Second-order cone programming formulation for two player zero-sum game with chance constraints

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Abstract

We consider a two player finite strategic zero-sum game where each player has stochastic linear constraints. We formulate the stochastic constraints of each player as chance constraints. We show the existence of a saddle point equilibrium if the row vectors of the random matrices, defining the stochastic constraints of each player, are elliptically symmetric distributed random vectors. We further show that a saddle point equilibrium can be obtained from the optimal solutions of a primal-dual pair of second-order cone programs.

Keywords: Stochastic programming, Chance constraints, Zero-sum game, Saddle point equilibrium, Second-order cone program.

1. Introduction

The equilibrium concept in game theory started with the paper by John von Neumann [18]. He showed that there exists a saddle point equilibrium for a finite strategic zero-sum game. In 1950, John Nash [17] showed that there always exists an equilibrium for a finite strategic general sum game with finite number of players. Later such equilibrium was called Nash equilibrium. It is well known that there is a substantial relationship between game theory and optimization theory. A saddle point equilibrium of a two player finite strategic zero-sum game can be obtained from the optimal solutions of a primal-dual pair of linear programs [1, 9], while a Nash equilibrium of a two player finite strategic general sum game can be obtained from a global

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maximum of a certain quadratic program [16]. The games discussed above are unconstrained games, i.e., the mixed strategies of each player are not further restricted by any constraints. Charnes [4] considered a two player constrained zero-sum game, where the mixed strategies of each player are constrained by linear inequalities. He showed that a saddle point equilibrium of a constrained zero-sum game can be obtained from the optimal solutions of a primal-dual pair of linear programs.

The above mentioned papers are deterministic in nature, i.e., the payoff functions and constraints (if any) are defined by real valued functions. However, in some practical cases the payoff functions or constraints are stochastic in nature due to various external factors. One way to handle stochastic Nash games is using expected payoff criterion. Ravat and Shanbhag [20] considered stochastic Nash games using expected payoff functions and expected value constraints. They showed the existence of a Nash equilibrium in various cases. The expected payoff criterion is more appropriate for the cases where the decision makers are risk neutral. The risk averse payoff criterion using the risk measures CVaR and variance has been considered in the literature [14, 20] and [8] respectively. Recently, a risk averse payoff criterion based on chance constraint programming has also received attention in the literature. The chance constraint programming based payoff criterion is appropriate for the cases where the players are interested in maximizing the random payoffs that can be obtained with certain confidence. Singh et al. have written a series of papers on chance-constrained games [21, 22, 23, 24, 25]. In [21, 22, 24, 25], they considered the case where the probability distribution of the payoff vector of each player is completely known. They showed the existence of a Nash equilibrium for elliptically symmetric distribution case [25], and they proposed some equivalent complementarity problems and mathematical programs to compute the Nash equilibria of these games [21, 22, 24]. In [23], they formulated the case of partially known distribution as a distributionally robust chance-constrained game. They showed the existence of a mixed strategy Nash equilibrium for these games and proposed some equivalent mathematical programs to compute it. There are some zero-sum chance-constrained games available in the literature [2, 3, 5, 6, 26].

The above mentioned papers on stochastic Nash games using chance constraint programming consider the case where the players' payoffs are random and there are no constraints for any player. In this paper, we consider a two player zero-sum constrained game introduced by Charnes [4]. We consider the case where the matrices defining the linear constraints of both the players are stochastic in nature. Such linear constraints could be viewed as budget constraints or resource constraints [7] of both the players where stochasticity is present through various external factors. We model the stochastic linear constraints as chance constraints. We show that there exists a mixed strategy saddle point equilibrium for a zero-sum game with chance constraints if the row vectors of these matrices are elliptically symmetric distributed random vectors. We further show that a saddle point equilibrium problem is equivalent to a primal-dual pair of second-order cone programs (SOCPs).

Now, we describe the structure of the rest of the paper. Section 2 contains the definition of a zero-sum game with chance constraints. We show the existence of a mixed strategy saddle point equilibrium in Section 3. Section 4 contains the second-order cone programming formulation. We present numerical results in Section 5. We conclude the paper in Section 6.

2. The model

A two player zero-sum game is described by an $m \times n$ matrix A, where m and n denote the number of actions of player 1 and player 2 respectively. The matrix A represents the payoffs of player 1 corresponding to different action pairs, and the payoffs of player 2 are given by $-A$. Let $I = \{1, 2, \cdots, m\}$ and $J = \{1, 2, \cdots, n\}$ be the action sets of player 1 and player 2 respectively. The actions belonging to sets I and J are also called pure strategies of player 1 and player 2 respectively. A mixed strategy of a player is defined by a probability distribution over his action set. Let $X = \left\{x \in \mathbb{R}^m \mid \sum_{i\in I} x_i = 1, x_i \geq 0,\ \forall\, i \in I\right\}$ and $Y = \left\{y \in \mathbb{R}^n \mid \sum_{j\in J} y_j = 1, y_j \geq 0,\ \forall\, j \in J\right\}$ be the sets of mixed strategies of player 1 and player 2 respectively. For a given strategy pair $(x, y) \in X \times Y$, the payoffs of player 1 and player 2 are given by $x^T Ay$ and $x^T(-A)y$. For a fixed strategy of one player, other player is interested in maximizing his payoff. Equivalently, for a fixed y player 1 maximizes $x^T Ay$, and for a fixed x player 2 minimizes $x^T Ay$. It is well known that a saddle point equilibrium of the above zero-sum game problem exists [18], and it can be obtained from the optimal solutions of a primal-dual pair of linear programs [1, 9]. Charnes [4] studied a constrained zero-sum game problem where the strategies of both players are further restricted by linear inequalities. A strategy pair (x, y) is said to be a saddle point equilibrium for a constrained zero sum game considered in [4] if and only if x and y simultaneously solve the following optimization problems:

$$\max_x x^T Ay \qquad\qquad \min_y x^T Ay$$

$$\text{s.t.} \qquad\qquad \text{s.t.}$$

$$Bx \geq b \qquad\qquad Dy \leq d$$

$$x \in X, \quad (2.1) \qquad\qquad y \in Y, \quad (2.2)$$

where $B \in \mathbb{R}^{p\times m}$, $D \in \mathbb{R}^{q\times n}$, $b \in \mathbb{R}^p$, $d \in \mathbb{R}^q$. Let index sets $\mathcal{I}_1 = \{1, 2\cdots, p\}$ and $\mathcal{I}_2 = \{1, 2\cdots, q\}$ denote the number of constraints of player 1 and player 2 respectively. Charnes [4] showed that a saddle point equilibrium of a constrained zero-sum game problem can be obtained from the optimal solutions of a primal-dual pair of linear programs. We consider the above constrained zero-sum game problem where the matrices defining the constraints are random matrices. Let B^w denotes a random matrix which defines the constraints of player 1, and D^w denotes a random matrix which defines the constraints of player 2; w denotes some uncertainty parameter. We consider the situation where each player is interested in maximizing his payoff such that each of his stochastic constraint is satisfied with a given probability. That is, the stochastic constraints of each player are replaced with the individual chance constraints [13, 19]. Then, a strategy pair (x, y) is called a saddle point equilibrium of a zero-sum game with individual chance constraints if and only if x and y simultaneously solve the following optimization problems:

$$\max_x x^T Ay \qquad\qquad \min_y x^T Ay$$

$$\text{s.t.} \qquad\qquad \text{s.t.}$$

$$P\{B_k^w x \geq b_k\} \geq \alpha_k^1,\ \forall\ k \in \mathcal{I}_1 \qquad\qquad P\{D_l^w y \leq d_l\} \geq \alpha_l^2,\ \forall\ l \in \mathcal{I}_2$$

$$x \in X, \quad (2.3) \qquad\qquad y \in Y, \quad (2.4)$$

where P is a probability measure, and $B_k^w = (B_{k1}^w, B_{k2}^w, \cdots, B_{km}^w)$, $k \in \mathcal{I}_1$ is a k^{th} row of matrix B^w, and $D_l^w = (D_{l1}^w, D_{l2}^w, \cdots, D_{ln}^w)$, $l \in \mathcal{I}_2$ is an l^{th} row of matrix D^w, and $\alpha_k^1 \in [0, 1]$ is the probability level for k^{th} constraint of player 1, and $\alpha_l^2 \in [0, 1]$ is the probability level for l^{th} constraint of player 2. Let $\alpha^1 = (\alpha_k^1)_{k=1}^p$ and $\alpha^2 = (\alpha_l^2)_{l=1}^q$, and $\alpha = (\alpha^1, \alpha^2)$. We denote the above zero-sum game with individual chance constraints by $G(\alpha)$. Denote,

$$S_1(\alpha^1) = \{x \in \mathbb{R}^m |\ x \in X,\ P\{B_k^w x \geq b_k\} \geq \alpha_k^1,\ \forall\ k \in \mathcal{I}_1\},$$

and

$$S_2(\alpha^2) = \{y \in \mathbb{R}^n |\ y \in Y,\ P\{D_l^w y \leq d_l\} \geq \alpha_l^2,\ \forall\ l \in \mathcal{I}_2\}.$$

The sets $S_1(\alpha^1)$ and $S_2(\alpha^2)$ are the feasible strategy sets of player 1 and player 2 respectively for the game $G(\alpha)$. Then, (x^*, y^*) is called a saddle point equilibrium of $G(\alpha)$ at $\alpha \in [0,1]^p \times [0,1]^q$, if the following inequality holds:

$$x^T A y^* \leq x^{*T} A y^* \leq x^{*T} A y,\ \forall\ x \in S_1(\alpha^1), y \in S_2(\alpha^2).$$

3. Existence of saddle point equilibrium

We consider the case where the row vectors of the random matrices B^w and D^w follow a multivariate elliptically symmetric distribution. The class of multivariate elliptically symmetric distributions generalize the multivariate normal distribution. Some famous multivariate distributions like normal, Cauchy, t, Laplace, and logistic distributions belong to the family of elliptically symmetric distributions.

Let B_k^w, $k \in \mathcal{I}_1$, follows a multivariate elliptically symmetric distribution with a location parameter $\mu_k^1 \in \mathbb{R}^m$ and a positive definite scale matrix $\Sigma_k^1 \in \mathbb{R}^{m \times m}$. We denote this by $B_k^w \sim Ellip\,(\mu_k^1, \Sigma_k^1)$. We denote a positive definite matrix Σ by $\Sigma \succ 0$. Let D_l^w, $l \in \mathcal{I}_2$, follows a multivariate elliptically symmetric distribution with a location parameter $\mu_l^2 \in \mathbb{R}^n$ and a positive definite scale matrix $\Sigma_l^2 \in \mathbb{R}^{n \times n}$. A linear combination of the components of a multivariate elliptically symmetric distributed random vector follows a univariate elliptically symmetric distribution [10]. Therefore, for a given $x \in X$, $B_k^w(-x) \sim Ellip\left(-x^T \mu_k^1, \sqrt{x^T \Sigma_k^1 x}\,\right)$, $k \in \mathcal{I}_1$, and for a given $y \in Y$, $D_l^w y \sim Ellip\left(y^T \mu_l^2, \sqrt{y^T \Sigma_l^2 y}\,\right)$, $l \in \mathcal{I}_2$. We can write $\sqrt{x^T \Sigma_k^1 x} = ||(\Sigma_k^1)^{\frac{1}{2}} x||$ and $\sqrt{y^T \Sigma_l^2 y} = ||(\Sigma_l^2)^{\frac{1}{2}} y||$ because $\Sigma_k^1 \succ 0$ and $\Sigma_l^2 \succ 0$. Then, $\xi_k^1 = \frac{-B_k^w x + x^T \mu_k^1}{||(\Sigma_k^1)^{\frac{1}{2}} x||}$, $k \in \mathcal{I}_1$, and $\xi_l^2 = \frac{D_l^w y - y^T \mu_l^2}{||(\Sigma_l^2)^{\frac{1}{2}} y||}$, $l \in \mathcal{I}_2$, follow a univariate spherically symmetric distribution with parameters 0 and 1. Now, we reformulate the strategy sets $S_1(\alpha^1)$ and $S_2(\alpha^2)$ by transforming the chance constraints into equivalent deterministic constraints. For instance, take a chance constraint defined by (2.3)

$$\begin{aligned} P\{B_k^w x \geq b_k\} \geq \alpha_k^1 &\Leftrightarrow P\{B_k^w(-x) \leq -b_k\} \geq \alpha_k^1 \\ &\Leftrightarrow P\left\{\frac{-B_k^w x + x^T\mu_k^1}{||(\Sigma_k^1)^{\frac{1}{2}}x||} \leq \frac{-b_k + x^T\mu_k^1}{||(\Sigma_k^1)^{\frac{1}{2}}x||}\right\} \geq \alpha_k^1 \\ &\Leftrightarrow -x^T\mu_k^1 + \Psi_{\xi_k^1}^{-1}(\alpha_k^1)||(\Sigma_k^1)^{\frac{1}{2}}x|| \leq -b_k, \end{aligned}$$

where $\Psi_{\xi_k^1}^{-1}(\cdot)$ is a quantile function of a spherically symmetric random variable ξ_k^1. Therefore, we can write the strategy set $S_1(\alpha^1)$ as

$$S_1(\alpha^1) = \left\{x \in \mathbb{R}^m \mid x \in X,\ -x^T\mu_k^1 + \Psi_{\xi_k^1}^{-1}(\alpha_k^1)||(\Sigma_k^1)^{\frac{1}{2}}x|| \leq -b_k,\ \forall\ k \in \mathcal{I}_1\right\}. \tag{3.1}$$

Similarly, we can write the strategy set $S_2(\alpha^2)$ as

$$S_2(\alpha^2) = \left\{y \in \mathbb{R}^n \mid y \in Y,\ y^T\mu_l^2 + \Psi_{\xi_l^2}^{-1}(\alpha_l^2)||(\Sigma_l^2)^{\frac{1}{2}}y|| \leq d_l,\ \forall\ l \in \mathcal{I}_2\right\}. \tag{3.2}$$

Assumption 1. *1. The set $S_1(\alpha^1)$ is strictly feasible, i.e., there exists an $x \in \mathbb{R}^m$ which is a feasible point of $S_1(\alpha^1)$ and the inequality constraints of $S_1(\alpha^1)$ are strictly satisfied by x.*

2. The set $S_2(\alpha^2)$ is strictly feasible, i.e., there exists an $y \in \mathbb{R}^n$ which is a feasible point of $S_2(\alpha^2)$ and the inequality constraints of $S_2(\alpha^2)$ are strictly satisfied by y.

Lemma 3.1. *For all $\alpha^1 \in (0.5, 1]^p$ and $\alpha^2 \in (0.5, 1]^q$, $S_1(\alpha^1)$ and $S_2(\alpha^2)$ are convex sets.*

Proof. It is enough to show that $h_k^1(x) = -x^T\mu_k^1 + \Psi_{\xi_k^1}^{-1}(\alpha_k^1)||(\Sigma_k^1)^{\frac{1}{2}}x||$ for all $k \in \mathcal{I}_1$, and $h_l^2(y) = y^T\mu_l^2 + \Psi_{\xi_l^2}^{-1}(\alpha_l^2)||(\Sigma_l^2)^{\frac{1}{2}}y||$ for all $l \in \mathcal{I}_2$, are convex functions of x and y respectively. For an $\alpha^1 \in (0.5, 1]^p$, $\Psi_{\xi_k^1}^{-1}(\alpha_k^1) \geq 0$ for all $k \in \mathcal{I}_1$, and for an $\alpha^2 \in (0.5, 1]^q$, $\Psi_{\xi_l^2}^{-1}(\alpha_l^2) \geq 0$ for all $l \in \mathcal{I}_2$. Then, from the property of norm, $h_k^1(x)$ and $h_l^2(y)$, for all k and l, are convex functions of x and y respectively.

Remark 3.2. *If the row vectors B_k^w, $k \in \mathcal{I}_1$ and D_l^w, $l \in \mathcal{I}_2$, have strictly positive density functions, Lemma 3.1 holds for all $\alpha^1 \in [0.5, 1]^p$ and $\alpha^2 \in [0.5, 1]^q$ [13].*

Now we show the existence of a saddle point equilibrium for game $G(\alpha)$.

Theorem 3.3. *Consider a constrained zero-sum matrix game where the matrices B^w and D^w, defining the constraints of both the players respectively, are random. Let the row vectors $B_k^w \sim Ellip(\mu_k^1, \Sigma_k^1)$, $k \in \mathcal{I}_1$, and $D_l^w \sim Ellip(\mu_l^2, \Sigma_l^2)$, $l \in \mathcal{I}_2$. For all k and l, $\Sigma_k^1 \succ 0$ and $\Sigma_l^2 \succ 0$. Then, there always exists a saddle point equilibrium for the game $G(\alpha)$ for all $\alpha \in (0.5, 1]^p \times (0.5, 1]^q$.*

Proof. For an $\alpha \in (0.5, 1]^p \times (0.5, 1]^q$, $S_1(\alpha^1)$ and $S_2(\alpha^2)$ are convex sets from Lemma 3.1. It is clear that $S_1(\alpha^1)$ and $S_2(\alpha^2)$ are closed sets and these sets are also bounded. The function $x^T Ay$ is a continuous function. Therefore, the existence of a saddle point equilibrium follows from the minimax theorem of Neumann [18].

Remark 3.4. *If the row vectors B_k^w, $k \in \mathcal{I}_1$ and D_l^w, $l \in \mathcal{I}_2$, have strictly positive density functions, Theorem 3.3 holds for all $\alpha^1 \in [0.5, 1]^p$ and $\alpha^2 \in [0.5, 1]^q$ [13].*

4. Second-order cone programming formulation

From minimax theorem (x^*, y^*) is a saddle point equilibrium for the game $G(\alpha)$ (which exists from Theorem 3.3) if and only if

$$x^{*T} Ay^* = \max_{x \in S_1(\alpha^1)} \min_{y \in S_2(\alpha^2)} x^T Ay = \min_{y \in S_2(\alpha^2)} \max_{x \in S_1(\alpha^1)} x^T Ay, \tag{4.1}$$

and

$$x^* \in \arg\max_{x \in S_1(\alpha^1)} \min_{y \in S_2(\alpha^2)} x^T Ay, \tag{4.2}$$

$$y^* \in \arg\min_{y \in S_2(\alpha^2)} \max_{x \in S_1(\alpha^1)} x^T Ay. \tag{4.3}$$

Denote, $X^+ = \{x | x_i \geq 0, \ \forall \ i \in I\}$ and $Y^+ = \{y | y_j \geq 0, \ \forall \ j \in J\}$. We first start with $\min_{y \in S_2(\alpha^2)} \max_{x \in S_1(\alpha^1)} x^T A y$ problem. For fixed y, the inner optimization problem can be equivalently written as

$$\left.\begin{aligned}
&\max_{x,(t_k^1)_{k=1}^p} \ x^T A y \\
&\text{s.t.} \\
&\quad (i) \ x^T \mu_k^1 - \Psi_{\xi_k^1}^{-1}(\alpha_k^1) ||t_k^1|| - b_k \geq 0, \ \forall \ k \in \mathcal{J}_1 \\
&\quad (ii) \ t_k^1 - (\Sigma_k^1)^{\frac{1}{2}} x = 0, \ \forall \ k \in \mathcal{J}_1 \\
&\quad (iii) \ \sum_{i \in I} x_i = 1 \\
&\quad (iv) \ x_i \geq 0, \ \forall \ i \in I.
\end{aligned}\right\} \tag{4.4}$$

Optimization problem (4.4) is a SOCP. Let $\lambda^1 = (\lambda_k^1)_{k=1}^p \in \mathbb{R}^p$, and $\delta_k^1 \in \mathbb{R}^m$, $k \in \mathcal{J}_1$, and ν^1, be the Lagrange multipliers corresponding to the constraints (i), (ii), and (iii) of (4.4) respectively. Then, the Lagrangian dual problem of (4.4) can be written as

$$\min_{\nu^1, (\delta_k^1)_{k=1}^p, \lambda^1 \geq 0} \ \max_{x \in X^+, (t_k^1)_{k=1}^p} \Bigg[x^T A y + \sum_{k=1}^p \lambda_k^1 \left(x^T \mu_k^1 - \Psi_{\xi_k^1}^{-1}(\alpha_k^1) ||t_k^1|| - b_k \right) + \sum_{k=1}^p \left(\delta_k^1\right)^T \left(t_k^1 - (\Sigma_k^1)^{\frac{1}{2}} x \right) + \nu^1 \left(1 - \sum_{i \in I} x_i \right) \Bigg],$$

where $\lambda^1 \geq 0$ means componentwise non-negativity. For fixed $\nu^1, (\delta_k^1)_{k=1}^p, \lambda^1 \geq 0$, we have

$$\begin{aligned}
&\max_{x \in X^+, (t_k^1)_{k=1}^p} \Bigg[x^T A y + \sum_{k=1}^p \lambda_k^1 \left(x^T \mu_k^1 - \Psi_{\xi_k^1}^{-1}(\alpha_k^1) ||t_k^1|| - b_k \right) \\
&\qquad + \sum_{k=1}^p \left(\delta_k^1\right)^T \left(t_k^1 - (\Sigma_k^1)^{\frac{1}{2}} x \right) + \nu^1 \left(1 - \sum_{i \in I} x_i \right) \Bigg] \\
&= \max_{x \in X^+} \left[x^T \left(A y + \sum_{k=1}^p \lambda_k^1 \mu_k^1 - \sum_{k=1}^p (\Sigma_k^1)^{\frac{1}{2}} \delta_k^1 - \nu^1 \mathbb{1}_m \right) \right] \\
&\quad + \max_{(t_k^1)_{k=1}^p} \left[\sum_{k=1}^p \left(\left(\delta_k^1\right)^T t_k^1 - \lambda_k^1 \Psi_{\xi_k^1}^{-1}(\alpha_k^1) ||t_k^1|| \right) \right] + \nu^1 - \sum_{k=1}^p \lambda_k^1 b_k.
\end{aligned}$$

First and second max above is ∞ unless

$$Ay + \sum_{k=1}^{p} \lambda_k^1 \mu_k^1 - \sum_{k=1}^{p} (\Sigma_k^1)^{\frac{1}{2}} \delta_k^1 \leq \nu^1 \mathbb{1}_m,$$
$$||\delta_k^1|| \leq \lambda_k^1 \Psi_{\xi_k^1}^{-1}(\alpha_k^1), \ \forall \ k \in \mathcal{I}_1,$$

where $\mathbb{1}_m$ denotes an $m \times 1$ vector of ones. Therefore, the Lagrangian dual of (4.4) is given by the following SOCP:

$$\left.\begin{array}{ll} \displaystyle\min_{\nu^1, (\delta_k^1)_{k=1}^p, \lambda^1} & \displaystyle\nu^1 - \sum_{k=1}^{p} \lambda_k^1 b_k \\ \text{s.t.} & \\ & (i)\ Ay + \displaystyle\sum_{k=1}^{p} \lambda_k^1 \mu_k^1 - \sum_{k=1}^{p} (\Sigma_k^1)^{\frac{1}{2}} \delta_k^1 \leq \nu^1 \mathbb{1}_m \\ & (ii)\ ||\delta_k^1|| \leq \lambda_k^1 \Psi_{\xi_k^1}^{-1}(\alpha_k^1), \ \forall \ k \in \mathcal{I}_1 \\ & (iii)\ \lambda_k^1 \geq 0, \ \forall \ k \in \mathcal{I}_1. \end{array}\right\} \quad (4.5)$$

Under Assumption 1 and due to the fact that $\nu^1, (\delta_k^1)_{k=1}^p$ are unrestricted variables, SOCPs (4.4) and (4.5) are strictly feasible. Therefore, the strong duality for the primal-dual pair of SOCPs (4.4) and (4.5) holds such that their optimal objective function values are same [15]. Hence, the $\min_{y \in S_2(\alpha^2)} \max_{x \in S_1(\alpha^1)} x^T A y$ problem is equivalent to the following SOCP:

$$\left.\begin{array}{ll} \displaystyle\min_{y, \nu^1, (\delta_k^1)_{k=1}^p, \lambda^1} & \displaystyle\nu^1 - \sum_{k=1}^{p} \lambda_k^1 b_k \\ \text{s.t.} & \\ & (i)\ Ay + \displaystyle\sum_{k=1}^{p} \lambda_k^1 \mu_k^1 - \sum_{k=1}^{p} (\Sigma_k^1)^{\frac{1}{2}} \delta_k^1 \leq \nu^1 \mathbb{1}_m \\ & (ii)\ y^T \mu_l^2 + \Psi_{\xi_l^2}^{-1}(\alpha_l^2) ||(\Sigma_l^2)^{\frac{1}{2}} y|| \leq d_l, \ \forall \ l \in \mathcal{I}_2 \\ & (iii)\ ||\delta_k^1|| \leq \lambda_k^1 \Psi_{\xi_k^1}^{-1}(\alpha_k^1), \ \forall \ k \in \mathcal{I}_1 \\ & (iv)\ \displaystyle\sum_{j \in J} y_j = 1 \\ & (v)\ y_j \geq 0, \ \forall \ j \in J \\ & (vi)\ \lambda_k^1 \geq 0, \ \forall \ k \in \mathcal{I}_1. \end{array}\right\} \quad \text{(P)}$$

Next, we start with $\max_{x \in S_1(\alpha^1)} \min_{y \in S_2(\alpha^2)} x^T A y$ problem. For fixed x, the inner optimization problem can be equivalently written as

$$\left.\begin{aligned} &\min_{y,(t_l^2)_{l=1}^q} \; x^T A y \\ \text{s.t.} \\ &\quad (i)\; y^T \mu_l^2 + \Psi_{\xi_l^2}^{-1}(\alpha_l^2)||t_l^2|| - d_l \leq 0, \;\forall\, l \in \mathcal{I}_2 \\ &\quad (ii)\; t_l^2 - (\Sigma_l^2)^{\frac{1}{2}} y = 0, \;\forall\, l \in \mathcal{I}_2 \\ &\quad (iii)\; \sum_{j \in J} y_j = 1 \\ &\quad (iv)\; y_j \geq 0, \;\forall\, j \in J. \end{aligned}\right\} \tag{4.6}$$

Optimization problem (4.6) is a SOCP. Let $\lambda^2 = (\lambda_l^2)_{l=1}^q \in \mathbb{R}^q$, and $\delta_l^2 \in \mathbb{R}^n$, $l \in \mathcal{I}_2$, and ν^2, be the Lagrange multipliers corresponding to the constraints (i), (ii), and (iii) of (4.6) respectively. Then, the Lagrangian dual problem of (4.6) can be written as

$$\max_{\nu^2,(\delta_l^2)_{l=1}^q,\lambda^2 \geq 0} \; \min_{y \in Y^+,(t_l^2)_{l=1}^q} \Bigg[x^T A y + \sum_{l=1}^q \lambda_l^2 \left(y^T \mu_l^2 + \Psi_{\xi_l^2}^{-1}(\alpha_l^2)||t_l^2|| - d_l \right) + \sum_{l=1}^q \left(\delta_l^2\right)^T \left(t_l^2 - (\Sigma_l^2)^{\frac{1}{2}} y \right) + \nu^2 \left(1 - \sum_{j \in J} y_j \right) \Bigg].$$

For fixed $\nu^2, (\delta_l^2)_{l=1}^q, \lambda^2 \geq 0$, we have

$$\begin{aligned} &\min_{y \in Y^+,(t_l^2)_{l=1}^q} \Bigg[x^T A y + \sum_{l=1}^q \lambda_l^2 \left(y^T \mu_l^2 + \Psi_{\xi_l^2}^{-1}(\alpha_l^2)||t_l^2|| - d_l \right) \\ &\qquad\qquad + \sum_{l=1}^q \left(\delta_l^2\right)^T \left(t_l^2 - (\Sigma_l^2)^{\frac{1}{2}} y \right) + \nu^2 \left(1 - \sum_{j \in J} y_j \right) \Bigg] \\ &= \min_{y \in Y^+} \left[y^T \left(A^T x + \sum_{l=1}^q \lambda_l^2 \mu_l^2 - \sum_{l=1}^q (\Sigma_l^2)^{\frac{1}{2}} \delta_l^2 - \nu^2 \mathbb{1}_n \right) \right] \\ &\quad + \min_{(t_l^2)_{l=1}^q} \left[\sum_{l=1}^q \left(\lambda_l^2 \Psi_{\xi_l^2}^{-1}(\alpha_l^2)||t_l^2|| + \left(\delta_l^2\right)^T t_l^2 \right) \right] + \nu^2 - \sum_{l=1}^q \lambda_l^2 d_l \end{aligned}$$

First and second min above is $-\infty$, unless

$$A^T x + \sum_{l=1}^{q} \lambda_l^2 \mu_l^2 - \sum_{l=1}^{q} (\Sigma_l^2)^{\frac{1}{2}} \delta_l^2 \geq \nu^2 \mathbb{1}_n$$
$$||\delta_l^2|| \leq \lambda_l^2 \Psi_{\xi_l^2}^{-1}(\alpha_l^2), \ \forall \ l \in \mathcal{I}_2.$$

Therefore, the Lagrangian dual of (4.6) is given by the following SOCP:

$$\left.\begin{array}{ll} \max\limits_{\nu^2, (\delta_l^2)_{l=1}^q, \lambda^2 \geq 0} & \nu^2 - \sum\limits_{l=1}^{q} \lambda_l^2 d_l \\ \text{s.t.} & \\ & (i) \ A^T x + \sum\limits_{l=1}^{q} \lambda_l^2 \mu_l^2 - \sum\limits_{l=1}^{q} (\Sigma_l^2)^{\frac{1}{2}} \delta_l^2 \geq \nu^2 \mathbb{1}_n \\ & (ii) \ ||\delta_l^2|| \leq \lambda_l^2 \Psi_{\xi_l^2}^{-1}(\alpha_l^2), \ \forall \ l \in \mathcal{I}_2 \\ & (iii) \ \lambda_l^2 \geq 0, \ \forall \ l \in \mathcal{I}_2. \end{array}\right\} \quad (4.7)$$

From the similar arguments used above, the strong duality for primal-dual pair of SOCPs (4.6) and (4.7) holds such that their optimal objective function values are same. Hence, the $\max\limits_{x \in S_1(\alpha^1)} \min\limits_{y \in S_2(\alpha^2)} x^T A y$ problem is equivalent to the following SOCP:

$$\left.\begin{array}{ll} \max\limits_{x, \ \nu^2, (\delta_l^2)_{l=1}^q, \lambda^2} & \nu^2 - \sum\limits_{l=1}^{q} \lambda_l^2 d_l \\ \text{s.t.} & \\ & (i) \ A^T x + \sum\limits_{l=1}^{q} \lambda_l^2 \mu_l^2 - \sum\limits_{l=1}^{q} (\Sigma_l^2)^{\frac{1}{2}} \delta_l^2 \geq \nu^2 \mathbb{1}_n \\ & (ii) \ -x^T \mu_k^1 + \Psi_{\xi_k^1}^{-1}(\alpha_k^1) ||(\Sigma_k^1)^{\frac{1}{2}} x|| \leq -b_k, \ \forall \ k \in \mathcal{I}_1 \\ & (iii) \ ||\delta_l^2|| \leq \lambda_l^2 \Psi_{\xi_l^2}^{-1}(\alpha_l^2), \ \forall \ l \in \mathcal{I}_2 \\ & (iv) \ \sum\limits_{i \in I} x_i = 1 \\ & (v) \ x_i \geq 0, \ \forall \ i \in I \\ & (vi) \ \lambda_l^2 \geq 0, \ \forall \ l \in \mathcal{I}_2. \end{array}\right\} \quad (D)$$

The SOCPs (P) and (D) are primal-dual pair of optimization problems. We show that a saddle point equilibrium of the game $G(\alpha)$ can be obtained from the optimal solutions of (P)-(D).

Theorem 4.1. *Consider a constrained zero-sum game where the matrices B^w and D^w, defining the constraints player 1 and player 2 respectively, are random. Let the row vector $B_k^w \sim Ellip(\mu_k^1, \Sigma_k^1)$, $k \in \mathcal{I}_1$, where $\Sigma_k^1 \succ 0$, and the row vector $D_l^w \sim Ellip(\mu_l^2, \Sigma_l^2)$, $l \in \mathcal{I}_2$, where $\Sigma_l^2 \succ 0$. Then, for a given $\alpha \in (0.5, 1]^p \times (0.5, 1]^q$, (x^*, y^*) is a saddle point equilibrium of the game $G(\alpha)$ if and only if there exists $(\nu^{1*}, (\delta_k^{1*})_{k=1}^p, \lambda^{1*})$ and $(\nu^{2*}, (\delta_l^{2*})_{l=1}^q, \lambda^{2*})$ such that $(y^*, \nu^{1*}, (\delta_k^{1*})_{k=1}^p, \lambda^{1*})$ and $(x^*, \nu^{2*}, (\delta_l^{2*})_{l=1}^q, \lambda^{2*})$ are optimal solutions of primal-dual pair of SOCPs* (P) *and* (D) *respectively.*

Proof. Let (x^*, y^*) be a saddle point equilibrium of the game $G(\alpha)$. Then, x^* and y^* are the solutions of (4.2) and (4.3) respectively. This together with Assumption 1 implies that there exist $(\nu^{1*}, (\delta_k^{1*})_{k=1}^p, \lambda^{1*})$ and $(\nu^{2*}, (\delta_l^{2*})_{l=1}^q, \lambda^{2*})$ such that $(y^*, \nu^{1*}, (\delta_k^{1*})_{k=1}^p, \lambda^{1*})$ and $(x^*, \nu^{2*}, (\delta_l^{2*})_{l=1}^q, \lambda^{2*})$ are optimal solutions of (P) and (D) respectively.

Let $(y^*, \nu^{1*}, (\delta_k^{1*})_{k=1}^p, \lambda^{1*})$ and $(x^*, \nu^{2*}, (\delta_l^{2*})_{l=1}^q, \lambda^{2*})$ be optimal solutions of (P) and (D) respectively. Under Assumption 1, (P) and (D) are strictly feasible. Therefore, strong duality holds for primal-dual pair (P)-(D) [15]. Then, we have

$$\nu^{1*} - \sum_{k=1}^{p} \lambda_k^{1*} b_k = \nu^{2*} - \sum_{l=1}^{q} \lambda_l^{2*} d_l. \tag{4.8}$$

Now take the constraint (i) of (P)

$$Ay^* + \sum_{k=1}^{p} \lambda_k^{1*} \mu_k^1 - \sum_{k=1}^{p} (\Sigma_k^1)^{\frac{1}{2}} \delta_k^{1*} \leq \nu^{1*} \mathbb{1}_m \tag{4.9}$$

Multiplying (4.9) by vector x^{*T} from left, we have

$$
\begin{aligned}
x^{*T}Ay^* &\leq \nu^{1*} - \sum_{k=1}^{p} \lambda_k^{1*} x^{*T} \mu_k^1 + \sum_{k=1}^{p} (\delta_k^{1*})^T (\Sigma_k^1)^{\frac{1}{2}} x^* \\
&\leq \nu^{1*} - \sum_{k=1}^{p} \lambda_k^{1*} x^{*T} \mu_k^1 + \sum_{k=1}^{p} \lambda_k^{1*} \Psi_{\xi_k^1}^{-1}(\alpha_k^1) ||(\Sigma_k^1)^{\frac{1}{2}} x^*|| \\
&\leq \nu^{1*} - \sum_{k=1}^{p} \lambda_k^{1*} b_k. \qquad (4.10)
\end{aligned}
$$

The second inequality above follows from Cauchy-Schwartz inequality and constraint (iii) of (P), and third inequality follows from (vi) of (P) and (ii) of (D). Now take the constraint (i) of (D)

$$
A^T x^* + \sum_{l=1}^{q} \lambda_l^{2*} \mu_l^2 - \sum_{l=1}^{q} (\Sigma_l^2)^{\frac{1}{2}} \delta_l^{2*} \geq \nu^{2*} \mathbb{1}_n \qquad (4.11)
$$

By using the similar approach as above, we have

$$
x^{*T}Ay^* \geq \nu^{2*} - \sum_{l=1}^{q} \lambda_l^{2*} d_l. \qquad (4.12)
$$

From (4.8), (4.10) and (4.12)

$$
x^{*T}Ay^* = \nu^{1*} - \sum_{k=1}^{p} \lambda_k^{1*} b_k = \nu^{2*} - \sum_{l=1}^{q} \lambda_l^{2*} d_l. \qquad (4.13)
$$

It is clear that (4.10) holds for all $x \in S_1(\alpha^1)$. Then, from (4.13), we have

$$
x^T Ay^* \leq x^{*T}Ay^*, \ \forall \ x \in S_1(\alpha^1). \qquad (4.14)
$$

Similarly, (4.12) holds for all $y \in S_2(\alpha^2)$ and by using (4.13) we have

$$
x^{*T}Ay \geq x^{*T}Ay^*, \ \forall \ y \in S_2(\alpha^2). \qquad (4.15)
$$

From (4.14) and (4.15), (x^*, y^*) is a saddle point equilibrium.

5. Numerical results

For illustration purpose we consider randomly generated instances of zero-sum game with random constraints. We compute the saddle point equilibria by solving the SOCPs (P) and (D). The SOCP is easy to solve and there are many free solvers available. To solve (P) and (D), we use CVX, a package for specifying and solving convex programs [12, 11]. Our numerical experiments were carried out on an Intel(R) 32-bit core(TM) i3-3110M CPU @ 2.40GHz×4 and 3.8 GiB of RAM machine.

Example 5.1. *We consider a zero-sum game which is described by* 4×4 *payoff matrix* A *and* 3×4 *random matrices* B^w *and* D^w. *The rows of matrices* B^w *and* D^w *follow a multivariate normal distribution. The game is described by the following randomly generated data:*

$$A = \begin{pmatrix} 2 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \ \mu_1^1 = \begin{pmatrix} 10 \\ 8 \\ 13 \\ 11 \end{pmatrix}, \ \mu_2^1 = \begin{pmatrix} 11 \\ 9 \\ 14 \\ 10 \end{pmatrix}, \ \mu_3^1 = \begin{pmatrix} 9 \\ 12 \\ 15 \\ 12 \end{pmatrix},$$

$$\mu_1^2 = \begin{pmatrix} 10 \\ 11 \\ 7 \\ 8 \end{pmatrix}, \mu_2^2 = \begin{pmatrix} 12 \\ 9 \\ 11 \\ 9 \end{pmatrix}, \mu_3^2 = \begin{pmatrix} 12 \\ 7 \\ 6 \\ 12 \end{pmatrix}, \ b = \begin{pmatrix} 5 \\ 6 \\ 4 \end{pmatrix}, \ d = \begin{pmatrix} 10 \\ 12 \\ 13 \end{pmatrix},$$

$$\Sigma_1^1 = \begin{pmatrix} 12 & 4 & 4 & 3 \\ 4 & 12 & 3 & 3 \\ 4 & 3 & 12 & 2 \\ 3 & 3 & 2 & 12 \end{pmatrix}, \Sigma_2^1 = \begin{pmatrix} 12 & 4 & 3 & 4 \\ 4 & 12 & 3 & 4 \\ 3 & 3 & 12 & 2 \\ 4 & 4 & 2 & 12 \end{pmatrix}, \Sigma_3^1 = \begin{pmatrix} 12 & 2 & 2 & 3 \\ 2 & 12 & 2 & 2 \\ 2 & 2 & 12 & 3 \\ 3 & 2 & 3 & 12 \end{pmatrix},$$

$$\Sigma_1^2 = \begin{pmatrix} 10 & 2 & 4 & 3 \\ 2 & 10 & 3 & 3 \\ 4 & 3 & 12 & 2 \\ 3 & 3 & 2 & 12 \end{pmatrix}, \Sigma_2^2 = \begin{pmatrix} 10 & 3 & 2 & 2 \\ 3 & 12 & 2 & 4 \\ 2 & 2 & 10 & 4 \\ 2 & 4 & 4 & 10 \end{pmatrix}, \Sigma_3^2 = \begin{pmatrix} 12 & 3 & 3 & 4 \\ 3 & 10 & 2 & 3 \\ 3 & 2 & 10 & 3 \\ 4 & 3 & 3 & 10 \end{pmatrix}.$$

Table 1 Summarizes the saddle point equilibria of the game $G(\alpha)$, corresponding to data given in Example 5.1, for various values of α. We also perform numerical experiments by considering various random instances of the game with different sizes. We generate the data using the integer random

Table 1: Saddle point equilibria for various values of α

α		Saddle Point Equilibrium		Objective function value	
α^1	α^2	x^*	y^*	(P)	(D)
(0.6, 0.6, 0.6)	(0.6, 0.6, 0.6)	$\left(\frac{1966}{10000}, \frac{1050}{10000}, \frac{5571}{10000}, \frac{1413}{10000}\right)$	$\left(0, 0, 1, 0\right)$	1	1
(0.7, 0.7, 0.7)	(0.7, 0.7, 0.7)	$\left(\frac{2254}{10000}, 0, \frac{7746}{10000}, 0\right)$	$\left(0, \frac{2038}{10000}, \frac{7866}{10000}, \frac{96}{10000}\right)$	1.22	1.22
(0.8, 0.8, 0.8)	(0.8, 0.8, 0.8)	$\left(\frac{207}{1000}, 0, \frac{793}{1000}, 0\right)$	$\left(0, \frac{1168}{10000}, \frac{4488}{10000}, \frac{4344}{10000}\right)$	1.55	1.55

number generator **randi**. We take A=**randi**(10, m, n). It generates an $m \times n$ integer matrix whose entries are not more than 10. We take mean vectors corresponding to the constraints of player 1 as μ_k^1=**randi**$\big([10m, 12m], m, 1\big)$, $k \in \mathcal{I}_1$. It generates an $m \times 1$ integer vector whose entries are within interval $[10m, 12m]$. We take the mean vectors corresponding to the constraints of player 2 as μ_l^2=**randi**$(n, n, 1)$, $l \in \mathcal{I}_2$. We generate the covariance matrices $\{\Sigma_k^1\}_{k=1}^p$ and $\{\Sigma_l^2\}_{l=1}^q$, corresponding to the constraints of player 1 and player 2 respectively, by setting $\Sigma_k^1 = Q_1 + Q_1^T + \theta_1 \cdot I_{m\times m}$ and $\Sigma_l^2 = Q_2 + Q_2^T + \theta_2 \cdot I_{n\times n}$, where Q_1=**randi**$(5, m)$ and Q_2=**randi**$(5, n)$. The matrix Q_1 is an $m \times m$ randomly generated integer matrix whose entries are not more than 5, and the matrix Q_2 is an $n \times n$ randomly generated integer matrix whose entries are not more than 5. For a given k, $I_{k\times k}$ is an $k \times k$ identity matrix. We set the parameters θ_1 and θ_2 sufficiently large so that the matrices $\{\Sigma_k^1\}_{k=1}^p$ and $\{\Sigma_l^2\}_{l=1}^q$ are positive definite. In our experiments, we take $\theta_1 = 2m$ and $\theta_2 = 2n$. We take the bounds defining the constraints of both the players as b=**randi**$(m, p, 1)$ and d = **randi**$\big([6n, 7n], q, 1\big)$. We generate the confidence levels for both the players' constraints within the interval $[0.5, 1]$ by taking $\alpha_k^1 = \alpha_l^2 = \frac{1}{2}(1+$**rand**$)$ for all k and l, where **rand** generates a random number within $[0, 1]$. Table 2 summarizes the average time for solving SOCPs (P) and (D).

Table 2: Average time for solving SOCPs (P)-(D)

No. of instances	Number of actions		Number of constraints		Average time (s)	
	m	n	p	q	(P)	(D)
10	50	60	20	25	5.14	4.74
10	100	120	40	50	50.26	43.97
10	150	150	60	60	120.29	141.92

6. Conclusions

We show the existence of a saddle point equilibrium for a two player zero-sum game with individual chance constraints if the row vectors of the random matrices, defining the constraints, are elliptically symmetric distributed random vectors. We show that the saddle point equilibria of these games can be obtained from the optimal solutions of a primal-dual pair of SOCPs. We compute the saddle point equilibria of randomly generated zero-sum games of different sizes by using CVX package for convex optimization problems in MATLAB.

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