Ostrowski et al (2012), tighter expressions of ramp constraints are introduced, and in Morales-Espana et al (2013a), the generation limit constraint is replaced by a tighter inequality which takes into account the minimum-up time and the ramp constraints. Yang et al (2015) project the generated power of each unit i from $[P^i_{min}, P^i_{max}]$ to $[0,1]$ to improve the linear relaxation of the formulation. In Pan et al (2016), the authors introduce valid inequalities for units with ramp rates greater than the minimum generation limit.

Even though the UCP has been subject to a large research activity (see survey Sheble and Fahd (1994), its update Padhy (2004) and more recently Tahanan et al (2015)), it still cannot be regarded as a well-solved problem. In this article, we focus on its combinatorial structure, by investigating polyhedral aspects of the MUCP.

For each unit $i \in \mathcal{N}$ and time period $t \in \mathcal{T}$, let us consider three variables: $x^i_t \in \{0,1\}$ indicates if unit i is up at time t; $u^i_t \in \{0,1\}$ indicates whether unit i starts up at time t; and $p^i_t \in \mathbb{R}$ is the quantity of power produced by unit i at time t. Without loss of generality we consider that $L^i, \ell^i \leq T$. The MUCP can be formulated as follows:

$$\min_{x,u,p} \quad \sum_{i=1}^{n}\sum_{t=1}^{T} c_f^i x_t^i + c_p^i p_t^i + c_0^i u_t^i$$

$$\text{s. t.} \quad \sum_{t'=t-L^i+1}^{t} u_{t'}^i \leq x_t^i \qquad \forall i \in \mathcal{N},\ \forall t \in \{L^i, ..., T\} \tag{1}$$

$$\sum_{t'=t-\ell^i+1}^{t} u_{t'}^i \leq 1 - x_{t-\ell^i}^i \qquad \forall i \in \mathcal{N},\ \forall t \in \{\ell^i, ..., T\} \tag{2}$$

$$u_t^i \geq x_t^i - x_{t-1}^i \qquad \forall i \in \mathcal{N},\ \forall t \in \{2, ..., T\} \tag{3}$$

$$\sum_{i=1}^{n} p_t^i \geq D_t \qquad \forall t \in \mathcal{T} \tag{4}$$

$$P_{min}^i x_t^i \leq p_t^i \leq P_{max}^i x_t^i \qquad \forall i \in \mathcal{N},\ \forall t \in \mathcal{T} \tag{5}$$

$$x_t^i, u_t^i \in \{0,1\} \qquad \forall i \in \mathcal{N},\ \forall t \in \mathcal{T} \tag{6}$$

Inequalities (1), (2) and (3), are introduced by Rajan and Takriti (2005). Inequality (1) is the minimum up-time constraint: it states that if unit i is down at time t, then it cannot have started up during the L^i previous periods. Inequality (2) is the minimum down-time constraint, which is symmetric to the minimum up-time constraint. Inequality (3) ensures that if unit i starts up at time t (i.e. $x_t^i - x_{t-1}^i = 1$) then the start-up variable u_t must be 1. Inequality (5) sets bounds to the quantity of power produced by each unit, and inequality (4) ensures that the demand is satisfied at each time period. For each unit i, initial conditions can be accounted for by adding the following inequalities:

$$x_t^i = e^i,\ \forall t \in [1, \tau^i] \quad \text{and} \quad u_1^i \geq x_1^i - e^i \tag{7}$$

We consider P_{UCP}^n, the polytope associated to the formulation (1) - (7). In order to study its combinatorial structure, we will focus on its projection on binary variables x and u:

Lemma 1. *The projection of polytope P_{UCP}^n on variables (x,u) is*

$$P_{x,u}^n = Conv\left\{ (x,u) \in \left(\{0,1\}^{\mathcal{N}\times\mathcal{T}}\right)^2 \text{ s. t. } (1),\ (2),\ (3),\ (7) \text{ and } \sum_{i=1}^{n} P_{max}^i x_t^i \geq D_t\ \forall t \in \mathcal{T} \right\}.$$

Proof Let $Proj_{x,u}(P_{UCP}^n)$ be the projection of polytope P_{UCP}^n on variables (x,u). Given $(\tilde{x},\tilde{u}) \in P_{x,u}^n$, set $\tilde{p}_t^i = P_{max}^i \tilde{x}_t^i$, $\forall i \in \mathcal{N}$, $\forall t \in \mathcal{T}$, then $(\tilde{x},\tilde{u},\tilde{p}) \in P_{UCP}^n$. Conversely, consider $(\overline{x},\overline{u}) \in Proj_{x,u}(P_{UCP}^n)$. There exists $\overline{p}$ such that $(\overline{x},\overline{u},\overline{p}) \in P_{UCP}^n$. Thus $\sum_{i=1}^n \overline{p}_t^i \geq D_t\ \forall t \in \mathcal{T}$ and $\overline{p}_t^i \leq P_{max}^i \overline{x}_t^i$. It follows $\sum_{i=1}^n P_{max}^i \overline{x}_t^i \geq D_t$, and thus $(\overline{x},\overline{u}) \in P_{x,u}^n$.

□

Several articles propose polyhedral studies for the UCP with only one production unit. The min-up and min-down constraints for one production unit are modeled in Takriti et al (2000) using binary variables x_t to indicate whether the unit is up at time t. In Lee et al (2004) the authors provide a complete description of the 1-unit polytope associated to these min-up/min-down constraints in the x variable space. In case the unit has a start-up cost, additional binary variables u_t are needed to indicate whether the unit starts up at time t. In Rajan and Takriti (2005), the authors study the 1-unit polytope associated to the min-up and min-down constraints in the (x, u) variable space: this polytope is exactly $P_{x,u}^1$ when $D_t = 0$, $t \in \mathcal{T}$ – note that $P_{x,u}^1$ is full-dimensional if and only if $D_t = 0$, $\forall t \in \mathcal{T}$. They prove that inequalities (1), (2) and

(3) completely describe this polytope. In Gentile et al (2016), the polytope of the 1-unit problem with min-up/down constraints and generation limits (including start-up and shut-down generation limits) is completely described. The polytope of the same problem with additional start-up and shut-down trajectories is completely described in Morales-Espana et al (2015). In Pan and Guan (2016), the polytope of the 1-unit problem with min-up/down constraints, generation limits and ramp constraints is completely described for $T = 3$, and additional inequalities, that are valid for any T, are introduced. Knueven et al (2016) propose a compact extended formulation of this last problem.

In this article, we analyze polyhedral aspects of the MUCP with n production units. In Section 2, some facial properties of inequalities (1), (2) and (3) are given. In Section 3, the rank of a subset $C \in \mathcal{N}$ at time $t \in \mathcal{T}$ is defined. On this basis, a large family of valid inequalities is described in Section 4, and facet defining cases are studied in Section 5. The Branch-&-Cut algorithm we devised based on our valid inequalities is presented in Section 6, along with some experimental results in Section 7.

2 Polyhedral study

In this section, we give some first polyhedral results on polytope $P^n_{x,u}$, for any number n of units.

For a given unit $i \in \mathcal{N}$, the value of u^i_1 can be deduced from i's initial conditions and the value of x^i_1. Then each solution of the MUCP can be fully described without variable u^i_1. For simplicity purpose, we will then omit initial conditions, variables u^i_1, $i \in \mathcal{N}$, together with inequalities (1) and (2) involving variables u_1. Thus, polytope $P^n_{x,u}$ can be restricted to a simpler polytope as follows:

$$P^n_{x,u} = Conv\left\{ (x,u) \in \{0,1\}^{\mathcal{N}\times\mathcal{T}} \times \{0,1\}^{\mathcal{N}\times\mathcal{T}\setminus\{1\}} \text{ s. t. } (1),\ (2),\ (3) \ \text{ and } \ \sum_{i=1}^{n} P^i_{max} x^i_t \geq D_t \ \forall t \in \mathcal{T} \right\}$$

Note that, since variables u^i_1 are omitted, inequalities (1) for $t = L^i$, $i \in \mathcal{N}$ and inequalities (2) for $t = \ell^i$, $i \in \mathcal{N}$ are dropped. Without loss of generality, we assume that $L^i, \ell^i \leq T - 1$.

We first introduce some vectors that will be useful in the polyhedral proofs. Note that a solution of $P^n_{x,u}$ is a couple $(x,u) \in \{0,1\}^{\mathcal{N}\times\mathcal{T}} \times \{0,1\}^{\mathcal{N}\times\mathcal{T}\setminus\{1\}}$ resulting in $n(2T-1)$ coordinates. Given a unit $i \in \mathcal{N}$ and a time period $t \in \mathcal{T}$, let $\chi^u_{i,t}$ (resp. $\chi^d_{i,t}$) be the vector such that unit i is down (resp. up) on $[1, t-1]$, starts up (resp. shuts down) at time t and remains up (resp. down) on $[t, T]$, and such that unit j is up at all times, for all $j \neq i$. Moreover, let $\chi_0 \in P^n_{x,u}$ be the vector in which all units are up at all times. To illustrate, the coordinates of vector χ^u_{i,t_0} are the following:

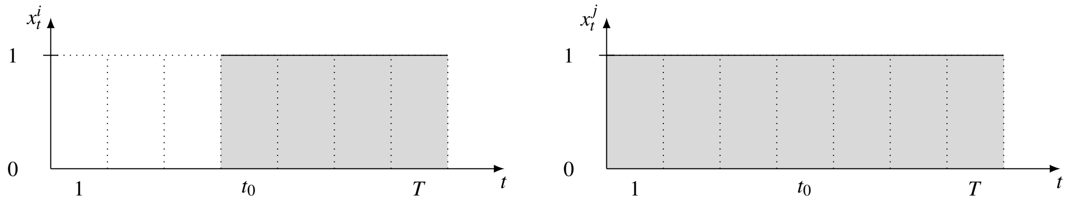
$$\begin{array}{cccccccccccccccc}
 & 1 & \cdots & & t_0 & & \cdots & T & & & & & & & \\
x^i = & (0, & ..., & 0, & 1, & 1, & ..., & 1\,) & \text{and} & x^j = & (1, & ..., & 1) & j \neq i \\
u^i = & (0, & ..., & 0, & 1, & 0, & ..., & 0\,) & \text{and} & u^j = & (0, & ..., & 0) & j \neq i
\end{array}$$

A simple way to present the "x" coordinates of vector χ^u_{i,t_0}, $t_0 \in \mathcal{T}\setminus\{1\}$, is the diagram of Figure 1.

The proofs of the two following theorems are extensions of results for the 1-unit polytope from Rajan and Takriti (2005).

Theorem 1. *The polytope $P^n_{x,u}$ is full-dimensional if and only if for all $i \in \mathcal{N}$ and for all $t \in \mathcal{T}$, there exists a solution $(\bar{x},\bar{u}) \in P^n_{x,u}$ such that $\bar{x}^i_t = 0$.*

Proof First, note that if there is no solution $(\bar{x},\bar{u})$ such that $\bar{x}^i_t = 0$, then the polytope is trivially not full-dimensional. Now, if the hypothesis holds, *i.e.* for each time t and each unit i, there exists a solution in which unit i is down at t, then, at each time t, the units of $\mathcal{N}\setminus\{i\}$ are sufficient to cover the demand, $\forall i \in \mathcal{N}$. Thus, given $i \in \mathcal{N}$, vectors ($\chi^d_{i,t}$, $t \in \mathcal{T}$) and vectors ($\chi^u_{i,t}$, $t \in \mathcal{T}\setminus\{1\}$) are $2T-1$ incident vectors



(a) Coordinates x^i (b) Coordinates x^j, for $j \neq i$

Fig. 1: Coordinates x of vector χ^u_{i,t_0}

of solutions in $P^n_{UCP}(x,u)$. Hence, vectors $\chi^u_{i,t}$, $\chi^d_{i,t}$ and vectors χ_0 constitute a set of $n(2T-1)+1$ affinely independent vectors of $P^n_{x,u}$, thus $P^n_{x,u}$ is full-dimensional.

□

In the following, we will consider the full-dimensionality condition is satisfied, *i.e.* $n-1$ units are always sufficient to meet the demand at any time. Note this assumption is required to come up with a reliable production plan.

Theorem 2. *Inequalities (2), (3) and trivial inequalities $u^i_t \geq 0$, $i \in \mathcal{N}$, $\forall t \in \mathcal{T}\backslash\{1\}$, describe facets of $P^n_{x,u}$.*

Proof Let $i_0 \in \mathcal{N}$ and $t_0 \in \mathcal{T}\backslash\{1\}$.

Vectors $(\chi^u_{i,t}, (i,t) \in (\mathcal{N}\times\mathcal{T}\backslash\{1\}) \setminus \{(i_0,t_0)\})$, vectors $(\chi^d_{i,t}, (i,t) \in \mathcal{N}\times\mathcal{T})$, and vector χ_0 are $n(2T-1)$ affinely independent vectors of $P^n_{x,u}$ satisfying $u^{i_0}_{t_0} = 0$. So the trivial inequality defines a facet of $P^n_{x,u}$.

Vectors $(\chi^u_{i,t}, (i,t) \in \mathcal{N}\times\mathcal{T}\backslash\{1\})$, vectors $(\chi^d_{i,t}, (i,t) \in (\mathcal{N}\times\mathcal{T})\backslash\{(i_0,t_0)\})$ and vector χ_0 are $n(2T-1)$ affinely independent vectors of $P^n_{x,u}$ satisfying $u^{i_0}_{t_0} = x^{i_0}_{t_0} - x^{i_0}_{t_0-1}$. So (3) defines a facet of $P^n_{x,u}$.

As inequality (2) has been proven to be facet defining for the 1-unit polytope $P^1_{x,u}$ (see Rajan and Takriti (2005)), there exist $2T-1$ affinely independent vectors $(x^i,u^i) \in P^1_{x,u}$ satisfying $\sum_{t=t_0-\ell^{i_0}+1}^{t_0} u^{i_0}_t = 1 - x^{i_0}_{t_0-\ell^{i_0}}$. From each vector (x^{i_0},u^{i_0}) we construct a vector $(\overline{x},\overline{u}) \in P^n_{x,u}$ satisfying $\sum_{t=t_0-\ell^{i_0}+1}^{t_0} \overline{u}^{i_0}_t = 1-\overline{x}^{i_0}_{t_0-\ell^{i_0}}$, by setting coordinates as follows: $(\overline{x}^j_t = 1,\, j \neq i, t \in \mathcal{T})$, $(\overline{u}^j_t = 0,\, j \neq i, t \in \mathcal{T}\backslash\{1\})$, and $\overline{x}^i_t = x^i_t$, $\overline{u}^i_t = u^i_t$, $\forall t$. These $2T-1$ vectors of $P^n_{x,u}$ alongside with the $(n-1)(2T-1)$ vectors $(\chi^u_{j,t},\, j \neq i, t \in \mathcal{T}\backslash\{1\})$, $(\chi^d_{j,t},\, j \neq i,\, t \in \mathcal{T})$, constitute a set of $n(2T-1)$ affinely independent vectors of $P^n_{x,u}$ satisfying inequality (2) with equality, which proves that (2) defines a facet of $P^n_{x,u}$.

□

Given a face F of $P^n_{x,u}$, a unit i and a time period t, Property $\Pi_{i,t}$ is as follows:

$$\text{Solution } (x,u) \in F \text{ satisfies } \Pi_{i,t} \iff \text{Unit } i \text{ is down on } [t,t+\ell^i] \text{ } i.e.\ x^i_{t'} = 0,\ \forall t' \in [t,t+\ell^j].$$

Let us consider the transformation $\Psi^i_{t_0,t_1}$ such that for any vector $\rho \in \{0,1\}^{\mathcal{N}\times\mathcal{T}} \times \{0,1\}^{\mathcal{N}\times\mathcal{T}\backslash\{1\}}$, $\Psi^i_{t_0,t_1}(\rho)$ is equal to vector ρ except for unit i which is down over $[t_0,t_1]$ and up the rest of the time.

We give a generic technical lemma.

Lemma 2. *Let $\sum_{j\in\mathcal{N}} a^j x^j + \sum_{j\in\mathcal{N}} b^j u^j \leq \gamma$ be a valid inequality for $P^n_{x,u}$, distinct from inequality (2). Let F be the associated face.*

(i) *If F is a facet, then for all $i \in \mathcal{N}$ and $t \in \mathcal{T}$, there exists $(x,u) \in F$ satisfying property $\Pi_{i,t}$.*

(*ii*) *For given $i \in \mathcal{N}$, if for all $t \in \mathcal{T}$ there exists $(x,u)_t \in F$ satisfying property $\Pi_{i,t}$, and if neither x^i nor u^i variables appear in $\sum_{j\in\mathcal{N}} a^j x^j + \sum_{j\in\mathcal{N}} b^j u^j$ (i.e. $a^i = b^i = 0$), then for all $t \in \mathcal{T}$, F contains the following solutions:*

- *Solution $\Psi^i_{t,t+\ell^j}((x,u)_t)$, where unit i is up on $[1,t-1]$, down on $[t,t+\ell^i]$ and up on $[t+\ell^i+1,T]$.*
- *Solution $\Psi^i_{t+1,t+\ell^j}((x,u)_t)$, where unit i is up on $[1,t]$, down on $[t+1,t+\ell^i]$ and up on $[t+\ell^i+1,T]$.*
- *Solution $\Psi^i_{t,t+\ell^j-1}((x,u)_t)$, where unit i is up on $[1,t-1]$, down on $[t,t+\ell^i-1]$ and up on $[t+\ell^i,T]$.*
- *Solution $\Psi^i_{1,t_0}((x,u)_1)$, for any $t_0 \in [1,\ell^j-1]$, where unit i is down on $[1,t_0]$ and up on $[t_0+1,T]$.*

Proof (*i*): Suppose $\Pi_{i,t}$ does not hold for given $i_0 \in \mathcal{N}$ and $t_0 \in \mathcal{T}$. Thus, for any given solution $(\tilde{x},\tilde{u}) \in F$, if unit i_0 is down at time t_0, it must start up before time $t_0+\ell^{i_0}$ (if $t_0 > T-\ell^{i_0}$ we can consider $t_0 = T-\ell^{i_0}$ w.l.o.g.). Then inequality $\sum_{t=t_0+1}^{t_0+\ell^{i_0}} \tilde{u}^{i_0}_t \geq 1-\tilde{x}^{i_0}_{t_0}$ is valid. As inequality (2) holds too, it follows $\sum_{t=t_0+1}^{t_0+\ell^{i_0}} \tilde{u}^{i_0}_t = 1-\tilde{x}^{i_0}_{t_0}$. Thus F is included in the face of inequality (2), which contradicts F is distinct from the face defined by (2).

(*ii*): For each $t \in \mathcal{T}$, from Property $\Pi_{i,t}$, $\exists (x,u)_t \in F$ such that $x^i_{t'} = 0\ \forall t' \in [t,t+\ell^i]$. Then vector $\Psi^i_{t,t+\ell^j}((x,u)_t)$ is still a solution as unit i remains down for ℓ^i+1 periods, thus satisfying the min-down constraints. Moreover, the demand is satisfied since unit i is up in vector $\Psi^i_{t,t+\ell^j}((x,u)_t)$ at least as often as in solution (x,u). As $a^i = b^i = 0$, solution $\Psi^i_{t,t+\ell^j}((x,u)_t)$ is a solution of F. Similarly, vectors $\Psi^i_{t+1,t+\ell^j}((x,u)_t)$, $\Psi^i_{t,t+\ell^j-1}((x,u)_t)$ and $\Psi^i_{1,t_0}((x,u)_1)$, for any $t_0 \in [1,\ell^j-1]$ are solutions of F.

□

We now prove that min-up inequalities (1) are facet defining under a mild condition.

Theorem 3 (Facet defining min-up inequalities)**.** *For $i \in \mathcal{N}$, for $t_1 \in \{L^i+1,...,T\}$, let F be the face defined by the min-up inequality (1) $\sum_{t=t_1-L^i+1}^{t_1} u^i_t \leq x^i_{t_1}$. F is a facet of $P^n_{x,u}$ if and only if for any unit $j \in \mathcal{N}\backslash\{i\}$ and time $t \in \mathcal{T}$, there exists a solution $(x,u) \in F$ satisfying $\Pi_{i,t}$.*

Proof The direct implication ($\Longrightarrow$) follows from Lemma 2 (*i*).

We prove the return implication ($\Longleftarrow$). Suppose that F is included in the face of an inequality $\sum_{j\in\mathcal{N}} \left(\sum_{t\in\mathcal{T}} a^j_t x^j_t + \sum_{t\in\mathcal{T}\backslash\{1\}} b^j_t u^j_t\right) \leq \gamma$. The claim is that $F = \{(x,u) \in P^n_{x,u} \,|\, \sum_{j\in\mathcal{N}} \left(\sum_{t\in\mathcal{T}} a^j_t x^j_t + \sum_{t\in\mathcal{T}\backslash\{1\}} b^j_t u^j_t\right) = \gamma\}$, which proves that F is a facet of $P^n_{x,u}$.

For any $j \in \mathcal{N}\backslash\{i\}$, there are no x^j nor u^j variables appearing in inequality $\sum_{t=t_1-L^i+1}^{t_1} u^i_t \leq x^i_{t_1}$. Since there exists $(x,u)_t \in F$ satisfying $\Pi_{j,t}$ for any $t \in \mathcal{T}$, it follows from Lemma 2 (*ii*) that for any $t \in \mathcal{T}$, $\Psi^i_{t,t+\ell^j}((x,u)_t) \in F$ and $\Psi^i_{t+1,t+\ell^j}((x,u)_t) \in F$. As these solutions differ only over variable x^i_t, we can conclude $a^j_t = 0, \forall j \in \mathcal{N}\backslash\{i\}, \forall t \in \mathcal{T}$. Moreover, Lemma 2 (*ii*) implies that $\Psi^i_{T-\ell^i,T}((x,u)_t) \in F$, which differs from $\Psi^i_{T-\ell^i,T-1}((x,u)_t) \in F$ only over x^i_T and u^i_T variables. As $a^i_T = 0$, it follows $b^i_T = 0$. Then, in the same way, we can see that $b^j_t = b^j_{t-1}, \forall j \in \mathcal{N}\backslash\{i\}, \forall t \in \mathcal{T}\backslash\{1,2\}$, by comparing vector $\Psi^i_{t,t+\ell^i-1}((x,u)_t) \in F$, $\forall t \in \mathcal{T}$ to vector $\Psi^i_{t,t+\ell^i}((x,u)_t) \in F$, $\forall t \in \mathcal{T}$, and vector $\Psi^i_{1,t'}((x,u)_1) \in F$, $t' \in [1,\ell^j-1]$, to vector $\Psi^i_{1,1+\ell^i}((x,u)_1) \in F$. It follows $b^j_t = 0, \forall j \in \mathcal{N}\backslash\{i\}, \forall t \in \mathcal{T}\backslash\{1\}$.

Vectors $\chi^d_{i,t_1} \in F$ and $\chi^d_{i,t_1-1} \in F$ differ only over variable $x^i_{t_1-1}$. Thus $a^i_{t_1-1} = 0$. Then, from vectors $\chi^d_{i,t} \in F$, $t \in [1,...,t_1]$, we can iteratively see that $a^i_t = 0$ for all $t \leq t_1-1$. We introduce vectors $\Theta^i_{t,t'} \in P^n_{x,u}$ such that i starts up at time t, stays up until t' and shuts down at time $t'+1$ (all other units are up at all times). As $P^n_{x,u}$ is full-dimensional, note that $n-1$ units are always sufficient to meet the demand at any time, thus for any $t > t_1$, vectors $\Theta^i_{t_1-L^i+1,t} \in F$. Moreover, vectors $\Theta^i_{t_1-L^i+1,t-1}$ and $\Theta^i_{t_1-L^i+1,t}$ differ only over variable x^i_t, which implies $a^i_t = 0$ for any $t > t_1$. Moreover, vector $\chi^d_{i,1} \in F$ and for any $t \geq t_1-L^i+1$, vector $\chi^u_{i,t} \in F$. Since vector $\chi^u_{i,t}$ differs from vector $\chi^d_{i,1}$ over variables $x^i_{t'}$, $t' \geq t$ and variable b^i_t, $b^i_t = -a^i_{t_1}$

for any $t \in [t_1 - L^i + 1, t_1]$, and $b^i_t = 0$ for any $t > t_1$. Finally, for any $t \leq t_1 - L^i$, vectors $\Theta^i_{t,t_1-1} \in F$, and Θ^i_{t,t_1-1} differs from $\chi^d_{i,1}$ over variables $x^i_{t'}$, $t' \in [t, t_1 - 1]$ and variable b^i_t, it follows that for any $t \leq t_1 - L^i$, $b^i_t = 0$.

The remaining inequality is then:

$$-a^i_{t_1} x^i_{t_1} + \sum_{t=t_1-L^i+1}^{t_1} a^i_{t_1} u^i_t = \gamma.$$

Since $\chi^d_{i,1} \in F$, $\gamma = 0$, which proves that F is a facet of $P^n_{x,u}$. □

3 Rank of unit subsets

In order to introduce new valid inequalities for the MUCP polytope, we define the rank of a unit subset.

For each subset of units $M \subset \mathcal{N}$, its *rank* $\alpha_t(M)$ is the smallest number of units that must be up in M at time t in a feasible solution. Since this rank is hard to compute, we will also consider an alternative definition which will be useful in practice. For each subset of units $M \subset \mathcal{N}$, its *static rank* is the smallest number of units that must be up in M at time t in order to satisfy the remaining demand $D_t - \sum_{j \notin M} P^j_{max}$.

As all feasible solutions meet the demand at time t, it is clear that:

$$\alpha_t(M) \geq \overline{\alpha}_t(M) \;\; \forall t \in \mathcal{T} \; \forall M \subset \mathcal{N}.$$

Consider an illustrative example of the MUCP with $T = 5$, $D = [50,\ 55,\ 35,\ 55,\ 35]$ and six units such that $P^1_{max} = 20$, $P^j_{max} = 12$, $j \in \{2,...,5\}$, $P^6_{max} = 8$ and $\ell^j = 2$, $j \in \{2,...,5\}$. Let $M = \{1,2\}$. Even if units 3, 4, 5 and 6 are up, their production is not sufficient to meet the demand at time 1. Indeed the remaining demand to be satisfied is $\overline{D}_1 = D_1 - \sum_{j=3}^{6} P^j_{max} = 6$. One unit in M must then necessarily be up to cover the demand. Thus $\overline{\alpha}_1(M) \geq 1$. Since only one unit of M is enough to cover the remaining demand $\overline{D}_1$ (here for example unit 1 is enough), $\overline{\alpha}_1(M) = 1$. As there exists a feasible solution in which only one unit of M is up at time 1 (for example the solution in which unit 2 is down at time 1 and starts up at time 2, while every other units are up at all times), $\alpha_1(M) = 1$. Let us now consider $M' = \{2,3,4,5\}$. The static rank of M' at time 3 is equal to 1: $\overline{\alpha}_3(M') = 1$, since one unit of M' is necessary and sufficient to satisfy the remaining demand $\overline{D}_3 = D_3 - P^1_{max} - P^6_{max} = 7$. However, in this example, $\overline{\alpha}_3(M') \neq \alpha_3(M')$. Indeed, there is no feasible solution in which only one unit of M' is up at time 3. First note $\overline{\alpha}_2(M') = \overline{\alpha}_4(M') = 3$ as at least three units of M' must be up up at times 2 and 4. Let us now assume there exists a feasible solution in which only one unit of M' is up at time 3. Thus, two units of M' must shut down at time 3. Since the minimum down time of each unit of M' is 2, the two units which shut down at time 3 cannot start up again at time 4. Yet there must be at least three units up in M' at time 4 to meet the demand, but only the two units which did not shut down at time 3 are available in M', which leads to a contradiction. As there exists a feasible solution with two units of M' up at time 3, $\alpha_3(M') = 2$.

Note that computing the rank of a unit subset is an optimization problem. It is to find the smallest number of units that must be up in M at time t in a feasible solution. In order to state the problem's complexity, let us consider its decision version: given an instance of the MUCP, a time period t_0, a unit subset M and an integer K, the question is whether there exists a feasible solution in which at most K units of M are up at time t_0, *i.e.* $\alpha_{t_0}(M) \leq K$.

Theorem 4. *Computing the rank of a unit subset is NP-hard for $T \geq 3$.*

Proof Let us consider an instance of the partition problem, with a set S of n positive integers $a_1, ..., a_n$. The question is whether S can be partitioned into two subsets S_1 and S_2 such that $\sum_{i\in S_1} a_i = \sum_{i\in S_2} a_i$.

First note that if such a partition exists, then $\sum_{i\in S_1} a_i = \sum_{i\in S_2} a_i = A$ where $A = \frac{1}{2}\sum_{i\in S} a_i$. Consider now the following instance of the rank decision problem: let $T = 3$ with $D = [A,\ 0,\ A]$, and n units such that $P^i_{max} = P^i_{min} = a^i$ and $\ell^i = 2$, $i \in \{1, ..., n\}$. The other characteristics are fixed arbitrarily. Set $t_0 = 2$, $M = \{1, ..., n\}$ and $K = 0$. Let us suppose there exists a solution to the latter instance. Let S_1 be the set of units up at time 1, and S_2 be the set of units up at time 3. We claim $(S_1,\ S_2)$ is a solution to the partition problem. Indeed, S_1 and S_2 are disjoint, as all units up at time 1 shut down at time 2, and stay down for a minimum of two time periods. Thus, all units up at time 1 are down at time 3. Moreover, the units in S_1 satisfy the demand at time 1, so $\sum_{i\in S_1} a_i \geq A$. Similarly, $\sum_{i\in S_2} a_i \geq A$. As S_1 and S_2 are disjoint, $2A \leq \sum_{i\in S_1} a_i + \sum_{i\in S_2} a_i \leq \sum_{i\in S} a_i = 2A$, we get $\sum_{i\in S_1} a_i = A$, $\sum_{i\in S_2} a_i = A$ and $S_1 \cup S_2 = S$.

Conversely, any solution to the partition problem can similarly be used to construct a solution to this instance of the rank computation problem.

□

Given this complexity result, we will use in practice the static rank instead of the rank. Indeed, the static rank can be computed in linear time (provided the units are sorted by decreasing order according to P^j_{max}) using Algorithm 1.

Algorithm 1 Computation of the static rank of set M at time t

```
Compute the remaining demand at time t: D̄_t = D_t − Σ_{j∉M} P^j_max
Sort units in M by decreasing order according to P_max
ρ = 0 and α = 0
while ρ < D̄_t do
    ρ += M[α]
    α++
end while
return α
```

For a given subset of units M, the definition of the rank at a given time t is extended to a given interval $\mathcal{I} = \{t_0,\ ...,\ t_1\}$. We denote by $\alpha_{\mathcal{I}}(M)$ the maximum rank of M over $\mathcal{I}$, *i.e.* $\alpha_{\mathcal{I}}(M) = \max_{t\in\mathcal{I}} \alpha_t(M)$. Similarly, the definition of the static rank at a given time t is extended to $\mathcal{I}$. We denote by $\overline{\alpha}_{\mathcal{I}}(M)$ the maximum static rank of M over $\mathcal{I}$.

Let t_{max} be the time period at which the demand is maximum on $\mathcal{I}$.

Lemma 3. $\alpha_{t_{max}}(C) = \alpha_{\mathcal{I}}(C)$.

Proof By definition of the rank of C at time t_{max}, there exists a solution $(x, u) \in P^n_{x,u}$ such that exactly $\alpha_{t_{max}}(C)$ units in C are up at time t_{max}. Let C_{max} be the set of units in C which are up at time t_{max} in solution (x, u). Using (x, u), we will iteratively construct a solution $(\tilde{x}, \tilde{u})$ such that any unit in $C\backslash C_{max}$ is down on the whole interval $\mathcal{I}$ and any unit in C_{max} is up on the whole interval $\mathcal{I}$: we first set $(\tilde{x}, \tilde{u})$ equal to (x, u), and we slightly modify the behavior of the units in C as follows:

- First, for any $j \in C_{max}$, $\forall t \in \mathcal{T}$, we set $\tilde{x}^j_t = 1$.
- For any $j \in C\backslash C_{max}$, if j starts up at $t \in \mathcal{I}$ in solution (x, u), we update coordinates $(\tilde{x}^j, \tilde{u}^j)$ of $(\tilde{x}, \tilde{u})$ such that j is down from time t to t_1, starts up at time $t_1 + 1$ and remains up until time T. Indeed, as the units in subset C_{max} meet the maximum demand $D_{t_{max}}$ on $\mathcal{I}$, for any $t \in \mathcal{I}$, the demand at time t is also met by units in C_{max}. For any $t \notin \mathcal{I}$, the units up at time t in $(\tilde{x}, \tilde{u})$ are exactly the units up at time t in solution (x, u). Thus, in solution $(\tilde{x}, \tilde{u})$, the demand is indeed satisfied at time t. Furthermore, by delaying the start-up of unit j, we respect its minimum down time, and since j remains up until the end of the time horizon, its minimum up time is satisfied.

– Similarly, for any $j \in C \backslash C_{max}$ such that j shuts down at time $t \in \mathcal{I}$, we set $(\tilde{x}^j, \tilde{u}^j)$ such that j is up from time 1 to $t_0 - 1$ and shuts down at time t_0. Similar arguments can prove that $(\tilde{x}, \tilde{u})$ remains feasible.

Consequently, $(\tilde{x}, \tilde{u})$ is a solution such that any unit in $C \backslash C_{max}$ is down on the whole interval $\mathcal{I}$ and any unit in C_{max} is up on the whole interval $\mathcal{I}$.

It follows $\alpha_t(C) \leq \alpha_{t_{max}}(C), \forall t \in \mathcal{I}$.

□

Let, for any $i \in M$, the *static i-rank* $\overline{\alpha}_t^i(M)$ be the smallest number of units that must be up in M at time t in order to satisfy the remaining demand $D_t - \sum_{j \notin M} P^j_{max}$, given that unit i is down at t. By Theorem 1, if $P^n_{UCP}(x,u)$ is full dimensional then the definition of $\overline{\alpha}_t^i(M)$ makes sense, as there exists a solution in which unit i is down at time t.

For example, by referring to the instance previously presented in this section, now consider unit subset $M'' = \{1,2,3,4\}$. The remaining demand at time 5 is $\overline{D}_5 = D_5 - P^5_{max} - P^6_{max} = 15$. Unit 1 is enough to meet the remaining demand, thus $\overline{\alpha}_5(M'') = 1$. However, if unit 1 is down at time 5, there must be at least two units of M'' up at time 5 to satisfy the remaining demand. It follows that $\overline{\alpha}_5^1(M'') = 2$.

4 Valid inequalities

In this section, we first define the up-set inequalities, which account for some of the combinatorial aspects induced by the knapsack structure of the MUCP. We will then introduce the interval up-set inequalities, as a generalization of the up-set inequalities. As they capture both knapsack constraints and minimum up and down times, they are more dedicated to the MUCP.

4.1 Up-set inequalities

By definition of the rank, for any subset $M \subset \mathcal{N}$ and time $t \in \mathcal{T}$, the *up-set inequality*, defined as follows, is valid:

$$\sum_{j \in M} x_t^j \geq \alpha_t(M) \tag{8}$$

This inequality is difficult to produce given that the rank $\alpha_t(M)$ is hard to compute. Thus, we also consider the *static up-set inequality*, a weaker version of inequality (8), defined as follows:

$$\sum_{j \in M} x_t^j \geq \overline{\alpha}_t(M) \tag{9}$$

In practice, if a lower bound α of $\overline{\alpha}_t(M)$ such that $\alpha \leq \alpha_t(M)$ is known, the corresponding inequality $\sum_{j \in M} x_t^j \geq \alpha$ can be used instead of (9).

These static up-set inequalities directly correspond to the extended cover inequalities for the knapsack polytope. We transpose the results of Balas (1975) into the MUCP.

For a given $t \in \mathcal{T}$, a subset C of $\mathcal{N}$ is called an *up-set* if $\overline{\alpha}_t(C) \geq 1$. In other words, C is an up-set if and only if the units in $\mathcal{N} \backslash C$ are not sufficient to meet the demand at time t.

An up-set C is called *minimal* if for all subset $Q \subsetneq C$, $\overline{\alpha}_t(Q) = 0$. For any minimal up-set C, we define $E(C) = C \cup C'$ as the *extension of C to* $\mathcal{N}$, where

$$C' = \{j \in \mathcal{N} \backslash C, P^j_{max} \geq P^{j_1}_{max}\} \text{ where } j_1 = \arg\max_{j \in C} P^j_{max}.$$

A minimal up-set C is called *strong* if for any minimal up-set A such that $|A| = |C|$ and $A \neq C$, $E(C) \not\subset E(A)$. For example, by referring to the MUCP instance defined in Section 3, if we consider the time period $t = 2$, subset $M = \{1,2\}$ is a minimal up-set as neither $\{1\}$ nor $\{2\}$ is an up-set. Since M contains the most powerful unit of $\mathcal{N}$ (unit 1), $E(M) = M$. However, subset M is not strong. Indeed, subset $A = \{2,3\}$ is a minimal up-set such that $|A| = |M|$, and $E(A) = \{1,2,3,4,5\}$ implying that $E(M) \subset E(A)$.

A subset M of $\mathcal{N}$ is said to be a *strong up-set extension* if there exists a strong up-set C such that:

(i) $M = E(C)$
(ii) $|C| = |M| - \overline{\alpha}_t(M) + 1$
(iii) $\overline{\alpha}_t(A) = 0$ where $A = C \backslash \{j_1, j_2\} \cup \{1\}$ and $j_2 = \arg\max_{j \in C \backslash \{j_1\}} P^j_{max}$.

Let $\mathcal{P}^n = Conv\{x_t \in \{0,1\}^n,\ \sum_{j\in\mathcal{N}} P^j_{max} x^j_t \geq D_t\}$ be the polytope of the MUCP considered at time period t only. We derive the following result from Balas (1975). The proof is given in the Appendix.

Theorem 5. *For any $M \subset \mathcal{N}$, the static up-set inequality (9) is facet defining for $\mathcal{P}^n$ if and only if M is a strong up-set extension.*

4.2 Interval Up-Set inequalities

Let $C \subset \mathcal{N}$ be a subset of units, with $i \in C$, and let $\mathcal{I} = \{t_0,...,t_1\} \subset \mathcal{T}$ be a time interval of length less than or equal to L^i, *i.e.* $t_1 - t_0 \leq L^i$. For all $\Delta \in \mathbb{N}$ we define the *interval up-set* inequality as follows:

$$\Delta + \sum_{t=t_0+1}^{t_1} u^i_t \leq x^i_{t_1} + \sum_{j\in C\backslash\{i\}} \left(x^j_{t_0} + \sum_{t'=t_0+1}^{t_1} u^j_t \right) \tag{10}$$

We first give a technical lemma:

Lemma 4. *For all $C \subset \mathcal{N}$, $\mathcal{I} = \{t_0,...,t_1\} \subset \mathcal{T}$ and $k \in \mathcal{I}$, the following holds:*

$$\sum_{j\in C} x^j_k \leq \sum_{j\in C} \left(x^j_{t_0} + \sum_{t=t_0+1}^{t_1} u^j_t \right)$$

Proof For all $j \in C$, the sum of inequalities (3) $x^j_t - x^j_{t-1} \leq u^j_t$ from t_0+1 to k yields $x^j_k \leq x^j_{t_0} + \sum_{t=t_0+1}^{k} u^j_t$.
Hence, summing over all $j \in C$ and using $u^j_t \geq 0\ \forall t \in \mathcal{T}$, we obtain $\sum_{j\in C} x^j_k \leq \sum_{j\in C} \left(x^j_{t_0} + \sum_{t=t_0+1}^{t_1} u^j_t \right)$.
□

Recall t_{max} denotes the time period at which the demand is maximum on $\mathcal{I}$, and by Lemma 3, $\alpha_{t_{max}}(C) = \alpha_{\mathcal{I}}(C)$.

Theorem 6. *For all $C \subset \mathcal{N}$, $i \in C$, $\mathcal{I} = \{t_0,...,t_1\}$ such that $t_1 - t_0 \leq L^i$,*

(i) *If $\Delta \leq \alpha_{\mathcal{I}}(C) - 1$, the interval up-set inequality is valid for $P^n_{x,u}$,*
(ii) *If $\Delta > \alpha_{\mathcal{I}}(C)$, the interval up-set inequality is not valid for $P^n_{x,u}$.*

Proof (i): As the length of $\mathcal{I}$ is less than or equal to L^i, from the min-up inequality (1) we have

$$\sum_{t=t_0+1}^{t_1} u^i_t \leq \sum_{t=t_1-L^i+1}^{t_1} u^i_t \leq x^i_{t_1}.$$

The up-set inequality for $C\backslash\{i\}$ at time t_{max}, alongside with Lemma 4 applied to $C\backslash\{i\}$ and $k = t_{max}$, yields:

$$\alpha_{\mathcal{I}}(C) - 1 = \alpha_{t_{max}}(C) - 1 \leq \alpha_{t_{max}}(C\backslash\{i\}) \leq \sum_{j\in C\backslash\{i\}} x^j_{t_{max}} \leq \sum_{j\in C\backslash\{i\}} \left(x^j_{t_0} + \sum_{t=t_0+1}^{t_1} u^j_t \right)$$

Thus, summing up these two inequalities, we directly obtain the interval up-set inequality (10) with $\Delta \leq \alpha_{\mathcal{I}}(C) - 1$.

(ii): By definition of the rank of C at time t_{max}, note that there exists a solution $(x,u) \in P^n_{x,u}$ such that exactly $\alpha_{t_{max}}(C)$ units in C are up at time t_{max}. Let C_{max} be the set of units in C which are up at time t_{max} in solution (x,u). As in the proof of Lemma 3, a solution $(\tilde{x},\tilde{u})$ can be constructed from (x,u): solution $(\tilde{x},\tilde{u})$ is such that any unit in $C\backslash C_{max}$ is down on the whole interval $\mathcal{I}$ and any unit in C_{max} is up on the whole interval $\mathcal{I}$.

Thus, there exists a solution $(\tilde{x},\tilde{u})$ such that any unit in $C\backslash C_{max}$ is down on the whole interval $\mathcal{I}$ and any unit in C_{max} is up on the whole interval $\mathcal{I}$. So the following holds:

$$\alpha_{\mathcal{I}}(C) + \sum_{t=t_0+1}^{t_1} \tilde{u}^i_t = \tilde{x}^i_{t_1} + \sum_{j\in C\backslash\{i\}} \left(\tilde{x}^j_{t_0} + \sum_{t=t_0+1}^{t_1} \tilde{u}^j_t \right).$$

So if $\Delta > \alpha_{\mathcal{I}}(C)$, the interval up-set inequality is violated by $(\tilde{x},\tilde{u})$. □

The proof of Theorem 6 (i) shows that in the case $\Delta \leq \alpha_{\mathcal{I}}(C) - 1$, the interval up-set inequality is valid, but above all is dominated by up-set inequalities (8), min-up inequalities (1) and inequalities (3).

As the interval up-set inequality is dominated when $\Delta < \alpha_{\mathcal{I}}(C)$ and is not valid when $\Delta > \alpha_{\mathcal{I}}(C)$, we will consider from now on $\Delta = \alpha_{\mathcal{I}}(C)$, the interval up-set inequality becoming:

$$\alpha_{\mathcal{I}}(C) + \sum_{t=t_0+1}^{t_1} u^i_t \leq x^i_{t_1} + \sum_{j\in C\backslash\{i\}} \left(x^j_{t_0} + \sum_{t'=t_0+1}^{t_1} u^j_t \right) \tag{11}$$

The following result provides a characterization of validity for the interval up-set inequality. For any interval $\mathcal{I} = \{t_0,...,t_1\} \subset \mathcal{T}$ and for any $i \in \mathcal{N}$, we define the subdivision $Y^i_{\mathcal{I}} = (y_t, t \geq 1)$ of interval $\mathcal{I}$ as follows:

$$\begin{cases} y_0 &= t_0 \\ y_{t+1} &= \arg\max_{t' \in \{y_t+1,...,\min(y_t+\ell^i,t_1)\}} D_{t'} \qquad \forall t \geq 0 \end{cases}$$

Recall that for each subset of units $C \subset \mathcal{N}$, for each time period $t \in \mathcal{T}$ and for each unit $i \in C$, the static i-rank $\overline{\alpha}^i_t(C)$ is the smallest number of units that must be up in M at time t in order to satisfy the remaining demand $D_t - \sum_{j\notin M} P^j_{max}$, given that unit i is down at t.

Theorem 7 (Validity characterization)**.** *Let $C \subset \mathcal{N}$, for any $i \in C$, for any interval $\mathcal{I} = \{t_0,...,t_1\} \subset \mathcal{T}$ such that $t_1 - t_0 \leq L^i$, the interval up-set inequality is valid for $P^n_{x,u}$ if and only if:*

$\forall y \in Y^i_{\mathcal{I}}$, if $\overline{\alpha}^i_y(C) < \alpha_{\mathcal{I}}(C)$, then there is no solution $(x,u) \in P^n_{x,u}$ such that

$$\begin{cases} x^i_y = 0 \\ \textit{and} \\ \sum_{j\in C\backslash\{i\}} \left(x^j_{t_0} + \sum_{t=t_0+1}^{t_1} u^j_t \right) < \alpha_{\mathcal{I}}(C) \end{cases} \tag{12}$$

Proof Suppose there exists $y \in Y^i_{\mathcal{I}}$ such that there is a solution (x,u) satisfying (12). As unit i is down at time y, if $x^i_{t_1} = 1$ then $\sum_{t=t_0+1}^{t_1} u^i_t = 1 = x^i_{t_1}$. Also, by the min-up inequality, if $x^i_{t_1} = 0$ then $\sum_{t=t_0+1}^{t_1} u^i_t = 0 = x^i_{t_1}$. So as $x^i_y = 0$, we have $\sum_{t=t_0+1}^{t_1} u^i_t = x^i_{t_1}$. Moreover, as $\sum_{j\in C\setminus\{i\}} \left(x^j_{t_0} + \sum_{t=t_0+1}^{t_1} u^j_t \right) < \alpha_{\mathcal{I}}(C)$, the interval up-set inequality is violated by solution (x,u). So the interval up-set inequality is not valid in this case.

Now suppose for all $y \in Y^i_{\mathcal{I}}$ such that $\overline{\alpha}^i_y(C) < \alpha_{\mathcal{I}}(C)$, there is no solution satisfying (12). Thus, for all $y \in Y^i_{\mathcal{I}}$ such that $\overline{\alpha}^i_y(C) < \alpha_{\mathcal{I}}(C)$, each solution (x,u) is such that:

$$\begin{cases} x^i_y = 1 \\ \text{or} \\ \sum_{j\in C\setminus\{i\}} \left(x^j_{t_0} + \sum_{t=t_0+1}^{t_1} u^j_t \right) \geq \alpha_{\mathcal{I}}(C) \end{cases}$$

Let (x,u) be a solution.

- **Case 1**: there exists $y \in Y^i_{\mathcal{I}}$ such that $\overline{\alpha}^i_y(C) < \alpha_{\mathcal{I}}(C)$ and $x^i_y = 0$.
  In this case we have $\sum_{j\in C\setminus\{i\}} \left(x^j_{t_0} + \sum_{t=t_0+1}^{t_1} u^j_t \right) \geq \alpha_{\mathcal{I}}(C)$ so the interval up-set inequality is valid.
- **Case 2**: there exists $y \in Y^i_{\mathcal{I}}$ such that $\overline{\alpha}^i_y(C) \geq \alpha_{\mathcal{I}}(C)$ and $x^i_y = 0$. By definition of $\overline{\alpha}^i_y(C)$, we know there are at least $\overline{\alpha}^i_y(C)$ units up in C at time y since unit i is down. Thus by Lemma 4, we get:

$$\overline{\alpha}^i_y(C) \leq \sum_{j\in C\setminus\{i\}} x^j_y \leq \sum_{j\in C\setminus\{i\}} \left(x^j_{t_0} + \sum_{t=t_0+1}^{t_1} u^j_t \right)$$

  Since $\alpha_{\mathcal{I}}(C) \leq \overline{\alpha}^i_y(C)$ we can conclude that the interval up-set inequality is also valid in this case.
- **Case 3**: unit i is up on the whole interval $\mathcal{I}$. By definition, there are at least $\alpha_{\mathcal{I}}(C)$ units in C up at time t_{max}. Thus there are at least $\alpha_{\mathcal{I}}(C) - 1$ units in $C\setminus\{i\}$ up at time t_{max}. By Lemma 4,

$$\alpha_{\mathcal{I}}(C) - 1 \leq \sum_{j\in C\setminus\{i\}} x^j_{t_{max}} \leq \sum_{j\in C\setminus\{i\}} \left(x^j_{t_0} + \sum_{t=t_0+1}^{t_1} u^j_t \right)$$

  As unit i is up on $\mathcal{I}$, $x^i_{t_1} = 1$ and $\sum_{t=t_0+1}^{t_1} u^i_t = 0$ so the interval up-set inequality is also valid in this case.
- Note that there are no cases left. Indeed, if $x^i_y = 1$ for all $y \in Y^i_{\mathcal{I}}$ then i is up on the whole interval $\mathcal{I}$. If i shuts down at some time $t \in \mathcal{I}$, it remains down at least for ℓ^i time periods. But the distance between two elements y of $Y^i_{\mathcal{I}}$ is at most ℓ^i, by construction of the subdivision $Y^i_{\mathcal{I}}$.

□

Note that the validity condition for the whole polytope from Theorem 7 may be hard to check. Let us consider the supporting instance restricted to interval $\mathcal{I}$, denoted by $Inst(\mathcal{I})$. As opposite to the general case, where $\alpha_{\mathcal{I}}(C)$ is hard to compute, the computation of the maximum rank over interval $\mathcal{I}$ for instance $Inst(\mathcal{I})$ is much easier. Indeed, $\alpha_{\mathcal{I}}(C) = \overline{\alpha}_{\mathcal{I}}(C)$, as the solution such that the $\overline{\alpha}_{\mathcal{I}}(C)$ most powerful units of C are up on $\mathcal{I}$, alongside with all units in $\mathcal{N}\setminus C$, is a solution to $Inst(\mathcal{I})$.

Let us define $P^n_{x,u}(\mathcal{I})$ the polytope associated to $Inst(\mathcal{I})$. If there exists $y_t \in Y^i_{\mathcal{I}}$ such that $\overline{\alpha}^i_{y_t}(C) < \alpha_{\mathcal{I}}(C)$, then we can easily construct a solution $(x,u) \in P^n_{x,u}(\mathcal{I})$ satisfying inequalities (12), if condition $\alpha_{\mathcal{I}}(C\setminus\{i\}) < \alpha_{\mathcal{I}}(C)$ holds. It suffices to set unit i down on interval $[y_{t-1}+1, y_{t-1}+\ell^i]$ and up at all other times, and to set the $\overline{\alpha}_{\mathcal{I}}(C) - 1$ most powerful units of $C\setminus\{i\}$ up on $\mathcal{I}$, alongside with all units in $\mathcal{N}\setminus C$. Consequently, the following result holds, thus providing the necessary and sufficient validity condition for the interval up-set inequality in polytope $P^n_{x,u}(\mathcal{I})$.

Theorem 8 (Validity characterization in $P^n_{x,u}(\mathcal{I})$)**.** *Let $C \subset \mathcal{N}$, for any $i \in C$, for any interval $\mathcal{I} = \{t_0,...,t_1\} \subset \mathcal{T}$ such that $t_1 - t_0 \leq L^i$ and $\alpha_{\mathcal{I}}(C\backslash\{i\}) < \alpha_{\mathcal{I}}(C)$, the interval up-set inequality is valid for $P^n_{x,u}(\mathcal{I})$ if and only if $\forall y \in Y^i_{\mathcal{I}}$, $\overline{\alpha}^i_y(C) \geq \alpha_{\mathcal{I}}(C)$.*

We will see in Theorem 9.2 that in the particular case $\alpha_{\mathcal{I}}(C\backslash\{i\}) = \alpha_{\mathcal{I}}(C)$, the interval up-set inequality is dominated by another valid inequality.

5 Facial study for interval up-set inequalities

We now explore the cases in which interval up-set inequalities are facet defining for $P^n_{x,u}$. In the following, for given $C \subset \mathcal{N}$, $i \in C$ and $\mathcal{I} = \{t_0,...,t_1\} \subset \mathcal{T}$ such that $t_1 - t_0 \leq L^i$ and validity conditions from Theorem 7 are satisfied, we denote by F the face defined by the interval up-set inequality:

$$F = \left\{ (x,u) \in P^n_{x,u} \mid \alpha_{\mathcal{I}}(C) + \sum_{t=t_0+1}^{t_1} u^i_t = x^i_{t_1} + \sum_{j \in C\backslash\{i\}} \left(x^j_{t_0} + \sum_{t=t_0+1}^{t_1} u^j_t \right) \right\}$$

5.1 Necessary facet conditions

The necessary facet conditions are presented with a series of Theorems 9.1 to 9.4.

Theorem 9.1. *If F is a facet of $P^n_{x,u}$, then:*

$$\forall y \in Y^i_{\mathcal{I}},\ \alpha^i_y(C) \leq \alpha_{\mathcal{I}}(C) \tag{13}$$

Proof Suppose there exists $y \in Y^i_{\mathcal{I}}$ such that $\alpha^i_y(C) > \alpha_{\mathcal{I}}(C)$.

Let us define the following inequality

$$\alpha_{\mathcal{I}}(C) + \beta^i_y(1 - x^i_y) + \sum_{t=t_0+1}^{t_1} u^i_t \leq x^i_{t_1} + \sum_{j \in C\backslash\{i\}} \left(x^j_{t_0} + \sum_{t=t_0+1}^{t_1} u^j_t \right) \tag{14}$$

where $\beta^i_y = \alpha^i_y(C) - \alpha_{\mathcal{I}}(C) > 0$.

We prove that this inequality is valid. Indeed, if $x^i_y = 1$ then inequality (14) transforms to the interval up-set inequality, so it is valid. If $x^i_y = 0$, *i.e.* i is down at time y, there are at least $\alpha^i_y(C)$ units up in $C\backslash\{i\}$ at time y. Hence

$$\alpha_{\mathcal{I}}(C) + \beta^i_y = \alpha^i_y(C) \leq \sum_{j \in C\backslash\{i\}} \left(x^j_{t_0} + \sum_{t=t_0+1}^{t_1} u^j_t \right).$$

So inequality (14) is valid.

Thus there exists a valid inequality that dominates the interval up-set inequality. □

Theorem 9.2. *If F is a facet of $P^n_{x,u}$, then:*

$$\alpha_{\mathcal{I}}(C\backslash\{i\}) < \alpha_{\mathcal{I}}(C) \tag{15}$$

Proof If $\alpha_{\mathcal{I}}(C\backslash\{i\}) = \alpha_{\mathcal{I}}(C)$ then by Lemma 4, we get the following inequality by summing inequalities (3) and up-set inequalities (8):

$$\alpha_{\mathcal{I}}(C) = \alpha_{\mathcal{I}}(C\backslash\{i\}) \leq \sum_{j \in C\backslash\{i\}} \left(x^j_{t_0} + \sum_{t=t_0+1}^{t_1} u^j_t \right)$$

Summing this inequality to the min-up inequality $\sum_{t=t_0+1}^{t_1} u_t^i \leq x_{t_1}^i$ we get the interval up-set inequality.

The interval up-set inequality is then dominated by up-set inequalities (8), min-up inequalities (1) and inequalities (3).

□

For any $j \in C\backslash\{i\}$, let $C_{\alpha-1}^{i,j}$ be the set of the $\alpha_{\mathcal{I}}(C)-1$ most powerful units of $C\backslash\{i,j\}$. Theorem 9.3 states that if F is a facet, then, for any $j \in C\backslash\{i\}$, unit i, units in $C_{\alpha-1}^{i,j}$ and units in $\mathcal{N}\backslash C$ are enough to meet to the demand at time t_1.

Theorem 9.3. *If F is a facet of $P_{x,u}^n$, then:*

$$P_{max}^i + \sum_{k \in C_{\alpha-1}^{i,j}} P_{max}^k + \sum_{k \in \mathcal{N}\backslash C} P_{max}^k \geq D_{t_1}, \ \forall j \in C\backslash\{i\} \tag{16}$$

Proof Let us define the following inequality:

$$\alpha_{\mathcal{I}}(C) + \sum_{t=t_0+1}^{t_1} u_t^i \leq x_{t_1}^i + x_{t_1}^j + \sum_{k \in C\backslash\{i,j\}} \left(x_{t_0}^k + \sum_{t=t_0+1}^{t_1} u_t^k \right) \tag{17}$$

We claim that this inequality is valid for $P_{x,u}^n$ in case condition (16) is not satisfied.
Consider $(x,u) \in P_{x,u}^n$.

- **Case 1**: $x_{t_1}^j = 1$
  In this case, inequality (17) transforms to

$$\alpha_{\mathcal{I}}(C) - 1 + \sum_{t=t_0+1}^{t_1} u_t^i \leq x_{t_1}^i + \sum_{k \in C\backslash\{i,j\}} \left(x_{t_0}^k + \sum_{t=t_0+1}^{t_1} u_t^k \right) \tag{18}$$

  Unit j is up at time t_1 in solution (x,u) but may be down at another time of $\mathcal{I}$. Thus we define solution $(\overline{x},\overline{u})$ as equal to solution (x,u), except unit j is up at all times in solution $(\overline{x},\overline{u})$. Solution $(\overline{x},\overline{u})$ remains feasible, and since (18) does not depend on j, if (x,u) violates inequality (18), so does $(\overline{x},\overline{u})$. However j is up at all times in $(\overline{x},\overline{u})$ so $\overline{x}_{t_0}^j + \sum_{t=t_0+1}^{t_1} \overline{u}_t^j = 1$. It follows:

$$\alpha_{\mathcal{I}}(C) + \sum_{t=t_0+1}^{t_1} \overline{u}_t^i > \overline{x}_{t_1}^i + \sum_{k \in C\backslash\{i\}} \left(\overline{x}_{t_0}^k + \sum_{t=t_0+1}^{t_1} \overline{u}_t^k \right)$$

  which is a contradiction, as the interval up-set inequality was supposed to be valid.
- **Case 2**: $x_{t_1}^j = 0$
  As unit j is down at time t_1, there are at least $\alpha_{\mathcal{I}}(C)$ units up in $C\backslash\{i,j\}$ since we assumed that the $\alpha_{\mathcal{I}}(C)-1$ most powerful units of $C\backslash\{i,j\}$ do not suffice to meet the demand at t_1, even if unit i and units in $\mathcal{N}\backslash C$ are up. So $\alpha_{\mathcal{I}}(C) \leq \sum_{k \in C\backslash\{i\}} \left(\overline{x}_{t_0}^k + \sum_{t=t_0+1}^{t_1} \overline{u}_t^k \right)$. With the min-up inequality (4.2) we can conclude that (17) is valid.

□

Theorem 9.4. *If F is a facet of $P_{x,u}^n$, then:*

$$\forall k \in \mathcal{N}\backslash\{i\},\ \forall t \in \mathcal{T}\backslash\{1\}\ , \begin{cases} \exists (x,u) \in F \text{ such that } u_t^k = 1 & \text{(19a)} \\ \exists (x,u) \in F \text{ such that } x_{t-1}^k - x_t^k = 1 & \text{(19b)} \\ \exists (x,u) \in F \text{ such that } x_{t'}^k = 0,\ \forall t' \in [t, t+\ell^k] & \text{(19c)} \end{cases}$$

First note that condition (19b) means that unit k shuts down at time t in solution (x,u), and that condition (19c) states that for any $k \in \mathcal{N}\backslash\{i\}$ and $\forall t \in \mathcal{T}$, there exists a solution $(x,u) \in F$ satisfying property $\Pi_{k,t}$.

Proof If for given time t and unit k, condition (19a) is not satisfied then all solutions $(x,u) \in F$ are such that $u_t^k = 0$. Thus the dimension of F is less than $dim(P_{x,u}^n) - 1$. Similarly, if condition (19b) does not hold then $F \subset \{u_t^k = x_t^k - x_{t-1}^k\}$ so F is not a facet. It has been proven in Lemma 2 that condition (19c) holds for any facet F.

□

If one of the conditions of the series of Theorems 9.1, 9.2, 9.3 and 9.4 does not hold, a valid inequality dominating the interval up-set inequality can be deduced. Note that conditions of Theorems 9.1, 9.2 and 9.3 are easy to check. The inequalities they produce can then be used as cuts in a Branch & Cut algorithm. Moreover, the inequality deduced from Theorem 9.1 can be extended to the following larger family of inequalities:

$$\alpha_{\mathcal{I}}(C) + \beta_{t_1}^i(1 - x_{t_1}^i) + \sum_{t=t_0+1}^{t_1} (1+\beta_t^i)u_t^i \leq x_{t_1}^i + \sum_{j \in C\backslash\{i\}} \left(x_{t_0}^j + \sum_{t=t_0+1}^{t_1} u_t^j \right)$$

where $\beta_y^i = \max_{t \in [\max(y-\ell^i, t_0), y-1]} (\alpha_t^i(C) - \alpha_{\mathcal{I}}(C)) > 0$. As the β coefficients depend on subset C, it would make the separation procedure hard to manage. Therefore we do not separate these inequalities in the Branch & Cut algorithm presented in Section 6.

5.2 Facet characterization in $P_{x,u}^n(\mathcal{I})$

Theorem 9 provides necessary facet conditions for F. We now discuss in which cases these conditions are necessary and sufficient. First, we give a technical lemma stating that, in any solution $(x,u) \in F$, each unit $j \in C\backslash\{i\}$ starts at most once on interval $\mathcal{I}$.

Lemma 5. *Let* $(x,u) \in F$. *For all* $j \in C\backslash\{i\}$, $x_{t_0}^j + \sum_{t=t_0+1}^{t_1} u_t^j \leq 1$.

Proof Suppose there exists $j_0 \in C\backslash\{i\}$ such that $x_{t_0}^j + \sum_{t=t_0+1}^{t_1} u_t^j \geq 2$. We define solution $(\bar{x},\bar{u})$ to be equal to (x,u), except that unit j_0 is up at all times in $(\bar{x},\bar{u})$. Obviously $(\bar{x},\bar{u}) \in P_{x,u}^n$. However, the following holds:

$$\alpha_{\mathcal{I}}(C) + \sum_{t=t_0+1}^{t_1} u_t^i - x_{t_1}^i - \sum_{j \in C\backslash\{i,j_0\}} \left(x_{t_0}^j + \sum_{t=t_0+1}^{t_1} u_t^j \right) \geq 2 \tag{20}$$

As $(\bar{x},\bar{u})$ is equal to (x,u) except on (x^j,u^j) coordinates, we can replace $(\bar{x},\bar{u})$ by (x,u) in inequality (20). Moreover, as j_0 is up at all times in $(\bar{x},\bar{u})$, we have $x_{t_0}^j + \sum_{t=t_0+1}^{t_1} u_t^j = 1$. Adding this equality to inequality (20) we get:

$$\alpha_{\mathcal{I}}(C) + \sum_{t=t_0+1}^{t_1} u_t^i > x_{t_1}^i + \sum_{j \in C\backslash\{i\}} \left(x_{t_0}^j + \sum_{t=t_0+1}^{t_1} u_t^j \right)$$

which means $(\bar{x},\bar{u})$ violates the interval up-set inequality. As it was assumed to be valid for $P_{x,u}^n$, we have here a contradiction.

□

Now recall $P^n_{x,u}(\mathcal{I})$, the polytope associated to the supporting instance $Inst(\mathcal{I})$ of interval $\mathcal{I}$. We denote by $F_\mathcal{I}$ the face of $P^n_{x,u}(\mathcal{I})$ associated to the interval up-set inequality.

Theorem 10. *$F_\mathcal{I}$ is a facet of $P^n_{x,u}(\mathcal{I})$ if and only if necessary facet conditions of Theorems 9.1, 9.2, 9.3 and 9.4 hold.*

Proof The direct implication has been proven in Theorems 9.1, 9.2, 9.3 and 9.4. We now prove the return implication.

First, for any subset $C_{up} \subset C\backslash\{i\}$ and $t,t' \in \mathcal{I}$, let $v(C_{up},[t,t'])$ be the vector such that units in subset C_{up} are up at all times of $\mathcal{I}$, units in $C\backslash(C_{up} \cup \{i\})$ are down at all times of $\mathcal{I}$, unit i is up on interval $[t,t']$ and down at all other times, and units in $\mathcal{N}\backslash C$ are up at all times.

Now suppose

$$F_\mathcal{I} \subset \{(x,u) \in P^n_{x,u}(\mathcal{I}) \mid \sum_{j\in\mathcal{N}} \left(\sum_{t\in\mathcal{I}} a^j_t x^j_t + \sum_{t\in\mathcal{I}\backslash\{t_0\}} b^j_t u^j_t \right) = \gamma \, (\star)\}$$

where $\gamma \in \mathbb{R}$, and $\forall j \in \mathcal{N}$, $\forall t \in \mathcal{I}$, $a^j_t \in \mathbb{R}$, $b^j_t \in \mathbb{R}$.

We claim that $F_\mathcal{I} = \{(x,u) \in P^n_{x,u}(\mathcal{I}) \mid \sum_{j\in\mathcal{N}} \left(\sum_{t\in\mathcal{I}} a^j_t x^j_t + \sum_{t\in\mathcal{I}\backslash\{1\}} b^j_t u^j_t \right) = \gamma\}$, which proves that F is a facet of $P^n_{x,u}$.

Let $k \in \mathcal{N}\backslash C$. There are neither x^k nor u^k variables appearing in the interval up-set inequality, and by condition (19c), for all $t \in \mathcal{T}$, there exists a solution $(x,u) \in F$ such that $\Pi_{k,t}$ is satisfied. So by Lemma 2 (ii), vectors $\Psi^k_{t,t+\ell^j}(x,u)$ and $\Psi^k_{t+1,t+\ell^j}(x,u)$ are solutions of $F_\mathcal{I}$. It follows $a^k_t = 0$, $t \geq 1$. Furthermore, by condition (19a), for any $t \geq t_0+1$ there is a solution $\chi^u_{k,t}(F_\mathcal{I}) \in F_\mathcal{I}$ such that unit k starts up at time t. We define $\tilde{\chi}^u_{k,t}(F_\mathcal{I})$ to be equal to $\chi^u_{k,t}(F_\mathcal{I})$ except that unit k is up at all times. As $\tilde{\chi}^u_{k,t}(F_\mathcal{I}) \in F_\mathcal{I}$, it follows $b^k_t = 0$, $t \geq t_0+1$.

By construction of the subdivision $Y^i_\mathcal{I}$, we must have $t_{max} \in Y^i_\mathcal{I}$. Thus, by condition (13), $\overline{\alpha}^i_{t_{max}} \leq \alpha_\mathcal{I}(C)$. So, if we denote by C^i_α the set of the $\alpha_\mathcal{I}(C)$ most powerful units of $C\backslash\{i\}$, the units in C^i_α are sufficient to satisfy the demand at time t_{max} (provided that units in $\mathcal{N}\backslash C$ are all up), thus, they are sufficient to satisfy the demand at any time $t \in \mathcal{I}$. Therefore vector $v(C^i_\alpha,\varnothing) \in F_\mathcal{I}$. Moreover, for any $t < t_1$, $v(C^i_\alpha,[t_0,t]) \in F_\mathcal{I}$. It follows $a^i_t = 0$, $t \in [t_0,t_1-1]$. For any $t > t_0$, we also have $v(C^i_\alpha,[t,t_1]) \in F_\mathcal{I}$. We thus get $b^i_t = b^i_{t_1} = -a^i_{t_1}$, $t \in [t_0+1,t_1]$.

Let $j \in C\backslash\{i\}$. By condition (19b), there exists a solution $\chi^d_{j,t_0+1}(F_\mathcal{I}) \in F_\mathcal{I}$ such that unit j shuts down at time t_0+1. By Lemma 5, if j shuts down at time t_0+1, it remains down on $[t_0+1,t_1]$ (otherwise we would have $x^j_{t_0} + \sum_{t=t_0+1}^{t_1} u^j_t > 1$). Thus we can define $\chi^d_{j,t}(F_\mathcal{I}) \in F_\mathcal{I}$, $t \in [t_0+1,t_1+1]$, as equal to $\chi^d_{j,t_0+1}(F)$ except that j shuts down at time t instead of time t_0+1 (if $t = t_1+1$ then j is up at all times). Thus we get $a^j_t = 0$, $t > t_0$. Similarly, by condition (19a), there exists a solution $\chi^u_{j,t_1}(F_\mathcal{I}) \in F_\mathcal{I}$ such that unit j starts up at time t_1. By Lemma 5, unit j is down on interval $[t_0,t_1-1]$ in solution $\chi^u_{j,t_1}(F)$. So we can define solutions $\chi^u_{j,t}(F_\mathcal{I}) \in F_\mathcal{I}$, $t \in [t_0+1,t_1]$, as equal to $\chi^u_{j,t_1}(F) \in F$ except that j starts up earlier (at time t instead of time t_1). With these vectors we get $a^j_{t_0} = b^j_{t_1} = b^j_t$, $t > t_0+1$.

Now equality $(\star)$ is proven to be of the form

$$\gamma + a^i \sum_{t=t_0+1}^{t_1} u^i_t = a^i x^i_{t_1} + \sum_{j\in C\backslash\{i\}} a^j \left(x^j_{t_0} + \sum_{t=t_0+1}^{t_1} u^j_t \right)$$

We prove that $a^i = a^j$, $j \in C$.

Let $j \in C^i_\alpha$ and $t \geq t_0+1$. By condition (19a), there exists a solution in F such that unit j starts up at time t. Thus, in this solution, j is down at time $t-1$, and there cannot be more than $\alpha_\mathcal{I}(C)-1$ units of $C\backslash\{i,j\}$ up at time $t-1$ (otherwise the interval up-set inequality would not be satisfied at equality).

This means that at time $t-1$, the demand can be satisfied by units of $C^{i,j}_{\alpha-1}$, unit i and units of $\mathcal{N}\backslash C$. Moreover, if $t = t_1$, the demand at time t_1 can also be satisfied by those units, as condition (16) holds. So $v(C^{i,j}_{\alpha-1},[t_0,t_1]) \in F_{\mathcal{I}}$. Considering vector $v(C^i_\alpha,\varnothing) \in F_{\mathcal{I}}$, we get $a^j = a^i$.

Let $j \in C\backslash(C^i_\alpha \cup \{i\})$. Recall vector $\chi^d_{j,t_1+1}(F_{\mathcal{I}}) \in F_{\mathcal{I}}$, in which unit j is up at all times. In the solution defined by $\chi^d_{j,t_1+1}(F_{\mathcal{I}})$, there are at most $\alpha_{\mathcal{I}}(C)-1$ units of $C\backslash\{i,j\}$ up on $\mathcal{I}$ (otherwise $\chi^d_{j,t_1+1}(F_{\mathcal{I}}) \notin F_{\mathcal{I}}$). So there exists a unit $k \in C^i_\alpha$ which is down at all times of $\mathcal{I}$ in solution $\chi^d_{j,t_1+1}(F_{\mathcal{I}})$. We define solution $\tilde{\chi}^d_{j,t_1+1}(F_{\mathcal{I}}) \in F$ as equal to $\chi^d_{j,t_1+1}(F_{\mathcal{I}})$ except that unit j is down at all times, and unit k is up at all times. It follows $a^j = a^k$. Since $a^k = a^i$, this concludes the proof.

□

Theorem 10 states that the interval up-set inequality is facet defining for $P^n_{x,u}(\mathcal{I})$, provided that necessary facet conditions of Theorems 9.1, 9.2, 9.3 and 9.4 hold.

An interesting problem is to extend this result to the whole polytope $P^n_{x,u}$. The difficulty is induced by some side effects happening at the outer edges of interval $\mathcal{I}$. However, we can provide some insights into how the result of Theorem 10 could be extended to $P^n_{x,u}$. Indeed, in most cases, any vector $(x,u) \in F_{\mathcal{I}}$ introduced in the proof of Theorem 10 can be extended to a vector of F. By defining C_{down} as the set of units in C which are down over interval $\mathcal{I}$ in solution (x,u), we can extend vector (x,u) to the whole time horizon T by gradually shutting down the units in C_{down} before time t_0, and then gradually start them up after t_1, so that their minimum-down time is satisfied and the demand is met. If such a kind of extension is possible for all the vectors introduced, then it proves that the considered interval up-set inequality defines also a facet of $P^n_{x,u}$, provided that additional vectors of F can be found, using condition of Theorem 9.4, to show that there are no variables x_t or u_t outside $\mathcal{I}$ defining F.

For example, let us consider $T = 4$, with $D = [20,\ 10,\ 10,\ 20]$, and three units such that $P^i_{max} = 10$, $L^i = 1$, $\ell^i = 2$, $i \in \{1,2,3\}$. The interval up-set inequality corresponding to $C = \{1,2,3\}$, $i = 1$ and $\mathcal{I} = \{2,3\}$ defines a facet of $P^n_{x,u}(\mathcal{I})$, from Theorem 10. This inequality also defines a facet of $P^n_{x,u}$, as the vectors introduced in Theorem 10 can be extended to vectors of $P^n_{x,u}(\mathcal{I})$.

However, in some singular cases, there may be no way to satisfy the demand outside interval $\mathcal{I}$ while satisfying the minimum-down times of units in C_{down}. In these cases, it is likely that interval up-set inequalities are dominated by some stronger inequalities taking into account the demand outside $\mathcal{I}$ which is higher than the demand inside $\mathcal{I}$.

6 Branch-&-Cut algorithm

In this section, we study the separation of up-set and interval up-set inequalities, in order to come up with a cutting plane generation procedure to be used in a Branch & Cut algorithm.

6.1 Separation of up-set inequalities

We first consider the separation problem of static up-set inequalities, defined as follows: given a set of units $\mathcal{N}$ with maximum power output P^j_{max}, $j \in \mathcal{N}$, a time horizon $\mathcal{T}$, a demand D_t, $t \in \mathcal{T}$, and a fractionnal solution $(\bar{x},\bar{u})$, the question is whether there exists a set $C \subset \mathcal{N}$ and a time period $t \in \mathcal{T}$ such that $\sum_{j \in \overline{C}} \bar{x}^j_t < \alpha_t(C)$.

The static up-set inequalities where $\overline{\alpha}_t(C) = 1$ correspond, in the context of the 0-1 knapsack problem, to the cover inequalities, which are known to be NP-complete to separate. The general static up-set inequalities correspond to the extended cover inequalities, whose separation problem's complexity is an open question (see Kaparis and Letchford (2010)). The following theorem states that the separation problem of static up-set inequalities is NP-complete.

Theorem 11. *The separation problem of static up-set inequalities is NP-complete.*

Proof The separation problem of up-set inequalities is obviously in NP, so we prove that the knapsack problem reduces to the separation problem.

Consider an instance of the knapsack problem with n objects associated with weights w^i and values $0 < a^i \leq 1$, $i \in \{1,...,n\}$. Let W be the capacity of the knapsack and let $K < \sum_{i\in\{1,...,n\}} a^i$, the question is whether there exists a subset S of objects such that the total weight of S is less than W, and the total value of S is greater than K.

Let us consider the following instance of the separation problem, where $A = \sum_{i\in\{1,...,n\}} a^i$, $\underline{a} = \min_{i\in\{1,...,n\}} a^i$, $\overline{a} = \max_{i\in\{1,...,n\}} a^i$ and $\lambda = \frac{\underline{a}}{\overline{a}(A-K)}$:

$$\begin{cases} \mathcal{N} = \{1,...,n+1\} \\ T = 1 \text{ and } D_1 = W \\ \forall i \in \{1,...,n\},\ P^i_{max} = w^i \text{ and } P^{n+1}_{max} = D_1 \\ \forall i \in \{1,...,n\},\ \overline{x}^i_1 = \lambda a^i \text{ and } \overline{x}^{n+1}_1 = 1 - \underline{a}/\overline{a} \end{cases}$$

Note that $\overline{x}^i_1 \in [0,1]$ for any i because it can be supposed w.l.o.g. that $K \leq A - \underline{a}$ (otherwise the only possible solution to the knapsack instance would be to include all objects in the knapsack).

Any subset $C \in \mathcal{N}$ of this instance has rank $\alpha_1(C)$ at most 1: indeed, if unit $n+1$ is in C then the demand is satisfied with one unit in C (unit $n+1$), and thus the corresponding rank is at most 1. If unit $n+1$ is not in C then the rank of C is zero since no unit in C is needed to meet the demand. Here, the separation problem of static up-set inequalities is to find a subset C (containing unit $n+1$) such that $\alpha_1(C) = 1$ and $\sum_{j\in C} \overline{x}^j_1 < 1$. It amounts to finding a subset $\overline{C} = \mathcal{N} \backslash C$ such that $\sum_{i\in\overline{C}} P^i_{max} < D_1$ and $\sum_{j\in\overline{C}} \overline{x}^j_1 > \left(\sum_{j\in\mathcal{N}} \overline{x}^j_1\right) - 1$, *i.e.* $\lambda \sum_{j\in\overline{C}} a^j > \lambda A + \overline{x}^{n+1}_1 - 1$, *i.e* $\sum_{j\in\overline{C}} a^j > K$. A solution to this separation problem is a solution to the knapsack instance, where the elements chosen in the knapsack would be exactly the elements in $\overline{C}$. Conversely a solution to the above knapsack instance is a solution to this separation problem.

□

In order to show that the separation of extended cover inequalities for the knapsack polytope (cf. Balas (1975)) is also NP-complete, it suffices to see that any instance of the knapsack problem can be transformed to an extended cover separation problem for instances with n objects, such that objects $\{2,...,n\}$ fit in the knapsack. Thus any cover C will contain object 1, and it follows $E(C) = C$.

We will see in Section 7 that, in practice, these inequalities are very effective. Classically, static up-set inequalities, or extended cover inequalities, are generated by a procedure that searches for cover inequalities and lifts them to stronger inequalities (Balas (1975)). Note that Kaparis and Letchford (2010) propose a heuristic for which the search is based on the construction of a cover set.

We propose an alternate separation algorithm for static up-set inequalities, taking advantage of the facet defining conditions we presented in Section 4.1.

Separation algorithm for up-set inequalities
Given a fractionnal solution $(\overline{x},\overline{u})$, for a given time period t, we first sort the units in non-decreasing order of $\frac{\overline{x}^j_t}{P^j_{max}}$ and store them in a list L. We then construct a set C by iteratively appending units of L, until the corresponding up-set inequality is violated. Hence, we first define the set S which contains the $|C| - \alpha_t(C) + 1$ less powerful units of C, *i.e.* units with smallest P_{max}. Finally we remove units from S one by one until obtaining a minimal up-set, and then we swap elements in and out of S to obtain a strong set. Finally, the separation procedure returns the extension of S.

6.2 Separation of interval up-set inequalities

In our Branch & Cut algorithm, we consider the following *static interval up-set* inequalities:

$$\overline{\alpha}_{\mathcal{I}}(C) + \sum_{t=t_0+1}^{t_1} u_t^i \leq x_{t_1}^i + \sum_{j \in C \setminus \{i\}} \left(x_{t_0}^j + \sum_{t'=t_0+1}^{t_1} u_t^j \right).$$

From Theorem 7, if validity condition $y \in Y_{\mathcal{I}}^i$, $\overline{\alpha}_y^i(C) \geq \overline{\alpha}_{\mathcal{I}}(C)$ holds, these inequalities are valid for $P_{x,u}^n$. Note that by Theorem 8, these inequalities correspond exactly to every valid interval up-set inequality for restricted polytope $P_{x,u}^n(\mathcal{I})$.

Static up-set inequalities are particular cases of static interval up-set inequalities where $t_0 = t_1$. Since from Theorem 11 the separation of static up-set inequalities is an NP-hard problem, we have the following result.

Theorem 12. *The separation of static interval up-set inequalities is an NP-hard problem.*

We propose the following separation algorithm for static interval up-set inequality, which is an extension of our algorithm to separate up-set inequalities.

Separation algorithm for interval up-set inequalities
Given a fractionnal solution $(\overline{x}, \overline{u})$, a time interval $[t_0, t_1]$ and a unit i such that $L^i \geq t_0 - t_1$, we first sort the units in non-decreasing order of $\frac{\overline{x}_{t_0}^j + \sum_{t=t_0+1}^{t_1} \overline{u}_t^j}{P_{max}^j}$ and store them in a list L. We then construct a set C by iteratively adding units of L in it, until the corresponding interval up-set inequality is violated. In this case, the separation procedure returns the corresponding set C.

7 Experimentation

In this section, some computational results relative to formulation (1) - (7) are presented. To evaluate the effectiveness of up-set and interval up-set inequalities, we separate them throughout a Branch & Cut tree, using Cplex 12.6.1 with default settings. All experiments were performed using one thread of a PC with a 64 bits Intel(R) Core(TM) i7-2600K processor running at 3.4GHz, and 16 GB of RAM memory. The problems are solved until optimality (defined within 10^{-6} of relative optimality tolerance) or until the time limit of 3600 seconds is reached.

We compared three methods to solve the MUCP:

- Cplex: Default Cplex used by its C++ API. In order to obtain non-biaised comparisons, we include an empty Branch & Cut instruction in the implementation.
- UP: Branch & Cut algorithm using only up-set cuts, separated with the algorithm given in Section 6.1. The cut generation is stopped whenever 300 inequalities have been produced.
- UP+IUP: Branch & Cut algorithm using up-set inequalities as described previously, and interval up-set inequalities. Interval up-set inequalities are separated with the algorithm given in Section 6.2 only at the root node when both Cplex and UP algorithm produce no more cuts.

7.1 Instances

For each instance, we generate a “2-peak per day” type demand with a large variation between peak and off-peak values: during one day, the typical demand in energy during one day has two peak periods, one in the morning and one in the evening. The amplitudes between peak and off-peak periods have similar characteristics to those in the dataset from Carrion and Arroyo (2006).

For all instances we randomly generate initial conditions e^i and set $\tau^i = 0$, for each unit i.

Note that many parameters are required to define an MUCP instance. Therefore, preliminary experiments were performed to emphasize which parameters affect the performances the most. The time horizon T has low impact on the computation time, as opposed to the number of units n. A fixed time horizon is thus considered for each instance while the number of units n varies depending on the instance class. We set the time horizon to $T = 96$ as it corresponds to the standard value of T in the short term UCP solved at EDF. The number of units n is chosen in the range $[10, 50]$. Moreover, the tightness of the production range $[P^i_{min}, P^i_{max}]$ for each unit i deeply affects the computation time of the MUCP. We use this observation to generate a particular class of instances.

We generate the following classes of instances:

- R : The realistic (R) instances are generated using data for real EDF units. We partitioned the units into three types, depending on their fuel: coal, gas and fuel oil. For each fuel type, we consider the characteristics (P_{min}, P_{max}, L, ℓ, c_f, c_0, c_p) of each real EDF unit, and we draw a correlation matrix between their characteristics. Moreover there is a typical range for each characteristic depending on the fuel. Thus, for each instance, we generate $\frac{n}{3}$ units with the characteristics based on the correlations and ranges of each fuel.
- L : The literature (L) instances are similarly generated, using the unit characteristics from the dataset presented in Carrion and Arroyo (2006). Note that in this class only one type of unit is considered, as the units characteristics appear to be similar to each other in the dataset from Carrion and Arroyo (2006).
- TPR : The "tight production range" (TPR) instances are generated as literature instances in which, for each unit i, we set P^i_{min} as a percent of P^i_{max}, namely 50 %, 75 % and 100 %. These classes are respectively denoted by TPR-50, TPR-75 and TPR-100. In these instances, P_{min} is closer to P_{max} than in the first two classes. As a basis for comparison, in the literature class, P_{min} is around 25 % of P_{max}, and varies from to 25 to 70% in the realistic class. Note that operating rules applying to the UCP lead to restrict the unit production range considered in the MUCP. A real-world fossil fuel unit must operate on a discrete set of productions instead of a continuous domain of production. Furthermore once a unit reaches a definite production, it must satisfy a *minimum operation time constraint*, *i.e.* the unit production must be constant or within a restricted range for a given time. Roughly speaking units in the UCP could be seen as units with a tight production range in the MUCP. Therefore, TPR instances are designed to give us an insight into the potential effectiveness of the interval up-set inequalities for the UCP.

It is well known that symmetries in the MUCP also deeply affect the computation time. In the dataset from Carrion and Arroyo (2006), symmetries are introduced by duplicating production units. We thus generate instances with symmetries (S) and instances without symmetries (NS) for each class R, L, TPR-50, TPR-75 and TPR-100. Units of NS instances are randomly generated according to the procedures previously described. Units of S instances are generated as follows: some units are randomly generated and then are duplicated d times, where d is randomly selected in $[1, \frac{n}{10}]$ for each unit, in order to obtain a total of n units.

7.2 Computational results

The result are presented with respect to instances partitioned into *categories* defined as the triplets (class, symmetries, size): classes R, L, TPR-50, TPR-75 or TPR-100; symmetry type NS or S; and size $n = 10$, $n = 20$ or $n = 50$. For each class and size considered, we generated 50 instances with symmetries (S) and

50 instances without symmetries (NS). For each class, we generated instances of various sizes: R and L classes of size $n = 20$ and $n = 50$ and TPR classes of size $n = 10$ and $n = 20$

As all R and L (resp. TPR-100) instances with $n = 20$ are already very well solved (resp. intractable) with Cplex, we generated instances with $n = 50$ (resp. $n = 10$). Note that size $n = 20$ has been considered for all instance classes.

As some instances are already very quickly solved by Cplex, adding new cuts for this kind of instances cannot compensate for the separation time it takes. Thus we want to discriminate between easy instances and hard instances. Then, inside a given instance category, an instance is said to be *hard* if it belongs to the 50% most difficult instances with respect to Cplex computation time, whenever this time exceeds 10 seconds.

The experimental results are presented in two tables as follows.

Table 1 displays a comparison between Cplex and UP+IUP for each category of instances. For this purpose, Table 1 indicates #H, the number of hard instances, and for each method:

Nodes (N): number of nodes in the Branch & Cut tree.
- Av.: average number
- Min: minimum number
- Max: maximum number

CPU: CPU time (in seconds)
- Av_{all}: average value for all instances
- Av_H: average value for hard instances
- Min: minimum value
- Min_H: minimum value for hard instances
- Max: maximum value

User cuts (only for UP+IUP): number of user cuts, totalizing up-set and interval up-set cuts
- Av.: average number
- Min: minimum number
- Max: maximum number.

Note that R and L instances of size $n = 20$ were all solved in less than 10 seconds, with an average CPU time less than 1 second. Similarly, R and L instances of size $n = 50$ without symmetries are easy instances, with an average CPU time of 5 seconds, a maximum CPU time of 41 seconds, and only eight instances with a CPU time exceeding 10 seconds. Another class of instance which is solved easily is TPR-50, with very few hard instances and a maximum CPU time of 30 seconds.

When the tightness of the production range increases to 75%, the instances are much harder than instances with a larger production range (like instances of classes R and L), but they remain tractable for $n = 20$. The TPR-100 instances appear to be very difficult to solve. For each instance of size $n = 20$, Cplex reaches the time limit of one hour. Instances of size $n = 10$ are already very hard, as the average CPU time is around 1000 seconds.

Note that for $n = 10$, symmetries do not seem to impact the resolution. However, for $n = 20$, and even more for $n = 50$, symmetries deeply affect the computation time.

Table 2 provides more details for the comparison of the three methods. As shown in Table 1, there is an important variability in the computation time within a given instance category. We then introduce the improvement score, which is a performance ratio comparing Cplex to one of our methods (UP or UP+IUP) denoted by B&C. The *improvement scores* relative to the number of nodes (N), the CPU time (CPU) and the linear relaxation value at the root (LR) are defined as follows.

$$I_N = 2\frac{N(Cplex)-N(B\&C)}{N(Cplex)+N(B\&C)} \quad I_{CPU} = 2\frac{CPU(Cplex)-CPU(B\&C)}{CPU(Cplex)+CPU(B\&C)} \quad I_{LR} = 2\frac{LR(B\&C)-LR(Cplex)}{LR(Cplex)+LR(B\&C)}$$

Table 1: Values

			Cplex								UP+IUP										
			Nodes (N)			CPU					Nodes (N)			User cuts			CPU				
		#H	**Av.**	**Min**	**Max**	**Av**$_{all}$	**Av**$_H$	**Min**	**Min**$_H$	**Max**	**Av.**	**Min**	**Max**	**Av.**	**Min**	**Max**	**Av**$_{all}$	**Av**$_H$	**Min**	**Min**$_H$	**Max**
R	(NS)	0	6.78	0	80	0.6386	0	0.29	0	1.75	3.88	0	38	1.84	0	13	0.6598	0	0.3	0	1.83
$n = 20$	(S)	0	16.92	0	318	0.6612	0	0.25	0	1.4	13.52	0	456	2.64	0	23	0.7036	0	0.29	0	1.65
R	(NS)	8	741.48	0	4.48e+03	4.809	14.57	1.11	10.09	24.78	587.18	0	6.68e+03	66.3	0	301	5.808	18	1.11	3.97	41.22
$n = 50$	(S)	25	174451	0	9.9e+05	804.5	1604	1.44	20.4	3600	110953	0	1.02e+06	175	0	466	542.1	1079	1.33	12.61	3600
L	(NS)	0	1.92	0	19	0.6926	0	0.31	0	2.04	1.48	0	15	0.48	0	8	0.743	0	0.33	0	2.18
$n = 20$	(S)	0	20.52	0	161	0.9008	0	0.38	0	2.73	19.02	0	199	2.12	0	14	0.9754	0	0.45	0	2.93
L	(NS)	2	55.88	0	617	2.826	10.09	1.46	10.07	10.12	48.9	0	675	3.02	0	50	3.428	11.29	1.53	10.16	12.41
$n = 50$	(S)	25	340619	0	2.15e+06	888.3	1770	1.85	23.69	3600	211246	0	1.4e+06	56.6	0	301	828.3	1650	1.95	4.06	3600
TPR-50	(NS)	0	5.66	0	99	0.6296	0	0.16	0	4.26	4.6	0	110	3.96	0	32	0.6316	0	0.15	0	4.28
$n = 10$	(S)	0	4.24	0	51	0.5864	0	0.21	0	2.28	2.6	0	39	2.84	0	25	0.602	0	0.21	0	2.12
TPR-50	(NS)	0	34.88	0	414	1.647	0	0.62	0	5.53	33.9	0	331	3.92	0	21	1.717	0	0.62	0	4.79
$n = 20$	(S)	3	339.32	0	3.64e+03	3.569	18.48	0.61	12.43	29.31	388.08	0	4.24e+03	25.6	0	163	4.003	21.61	0.62	14.43	35
TPR-75	(NS)	14	3995.84	0	1.45e+05	95.28	328.5	0.4	11.78	3600	3986.84	0	1.54e+05	86.1	0	394	93.13	321	0.41	8.59	3600
$n = 10$	(S)	13	528.84	0	6.61e+03	11.46	34.2	0.38	11.39	103.4	278.88	0	2.32e+03	82	0	259	8.249	23.16	0.39	8.62	43.38
TPR-75	(NS)	25	2097.36	13	1.51e+04	52.18	89.91	2.74	33.72	332.7	1899.7	19	2.21e+04	186	7	323	53.35	93.14	2.84	25.24	519.42
$n = 20$	(S)	25	28183.1	3	7.11e+05	281.2	541.7	2.06	48.62	3600	27516.4	2	6.54e+05	213	5	334	280.4	538.1	2.73	27.21	3600
TPR-100	(NS)	25	53653.4	0	2.04e+05	1166	2270	1.19	256.38	3600	47874.3	0	1.8e+05	310	11	463	1188	2297	1.22	171.59	3600
$n = 10$	(S)	25	43384.7	74	1.81e+05	975.4	1881	5.28	233.2	3600	41068	29	1.47e+05	324	99	442	994.9	1938	5.91	327.79	3600
TPR-100	(NS)	25	96232.7	51100	1.34e+05	3600	-	3600	-	3600	92139.6	63800	1.19e+05	311	300	365	3600	-	3600	-	3600
$n = 20$	(S)	25	101096	58885	1.5e+05	3600	-	3600	-	3600	99482.7	50400	1.48e+05	309	0	342	3600	-	3600	-	3600

Table 2: Average improvement score for various indicators

		UP				UP+IUP			
		I_N	I_{LR}	$I_{CPU}(all)$	$I_{CPU}(hard)$	I_N	I_{LR}	$I_{CPU}(all)$	$I_{CPU}(hard)$
(R) $n = 50$	**(S)**	62.3%	8.3%	34.1%	44.5%	61.8%	8.4%	19.0%	42.4%
(L) $n = 50$	**(S)**	42.3%	5.5%	18.9%	25.7%	40.4%	5.6%	9.5%	22.9%
TPR-75	**(NS)**	21.6%	4.0%	7.9%	17.2%	46.1%	9.3%	6.4%	19.9%
$n = 10$	**(S)**	25.1%	-0.9%	6.4%	9.3%	44.6%	6.6%	14.9%	30.5%
TPR-75	**(NS)**	16.2%	-0.8%	5.1%	6.7%	27.6%	-1.9%	7.0%	9.9%
$n = 20$	(S)	25.0%	0.2%	8.3%	6.3%	27.7%	2.4%	6.4%	14.6%
TPR-100	**(NS)**	9.7%	0.07%	-3.7 %	-6.2 %	29%	2.0%	-1.2%	0.6%
$n = 10$	**(S)**	7.4%	0.07%	-2.3%	-7.9%	26.2%	2.0%	2.7%	-9.3%

Note that the reference used is the average between the value from Cplex performance and the value from UP or UP+IUP performance. As we consider a minimization problem, the higher the linear relaxation, the better the lower bound on the optimal solution. Hence for any indicator (N, CPU or LR), the improvement score is positive whenever our method B&C outperforms Cplex with respect to the considered indicator.

Tables 2 presents, for both UP and UP+IUP and for various instance categories, the average improvement scores:

I_N: average improvement score relative to the number of nodes in the Branch & Cut tree.
I_{LR}: average improvement score relative to the linear relaxation at the root.
$I_{CPU}(all)$: average improvement score relative to the CPU time, for all instances.
$I_{CPU}(hard)$: average improvement score relative to the CPU time, for the hard instances.

This table only displays values for the instance categories on which a comparison makes sense, *i.e.* instances which are not too easy, but still tractable within the time limit.

Observe that both UP and UP+IUP perform very well on L and R instances with symmetries and size $n = 50$. However, as opposed to the TPR instances where interval up-set inequalities dramatically improves the performance of UP, the improvement for L and R instances appears to come from the separation of up-set inequalities. This may seem weird as Cplex and our UP algorithm can produce similar inequalities. This shows that even though the cut generation integrated in Cplex is supposed to be able to produce up-set cuts, our heuristic clearly performs better, as it finds useful cuts Cplex does not.

Finally note that both our methods, UP and UP+IUP, globally outperfom Cplex for these hard categories. One objective of this article is to solve TPR-75 and TPR-100 instances. Interestingly, the TPR-75 instances are solved more efficiently with UP, and even more with UP+IUP. This remark is also true for TPR-100 instances with respect to the relaxation value and the number of nodes, even though it does not show on the CPU time. There may be too many user cuts generated for TPR-100 instances of size $n = 10$: a more dedicated implementation of our Branch&Cut method for this category would be useful.

8 Conclusion

In this article, we propose a first polyhedral study of the MUCP with n production units. We first translate the classical extended cover inequalities of the knapsack polytope to obtain the *up-set inequalities*. We generalize these up-set inequalities to obtain the *interval up-set inequalities*. This new class of valid inequalities captures both knapsack constraints and min-up/min-down constraints, thus being more dedicated to the MUCP.

We completely describe the cases in which these inequalities are valid, and we also characterize the facet defining cases in a restricted polytope. We devise a Branch & Cut algorithm in which up-set and interval up-set inequalities are separated. Compared to default Cplex, up-set and interval up-set inequalities used as cuts are particularly efficient for the difficult categories of instances, namely realistic and literature instances with symmetries and a large number of units, and also instances where the production range is tighter.

A possible future work would be to consider a problem including additional technical constraints, such as ramp and minimum operation time constraints. This problem would be close, in its structure, to the TPR instances of the MUCP. Since interval up-set inequalities are particularly effective on the TPR-75 instances and to a lesser extent on the TPR-100, we hope that they will be also useful to solve the real-world UCP.

Our experiments also highlight how symmetries dramatically affect the computation time. Thus, future works could explore how to break symmetries in the UCP, using orbital branching as in Ostrowski et al (2015). Another promising perspective to deal with symmetries in the UCP would be to use a decomposition scheme where interval up-set structures would play an important role.

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Appendix

Let $\mathcal{N}$ and all of its subsets to be considered below be ordered so that $P^j_{max} \geq P^{j+1}_{max}$, $j \in \{1,...,n-1\}$. As we place ourselves here at a given time period $t \in \mathcal{T}$, for simplicity, we drop the index t from all variables and quantities.

For any subset $M \subset \mathcal{N}$, we denote by Up-Set(M) the corresponding inequality (9).

8.1 Proof of theorem 5

Lemma 6. *For any up-set $C \in \mathcal{N}$:*

$$\overline{\alpha}(E(C)) \geq |E(C)\backslash C| + 1.$$

Furthermore, if C is minimal, then $\overline{\alpha}(E(C)) = |E(C)\backslash C| + 1$.

Proof Consider a vector $\bar{x}$ such that $\sum_{j \in E(C)} \bar{x}^j \leq |E(C)\backslash C| = p$. In order to maximize the power production in solution $\bar{x}$, we need the p most powerful units in $E(C)$ to be up. The other units in $E(C)$ will be down, as $\sum_{j \in E(C)} \bar{x}^j \leq p$. Thus, the units up in $E(C)$ are exactly those in $|E(C)\backslash C|$. But as C is an up-set, the demand will not be met. So there is no vector $\bar{x}$ such that $\sum_{j \in E(C)} \bar{x}^j \leq |E(C)\backslash C|$ in $\mathcal{P}^n$.

Lemma 7. *A minimal up-set C is strong if and only if the set $R = C\backslash\{j_1\} \cup \{i_1\}$ is not an up-set, where i_1 is the most powerful unit of $\mathcal{N}\backslash E(C)$.*

Proof ($\Rightarrow$) If R is an up-set, then, following from the minimality of C, R is also minimal. Furthermore, $|R| = |C|$, $R \neq C$ and $E(C) \subset E(R)$ since $i_1 \leq j_1$. So C is not strong.

($\Leftarrow$) If C is not strong, there exists a minimal up-set A such that $|A| = |C|$, $A \neq C$ and $E(C) \subset E(A)$. We can write $A = (C \backslash C_A) \cup A_C$, where $C_A = C \backslash A$ and $A_C = A \backslash C$. It follows from $|A| = |C|$ and $E(C) \subset E(A)$ that $|C_A| = |A_C|$ and $j_1 \in C_A$.

A is an up-set so $\overline{A} = \overline{C} \cup C_A \backslash A_C$ does not suffice to meet the demand by itself. As every unit in C_A is more powerful than each unit in A_C (or else we would not have $E(C) \subset E(A)$), we can deduce that $\overline{C} \cup \{j_1\} \backslash \{i_1\}$ does not suffice either to meet the demand. So R is an up-set.

□

For any $M \subset \mathcal{N}$, we define $E^{-1}(M)$ the set of the $|M| - \overline{\alpha}(M) + 1$ less powerful units of M.

Lemma 8. *For any $M \subset \mathcal{N}$, M is a strong up-set extension if and only if $E^{-1}(M)$ is a strong up-set which satisfies (iii) and such that $E(E^{-1}(M)) = M$.*

Proof The set of the $|M| - \overline{\alpha}(M) + 1$ less powerful units of M is the only set that can possibly satisfy (i), (ii) and (iii). Hence the direct implication. The return implication follows from the definition of a strong up-set extension.

□

Lemma 9. *Let $M \subset \mathcal{N}$. If $E^{-1}(M)$ is an up-set, then either $M = E(E^{-1}(M))$ (and in that case $\sum_{j \in M} x^j \geq \overline{\alpha}(M)$ is exactly Up-Set($E(E^{-1}(M))$)) or the inequality $\sum_{j \in M} x^j \geq \overline{\alpha}(M)$ is dominated by Up-Set($E(E^{-1}(M))$).*

Proof First, it is clear that $M \subset E(E^{-1}(M))$. Indeed, if there was $j \in M$ such that $j \notin E(E^{-1}(M))$, we would have $j < j_1$, otherwise $j \in E(E^{-1}(M))$. But $j < j_1$ contradicts the definition of $E^{-1}(M)$.

Up-Set($E(E^{-1}(M))$) can be written:

$$\sum_{j \in E(E^{-1}(M)) \backslash M} x^j + \sum_{j \in M} x^j \geq |E(E^{-1}(M)) \backslash M| + |M| - |E^{-1}(M)| + 1.$$

Summed up to the trivial inequality

$$|E(E^{-1}(M)) \backslash M| \geq \sum_{j \in E(E^{-1}(M)) \backslash M} x^j$$

we directly obtain $\sum_{j \in M} x^j \geq \overline{\alpha}(M)$.

□

Proof of Theorem 5 ($\Rightarrow$) Let suppose M is not a strong up-set extension, i.e. $E^{-1}(M)$ is not a strong up-set, or does not satisfy the condition $E(E^{-1}(M)) = M$, or does not satisfy condition (iii).

Let first suppose $E^{-1}(M)$ is not an up-set. Then $\overline{E^{-1}(M)}$ is enough to meet the demand, and $|M \cap \overline{E^{-1}(M)}| = \overline{\alpha}(M) - 1$ so $\sum_{j \in M} x^j \geq \overline{\alpha}(M)$ is not valid.

Let now suppose $E^{-1}(M)$ is an up-set, but is not minimal. There exists a unit i such that $E^{-1}(M) \backslash \{i\}$ is still an up-set. Then, either $E(E^{-1}(M) \backslash \{i\}) = E(E^{-1}(M))$ or $E(E^{-1}(M) \backslash \{i\}) = E(E^{-1}(M)) \backslash \{i\}$. In both cases, Up-Set($E(C_M \backslash \{i\})$) dominates Up-Set($E(^{-1}(M))$). By Lemma 9, we can conclude that (8) is not a facet of $\mathcal{P}^n$.

Let suppose now that $E^{-1}(M)$ is not strong. By Proposition 7, $R = E^{-1}(M) \backslash \{j_1\} \cup \{i_1\}$ is an up-set. As $i_1 \leq j_1$, $E(E^{-1}(M)) \cup \{i_1\} \subset E(R)$. It can be easily checked that Up-Set($E(R)$) dominates Up-Set($E(E^{-1}(M))$). By Lemma 9, we can conclude that (8) is not a facet of $\mathcal{P}^n$.

Let now suppose that $M \neq E(E^{-1}(M))$. By Lemma 9, we can conclude that (8) is not a facet of $\mathcal{P}^n$.

Let now suppose $E^{-1}(M)$ does not satisfy condition (iii), which is to say $T = E^{-1}(M)\backslash\{j_1, j_2\} \cup \{1\}$ is an up-set. If $j_1 = 1$ then it means that $T = E^{-1}(M)\backslash\{j_2\}$ is an up-set so $E^{-1}(M)$ is not minimal. We have seen that in this case, (8) cannot be a facet of $\mathcal{P}^n$. Otherwise, $j_1 \neq 1$. Then we have:

$$\overline{T} = (\mathcal{N}\backslash M) \cup \{2,3,...,\overline{\alpha}(M)-1, j_1, j_2\}.$$

We consider a vector $\bar{x}$ satisfying (8) to equality: $\sum_{j\in M} \bar{x}^j = \overline{\alpha}(M)$. We show that if $\bar{x}^1 = 0$ then $\bar{x}$ is not feasible. Indeed, if the $\overline{\alpha}(M)$ units of M which are up in $\bar{x}$ are the $\alpha(M)$ most powerful units of $M\backslash\{1\}$, i.e. $\{2,3,...,\overline{\alpha}(M)-1, j_1, j_2\}$, then, even if all units in $N\backslash M$ are up, the demand cannot be met, as T is an up-set. So each vector $x \in P$ satisfying $\sum_{j\in M} x^j = \overline{\alpha}(M)$ is such that $x^1 = 1$, so there are less than n linearly independant vertices satisfying $\sum_{j\in M} x^j = \overline{\alpha}(M)$. Thus, (8) is not a facet of $\mathcal{P}^n$.

($\Leftarrow$) Let suppose M is a strong up-set extension. We will prove constructively that the hyperplane defined by (8) contains n linearly independant vertices of $\mathcal{P}^n$. Let $c = |E^{-1}(M)|$ and $m = |M|$.

Consider the $n \times n$ matrix:

$$X = \begin{bmatrix} U & B_1 & C_{n-m} \\ U & I_c & U \\ C_{m-c} & B_2 & U \end{bmatrix}$$

where B_1 and B_2 are $(n-m)\times c$ and $(m-c)\times c$ respectively, each of them having identical rows of the form

$$b_1 = (1,0,0,...,0),\, b_2 = (1,1,0,...,0).$$

I_c is the identity matrix of order c, U stands for matrices of ones of appropriate dimension. C_p is the $p\times p$ matrix such that C contains zeros on the diagonal, and ones everywhere else.

Each row of X corresponds to a vertex of P. The first $m-c$ columns of X correspond to the units in $M\backslash E^{-1}(M)$, the following c columns correspond to the units in $E^{-1}(M)$ and the last $n-m$ columns correspond to the units in $\mathcal{N}\backslash M$. Each row satisfies (8) to equality: among its first m entries, each row has exactly $\overline{\alpha}(M)$ entries equal to 1.

We now prove that each row is a feasible solutions, i.e. belongs to $\mathcal{P}^n$.

As $E^{-1}(M)$ is strong, $E^{-1}(M)\backslash\{j_1\}\cup\{i_1\}$ is not an up-set, so the units in $\overline{E^{-1}(M)}\cup\{j_1\}\backslash\{i_1\}$ suffice to meet the demand. In particular, for any $i \in \mathcal{N}\backslash M$, the units in $\overline{E^{-1}(M)}\cup\{j_1\}\backslash\{i\}$ suffice to meet the demand. Hence the feasibility of the first $n-m$ rows.

As $E^{-1}(M)$ is minimal, for all $i \in E^{-1}(M)$, $E^{-1}(M)\backslash\{i\}$ is not an up-set, so the units in $\overline{E^{-1}(M)}\cup\{i\}$ suffice to cover the demand. Hence the feasibility of the following c rows.

By condition (iii), $A = E^{-1}(M)\backslash\{j_1, j_2\}\cup\{1\}$ is not an up-set. Thus $\overline{A} = \overline{E^{-1}(M)}\cup\{j_1, j_2\}\backslash\{1\}$ is enough to cover the demand by itself. In particular, for all $i \in M\backslash E^{-1}(M)$, the units in $\overline{E^{-1}(M)}\cup\{j_1, j_2\}\backslash\{i\}$ suffice to meet the demand. Hence the feasibility of the last $m-c$ rows.

□