The structure of the infinite models in integer programming

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Abstract

The infinite models in integer programming can be described as the convex hull of some points or as the intersection of half-spaces derived from valid functions. In this paper we study the relationships between these two descriptions. Our results have implications for finite dimensional corner polyhedra. One consequence is that nonnegative continuous functions suffice to describe finite dimensional corner polyhedra with rational data. We also discover new facts about corner polyhedra with non-rational data.

1 Introduction

Let $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$. The *mixed-integer infinite group relaxation* M_b is the set of all pairs of functions (s, y) with $s : \mathbb{R}^n \to \mathbb{R}_+$ and $y : \mathbb{R}^n \to \mathbb{Z}_+$ having finite support (that is, $\{r : s(r) > 0\}$ and $\{p : y(p) > 0\}$ are finite sets) satisfying

$$\sum_{r \in \mathbb{R}^n} r s(r) + \sum_{p \in \mathbb{R}^n} p y(p) \in b + \mathbb{Z}^n. \tag{1.1}$$

M_b is a subset of the infinite-dimensional vector space $\mathbb{R}^{(\mathbb{R}^n)} \times \mathbb{R}^{(\mathbb{R}^n)}$, where $\mathbb{R}^{(\mathbb{R}^n)}$ denotes the set of finite support functions from $\mathbb{R}^n$ to $\mathbb{R}$. (Similarly, $\mathbb{R}_+^{(\mathbb{R}^n)}$ will denote the set of finite support functions from $\mathbb{R}^n$ to $\mathbb{R}$ that are nonnegative.) We will work with this vector space throughout the paper. A tuple (ψ, π, α), where $\psi, \pi : \mathbb{R}^n \to \mathbb{R}$ and $\alpha \in \mathbb{R}$, is a *valid tuple for* M_b if

$$\sum_{r \in \mathbb{R}^n} \psi(r) s(r) + \sum_{p \in \mathbb{R}^n} \pi(p) y(p) \geq \alpha \text{ for every } (s, y) \in M_b. \tag{1.2}$$

Since for $\lambda > 0$ the inequalities (1.2) associated with (ψ, π, α) and $(\lambda\psi, \lambda\pi, \lambda\alpha)$ are equivalent, from now on we assume $\alpha \in \{-1, 0, 1\}$.

The set of functions $y : \mathbb{R}^n \to \mathbb{Z}_+$ such that $(0, y) \in M_b$ will be called the *pure integer infinite group relaxation* I_b. In other words, $I_b = \{y : (0, y) \in M_b\}$. By definition, $I_b \subseteq \mathbb{R}^{(\mathbb{R}^n)}$.

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However, when convenient we will see I_b as a subset of M_b. A tuple (π, α), where $\pi : \mathbb{R}^n \to \mathbb{R}$ and $\alpha \in \mathbb{R}$, is called a *valid tuple for I_b* if

$$\sum_{p \in \mathbb{R}^n} \pi(p) y(p) \geq \alpha \text{ for every } y \in I_b. \tag{1.3}$$

Again, we will assume $\alpha \in \{-1, 0, 1\}$.

Models M_b and I_b were defined by Gomory and Johnson in a series of papers [11–13, 16] as a template to generate valid inequalities, derived from (1.2) and (1.3), for general integer programs. They have been the focus of extensive research, as summarized, e.g., in [3, 4], [6, Chapter 6].

Our Results. One would expect that the intersection of all valid tuples for M_b would be equal to $\mathrm{conv}(M_b)$, where $\mathrm{conv}(\cdot)$ denotes the convex hull operator. However, this is not true: this intersection is a strict superset of $\mathrm{conv}(M_b)$. One of our main results (Theorem 2.14) shows that the intersection of all valid tuples for M_b is, in fact, the closure of $\mathrm{conv}(M_b)$ under a norm topology on $\mathbb{R}^{(\mathbb{R}^n)} \times \mathbb{R}^{(\mathbb{R}^n)}$ that was first defined by Basu et al. [2]. We then give an explict characterization that shows that this closure coincides with $\mathrm{conv}(M_b) + (\mathbb{R}_+^{(\mathbb{R}^n)} \times \mathbb{R}_+^{(\mathbb{R}^n)})$. A similar phenomenon happens for I_b (Theorem 2.15).

A valid tuple (ψ, π, α) for M_b is minimal if there does not exist a pair of functions (ψ', π') different from (ψ, π), with $(\psi', \pi') \leq (\psi, \pi)$, such that (ψ', π', α) is a valid tuple for M_b. Our main tool is a characterization of the minimal tuples (Theorem 2.4) that extends a result of Johnson (see, e.g., Theorem 6.34 in [6]), that was obtained under the assumption that $\pi \geq 0$. The main novelty of our result over Johnson's is that minimality of a valid tuple (ψ, π, α) *implies* nonnegativity of π (no need to assume it). Moreover, π has to be continuous (in fact, it is Lipschitz continuous.)

Most of the prior literature on valid tuples (π, α) for I_b proceeds under the restrictive assumption that π is nonnegative (in fact, Gomory and Johnson included the assumption $\pi \geq 0$ in their original definition of valid tuple for I_b). This assumption has been criticized in more recent work on I_b, as there are valid functions not satisfying $\pi \geq 0$. In this paper, we prove that every valid tuple for I_b has an equivalent representation (π, α) where $\pi \geq 0$. More specifically, we show that for every valid tuple (π, α), there exist $\theta : \mathbb{R}^n \to \mathbb{R}$ and $\beta \in \mathbb{R}$ such that both (θ, β), $(-\theta, -\beta)$ are valid tuples and the valid tuple $(\pi', \alpha') = (\pi + \theta, \alpha + \beta)$ satisfies $\pi' \geq 0$ (Theorem 3.7). This settles an open question in [3, Open Question 2.5]. Being able to restrict to nonnegative valid tuples without loss has the added advantage that nonnegative valid tuples form a *compact, convex set* under the natural product topology on functions. Thus, one approach to understanding valid tuples is to understand the extreme points of this compact convex set, which are termed *extreme functions/tuples* in the literature. While this approach was standard for the area, our result about nonnegative valid tuples now gives a rigorous justification for this.

A valid tuple (π, α) for I_b is liftable if there there exists $\psi : \mathbb{R}^n \to \mathbb{R}$ such that (ψ, π, α) is a valid tuple for M_b. Minimal valid tuples (π, α) that are liftable are a strict subset of valid tuples, as we show that such π have to be nonnegative and Lipschitz continuous (Proposition 2.6 and Remark 2.7). This has some consequences for finite-dimensional corner polyhedra that have rational data, which are sets of the form $\mathrm{conv}(I_b) \cap \{y_r = 0, r \in \mathbb{R}^n \setminus P\}$, where $P \cup \{b\}$ is a finite subset of $\mathbb{Q}^n$. Theorem 4.3 shows that inequalities (1.3) associated with liftable tuples, when restricted to the space $\{y_r = 0, r \in \mathbb{R}^n \setminus P\}$, suffice to provide a complete inequality description for such corner polyhedra. Literature on valid tuples contains constructions of families of extreme valid tuples (π, α) such that π is discontinuous [8, 9, 14, 17, 19, 20] (or continuous but not Lipschitz continuous [17]). Our result above shows that such functions may be disregarded, if one is interested in valid inequalities or facets of rational corner polyhedra. Similarly, valid tuples (π, α) where $\pi \not\geq 0$ are also superfluous for such polyhedra. This is interesting, in our opinion, because it shows that such *extreme tuples* are redundant within the set of valid tuples, as far as rational corner polyhedra are concerned.

Crucial to the proof of the above result on rational corner polyhedra, is our characterization of the equations defining the affine hull of $\operatorname{conv}(I_b)$, which extends a result in [3]. This characterization is also essential in understanding the recession cone of $\operatorname{conv}(I_b) \cap \{y_r = 0, r \in \mathbb{R}^n \setminus P\}$, where P is a finite subset of $\mathbb{R}^n$. We use this to prove that $\operatorname{conv}(I_b) \cap \{y_r = 0, r \in \mathbb{R}^n \setminus P\}$ is a polyhedron, even if $P \cup \{b\}$ contains non-rational vectors (Theorem 4.2).

2 The structure of $\operatorname{conv}(M_b)$ and $\operatorname{conv}(I_b)$

A valid tuple (ψ, π, α) for M_b is said to be *minimal* if there does not exist a pair of functions (ψ', π') different from (ψ, π), with $(\psi', \pi') \leq (\psi, \pi)$, such that (ψ', π', α) is a valid tuple for M_b. Similarly, we say that a valid tuple (π, α) for I_b is *minimal* if there does not exist a function π' different from π, with $\pi' \leq \pi$, such that (π', α) is a valid tuple for I_b.

REMARK 2.1. *An application of Zorn's lemma (see, e.g., [5, Proposition A.1]) shows that, given a valid tuple (ψ, π, α) for M_b, there exists a minimal valid tuple (ψ', π', α) for M_b with $\psi' \leq \psi$ and $\pi' \leq \pi$. Similarly, given a valid tuple (π, α) for I_b, there exists a minimal valid tuple (π', α) for I_b with $\pi' \leq \pi$. We will use this throughout the paper.*

Given a tuple (ψ, π, α), we define

$$H_{\psi,\pi,\alpha} = \left\{ (s, y) \in \mathbb{R}^{(\mathbb{R}^n)} \times \mathbb{R}^{(\mathbb{R}^n)} : \sum_{r \in \mathbb{R}^n} \psi(r)s(r) + \sum_{p \in \mathbb{R}^n} \pi(p)y(p) \geq \alpha \right\}.$$

A valid tuple (ψ, π, α) for M_b is *trivial* if $\mathbb{R}_+^{(\mathbb{R}^n)} \times \mathbb{R}_+^{(\mathbb{R}^n)} \subseteq H_{\psi,\pi,\alpha}$. This happens if and only if $\psi \geq 0$, $\pi \geq 0$ and $\alpha \in \{0, -1\}$. Similarly, a valid tuple (π, α) for I_b is *trivial* if $\pi \geq 0$ and $\alpha \in \{0, -1\}$.

A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is *subadditive* if $\phi(r_1) + \phi(r_2) \geq \phi(r_1 + r_2)$ for every $r_1, r_2 \in \mathbb{R}^n$, and is *positively homogenous* if $\phi(\lambda r) = \lambda\phi(r)$ for every $r \in \mathbb{R}^n$ and $\lambda \geq 0$. If ϕ is subadditive and positive homogenous, then ϕ is called *sublinear*. The following proposition is well-known and its proof is relegated to the Appendix.

PROPOSITION 2.2. *Let (ψ, π, α) be a minimal valid tuple for M_b. Then ψ is sublinear and $\pi \leq \psi$.*

LEMMA 2.3. *Suppose $\pi : \mathbb{R}^n \to \mathbb{R}$ is subadditive and $\sup_{\varepsilon > 0} \frac{\pi(\varepsilon r)}{\varepsilon} < \infty$ for all $r \in \mathbb{R}^n$. Define $\psi(r) = \sup_{\varepsilon > 0} \frac{\pi(\varepsilon r)}{\varepsilon}$. Then ψ is sublinear and $\pi \leq \psi$.*

Proof. Since π is subadditive, ψ is readily checked to be subadditive as well. The fact that $\pi \le \psi$ follows by taking $\varepsilon = 1$. Finally, positive homogeneity of ψ follows from the definition of ψ. $\square$

The following theorem follows from the more general Theorem 37 in [21]. For completeness, we provide an alternate proof in the Appendix.

THEOREM 2.4. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$, $\pi : \mathbb{R}^n \to \mathbb{R}$ be any functions, and $\alpha \in \{-1, 0, 1\}$. Then (ψ, π, α) is a nontrivial minimal valid tuple for M_b if and only if the following hold:*

(a) π is subadditive;

(b) $\psi(r) = \sup_{\varepsilon>0} \frac{\pi(\varepsilon r)}{\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{\pi(\varepsilon r)}{\varepsilon} = \limsup_{\varepsilon\to 0^+} \frac{\pi(\varepsilon r)}{\varepsilon}$ for every $r \in \mathbb{R}^n$;

(c) π is Lipschitz continuous with Lipschitz constant $L := \max_{\|r\|=1} \psi(r)$;

(d) $\pi \ge 0$, $\pi(z) = 0$ for every $z \in \mathbb{Z}^n$, and $\alpha = 1$;

(e) (symmetry condition) π satisfies $\pi(r) + \pi(b - r) = 1$ for all $r \in \mathbb{R}^n$.

COROLLARY 2.5. *Let (π, α) be a nontrivial minimal valid tuple for I_b such that $\sup_{\varepsilon>0} \frac{\pi(\varepsilon r)}{\varepsilon} < \infty$ for every $r \in \mathbb{R}^n$. Define $\psi(r) = \sup_{\varepsilon>0} \frac{\pi(\varepsilon r)}{\varepsilon}$. Then (ψ, π, α) satisfies conditions (a)–(e) of Theorem 2.4 and therefore is a nontrivial minimal valid tuple for M_b.*

Conversely, if (ψ, π, α) is a nontrivial minimal valid tuple for M_b, then (π, α) is a nontrivial minimal valid tuple for I_b.

Proof. Since (π, α) is minimal, the same argument as in the proof of Proposition 2.2 shows that π is subadditive. Let ψ be defined as above. Following the proof of Theorem 2.4 it can be checked that minimality and nontriviality of (π, α) suffice to show that (ψ, π, α) satisfies (a)–(e), and therefore (ψ, π, α) is a nontrivial minimal valid tuple for M_b.

For the converse, we use a theorem of Gomory and Johnson (see, e.g., [6, Theorem 6.22]) stating that if $(\pi, 1)$ is a nontrivial valid tuple with $\pi \ge 0$, then $(\pi, 1)$ is minimal if and only if π is subadditive, $\pi(z) = 0$ for every $z \in \mathbb{Z}^n$, and π satisfies the symmetry condition. Let (ψ, π, α) be a nontrivial minimal valid tuple for M_b. By Theorem 2.4, $\pi \ge 0$, $\alpha = 1$, π is subadditive, $\pi(z) = 0$ for every $z \in \mathbb{Z}^n$, and π satisfies the symmetry condition. Therefore, by the above theorem, (π, α) is a nontrivial minimal valid tuple for I_b. $\square$

A valid tuple (π, α) for I_b is called *liftable* if there exists a function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that (ψ, π, α) is a valid tuple for M_b.

PROPOSITION 2.6. *Let (π, α) be a nontrivial valid tuple for I_b. Then (π, α) is liftable if and only if there exists a minimal valid tuple (π', α) such that $\pi' \le \pi$ and $\sup_{\varepsilon>0} \frac{\pi'(\varepsilon r)}{\varepsilon} < \infty$ for every $r \in \mathbb{R}^n$. In this case, defining $\psi(r) = \sup_{\varepsilon>0} \frac{\pi'(\varepsilon r)}{\varepsilon}$ gives a valid tuple (ψ, π', α) for M_b satisfying conditions (a)–(e) of Theorem 2.4.*

Proof. If (π, α) is nontrivial and liftable, then there exists ψ such that (ψ, π, α) is a valid tuple for M_b. Let (ψ', π', α) be a minimal valid tuple with $\psi' \le \psi$ and $\pi' \le \pi$. Since (π, α) is nontrivial, so is (ψ', π', α). By Theorem 2.4, $\alpha = 1$, $\pi' \ge 0$, and $\sup_{\varepsilon>0} \frac{\pi'(\varepsilon r)}{\varepsilon} < \infty$ for every $r \in \mathbb{R}^n$. By Corollary 2.5, (π', α) is minimal.

Conversely, let (π, α) be a nontrivial valid tuple for I_b, and let $\pi' \le \pi$ be such that (π', α) is minimal (and nontrivial) and $\psi(r) := \sup_{\varepsilon>0} \frac{\pi'(\varepsilon r)}{\varepsilon}$ is finite for every $r \in \mathbb{R}^n$. By Corollary 2.5, (ψ, π', α) is a nontrivial minimal valid tuple for M_b, and therefore (π', α) is liftable. Since $\pi \ge \pi'$, (π, α) is liftable as well. $\square$

Remark 2.7. *Let (π, α) be a nontrivial minimal valid tuple for I_b that is liftable. It follows from Proposition 2.6 (with $\pi' = \pi$) that $\psi(r) := \sup_{\varepsilon>0} \frac{\pi(\varepsilon r)}{\varepsilon}$ is finite for all $r \in \mathbb{R}^n$, and (ψ, π, α) is a minimal valid tuple for M_b that satisfies conditions (a)–(e) of Theorem 2.4. Therefore π is Lipschitz continuous and $\pi \ge 0$. There are nontrivial minimal valid tuples (π, α) for I_b for which π is not continuous, or π is continuous but not Lipschitz continuous, see the construction in [17, Section 5]. There are also nontrivial minimal valid tuples (π, α) for I_b with $\pi \not\ge 0$. None of these minimal tuples is liftable.*

2.1 The closure of $\operatorname{conv}(M_b)$

Lemma 2.8. *The following sets coincide:*

(a) $\left(\mathbb{R}_+^{(\mathbb{R}^n)} \times \mathbb{R}_+^{(\mathbb{R}^n)}\right) \cap \bigcap \{H_{\psi,\pi,\alpha} : (\psi, \pi, \alpha)$ *valid tuple*$\}$

(b) $\left(\mathbb{R}_+^{(\mathbb{R}^n)} \times \mathbb{R}_+^{(\mathbb{R}^n)}\right) \cap \bigcap \{H_{\psi,\pi,\alpha} : (\psi, \pi, \alpha)$ *nontrivial valid tuple*$\}$

(c) $\left(\mathbb{R}_+^{(\mathbb{R}^n)} \times \mathbb{R}_+^{(\mathbb{R}^n)}\right) \cap \bigcap \{H_{\psi,\pi,\alpha} : (\psi, \pi, \alpha)$ *minimal nontrivial valid tuple*$\}$

(d) $\left(\mathbb{R}_+^{(\mathbb{R}^n)} \times \mathbb{R}_+^{(\mathbb{R}^n)}\right) \cap \bigcap \{H_{\psi,\pi,\alpha} : (\psi, \pi, \alpha)$ *minimal nontrivial valid tuple,* $\psi, \pi \ge 0,\ \alpha = 1\}$

Proof. The equivalence of (a) and (b) follows from the definition of nontrivial valid tuple. The sets (b) and (c) coincide by Remark 2.1. Finally, Theorem 2.4 shows that (c) is equal to (d). $\square$

From now on, we denote by Q_b the set(s) of Lemma 2.8.

While $\operatorname{conv}(M_b) \subseteq Q_b$, this containment is strict, as shown in Remark 4.5. However, Theorem 2.14 below proves that, under an appropriate topology, the closure of $\operatorname{conv}(M_b)$ is exactly Q_b. In order to show this result, we need the following lemma, that may be of independent interest.

Lemma 2.9. *If $C \subseteq \mathbb{R}^n_+$ is closed, then so is $\operatorname{conv}(C) + \mathbb{R}^n_+$.*

Proof. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of points in $\operatorname{conv}(C) + \mathbb{R}^n_+$ that converges to some $\bar{x} \in \mathbb{R}^n$. We need to show that $\bar{x} \in \operatorname{conv}(C) + \mathbb{R}^n_+$.

By Carathéodory theorem, for every $i \in \mathbb{N}$ we can write

$$x_i = \textstyle\sum_{t=1}^{n+1} \lambda_i^t x_i^t + r_i, \tag{2.1}$$

where $x_i^t \in C$ for all t, $\lambda_i^t \ge 0$ for all t, $\sum_t \lambda_i^t = 1$, and $r_i \in \mathbb{R}^n_+$.

Since C is a closed set, by repeatedly taking subsequences of the original sequence $(x_i)_{i\in\mathbb{N}}$, we can assume that for every $t = 1, \dots, n+1$ the following conditions hold:

(a) either the sequence $(x_i^t)_{i\in\mathbb{N}}$ is unbounded or it converges to some $\bar{x}^t \in C$;

(b) the sequence $(\lambda_i^t)_{i\in\mathbb{N}}$ converges to some number $\bar{\lambda}^t \in [0,1]$.

Note that $\sum_{t=1}^n \bar{\lambda}_t = 1$.

Let $T_1 \subseteq \{1,\dots,n+1\}$ be the set of indices such that the sequence $(x_i^t)_{i\in\mathbb{N}}$ converges to $\bar{x}^t$, and let $T_2 = \{1,\dots,n+1\} \setminus T_1$. For $i \in \mathbb{N}$ we rewrite (2.1) as

$$x_i - \sum_{t\in T_1} \lambda_i^t x_i^t = \sum_{t\in T_2} \lambda_i^t x_i^t + r_i. \tag{2.2}$$

Since the left-hand side of (2.2) converges to

$$\bar{r} := \bar{x} - \sum_{t\in T_1} \bar{\lambda}^t \bar{x}^t, \tag{2.3}$$

the right-hand side must also converge to $\bar{r}$. Note that $\bar{r} \in \mathbb{R}^n_+$, as the right-hand side of (2.2) is a nonnegative vector for all $i \in \mathbb{N}$. Furthermore, $\bar{\lambda}^t = 0$ for every $t \in T_2$, otherwise the right-hand side of (2.2) would not converge. This implies that $\sum_{t\in T_1} \bar{\lambda}^t = 1$ and thus equation (2.3) proves that $\bar{x} \in \operatorname{conv}(C) + \mathbb{R}^n_+$. $\square$

Define the following norm on $\mathbb{R}^{(\mathbb{R}^n)} \times \mathbb{R}^{(\mathbb{R}^n)}$, which was first introduced in [2]:

$$|(s,y)|_* := |s(0)| + \sum_{r\in\mathbb{R}^n} \|r\||s(r)| + |y(0)| + \sum_{p\in\mathbb{R}^n} \|p\||y(p)|.$$

For any two functions $\psi : \mathbb{R}^n \to \mathbb{R}$, $\pi : \mathbb{R}^n \to \mathbb{R}$, we define a linear functional $F_{\psi,\pi}$ on the space $\mathbb{R}^{(\mathbb{R}^n)} \times \mathbb{R}^{(\mathbb{R}^n)}$ as follows:

$$F_{\psi,\pi}(s,y) = \sum_{r\in\mathbb{R}^n} \psi(r)s(r) + \sum_{p\in\mathbb{R}^n} \pi(p)y(p). \tag{2.4}$$

Lemma 2.10. *Under the $|(\cdot,\cdot)|_*$ norm, the linear functional $F_{\psi,\pi}$ is continuous if $(\psi,\pi,1)$ is a nontrivial minimal valid tuple for M_b.*

Proof. Since $(\psi,\pi,1)$ is a nontrivial minimal valid tuple for M_b, conditions (a)–(e) of Theorem 2.4 are satisfied. In order to show that $F_{\psi,\pi}$ is continuous, it is equivalent to show that $F_{\psi,\pi}$ is bounded, i.e., there exists a number M such that $|F_{\psi,\pi}(s,y)| \le M$ for all (s,y) satisfying $|(s,y)|_* = 1$ (see Conway [7]).

We claim that M can be chosen to be $\max_{\|r\|=1} \psi(r)$. (The maximum exists because by condition (a) in Theorem 2.4, ψ is sublinear and therefore continuous on $\mathbb{R}^n$.) Consider (s,y) such that $|(s,y)|_* = 1$. Using $\pi \le \psi$ (Proposition 2.2) and $\psi \ge 0$ (Theorem 2.4), we have

$$\begin{aligned}
|F_{\psi,\pi}(s,y)| &= \left|\sum_{r\in\mathbb{R}^n} \psi(r)s(r) + \sum_{p\in\mathbb{R}^n} \pi(p)y(p)\right| \\
&\le \sum_{r\in\mathbb{R}^n} \psi(r)|s(r)| + \sum_{p\in\mathbb{R}^n} \psi(p)|y(p)| \\
&= \sum_{r\in\mathbb{R}^n} \psi\left(\tfrac{r}{\|r\|}\right) \|r\||s(r)| + \sum_{p\in\mathbb{R}^n} \psi\left(\tfrac{p}{\|p\|}\right) \|p\||y(p)| \\
&\le M\left(\sum_{r\in\mathbb{R}^n} \|r\||s(r)| + \sum_{p\in\mathbb{R}^n} \|p\||y(p)|\right) \\
&\le M\left(|s(0)| + \sum_{r\in\mathbb{R}^n} \|r\||s(r)| + |y(0)| + \sum_{p\in\mathbb{R}^n} \|p\||y(p)|\right) = M.
\end{aligned}$$

$\square$

LEMMA 2.11. *Under the $|(\cdot,\cdot)|_*$ norm, the linear functional $F_{\psi,\pi}$ is continuous if ψ and π have finite support.*

Proof. Let $R, P \subseteq \mathbb{R}^n$ be the supports of ψ, π respectively. Define

$$N = \max\left\{\max_{r\in R\setminus\{0\}} \frac{1}{\|r\|},\ \max_{p\in P\setminus\{0\}} \frac{1}{\|p\|}\right\}, \quad L = \max\left\{\max_{r\in R} |\psi(r)|,\ \max_{p\in P} |\pi(p)|\right\},$$

and $M = N \cdot L$. One now checks that

$$\begin{aligned} |F_{\psi,\pi}(s,y)| &= \left|\textstyle\sum_{r\in R} \psi(r)s(r) + \sum_{p\in P} \pi(p)y(p)\right| \\ &\le L\left(\textstyle\sum_{r\in R} |s(r)| + \sum_{p\in P} |y(p)|\right) \\ &\le LN\left(|s(0)| + \textstyle\sum_{r\in R\setminus\{0\}} \|r\||s(r)| + |y(0)| + \sum_{p\in P\setminus\{0\}} \|p\||y(p)|\right) \\ &= M|(s,y)|_*. \end{aligned}$$

This shows that $F_{\psi,\pi}$ is a bounded linear functional, and hence continuous. □

Define $\mathrm{cl}(\cdot)$ as the closure operator with respect to the topology induced by $|(\cdot,\cdot)|_*$.

LEMMA 2.12. *Under the topology induced by $|(\cdot,\cdot)|_*$, the set Q_b is closed.*

Proof. Since $\mathbb{R}_+^{(\mathbb{R}^n)} \times \mathbb{R}_+^{(\mathbb{R}^n)}$ is defined by a family of halfspaces with finite support, by Lemma 2.11, this set is closed. Furthermore, Lemma 2.10 implies that the set $H_{\psi,\pi,1}$ is closed whenever $(\psi,\pi,1)$ is a minimal valid tuple for M_b. The thesis now follows as Q_b can be defined as set (d) in Lemma 2.8. □

For any subsets $R, P \subseteq \mathbb{R}^n$, define

$$V_{R,P} = \left\{(s,y) \in \mathbb{R}^{(\mathbb{R}^n)} \times \mathbb{R}^{(\mathbb{R}^n)} : s(r) = 0\ \forall r \notin R,\ y(p) = 0\ \forall p \notin P\right\}.$$

When convenient, we will see $V_{R,P}$ as a subset of $\mathbb{R}^R \times \mathbb{R}^P$ by dropping the variables set to 0.

LEMMA 2.13. *For any $R, P \subseteq \mathbb{R}^n$, $V_{R,P}$ is a closed subspace of $\mathbb{R}^{(\mathbb{R}^n)} \times \mathbb{R}^{(\mathbb{R}^n)}$.*

Proof. For every $r \in \mathbb{R}^n$, define the subspace $X_r = \{(s,y) : s(r) = 0\}$. Similarly, for $p \in \mathbb{R}^n$ define $Y_p = \{(s,y) : y(p) = 0\}$. For any fixed $\bar r \in \mathbb{R}^n$, by defining $\psi(\bar r) = 1$ and $\psi(r) = 0$ for all $r \neq \bar r$, and defining $\pi = 0$, we observe that $X_{\bar r}$ is the kernel of $F_{\psi,\pi}$ and thus, by Lemma 2.11, $X_{\bar r}$ is closed. Similarly, each Y_p is closed. The result now follows form the fact that $V_{R,P} = \bigcap_{r\notin R} X_r \cap \bigcap_{p\notin P} Y_p$. □

THEOREM 2.14. $Q_b = \mathrm{cl}(\mathrm{conv}(M_b)) = \mathrm{conv}(M_b) + (\mathbb{R}_+^{(\mathbb{R}^n)} \times \mathbb{R}_+^{(\mathbb{R}^n)})$.

Proof. We first show that $Q_b \supseteq \mathrm{cl}(\mathrm{conv}(M_b))$. Since, under the topology induced by $|(\cdot,\cdot)|_*$, Q_b is a closed convex set by Lemma 2.12, it suffices to show that $Q_b \supseteq M_b$. This follows from the fact that $M_b \subseteq \mathbb{R}_+^{(\mathbb{R}^n)} \times \mathbb{R}_+^{(\mathbb{R}^n)}$ and every inequality that defines Q_b is valid for M_b.

We next show that $Q_b \subseteq \mathrm{cl}(\mathrm{conv}(M_b))$. Consider a point $(s,y) \notin \mathrm{cl}(\mathrm{conv}(M_b))$. By the Hahn-Banach theorem, there exists a continuous linear functional that separates (s,y) from cl(conv(M_b)). In other words, there exist two functions $\psi, \pi : \mathbb{R}^n \to \mathbb{R}$ and a real number α such that $F_{\psi,\pi}(s, y) < \alpha$ and $\mathrm{cl}(\mathrm{conv}(M_b)) \subseteq H_{\psi,\pi,\alpha}$, implying that (ψ, π, α) is a valid tuple for M_b. Thus $(s, y) \notin Q_b$.

We now show that $\mathrm{conv}(M_b)+(\mathbb{R}_+^{(\mathbb{R}^n)} \times \mathbb{R}_+^{(\mathbb{R}^n)}) \subseteq Q_b$. Consider any point $(s_1, y_1)+(s_2, y_2)$, where $(s_1, y_1) \in \mathrm{conv}(M_b)$ and $s_2 \geq 0, y_2 \geq 0$. Since Q_b can be written as the set (d) in Lemma 2.8 and $\mathrm{conv}(M_b) \subseteq \mathbb{R}_+^{(\mathbb{R}^n)} \times \mathbb{R}_+^{(\mathbb{R}^n)}$, we just need to verify that $(s_1, y_1) + (s_2, y_2) \in H_{\psi,\pi,1}$ for all $\psi, \pi \geq 0$. This follows because $(s_1, y_1) \in H_{\psi,\pi,1}$ and (s_2, y_2) and ψ, π are all nonnegative.

We finally show that $\mathrm{conv}(M_b) + (\mathbb{R}_+^{(\mathbb{R}^n)} \times \mathbb{R}_+^{(\mathbb{R}^n)}) \supseteq Q_b$. Consider $(s^*, y^*) \notin \mathrm{conv}(M_b) + (\mathbb{R}_+^{(\mathbb{R}^n)} \times \mathbb{R}_+^{(\mathbb{R}^n)})$. We prove that $(s^*, y^*) \notin Q_b$. This is obvious when $(s^*, y^*) \notin \mathbb{R}_+^{(\mathbb{R}^n)} \times \mathbb{R}_+^{(\mathbb{R}^n)}$. Therefore we assume $s^* \geq 0$, $y^* \geq 0$. Let $R \subseteq \mathbb{R}^n$ be a finite set containing the support of s^* and satisfying $\mathrm{cone}(R) = \mathbb{R}^n$, and let $P \subseteq \mathbb{R}^n$ be a finite set containing the support of y^*. Then $(s^*, y^*) \notin \mathrm{conv}(M_b \cap V_{R,P}) + (\mathbb{R}_+^R \times \mathbb{R}_+^P)$. (We use the same notation (s^*, y^*) to indicate the restriction of (s^*, y^*) to $\mathbb{R}^R \times \mathbb{R}^P$.) Since $M_b \cap V_{R,P}$ is the inverse image of the closed set $b + \mathbb{Z}^n$ under the linear transformation given by the matrix (R, P), $M_b \cap V_{R,P}$ is closed in the usual finite dimensional topology of $V_{R,P}$. Therefore, by Lemma 2.9, $\mathrm{conv}(M_b \cap V_{R,P}) + (\mathbb{R}_+^R \times \mathbb{R}_+^P)$ is closed as well. This implies that there exists a valid inequality in $\mathbb{R}^R \times \mathbb{R}^P$ separating (s^*, y^*) from $\mathrm{conv}(M_b \cap V_{R,P}) + (\mathbb{R}_+^R \times \mathbb{R}_+^P)$. Since the recession cone of $\mathrm{conv}(M_b \cap V_{R,P}) + (\mathbb{R}_+^R \times \mathbb{R}_+^P)$ contains $(\mathbb{R}_+^R \times \mathbb{R}_+^P)$ and because $s^*, y^* \geq 0$, this valid inequality is of the form $\sum_{r\in R} h(r)s(r) + \sum_{p\in P} d(p)y(p) \geq 1$ where $h(r) \geq 0$ for $r \in R$ and $d(p) \geq 0$ for $p \in P$.

Now define the functions

$$\psi(r) = \inf \left\{ \textstyle\sum_{r'\in R} h(r')s(r') : r = \sum_{r'\in R} r's(r'),\ s : R \to \mathbb{R}_+ \right\},$$

$$\begin{aligned}\pi(r) = \inf \Big\{ & \textstyle\sum_{r'\in R} h(r')s(r') + \sum_{p'\in P} d(p')y(p') : \\ & r = \textstyle\sum_{r'\in R} r's(r') + \sum_{p'\in P} p'y(p'),\ s : R \to \mathbb{R}_+,\ y : P \to \mathbb{Z}_+ \Big\}.\end{aligned}$$

Since $\mathrm{cone}(R) = \mathbb{R}^n$, ψ and π are well-defined functions. As the sum only involves nonnegative terms, $\psi, \pi \geq 0$. It can be checked that $(\psi, \pi, 1)$ is a valid tuple for M_b, and since $(s^*, y^*) \notin H_{\psi,\pi,1}$, we have $(s^*, y^*) \notin Q_b$. □

2.2 The closure of conv(I_b)

In the following, we see $\mathbb{R}^{(\mathbb{R}^n)}$ as a topological vector subspace of the space $\mathbb{R}^{(\mathbb{R}^n)} \times \mathbb{R}^{(\mathbb{R}^n)}$ endowed with the topology induced by the norm $|(\cdot, \cdot)|_*$. With a slight abuse of notation, for any $y \in \mathbb{R}^{(\mathbb{R}^n)}$, $|y|_* = \sum_{p\in\mathbb{R}^n} \|p\||y(p)| + |y(0)|$. Also, given $\pi : \mathbb{R}^n \to \mathbb{R}$ and $\alpha \in \mathbb{R}$, we let $H_{\pi,\alpha} = \left\{y \in \mathbb{R}^{(\mathbb{R}^n)} : \sum_{p\in\mathbb{R}^n} \pi(p)y(p) \geq \alpha\right\}$.

We define $G_b = \{y \in \mathbb{R}^{(\mathbb{R}^n)} : (0, y) \in Q_b\}$. Since Q_b can be written as the set (d) in Lemma 2.8, by Corollary 2.5 we have that

$$G_b = \mathbb{R}_+^{(\mathbb{R}^n)} \cap \bigcap\{H_{\pi,\alpha} : (\pi, \alpha) \text{ minimal nontrivial liftable tuple}\}. \tag{2.5}$$

Similar to the mixed-integer case, $\mathrm{conv}(I_b) \subsetneq G_b$ (this will be shown in Remark 3.5).

Theorem 2.15. $G_b = \mathrm{cl}(\mathrm{conv}(I_b)) = \mathrm{conv}(I_b) + \mathbb{R}^{(\mathbb{R}^n)}_+$.

Proof. By Theorem 2.14, $Q_b = \mathrm{conv}(M_b) + (\mathbb{R}^{(\mathbb{R}^n)}_+ \times \mathbb{R}^{(\mathbb{R}^n)}_+)$. Since the inequality $s \geq 0$ is valid for M_b, by taking the intersection with the subspace $\{(s, y) : s = 0\}$ we obtain the equality $G_b = \mathrm{conv}(I_b) + \mathbb{R}^{(\mathbb{R}^n)}_+$. Furthermore, since G_b coincides with the intersection of the closed set Q_b with the closed subspace defined by $s = 0$ (this subspace is closed by Lemma 2.13), G_b is a closed set. Therefore, $\mathrm{conv}(I_b) + \mathbb{R}^{(\mathbb{R}^n)}_+$ is a closed set, and we have $\mathrm{cl}(\mathrm{conv}(I_b)) \subseteq \mathrm{conv}(I_b) + \mathbb{R}^{(\mathbb{R}^n)}_+$.

It remains to show that $\mathrm{conv}(I_b) + \mathbb{R}^{(\mathbb{R}^n)}_+ \subseteq \mathrm{cl}(\mathrm{conv}(I_b))$. To prove this, it suffices to show that for every $\bar{y} \in I_b$ and $r \in \mathbb{R}^n$, the point $\bar{y} + \hat{y}_r$, where $\hat{y}_r(r) = 1$ and $\hat{y}_r(p) = 0$ for $p \neq r$, is the limit of a sequence of points in $\mathrm{conv}(I_b)$ with respect to our topology. So fix $\bar{y} \in I_b$ and $r \in \mathbb{R}^n$. For every integer $k \geq 1$, there exist $q_k \in \mathbb{Z}^n$ and a real number $\lambda_k \geq 1$ such that $\|q_k - \lambda_k r\| < \frac{1}{k}$. Define y_k by setting $y_k(r) = y_k\big(\frac{q_k - \lambda_k r}{\lambda_k}\big) = 1$, and $y_k(p) = 0$ for $p \neq r$. Since $\sum_{p \in \mathbb{R}^n} p \cdot (\lambda_k y_k(p)) = q_k \in \mathbb{Z}^n$, every point of the form $\bar{y} + \lambda_k y_k$ is in I_b. Since $\lambda_k \geq 1$ for every $k \geq 1$, we have $\bar{y} + y_k = \frac{\lambda_k - 1}{\lambda_k}\bar{y} + \frac{1}{\lambda_k}(\bar{y} + \lambda_k y_k) \in \mathrm{conv}(I_b)$. Furthermore, $\|y_k - \hat{y}_r\|_* = \big\|\frac{q_k - \lambda_k r}{\lambda_k}\big\| < \frac{1}{k}$. Therefore, the sequence of points $\bar{y} + y_k$ converges to $\bar{y} + \hat{y}_r$ as $k \to \infty$. □

3 Hamel bases, affine hulls and nonnegative representation of valid tuples

In finite dimensional spaces, the affine hull of any subset C can be equivalently described as the set of affine combinations of points in C or the intersection of all hyperplanes containing C. Lemma 3.1 (which is probably well known) shows that the same holds in infinite dimension.

Before stating and proving the lemma, we give a precise definition of hyperplane in infinite dimensional vector spaces. Given a vector space V over a field $\mathbb{F}$, a subset $H \subseteq V$ is said to be a *hyperplane* in V if there exist a linear functional $F : V \to \mathbb{F}$ and a scalar $\delta \in \mathbb{F}$ such that $H = \{v \in V : F(v) = \delta\}$.

Lemma 3.1. *Let V be a vector space over a field $\mathbb{F}$. For every $C \subseteq V$, the set of affine combinations of points in C is equal to the intersection of all hyperplanes containing C.*

Proof. By possibly translating C, we assume w.l.o.g. that $L := \mathrm{aff}(C)$ is a linear subspace. If $x \in C$ then x belongs to every hyperplane containing C, and therefore L is contained in the intersection of all hyperplanes containing C.

For the reverse inclusion, let $\bar{x}$ be a point not in L. By the axiom of choice, there exists a basis B of V containing $\bar{x}$ such that $B \cap L$ is a basis of L. Let F be the linear functional that takes value 1 on $\bar{x}$ and 0 on every element in $B \setminus \{\bar{x}\}$. Then $L \subseteq \{x : F(x) = 0\}$, but $F(\bar{x}) = 1$. □

The next proposition shows that there is no hyperplane containing M_b.

Proposition 3.2. $\mathrm{aff}(M_b) = \mathbb{R}^{(\mathbb{R}^n)} \times \mathbb{R}^{(\mathbb{R}^n)}$.

Proof. Assume by contradiction that $\operatorname{aff}(M_b) \subsetneq \mathbb{R}^{(\mathbb{R}^n)} \times \mathbb{R}^{(\mathbb{R}^n)}$. By Lemma 3.1, there exists an equation $\sum_{r\in\mathbb{R}^n} \gamma(r)s(r) + \sum_{p\in\mathbb{R}^n} \theta(p)y(p) = \alpha$ satisfied by all points in M_b, where $(\gamma, \theta, \alpha) \neq (0, 0, 0)$. As $\mathbb{R}^{(\mathbb{R}^n)} \times \mathbb{R}^{(\mathbb{R}^n)}$ is not contained in any hyperplane, either the valid tuple (γ, θ, α) or the valid tuple $(-\gamma, -\theta, -\alpha)$ is nontrivial. W.l.o.g., we assume that (γ, θ, α) is nontrivial. Let $(\gamma', \theta', \alpha)$ be a minimal valid tuple with $\gamma' \leq \gamma$ and $\theta' \leq \theta$. Note that $(\gamma', \theta') \neq (0, 0)$, as $(\gamma', \theta', \alpha)$ is nontrivial. Since $(\gamma', \theta', \alpha)$ is minimal and nontrivial, Theorem 2.4 implies that γ' and θ' are continuous nonnegative functions. Therefore, as $(\gamma', \theta') \neq (0, 0)$, there exists $\bar{r} \in \mathbb{Q}^n$ such that $\gamma'(\bar{r}) > 0$ or $\theta'(\bar{r}) > 0$. Assume $\gamma'(\bar{r}) > 0$ (the other case is similar) and let $(\bar{s}, \bar{y}) \in M_b$. Then there exists an integer $k > 0$ such that the point $(s', \bar{y})$ defined by $s'_{\bar{r}} = \bar{s}_{\bar{r}} + k$ and $s'_r = 0$ for $r \neq \bar{r}$ is in M_b. Therefore

$$\sum_{r\in\mathbb{R}^n} \gamma(r)s'(r) + \sum_{p\in\mathbb{R}^n} \theta(p)\bar{y}(p) \geq \sum_{r\in\mathbb{R}^n} \gamma'(r)s'(r) + \sum_{p\in\mathbb{R}^n} \theta'(p)\bar{y}(p) > \alpha,$$

contradicting the assumption that $\sum_{r\in\mathbb{R}^n} \gamma(r)s(r) + \sum_{p\in\mathbb{R}^n} \theta(p)y(p) = \alpha$ for all $(s, y) \in M_b$. □

The characterization of $\operatorname{aff}(I_b)$ is more involved and requires some preliminary notions.

3.1 Hamel bases and the solutions to Cauchy functional equation

A function $\theta : \mathbb{R}^n \to \mathbb{R}$ is *additive* if it satisfies the following *Cauchy functional equation in* $\mathbb{R}^n$:

$$\theta(u + v) = \theta(u) + \theta(v) \text{ for all } u, v \in \mathbb{R}^n. \tag{3.1}$$

Note that if θ is an additive function, then

$$\theta(qx) = q\theta(x) \text{ for every } x \in \mathbb{R}^n \text{ and } q \in \mathbb{Q}. \tag{3.2}$$

Equation (3.1) has been extensively studied, see e.g. [1]. We summarize here the main results that we will employ.

Given any $c \in \mathbb{R}^n$, the linear function $\theta(x) = c^T x$ is obviously a solution to the equation. However, these are not the only solutions. Below we describe all solutions to the equation.

A *Hamel basis* for $\mathbb{R}^n$ is a basis of the vector space of $\mathbb{R}^n$ over the field $\mathbb{Q}$. In other words a Hamel basis is a subset $B \subseteq \mathbb{R}^n$ such that, for every $x \in \mathbb{R}^n$, there exists a unique choice of a finite subset $\{\beta_1, \ldots, \beta_t\} \subseteq B$ (where t depends on x) and nonzero rational numbers $\lambda_1, \ldots, \lambda_t$ such that

$$x = \sum_{i=1}^t \lambda_i \beta_i. \tag{3.3}$$

The existence of B is guaranteed under the axiom of choice.

For every $\beta \in B$, let $c(\beta)$ be a real number. Define θ as follows: for every $x \in \mathbb{R}^n$, if (3.3) is the unique decomposition of x, set

$$\theta(x) = \sum_{i=1}^t \lambda_i c(\beta_i). \tag{3.4}$$

It is easy to check that a function of this type is additive. The following theorem proves that all additive functions are of this form.

Theorem 3.3. *Let B a Hamel basis of $\mathbb{R}^n$. Then every additive function is of the form* (3.4) *for some choice of real numbers $c(\beta)$, $\beta \in B$.*

3.2 The affine hull of I_b

The following result is an immediate extension of a result of Basu, Hildebrand and Köppe (see [3, Propositions 2.2–2.3]).

Proposition 3.4. *The affine hull of I_b is described by the equations*

$$\textstyle\sum_{p\in\mathbb{R}^n}\theta(p)y(p)=\theta(b) \tag{3.5}$$

for all additive functions $\theta:\mathbb{R}^n\to\mathbb{R}$ such that $\theta(p)=0$ for every $p\in\mathbb{Q}^n$.

Proof. By Lemma 3.1, the affine hull of I_b is the intersection of all hyperplanes in $\mathbb{R}^{(\mathbb{R}^n)}$ containing I_b.

We first show that any equation of the form (3.5) gives a hyperplane that contains I_b. If $y\in I_b$, then there exists $k\in\mathbb{Z}$ such that $\sum_{p\in\mathbb{R}^n}py_p=b+k$. This implies that

$$\textstyle\sum_{p\in\mathbb{R}^n}\theta(p)y(p)=\theta\left(\sum_{p\in\mathbb{R}^n}py(p)\right)=\theta(b+k)=\theta(b),$$

where the first equation comes from the additivity of θ and the integrality of $y(p)$, and the last equation from $\theta(k)=0$. This shows that every equation of the form (3.5) is valid for I_b.

Next, we show that any hyperplane in $\mathbb{R}^{(\mathbb{R}^n)}$ containing I_b has the form (3.5). Let $\sum_{p\in\mathbb{R}^n}\theta(p)y(p)=\alpha$ be a hyperplane containing I_b. We show that θ is an additive function. Given $p\in\mathbb{R}^n$, let e_p denote the function such that $e_p(p)=1$ and $e_p(p')=0$ for $p'\neq p$. Given $p_1,p_2\in\mathbb{R}^n$, define $y_1=e_{p_1+p_2}+e_{b-p_1-p_2}$ and $y_2=e_{p_1}+e_{p_2}+e_{b-p_1-p_2}$. Since $y_1,y_2\in I_b$, $\alpha=\sum_{p\in\mathbb{R}^n}\theta(p)y_1(p)=\sum_{p\in\mathbb{R}^n}\theta(p)y_2(p)$. This shows that $\theta(p_1+p_2)=\theta(p_1)+\theta(p_2)$. Therefore θ is additive.

Since (θ,α) and $(-\theta,-\alpha)$ are valid tuples, and valid tuples are nonnegative on the rationals, it follows that $\theta(p)=0$ for every $p\in\mathbb{Q}^n$. Finally, since $e_b\in I_b$, we have that $\alpha=\theta(b)$. $\square$

Remark 3.5. *Since, by the above proposition,* $\operatorname{conv}(I_b)$ *is contained in some hyperplane,* $\operatorname{conv}(I_b)\subsetneq\operatorname{conv}(I_b)+\mathbb{R}_+^{(\mathbb{R}^n)}=G_b$*, where the equality follows from Theorem 2.15.*

In the following, $e_1,\dots,e_n$ denote the vectors of the standard basis of $\mathbb{R}^n$.

Proposition 3.6. *Let P be a finite subset of $\mathbb{R}^n$. Then* $\operatorname{aff}(I_b)\cap V_P$ *is a rational affine subspace of $\mathbb{R}^P$, i.e., there exist a natural number $m\le|P|$, a rational matrix $\Theta\in\mathbb{Q}^{m\times|P|}$ and a vector $d\in\mathbb{R}^m$ such that* $\operatorname{aff}(I_b)\cap V_P=\{s\in\mathbb{R}^P:\Theta s=d\}$*. Moreover,* $\operatorname{aff}(I_b)\cap V_P=V_P$ *if and only if $P\subseteq\mathbb{Q}^n$.*

Proof. Let $I=\{p_1,\dots,p_k\}$ be a maximal subset of vectors in P such that $I\cup\{e_1,\dots,e_n\}$ is linearly independent over $\mathbb{Q}$, and let B a Hamel basis of $\mathbb{R}^n$ containing $I\cup\{e_1,\dots,e_n\}$. Note that $I=\emptyset$ if and only if $P\subseteq\mathbb{Q}^n$.

For every $i=1,\dots,k$, let θ_i be the additive function defined by $\theta_i(p_i)=1$ and $\theta_i(p)=0$ for every $p\in B\setminus\{p_i\}$. Note that every θ_i is an additive function that takes value 0 on the rationals, since $\{e_1,\dots,e_n\}\subseteq B$. Moreover, $\theta_i(p)\in\mathbb{Q}$ for all $p\in P$. Therefore, by Proposition 3.4, $\sum_{p\in P}\theta_i(p)s(p)=\theta_i(b)$ is an equation satisfied by $\operatorname{aff}(I_b)\cap V_P$ with rational coefficients on the left hand side. Thus, again by Proposition 3.4, it suffices to show that for every additive function θ that takes value 0 on the rationals, there exist $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ such that $\theta(p) = \sum_{i=1}^k \lambda_i \theta_i(p)$ for every $p \in P$.

Let θ be an additive function that takes value 0 on the rationals, and define $\lambda_i = \theta(p_i)$ for $i = 1, \dots, k$. For every $p \in P$, there exist $\bar{q} \in \mathbb{Q}^n$ and $q_1, \dots, q_k \in \mathbb{Q}$ such that $p = \bar{q} + \sum_{i=1}^k q_i p_i$. Then, since θ_i is additive and $\theta_i(\bar{q}) = 0$, we have $\theta_i(p) = \theta_i(\sum_{j=1}^k q_j p_j) = \sum_{j=1}^k q_j \theta_i(p_j) = q_i$ for every $i = 1, \dots, k$. It follows that

$$\theta(p) = \theta\left(\sum_{i=1}^k q_i p_i\right) = \sum_{i=1}^k q_i \theta(p_i) = \sum_{i=1}^k \theta_i(p)\lambda_i.$$

We finally observe that in the above arguments, if $I \neq \emptyset$, then we get at least one nontrivial equation corresponding to θ_i, $i \in I$. Therefore, $\operatorname{aff}(I_b) \cap V_P = V_P$ if and only if $I \neq \emptyset$, which is equivalent to $P \subseteq \mathbb{Q}^n$. $\square$

3.3 Sufficiency of nonnegative functions to describe $\operatorname{conv}(I_b)$

As mentioned in the introduction, to the best of our knowledge the study of valid tuples for I_b in prior literature is restricted to nonnegative valid tuples, with the exception of [4]. The standard justification behind this assumption is the fact that valid tuples are nonnegative on the rational vectors. Since in practice we are interested in finite dimensional faces of $\operatorname{conv}(I_b)$ that correspond to rational vectors, such an assumption seems reasonable. However, no mathematical evidence exists in the literature that a complete inequality description of these faces can be obtained from the nonnegative valid tuples only.[1] We prove below that any valid tuple is equivalent to a nonnegative valid tuple, modulo the affine hull. This gives the first proof of the above assertion and puts the nonnegativity assumption on a sound mathematical foundation. Later we will show that even a smaller class of nonnegative valid tuples suffices to describe the finite dimensional faces of $\operatorname{conv}(I_b)$ that correspond to rational vectors, in particular the nontrivial minimal liftable tuples suffice.

Theorem 3.7. *For every valid tuple (π, α) for I_b, there exists a unique additive function $\theta : \mathbb{R}^n \to \mathbb{R}$ such that $\theta(p) = 0$ for ever $p \in \mathbb{Q}^n$ and the valid tuple $(\pi', \alpha') = (\pi + \theta, \alpha + \theta(b))$ satisfies $\pi' \geq 0$.*

This answers Open Question 2.5 in [3].

Note that if B is a Hamel basis of $\mathbb{R}^n$ such that $e_i \in B$ for all $i \in [n]$ and θ is an additive function as in (3.4), the requirement that $\theta(p) = 0$ for every $p \in \mathbb{Q}^n$ is equivalent to $c(e_i) = 0$ for $i \in [n]$. Therefore, in order to prove the theorem, we show that given a valid tuple (π, α), there exists a unique additive function θ such that $\theta(e_i) = 0$ for all $i \in [n]$ and $\pi + \theta$ is a nonnegative function.

We remark that it is sufficient to show the result for a minimal tuple (π, α). This is because if (π, α) is a valid tuple, then there is a minimal valid tuple (π', α) with $\pi' \leq \pi$. Now, note that $(\pi' + \theta, \alpha + \theta(b))$ is still a minimal tuple, and if $\pi' + \theta$ is nonnegative then so is $\pi + \theta$. Thus in the following we assume that (π, α) is minimal.

[1] Such results are obtainable in the case $n = 1$ by more elementary means such as interpolation. We are unaware of a way to establish these results for general $n \geq 2$ without using the technology developed in this paper.

Lemma 3.8. *If (π, α) is a minimal valid tuple, then π is subadditive, $\pi(z) = 0$ for every $z \in \mathbb{Z}^n$, and π is periodic modulo $\mathbb{Z}^n$.*

Proof. Subadditivity can be shown as usual (the proof does not require the nonnegativity of π). On the contrary, the usual proof that $\pi(z) = 0$ for every $z \in \mathbb{Z}^n$ requires the nonnegativity of π. However, one immediately observes that π must be nonnegative on the rationals (and thus on the integers), and this is enough to apply the usual proof. Periodicity now follows as usual. □

Some useful results from [21] We will need some results of Yıldız and Cornuéjols [21], which need to be slightly generalized, as only valid/minimal tuples with $\alpha = 1$ are considered in [21].

Let $(\pi, 1)$ be a minimal tuple. By Lemma 12 in [21] (with $f = -b$ and $S = \mathbb{Z}^n$), π satisfies the generalized symmetry condition (equation (4) in [21]), which, by periodicity of π modulo $\mathbb{Z}^n$, reads as follows:

$$\pi(p) = \sup_{k \in \mathbb{Z}_{>0}} \left\{ \frac{1 - \pi(b - kr)}{k} \right\} \quad \text{for all } p \in \mathbb{R}^n. \tag{3.6}$$

Then, by Proposition 17 in [21] (with $f = -b$, $S = \mathbb{Z}^n$, $X = \{0\}$), the supremum in (3.6) is attained if and only if $\pi(r) + \pi(b - r) = 1$. Proposition 18 in [21] then implies the following: if $p \in \mathbb{R}^n$ is such that $\pi(p) + \pi(b - p) > 1$, then

$$\limsup_{k \in \mathbb{Z}_{>0}, k \to \infty} \frac{\pi(kp)}{k} = \limsup_{k \in \mathbb{Z}_{>0}, k \to \infty} \frac{-\pi(-kp)}{k}.$$

One straightforwardly (and patiently) verifies that when (π, α) is a minimal tuple with α not restricted to be 1, the above result generalizes as follows:

Proposition 3.9. *Let (π, α) be a minimal valid tuple. If $p \in \mathbb{R}^n$ is such that $\pi(p) + \pi(b - p) > \alpha$, then*

$$\limsup_{k \in \mathbb{Z}_{>0}, k \to \infty} \frac{\pi(kp)}{k} = \limsup_{k \in \mathbb{Z}_{>0}, k \to \infty} \frac{-\pi(-kp)}{k}.$$

Construction of θ Let B be a Hamel basis of $\mathbb{R}^n$ containing the unit vectors $e_1, \ldots, e_n$. For every $\beta \in B$ define

$$c(\beta) = \inf_{k \in \mathbb{Z}_{>0}} \frac{\pi(k\beta)}{k}. \tag{3.7}$$

We will show that this is the correct choice for the constant $c(\beta)$.

Lemma 3.10. *The value of $c(\beta)$ is finite and*

$$c(\beta) = \inf_{k \in \mathbb{Z}_{>0}} \frac{\pi(k\beta)}{k} = \sup_{k \in \mathbb{Z}_{>0}} \frac{-\pi(-k\beta)}{k}.$$

Proof. We prove a sequence of claims.

Claim 3.11. *The inequality "$\inf \geq \sup$" holds and both terms are finite.*

Proof of Claim. Let h, k be positive integers. Then, by subadditivity, $h\pi(k\beta) + k\pi(-h\beta) \geq \pi(0) = 0$, thus $\frac{\pi(k\beta)}{k} \geq -\frac{\pi(-h\beta)}{h}$. Since this holds for all positive integers h, k, the claim is proven. ◇

We now assume by contradiction that

$$\inf_{k\in\mathbb{Z}_{>0}} \frac{\pi(k\beta)}{k} - \sup_{k\in\mathbb{Z}_{>0}} \frac{-\pi(-k\beta)}{k} \geq \varepsilon$$

for some $\varepsilon > 0$. In other words,

$$\inf_{k\in\mathbb{Z}_{>0}} \frac{\pi(k\beta)}{k} + \inf_{k\in\mathbb{Z}_{>0}} \frac{\pi(-k\beta)}{k} \geq \varepsilon. \tag{3.8}$$

Claim 3.12. *The following equation holds:*

$$\inf_{k\in\mathbb{Z}_{>0}} \frac{\pi(k\beta)}{k} + \inf_{k\in\mathbb{Z}_{>0}} \frac{\pi(-k\beta)}{k} = \inf_{k\in\mathbb{Z}_{>0}} \frac{\pi(k\beta)+\pi(-k\beta)}{k}. \tag{3.9}$$

Proof of Claim. Since the inequality "$\leq$" is obvious, we prove the reverse inequality. To do so, it is sufficient to show that given positive integers h, k, there exists a positive integer ℓ such that

$$\frac{\pi(h\beta)}{h} + \frac{\pi(-k\beta)}{k} \geq \frac{\pi(\ell\beta)+\pi(-\ell\beta)}{\ell}. \tag{3.10}$$

Choose $\ell = hk$. Then, by subadditivity,

$$k\pi(h\beta) + h\pi(-k\beta) \geq \pi(\ell\beta) + \pi(-\ell\beta).$$

After dividing by $\ell = hk$, we obtain (3.10) and the claim is proven. ◇

By the previous claim, assumption (3.8) is equivalent to

$$\pi(k\beta) + \pi(-k\beta) \geq \varepsilon k \ \text{ for all positive integers } k.$$

Claim 3.13. *There exists a nonzero $k \in \mathbb{Z}$ such that $\pi(k\beta) + \pi(b - k\beta) > \alpha$.*

Proof of Claim. By subadditivity, for every integer k we have

$$\pi(b - k\beta) \geq \pi(-k\beta) - \pi(-b), \qquad \pi(b + k\beta) \geq \pi(k\beta) - \pi(-b).$$

It follows that

$$\pi(k\beta) + \pi(-k\beta) + \pi(b - k\beta) + \pi(b + k\beta) \geq 2(\pi(k\beta) + \pi(-k\beta) - \pi(-b)) \geq 2(\varepsilon k - \pi(-b)).$$

The right-hand side is greater than 2α if $k > \frac{\alpha+\pi(-b)}{\varepsilon}$. For this choice of k, we conclude that either $\pi(k\beta) + \pi(b - k\beta) > \alpha$ or $\pi(-k\beta) + \pi(b + k\beta) > \alpha$ (or both), and the claim is proven. ◇

Without loss of generality, we assume that $\pi(\bar{k}\beta) + \pi(b - \bar{k}\beta) > \alpha$ for some integer $\bar{k} > 0$. (If $\bar{k} < 0$, one can replace β with $-\beta$ in the Hamel basis.)

Define $p = \bar{k}\beta$. Since $\pi(p) + \pi(b - p) > \alpha$, by Proposition 3.9

$$\limsup_{k \in \mathbb{Z}_{>0}, k \to \infty} \frac{\pi(kp)}{k} = \limsup_{k \in \mathbb{Z}_{>0}, k \to \infty} \frac{-\pi(-kp)}{k}.$$

To conclude the proof of the lemma, it is sufficient to show the following:

Claim 3.14. *The following equations hold:*

$$\limsup_{k \in \mathbb{Z}_{>0}, k \to \infty} \frac{\pi(kp)}{k} = \bar{k} \cdot \inf_{k \in \mathbb{Z}_{>0}} \frac{\pi(k\beta)}{k}, \qquad \limsup_{k \in \mathbb{Z}_{>0}, k \to \infty} \frac{-\pi(-kp)}{k} = \bar{k} \cdot \inf_{k \in \mathbb{Z}_{>0}} \frac{-\pi(-k\beta)}{k}$$

Proof of Claim. We only prove the first equation, as the other one is analogous. Note that since the lim sup is always at least as large as the inf,

$$\limsup_{k \in \mathbb{Z}_{>0}, k \to \infty} \frac{\pi(kp)}{k} \geq \inf_{k \in \mathbb{Z}_{>0}} \frac{\pi(kp)}{k} = \bar{k} \cdot \inf_{k \in \mathbb{Z}_{>0}} \frac{\pi(k\bar{k}\beta)}{k\bar{k}} \geq \bar{k} \cdot \inf_{h \in \mathbb{Z}_{>0}} \frac{\pi(h\beta)}{h}$$

and thus the inequality "$\geq$" is verified.

In order show that the inequality "$\leq$" holds, we prove that for every $\varepsilon > 0$ and every integer $k > 0$ there exists an integer $h > 0$ such that

$$\frac{\pi(\ell p)}{\ell} \leq \frac{\bar{k}\pi(k\beta)}{k} + \varepsilon \text{ for all } \ell \geq h.$$

Choose

$$h = \left\lceil \max_{m \in \{0,\ldots,k-1\}} \left\{ \frac{\bar{k}\pi(m\beta)}{\varepsilon} \right\} \right\rceil.$$

Given any $\ell \geq h$, write $\ell = tk + m$, where $t \in \mathbb{Z}$ and $m \in \{0, \ldots, k-1\}$. Then

$$\begin{aligned} \frac{\pi(\ell p)}{\ell} = \frac{\pi(\ell\bar{k}\beta)}{\ell} \leq \frac{\bar{k}\pi(\ell\beta)}{\ell} = \frac{\bar{k}\pi((tk+m)\beta)}{\ell} &\leq \frac{\bar{k}(\pi(tk\beta) + \pi(m\beta))}{\ell} \\ &\leq \frac{\bar{k}\pi(tk\beta)}{tk} + \frac{\bar{k}\pi(m\beta)}{\ell} \leq \frac{\bar{k}\pi(k\beta)}{k} + \frac{\bar{k}\pi(m\beta)}{\ell}. \end{aligned} \tag{3.11}$$

The conclusion follows as $\frac{\bar{k}\pi(m\beta)}{\ell} \leq \varepsilon$, since $\ell \geq h$. $\diamond$

This concludes the proof of the lemma. $\square$

Now let θ be defined as in (3.4), where the constants $c(\beta)$ for $\beta \in B$ are chosen as in (3.7). In the next two lemmas we prove that $\theta(e_i) = 0$ for all $i \in [n]$ and $\pi - \theta$ is nonnegative.

Lemma 3.15. *$\theta(e_i) = 0$ for all $i \in [n]$.*

Proof. Fix $i \in [n]$. Since $e_i \in B$, it is sufficient to check that $c(e_i) = 0$. By Lemma 3.10, $c(e_i) = \inf_{k \in \mathbb{Z}_{>0}} \frac{\pi(ke_i)}{k}$. Since $\pi(ke_i) = 0$ for all $k \in \mathbb{Z}$, we immediately see that $c(e_i) = 0$. $\square$

LEMMA 3.16. *If θ is defined as in* (3.4)*, with the constants $c(\beta)$ given in* (3.7)*, then the function $\pi - \theta$ is nonnegative.*

Proof. Let $x \in \mathbb{R}^n$. Then there exist $\beta_1, \dots, \beta_t \in B$ and nonzero rational numbers $\lambda_1, \dots, \lambda_t$ such that $x = \sum_{i=1}^t \lambda_i \beta_i$, and we have $\theta(x) = \sum_{i=1}^t \lambda_i c(\beta_i)$. We prove that $\pi(x) - \theta(x) \geq 0$.

For every $i \in \{1, \dots, t\}$, we can write $\lambda_i = \frac{p_i}{q_i}$, where every p_i is a nonzero integer and every q_i is a positive integer. Define $Q = q_1 \cdots q_t$. Take arbitrary positive integers $k_1, \dots, k_t$ (these numbers will be fixed later) and define $K = k_1 \cdots k_t$. Since $\frac{Q}{q_i}$ and $\frac{K}{k_i}$ are positive integers for every i, by subadditivity we have

$$QK\pi(x) + \sum_{i=1}^t \frac{QK}{q_i k_i} \pi(-k_i p_i \beta_i) \geq \pi\left(QKx - \sum_{i=1}^t QK\lambda_i \beta_i\right) = \pi(0) = 0.$$

This implies that

$$\pi(x) \geq \sum_{i=1}^t \lambda_i \, \frac{-\pi(-k_i p_i \beta_i)}{k_i p_i}. \tag{3.12}$$

Now fix $\varepsilon > 0$. If i is an index such that $p_i > 0$, by Lemma 3.10 we can choose k_i such that $\frac{-\pi(-k_i\beta_i)}{k_i} \geq c(\beta_i) - \varepsilon$. Then by subadditivity

$$\frac{-\pi(-k_i p_i \beta_i)}{k_i p_i} \geq \frac{-\pi(-k_i \beta_i)}{k_i} \geq c(\beta_i) - \varepsilon.$$

If i is an index such that $p_i < 0$, by Lemma 3.10 we can choose k_i such that $\frac{\pi(k_i\beta_i)}{k_i} \leq c(\beta_i) + \varepsilon$. Then by subadditivity

$$\frac{-\pi(-k_i p_i \beta_i)}{k_i p_i} \leq \frac{\pi(k_i \beta_i)}{k_i} \leq c(\beta_i) + \varepsilon.$$

Then, rembering that $\lambda_i > 0$ if and only if $p_i > 0$, equation (3.12) gives $\pi(x) \geq \sum_{i=1}^t \lambda_i c(\beta_i) - t\varepsilon$. Since this holds for every $\varepsilon > 0$, we have $\pi(x) \geq \sum_{i=1}^t \lambda_i c(\beta_i)$ and thus $\pi(x) - \theta(x) \geq 0$. $\square$

This concludes the proof of the existence of θ. One easily verifies that in the above proof the choice of θ is unique, thus the proof of Theorem 3.7 is complete.

4 Recession cones and canonical faces

A *canonical face* of $\operatorname{conv}(M_b)$ is a face of the form $F = \operatorname{conv}(M_b) \cap V_{R,P}$ for some $R, P \subseteq \mathbb{R}^n$. If R and P are finite, F is a *finite canonical face* of $\operatorname{conv}(M_b)$. The same definitions can be given for $\operatorname{conv}(I_b)$. The corner polyhedra defined by Gomory and Johnson [11–13] are precisely the finite canonical faces of $\operatorname{conv}(I_b)$.

The notion of recession cone of a closed convex set is standard (see, e.g., [18]). We extend it to general convex sets in general vector spaces (possibly infinite-dimensional) in the following way. Let V be a vector space and let $C \subseteq V$ be a convex set. For any $x \in C$, define

$$C_\infty(x) = \{r \in V : x + \lambda r \in C \text{ for all } \lambda \geq 0\}.[2]$$

[2] Using the Hahn-Banach separation theorem, it can be shown that if V is a topological vector space and C is a closed convex subset, then $C_\infty(x) = C_\infty(x')$ for all $x, x' \in C$.

We define the *recession cone* of C as $\operatorname{rec}(C) = \bigcap_{x \in C} C_\infty(x)$. Theorem 2.14 yields the following result.

COROLLARY 4.1. *Let $F = \operatorname{conv}(M_b) \cap V_{R,P}$ be a canonical face of $\operatorname{conv}(M_b)$. Then F is a face of $\operatorname{cl}(\operatorname{conv}(M_b))$ if and only if $F + (\mathbb{R}^R_+ \times \mathbb{R}^P_+) = F$, i.e., $\operatorname{rec}(F)$ is the nonnegative orthant.*

Proof. By Theorem 2.14,

$$\begin{aligned}\operatorname{cl}(\operatorname{conv}(M_b)) \cap V_{R,P} &= \big(\operatorname{conv}(M_b) + (\mathbb{R}^{(\mathbb{R}^n)}_+ \times \mathbb{R}^{(\mathbb{R}^n)}_+)\big) \cap V_{R,P} \\ &= (\operatorname{conv}(M_b) \cap V_{R,P}) + (\mathbb{R}^R_+ \times \mathbb{R}^P_+) \\ &= F + (\mathbb{R}^R_+ \times \mathbb{R}^P_+)\end{aligned}$$

The results follows from the observation that F is a face of $\operatorname{cl}(\operatorname{conv}(M_b))$ if and only if $F = \operatorname{cl}(\operatorname{conv}(M_b) \cap V_{R,P}$. $\square$

Define L to be the linear space parallel to the affine hull of $\operatorname{conv}(I_b)$; Proposition 3.4 shows that L is the set of all $y \in \mathbb{R}^{(\mathbb{R}^n)}$ that satisfy $\sum_{p \in \mathbb{R}^n} \theta(p) y(p) = 0$ for all additive functions $\theta : \mathbb{R}^n \to \mathbb{R}$ such that $\theta(p) = 0$ for all $p \in \mathbb{Q}^n$. For any $P \subseteq \mathbb{R}^n$, define the face $C^P = \operatorname{conv}(I_b) \cap V_P$ of $\operatorname{conv}(I_b)$.

THEOREM 4.2. *For every finite subset $P \subseteq \mathbb{R}^n$, the following are all true:*

(a) the face $C^P = \operatorname{conv}(I_b) \cap V_P$ is a rational polyhedron in $\mathbb{R}^P$;

(b) every extreme ray of C^P is spanned by some $r \in \mathbb{Z}^P_+$ such that $\sum_{p \in P} p r(p) \in \mathbb{Z}^n$;

(c) $\operatorname{rec}(C^P) = L \cap \mathbb{R}^{(\mathbb{R}^n)}_+ \cap V_P = (L \cap V_P) \cap \mathbb{R}^P_+$.

Proof. By dropping variables set to zero, $I_b \cap V_P$ is the set of vectors $y \in \mathbb{Z}^P_+$ such that $\sum_{p \in P} p y(p) \in b + \mathbb{Z}^n$. We say that a feasible point $y \in I_b \cap V_P$ is minimal if there is no feasible point $y' \neq y$ such that $y' \leq y$. Every vector $d \in \mathbb{Z}^P_+$ such that $\sum_{p \in P} p d(p) \in \mathbb{Z}^n$ is called a *ray*. A ray d is minimal if there is no ray $d' \neq d$ such that $d' \leq d$.

We claim that every feasible point y is the sum of a minimal feasible point and a nonnegative integer combination of minimal rays. To see this, as long as there is a ray d such that $d \leq y$, replace y with $y - d$. Note that this operation can be repeated only a finite number of times. Denote by $\bar{y}$ the feasible point obtained at the end of this procedure. Then y is the sum of $\bar{y}$ and a nonnegative integer combination of rays. We observe that $\bar{y}$ is minimal: if not, there would exist a feasible point $y' \neq \bar{y}$ such that $y' \leq \bar{y}$; but then the vector $d := \bar{y} - y'$ would be a ray satisfying $d \leq \bar{y}$, contradicting the fact that the procedure has terminated. Therefore y is the sum of a minimal feasible point $\bar{y}$ and a nonnegative integer combination of rays. Since every ray is a nonnegative integer combination of minimal rays (argue as above), we conclude that y is the sum of a minimal feasible point and a nonnegative integer combination of minimal rays.

By Gordon–Dickson lemma (see, e.g., [10]), the set of minimal feasible points and the set of minimal rays are both finite. Let Y be the set of points that are the sum of a minimal feasible point and a nonnegative integer combination of minimal rays. Thus, there exist finite sets $E \subseteq \mathbb{Z}^P_+$ and $R \subseteq \mathbb{Z}^P_+$ such that $Y = E + \operatorname{integ.cone}(R)$, where $\operatorname{integ.cone}(R)$ denotes the set of all nonnegative integer combinations of vectors in R. So $\operatorname{conv}(Y) =$

$\operatorname{conv}(E + \operatorname{integ.cone}(R)) = \operatorname{conv}(E) + \operatorname{conv}(\operatorname{integ.cone}(R)) = \operatorname{conv}(E) + \operatorname{cone}(R)$, where $\operatorname{cone}(R)$ denotes the conical hull of R. Hence, $\operatorname{conv}(Y)$ is a rational polyhedron, by the Minkowski-Weyl Theorem [6, Theorem 3.13]. The above observation proves that $I_b \cap V_P \subseteq Y$. On the other hand, by using the fact that if y is a feasible point and d is a ray then $y + d$ is a feasible point, one readily verifies that $Y \subseteq I_b \cap V_P$. Then $I_b \cap V_P = Y$ and therefore $\operatorname{conv}(I_b) \cap V_P = \operatorname{conv}(I_b \cap V_P) = \operatorname{conv}(Y)$. Hence, $\operatorname{conv}(I_b) \cap V_P$ is a rational polyhedron.

The above analysis proves (a) and (b) simultaneously. We now prove (c).

We first show that $\operatorname{rec}(C^P) \subseteq L \cap \mathbb{R}_+^{(\mathbb{R}^n)} \cap V_P$. Consider any $\bar{d} \in \operatorname{rec}(C^P)$. By part (ii), $\bar{d}$ is a nonnegative combination of vectors $d \in \mathbb{Z}_+^P$ such that $\sum_{p \in P} pr(p) \in \mathbb{Z}^n$. Observe that each such $d \in L$. Thus, $\bar{d} \in L$ since L is a linear space. Therefore, $\operatorname{rec}(C^P) \subseteq L \cap \mathbb{R}_+^{(\mathbb{R}^n)} \cap V_P$.

We now want to establish that $L \cap \mathbb{R}_+^{(\mathbb{R}^n)} \cap V_P \subseteq \operatorname{rec}(C^P)$. First, consider any $d \in L \cap \mathbb{R}_+^{(\mathbb{R}^n)} \cap V_P$ such that $d \in \mathbb{Q}^P$, i.e., d has only rational coordinates. Let $\lambda > 0$ be such that $\bar{d} = \lambda d \in \mathbb{Z}_+^P$. We claim that $\sum_{p \in P} p\bar{d}(p) \in \mathbb{Q}^n$. Otherwise, there exists[3] an additive function $\theta : \mathbb{R}^n \to \mathbb{R}$ such that $0 \neq \theta(\sum_{p \in P} p\bar{d}(p)) = \sum_{p \in P} \theta(p)\bar{d}(p) = \lambda \sum_{p \in P} \theta(p)d(p)$, which violates the hypothesis that $d \in L$. Since $\sum_{p \in P} p\bar{d}(p) \in \mathbb{Q}^n$, this implies that there exists a positive scaling $\tilde{d}$ of d such that $\sum_{p \in P} p\tilde{d}(p) \in \mathbb{Z}^n$. It is easy to verify that $\tilde{d} \in \operatorname{rec}(C^P)$ and therefore $d \in \operatorname{rec}(C^P)$. This shows that all rational vectors in $L \cap \mathbb{R}_+^{(\mathbb{R}^n)} \cap V_P$ are in $\operatorname{rec}(C^P)$. Since, by Proposition 3.6, $L \cap V_P$ is a rational subspace, $L \cap \mathbb{R}_+^{(\mathbb{R}^n)} \cap V_P \subseteq \operatorname{rec}(C^P)$. □

THEOREM 4.3. *Let $P \subseteq \mathbb{R}^n$ be finite. Then the following are equivalent:*

(a) $P \subseteq \mathbb{Q}^n$;

(b) $\operatorname{rec}(C^P) = \mathbb{R}_+^P$;

(c) the dimension of C^P is $|P|$;

(d) $C^P = G_b \cap V_P$.

Proof. (a) is equivalent to (b) by Proposition 3.6 and Theorem 4.2. (b) is equivalent to (c) by Proposition 3.6. The equivalence of (a) and (d) follows from the equivalence of (a) and (b), Corollary 4.1 and Theorem 2.15.

□

By (2.5), condition (d) in the above theorem states that the finite dimensional corner polyhedron C^P has a complete inequality description given by the restriction of liftable valid tuples.

EXAMPLE 4.4. *There are finite dimensional faces of* $\operatorname{conv}(M_b)$ *that are not closed. Let $n = 1$, $b \in \mathbb{Q}$, $\omega \in \mathbb{R} \setminus \mathbb{Q}$, $R = \{-1\}$, $P = \{b, \omega\}$. Consider the point $(\bar{s}, \bar{y})$ defined by $\bar{s}(-1) = 0$ and $\bar{y}(b) = \bar{y}(\omega) = 1$. Note that $(\bar{s}, \bar{y}) \notin$* $\operatorname{conv}(M_b) \cap V_{R,P}$*, as the only point in M_b satisfying $s(-1) = 0$ and $y(b) \leq 1$ has $y(b) = 1$, $y(\omega) = 0$.*

We now show that $(\bar{s}, \bar{y}) \in$ $\operatorname{cl}(\operatorname{conv}(M_b) \cap V_{R,P})$ *by constructing for every $\varepsilon > 0$ a point in* $\operatorname{conv}(M_b) \cap V_{R,P}$ *whose Euclidean distance from $(\bar{s}, \bar{y})$ is at most ε. So fix $\varepsilon > 0$. Let $\hat{y}(\omega)$*

[3] Such an additive function can be constructed by first constructing a Hamel basis of $\mathbb{R}^n$ over $\mathbb{Q}$ containing $\sum_{p \in P} p\bar{d}(p), e^1, \ldots, e^n$, and setting θ to be 1 on $\sum_{p \in P} p\bar{d}(p)$ and 0 everywhere else on this basis.

be a positive integer such that the fractional part of $\omega\hat{y}(\omega)$ is at most ε. Let $\hat{s}(-1)$ be equal to this fractional part, and $\hat{y}(b) = 1$. Then $(\hat{s}, \hat{y}) \in M_b \cap V_{R,P}$. By taking a suitable convex combination of $(\hat{s}, \hat{y})$ and the point of $M_b \cap V_{R,P}$ defined by $y(b) = 1$, $s(-1) = y(\omega) = 0$, we find a point in $\operatorname{conv}(M_b) \cap V_{R,P}$ whose distance from $(\bar{s}, \bar{y})$ is at most ε.

REMARK 4.5. *Since $Q_b = \operatorname{cl}(\operatorname{conv}(M_b))$ by Theorem 2.14, for every $R, P \subseteq \mathbb{R}^n$ the set $Q_b \cap V_{R,P}$ is closed by Lemma 2.13. The previous example gives sets R, P such that $\operatorname{conv}(M_b) \cap V_{R,P}$ is not closed. Thus $\operatorname{conv}(M_b)$ is a strict subset of Q_b.*

COROLLARY 4.6. $\operatorname{conv}(I_b)_\infty(x) = L \cap \mathbb{R}_+^{(\mathbb{R}^n)}$ *for every $x \in \operatorname{conv}(I_b)$. Consequently,* $\operatorname{rec}(\operatorname{conv}(I_b)) = L \cap \mathbb{R}_+^{(\mathbb{R}^n)}$.

Proof. Given any $x \in \operatorname{conv}(I_b)$, we show that $\operatorname{conv}(I_b)_\infty(x) = L \cap \mathbb{R}_+^{(\mathbb{R}^n)}$.

($\subseteq$) Consider any vector $y \in \operatorname{conv}(I_b)_\infty(x)$. Let P denote the union of the support of x, y. This implies that $y \in \operatorname{rec}(C^P)$ and by Theorem 4.2, $y \in L \cap \mathbb{R}_+^{(\mathbb{R}^n)} \cap V_P \subseteq L \cap \mathbb{R}_+^{(\mathbb{R}^n)}$.

($\supseteq$) Consider any $y \in L \cap \mathbb{R}_+^{(\mathbb{R}^n)}$. Let P denote the union of the support of x, y. Then $y \in L \cap \mathbb{R}_+^{(\mathbb{R}^n)} \cap V_P$. By Theorem 4.2 (c), we obtain that $y \in C_\infty^P(x)$. This implies that $y \in \operatorname{conv}(I_b)_\infty(x)$. □

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A Missing proofs

Proof of Proposition 2.2. Assume that ψ is not subadditive. Then $\psi(r_1+r_2) > \psi(r_1)+\psi(r_2)$ for some $r_1, r_2 \in \mathbb{R}^n$. Let $\psi' : \mathbb{R}^n \to \mathbb{R}$ be defined as $\psi'(r_1+r_2) = \psi(r_1)+\psi(r_2)$ and $\psi'(r) = \psi(r)$ for every $r \neq r_1+r_2$. Then (ψ', π, α) is easily seen to be a valid tuple, a contradiction to the minimality of (ψ, π, α).

Now assume that ψ is not positively homogenous. Then $\psi(\lambda r_1) < \lambda\psi(r_1)$ for some $r_1 \in \mathbb{R}^n$ and $\lambda > 0$. Let $\psi' : \mathbb{R}^n \to \mathbb{R}$ be defined as $\psi'(r_1) = \frac{\psi(\lambda r_1)}{\lambda}$ and $\psi'(r) = \psi(r)$ for every $r \neq r_1$. Again, (ψ', π, α) is a valid tuple, a contradiction to the minimality of (ψ, π, α). Thus ψ is sublinear.

Finally, assume that $\pi(r_1) > \psi(r_1)$ for some $r_1 \in \mathbb{R}^n$. Let $\pi' : \mathbb{R}^n \to \mathbb{R}$ be defined as $\pi'(r_1) = \psi(r_1)$ and $\pi'(r) = \pi(r)$ for every $r \neq r_1$. The tuple (ψ, π', α) is valid, and this shows that $\pi \leq \psi$. □

Proof of Theorem 2.4. (⇐) Theorem 6.34 in [6] shows that if conditions (a)–(e) are satisfied, then (ψ, π, α) is a minimal valid tuple for M_b. Since $\alpha = 1$, the tuple is nontrivial.

(⇒) Suppose that (ψ, π, α) is a nontrivial minimal valid tuple for M_b.

(a) This proof is the same as the subadditivity proof in Proposition 2.2.

(b) We first establish the following claim.

CLAIM A.1. $\psi(r) \geq \sup_{\varepsilon>0} \frac{\pi(\varepsilon r)}{\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{\pi(\varepsilon r)}{\varepsilon} = \limsup_{\varepsilon\to 0^+} \frac{\pi(\varepsilon r)}{\varepsilon}$.

Proof of Claim. Since (ψ, π, α) is minimal, from Proposition 2.2 ψ is sublinear and $\pi \leq \psi$. Hence for $\varepsilon > 0$ and $r \in \mathbb{R}^n$ we have that $\frac{\pi(\varepsilon r)}{\varepsilon} \leq \frac{\psi(\varepsilon r)}{\varepsilon} = \psi(r)$. Thus, $\sup_{\varepsilon>0} \frac{\pi(\varepsilon r)}{\varepsilon} \leq \psi(r)$ and this implies that $\sup_{\varepsilon>0} \frac{\pi(\varepsilon r)}{\varepsilon}$ is a finite real number. By Theorem 7.11.1 in [15] and the subadditivity of π, this implies that $\sup_{\varepsilon>0} \frac{\pi(\varepsilon r)}{\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{\pi(\varepsilon r)}{\varepsilon} = \limsup_{\varepsilon\to 0^+} \frac{\pi(\varepsilon r)}{\varepsilon}$. ◇

The above claim shows that the function $\psi'(r) := \lim_{\varepsilon\to 0^+} \frac{\pi(\varepsilon r)}{\varepsilon}$ is well defined, and $\psi' \leq \psi$. Furthermore, by Lemma 2.3, ψ' is sublinear. We prove below that (ψ', π, α) is a valid tuple. Therefore, since (ψ, π, α) is minimal, validity of (ψ', π, α) will imply that $\psi = \psi'$.

Assume by contradiction that (ψ', π, α) is not valid. Then there exists $(s, y) \in M_b$ such that

$$\sum_{r\in\mathbb{R}^n} \psi'(r)s_r + \sum_{p\in\mathbb{R}^n} \pi(p)y_p = \alpha - \delta$$

for some $\delta > 0$. Define $\tilde{r} = \sum_{r\in\mathbb{R}^n} r s_r$. Since $\psi'(r) = \lim_{\varepsilon\to 0^+} \frac{\pi(\varepsilon r)}{\varepsilon}$, there exists some $\beta > 0$ such that

$$\frac{\pi(\varepsilon\tilde{r})}{\varepsilon} < \psi'(\tilde{r}) + \delta \quad \text{for all} \quad 0 < \varepsilon < \beta.$$

Let $D \in \mathbb{Z}_{>0}$ be such that $1/D \leq \beta$ and define $\tilde{y}$ to be

$$\tilde{y}_r = \begin{cases} y_r + D & \text{if } r = \tilde{r}/D, \\ y_r & \text{if } r \neq \tilde{r}/D. \end{cases}$$

Note that

$$\sum_{r\in\mathbb{R}^n} r\tilde{y}_r = \sum_{r\in\mathbb{R}^n} r s_r + \sum_{p\in\mathbb{R}^n} p y_p \in b + \mathbb{Z}^n,$$

and so $(0,\tilde{y}) \in M_b$. Hence $\sum_{p\in\mathbb{R}^n} \pi(p)\tilde{y}_p \geq \alpha$. However,

$$\begin{aligned}
\sum_{p\in\mathbb{R}^n} \pi(p)\tilde{y}_p &= \frac{\pi(\tilde{r}/D)}{1/D} + \sum_{p\in\mathbb{R}^n} \pi(p)y_p \\
&< \psi'(\tilde{r}) + \delta + \sum_{p\in\mathbb{R}^n} \pi(p)y_p && \text{by definition of } \delta \\
&\leq \sum_{r\in\mathbb{R}^n} \psi'(r)s_r + \delta + \sum_{p\in\mathbb{R}^n} \pi(p)y_p && \text{by sublinearity of } \psi' \\
&= \alpha,
\end{aligned}$$

which is a contradiction.

(c) We now show that π is Lipschitz continuous with Lipschitz constant $L := \max_{\|r\|=1}\psi(r)$. By Proposition 2.2, ψ is sublinear; thus, it is continuous. Therefore $\max_{\|r\|=1}\psi(r)$ is attained. Moreover, by subadditivity of π, we obtain that $\pi(x) - \pi(y) \leq \pi(x-y)$ for all $x, y \in \mathbb{R}^n$. Therefore, $|\pi(x) - \pi(y)| \leq \max\{\pi(x-y), \pi(y-x)\}$. Thus, for all $x \neq y$,

$$\frac{|\pi(x) - \pi(y)|}{\|x-y\|} \leq \frac{\max\{\pi(x-y), \pi(y-x)\}}{\|x-y\|} \leq \frac{\max\{\psi(x-y), \psi(y-x)\}}{\|x-y\|} \leq L,$$

where the second inequality follows from Proposition 2.2.

(d) We prove this with a sequence of claims.

Claim A.2. *$\pi(r) \geq 0$ for all $r \in \mathbb{R}^n$.*

Proof of Claim. Let $p^* \in \mathbb{Q}^n$. Then there exists $D \in \mathbb{Z}_{>0}$ such that $Dp^* \in \mathbb{Z}^n$. Let $(s,y) \in M_b$ and, for some $k \in Z_+$, define $(s,\tilde{y})$ by setting $\tilde{y}_{p^*} = y_{p^*} + kD$ and $\tilde{y}_p = y_p$ for $p \neq p^*$. Note that $(s,\tilde{y}) \in M_b$ for every $k \in \mathbb{Z}_+$. This shows that $\pi(p^*) \geq 0$ for every $p^* \in \mathbb{Q}^n$. Since π is Lipschitz continuous by part (c) above, we must have $\pi \geq 0$ everywhere. ◇

Claim A.3. *$\pi(z) = 0$ for all $z \in \mathbb{Z}^n$.*

Proof of Claim. Assume to the contrary that there is some $z \in \mathbb{Z}^n$ such that $\pi(z) \neq 0$. By the previous claim, $\pi(z) > 0$. Define π' to be $\pi'(z) = 0$ and $\pi'(p) = \pi(p)$ for $p \neq z$. Then (ψ, π', α) is easily seen to be a valid tuple. This contradicts the minimality of (ψ, π, α). ◇

We now show that $\alpha = 1$. Since $\psi, \pi \geq 0$ by parts (b) and (d), if $\alpha = 0$ or $\alpha = -1$, then this would contradict the fact that the tuple is nontrivial.

(e) The proof is identical to part (d) of the proof of Theorem 6.22 in [6].

This concludes the proof of the theorem. □