

Optimality conditions for problems over symmetric cones and a simple augmented Lagrangian method

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Abstract

In this work we are interested in nonlinear symmetric cone problems (NSCPs), which contain as special cases nonlinear semidefinite programming, nonlinear second order cone programming and the classical nonlinear programming problems. We explore the possibility of reformulating NSCPs as common nonlinear programs (NLPs), with the aid of squared slack variables. Through this connection, we show how to obtain second order optimality conditions for NSCPs in an easy manner, thus bypassing a number of difficulties associated to the usual variational analytical approach. We then discuss several aspects of this connection. In particular, we show a “sharp” criterion for membership in a symmetric cone that also encodes rank information. Also, we discuss the possibility of importing convergence results from nonlinear programming to NSCPs, which we illustrate by discussing a simple augmented Lagrangian method for nonlinear symmetric cones. We show that, employing the slack variable approach, we can use the results for NLPs to prove convergence results, thus extending a special case (i.e., the case with strict complementarity) of an earlier result by Sun, Sun and Zhang for nonlinear semidefinite programs.

Keywords: symmetric cone, optimality conditions, augmented Lagrangian.

1 Introduction

In this paper, we analyze optimality conditions for the following problem:

$$\begin{aligned} & \underset{x}{\text{minimize}} && f(x) \\ & \text{subject to} && h(x) = 0, \\ & && g(x) \in \mathcal{K}, \\ & && x \in \mathbb{R}^n, \end{aligned} \tag{P1}$$

where $f: \mathbb{R}^n \rightarrow \mathbb{R}$, $h: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $g: \mathbb{R}^n \rightarrow \mathcal{E}$ are twice continuously differentiable functions. We assume that \mathcal{E} is a finite dimensional space equipped with an inner product $\langle \cdot, \cdot \rangle$. Here, $\mathcal{K} \subseteq \mathcal{E}$ is a *symmetric cone*, that is

1. \mathcal{K} is *self-dual*, i.e., $\mathcal{K} := \mathcal{K}^* = \{x \mid \langle x, y \rangle \geq 0, \forall y \in \mathcal{K}\}$,

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2. \mathcal{K} is *full-dimensional*, i.e., the interior of \mathcal{K} is not empty,
3. \mathcal{K} is *homogeneous*, i.e., for every pair of points x, y in the interior of \mathcal{K} , there is a linear bijection T such that $T(x) = y$ and $T(\mathcal{K}) = \mathcal{K}$. In short, the group of automorphism of \mathcal{K} acts transitively on the interior of \mathcal{K} .

Then, problem (P1) is called a *nonlinear symmetric cone problem* (NSCP). One of the advantages of dealing with NSCPs is that it provides an unified framework for a number of different problems including classical nonlinear programs (NLPs), nonlinear semidefinite programs (NSDPs), nonlinear second order cone programs (NSOCPs) and any mixture of those three.

Since \mathcal{K} is a symmetric cone, we may assume that \mathcal{E} is equipped with a bilinear map $\circ : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$ such that \mathcal{K} is the corresponding cone of squares, that is,

$$\mathcal{K} = \{y \circ y \mid y \in \mathcal{E}\}.$$

Furthermore, we assume that \circ possesses the following three properties:

1. $y \circ z = z \circ y$,
2. $y \circ (y^2 \circ z) = y^2 \circ (y \circ z)$, where $y^2 = y \circ y$,
3. $\langle y \circ z, w \rangle = \langle y, z \circ w \rangle$,

for all $y, w, z \in \mathcal{E}$. Under these conditions, (\mathcal{E}, \circ) is called an *Euclidean Jordan algebra*. It can be shown that every symmetric cone arises as the cone of squares of some Euclidean Jordan algebra, see Theorems III.2.1 and III.3.1 in [12]. We will use $\|\cdot\|$ to indicate the norm induced by $\langle \cdot, \cdot \rangle$. We remark that no previous knowledge on Jordan algebras will be assumed and we will try to be as self-contained as possible.

While the analysis of first order conditions for (P1) is relatively straightforward, it is challenging to obtain a workable description of second order conditions for (P1). We recall that for NSOCPs and NSDPs, in order to obtain the so-called “zero-gap” optimality conditions, it is not enough to build the Lagrangian and require it to be positive semidefinite/definite over the critical cone. In fact, an extra term is needed which, in the literature, is known as the *sigma-term* and is said to model the curvature of the underlying cone. The term “zero-gap” alludes to the fact that the change from “necessary” to “sufficient” should involve, apart from minor technicalities, only a change from “ \geq ” to “ $>$ ”, as it is the case for classical nonlinear programming (see Theorems 12.5 and 12.6 in [26] or Section 3.3 in [3]).

Typically, there are two approaches for obtaining “zero-gap” second order conditions. The first is to compute directly the so-called *second order tangent sets* of \mathcal{K} . This was done, for instance, by Bonnans and Ramírez in [7] for NSOCP. Another approach is to express the cone as

$$\mathcal{K} = \{z \in \mathcal{E} \mid \varphi(z) \leq 0\},$$

where φ is some convex function. Then, the second order tangent sets can be computed by examining the second order directional derivatives of φ . This is the approach favored by Shapiro in [29] for NSDPs.

For \mathcal{K} , there is a natural candidate for φ . Over an Euclidean Jordan algebra, we have a “minimum eigenvalue function” σ_{\min} , for which $x \in \mathcal{K}$ if and only if $\sigma_{\min}(x) \geq 0$, in analogy to the positive semidefinite cone case. We then take $\varphi = -\sigma_{\min}$. Unfortunately, as far as we know, it is still an open problem to give explicit descriptions of higher order directional derivatives for $-\sigma_{\min}$. In addition, it seems complicated to describe the second order tangent sets of \mathcal{K} directly.

In fact, the only reference we could find that discussed second order conditions for NSCPs is the work by Kong, Tunçel and Xiu [24], where they define the strong second order sufficient condition for a linear symmetric cone program, see Definition 3.3 therein. Here, we bypass all these difficulties by exploring the Jordan algebraic connection and transforming (P1) into an ordinary nonlinear program with equality constraints:

$$\begin{aligned} & \underset{x,y}{\text{minimize}} && f(x) \\ & \text{subject to} && h(x) = 0, \\ & && g(x) = y \circ y, \\ & && x \in \mathbb{R}^n, y \in \mathcal{E}. \end{aligned} \tag{P2}$$

We will then use (P2) to derive optimality conditions for (P1). By writing down the second order conditions for (P2) and eliminating the slack variable y , we can obtain second order conditions for (P1). This is explained in more detail in Section 6. The drawback of this approach is that the resulting second order conditions require strict complementarity. How serious this drawback is depends, of course, on the specific applications one has in mind. Still, we believe the connection between the two formulations can bring some new insights.

In particular, through this work we found a “sharp” characterization of membership in a symmetric cone. Note that since \mathcal{K} is self-dual, a necessary and sufficient condition for some $\lambda \in \mathcal{E}$ to belong to \mathcal{K} is that $\langle z, \lambda \rangle \geq 0$ holds for all $z \in \mathcal{K}$, or equivalently, that $\langle w \circ w, \lambda \rangle \geq 0$ for all $w \in \mathcal{E}$. This, however, gives no information on the rank of λ . In contrast, Theorem 3.3 shows that if we instead require that $\langle w \circ w, \lambda \rangle > 0$ for all nonzero w in some special subspace of \mathcal{E} , this not only guarantees that $\lambda \in \mathcal{K}$, but also reveals information about its rank. This generalizes Lemma 1 in [25] for all symmetric cones.

Moreover, our analysis opens up the possibility of importing convergence results from the NLP world to the NSCP world, instead of proving them from scratch. In Section 7, we illustrate this by extending a result of Sun, Sun and Zhang [32] on the quadratic augmented Lagrangian method.

The paper is organized as follows. In Section 2, we review basic notions related to Euclidean Jordan algebras, KKT points and second order conditions. In Section 3, we prove a criterion for membership in a symmetric cone. In Section 4, we provide sufficient conditions that guarantee equivalence between KKT points of (P1) and (P2). In Section 5, we discuss the relation between constraint qualifications of those two problems. In Section 6, we present second order conditions for (P1). In Section 7, we discuss a simple augmented Lagrangian method. We conclude in Section 8, with final remarks and a few suggestions for further work.

2 Preliminaries

2.1 Euclidean Jordan algebra

We first review a few aspects of the theory of Euclidean Jordan algebras. More details can be found in the book by Faraut and Korányi [12] and also in the survey paper by Faybusovich [14]. First of all, any symmetric cone \mathcal{K} arises as the cone of squares of some Euclidean Jordan algebra (\mathcal{E}, \circ) . Furthermore, we can assume that \mathcal{E} has an unit element e satisfying $y \circ e = y$, for all $y \in \mathcal{E}$. Reciprocally, given an Euclidean Jordan algebra (\mathcal{E}, \circ) , it can be shown that the corresponding cone of squares is a symmetric cone. See Theorems III.2.1 and III.3.1 in [12], for more details.

Given $y \in \mathcal{E}$, we denote by L_y the linear operator such that

$$L_y(w) = y \circ w,$$

for all $w \in \mathcal{E}$.

In what follows, we say that c is an *idempotent* if $c \circ c = c$. Moreover, c is *primitive* if it is nonzero and there is no way of writing $c = a + b$, with nonzero idempotents a and b satisfying $a \circ b = 0$.

Theorem 2.1 (Spectral Theorem, see Theorem III.1.2 in [12]). *Let (\mathcal{E}, \circ) be an Euclidean Jordan algebra and let $y \in \mathcal{E}$. Then there are primitive idempotents c_1, \dots, c_r satisfying*

$$c_i \circ c_j = 0 \quad \text{for } i \neq j, \quad (2.1)$$

$$c_i \circ c_i = c_i, \quad i = 1, \dots, r, \quad (2.2)$$

$$c_1 + \dots + c_r = e, \quad i = 1, \dots, r, \quad (2.3)$$

and unique real numbers $\sigma_1, \dots, \sigma_r$ satisfying

$$y = \sum_{i=1}^r \sigma_i c_i. \quad (2.4)$$

We say that c_1, \dots, c_r in Theorem 2.1 form a *Jordan frame* for y , and $\lambda_1, \dots, \lambda_r$ are the *eigenvalues* of y . We remark that r only depends on the algebra \mathcal{E} . Given $y \in \mathcal{E}$, we define its trace by

$$\text{tr}(y) = \sigma_1 + \dots + \sigma_r,$$

where $\sigma_1, \dots, \sigma_r$ are the eigenvalues of y . As in the case of matrices, it turns out that the trace function is linear. It can also be used to define an inner product compatible with the Jordan product, and so henceforth we will assume that $\langle x, y \rangle = \text{tr}(x \circ y)$. In the case of symmetric matrices, $\langle \cdot, \cdot \rangle$ turns out to be the Frobenius inner product.

For an element $y \in \mathcal{E}$, we define the *rank* of y as the number of nonzero λ_i 's that appear in (2.4). Then, the rank of \mathcal{K} is defined by

$$\text{rank } \mathcal{K} = \max\{\text{rank } y \mid y \in \mathcal{K}\} = r = \text{tr}(e).$$

We will also say that the rank of \mathcal{E} is $r = \text{tr}(e)$.

For the next theorem, we need the following notation. Given $y \in \mathcal{E}$ and $a \in \mathbb{R}$, we write

$$V(y, a) = \{z \in \mathcal{E} \mid y \circ z = az\}.$$

For any $V, V' \subseteq \mathcal{E}$, we write $V \circ V' = \{y \circ z \mid y \in V, z \in V'\}$.

Theorem 2.2 (Peirce Decomposition – 1st version, see Proposition IV.1.1 in [12]). *Let $c \in \mathcal{E}$ be an idempotent. Then \mathcal{E} is decomposed as the orthogonal direct sum*

$$\mathcal{E} = V(c, 1) \oplus V\left(c, \frac{1}{2}\right) \oplus V(c, 0).$$

In addition, $V(c, 1)$ and $V(c, 0)$ are Euclidean Jordan algebras satisfying $V(c, 1) \circ V(c, 0) = \{0\}$. Moreover, $(V(c, 1) + V(c, 0)) \circ V(c, 1/2) \subseteq V(c, 1/2)$ and $V(c, 1/2) \circ V(c, 1/2) \subseteq V(c, 1) + V(c, 0)$.

The Peirce decomposition has another version, with detailed information on the way that the algebra is decomposed.

Theorem 2.3 (Peirce Decomposition – 2nd version, see Theorem IV.2.1 in [12]). *Let c_1, \dots, c_r be a Jordan frame for $y \in \mathcal{E}$. Then \mathcal{E} is decomposed as the orthogonal sum*

$$\mathcal{E} = \bigoplus_{1 \leq i \leq j \leq r} V_{ij},$$

where

$$V_{ii} = V(c_i, 1) = \{\alpha c_i \mid \alpha \in \mathbb{R}\},$$

$$V_{ij} = V\left(c_i, \frac{1}{2}\right) \cap V\left(c_j, \frac{1}{2}\right), \quad \text{for } i \neq j.$$

Moreover

(i) the V_{ii} 's are subalgebras of \mathcal{E} ,

(ii) the following relations hold:

$$V_{ij} \circ V_{ij} \subseteq V_{ii} + V_{jj} \quad \forall i, j, \quad (2.5)$$

$$V_{ij} \circ V_{jk} \subseteq V_{ik} \quad \text{if } i \neq k, \quad (2.6)$$

$$V_{ij} \circ V_{kl} = \{0\} \quad \text{if } \{i, j\} \cap \{k, l\} = \emptyset. \quad (2.7)$$

The algebra (\mathcal{E}, \circ) is said to be *simple* if there is no way to write $\mathcal{E} = V \oplus W$, where V and W are both nonzero subalgebras of \mathcal{E} . We will say that \mathcal{K} is *simple* if it is the cone of squares of a simple algebra. It turns out that every Euclidean Jordan algebra can be decomposed as a direct sum of simple Euclidean Jordan algebras, which then induces a decomposition of \mathcal{K} in simple symmetric cones. This means that we can write

$$\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_\ell,$$

$$\mathcal{K} = \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_\ell,$$

where the \mathcal{E}_i 's are simple Euclidean Jordan algebras of rank r_i and \mathcal{K}_i is the cone of squares of \mathcal{E}_i . Note that orthogonality expressed by this decomposition is not only with respect to the inner product $\langle \cdot, \cdot \rangle$ but also with respect to the Jordan product \circ . There is a classification of the simple Euclidean Jordan algebras and, up to isomorphism, they fall in four infinite families and a single exceptional case.

Due to the decomposition results, some articles only deal with simple Jordan algebras (such as [31, 13]), while others prove results in full generality (such as [1]). The extension from the simple case to the general case is usually straightforward but must be done carefully.

We recall the following properties of \mathcal{K} . The results follow from various propositions that appear in [12], such as Proposition III.2.2 and Exercise 3 in Chapter III. See also Equation (10) in [30].

Proposition 2.4. *Let $y, w \in \mathcal{E}$.*

(i) $y \in \mathcal{K}$ if and only if the eigenvalues of y are nonnegative.

(ii) $y \in \text{int } \mathcal{K}$ if and only if the eigenvalues of y are positive.

(iii) $y \in \text{int } \mathcal{K}$ if and only if $\langle y, w \circ w \rangle > 0$ for all nonzero $w \in \mathcal{E}$.

(iv) Suppose $y, w \in \mathcal{K}$. Then, $y \circ w = 0$ if and only if $\langle y, w \rangle = 0$.

From item (iv) of Proposition 2.4, we have that if c and c' are two idempotents belonging to distinct blocks, we also have $c \circ c' = 0$ in addition to $\langle c, c' \rangle = 0$. Since this holds for all idempotents, we have $\mathcal{E}_i \circ \mathcal{E}_j = 0$, whenever $i \neq j$.

Due to Proposition 2.4, if $y \in \mathcal{K}$, then the eigenvalues of y are nonnegative, and so we can define the square root of y as

$$\sqrt{y} = \sum_{i=1}^r \sqrt{\sigma_i} c_i,$$

where $\{c_1, \dots, c_r\}$ is a Jordan frame for y .

2.2 Projection on a symmetric cone

Denote by $P_{\mathcal{K}}$ the orthogonal projection on \mathcal{K} . Given $y \in \mathcal{E}$, $P_{\mathcal{K}}(y)$ satisfies

$$P_{\mathcal{K}}(y) = \operatorname{argmin}_{z \in \mathcal{K}} \|y - z\|.$$

In analogy to the case of a positive semidefinite cone, to project x on \mathcal{K} it is enough to zero the negative eigenvalues of x . We register this well-known fact as a proposition.

Proposition 2.5. *Let $y \in \mathcal{E}$ and consider a Jordan decomposition of y*

$$y = \sum_{i=1}^r \sigma_i c_i,$$

where $\{c_1, \dots, c_r\}$ is a Jordan frame for y . Then, its projection is given by

$$P_{\mathcal{K}}(y) = \sum_{i=1}^r \max(\sigma_i, 0) c_i. \quad (2.8)$$

Proof. Let $z \in \mathcal{K}$. In view of Theorem 2.3, we can write

$$z = \sum_{i=1}^r v_{ii} + \sum_{1 \leq i < j \leq r} v_{ij},$$

where $v_{ij} \in V_{ij}$ for all i and j . As $V_{ii} = \{\alpha c_i \mid \alpha \in \mathbb{R}\}$, we have

$$z = \sum_{i=1}^r \alpha_i c_i + \sum_{1 \leq i < j \leq r} v_{ij},$$

for some constants $\alpha_i \in \mathbb{R}$. Recall that the subspaces V_{ij} are all orthogonal among themselves and that \mathcal{K} is self-dual. Then, since $c_i \in \mathcal{K}$ and $\langle z, c_i \rangle \geq 0$, we have $\alpha_i \geq 0$ for all i . Furthermore, we have

$$\|y - z\|^2 = \sum_{i=1}^r (\sigma_i - \alpha_i)^2 \|c_i\|^2 + \sum_{1 \leq i < j \leq r} \|v_{ij}\|^2.$$

Therefore, if we wish to minimize $\|y - z\|^2$, the best we can do is to set each v_{ij} to zero and each α_i to $\max(\sigma_i, 0)$. This shows that (2.8) holds. \square

In analogy to the symmetric matrices, we will use the following notation:

$$P_{\mathcal{K}}(y) = [y]_+.$$

The following observation will also be helpful.

Lemma 2.6. *Let \mathcal{K} be a symmetric cone and $v \in \mathcal{E}$. Then,*

$$v - [v]_+ = -[-v]_+.$$

Proof. The Moreau decomposition (see, e.g., Theorem 3.2.5 in [18]) tells us that $v - [v]_+ = P_{\mathcal{K}^\circ}(v)$, where \mathcal{K}° is the polar cone of \mathcal{K} . As \mathcal{K} is self-dual, we have $\mathcal{K}^\circ = -\mathcal{K}$. Therefore, $P_{\mathcal{K}^\circ}(v) = -P_{\mathcal{K}}(-v) = -[-v]_+$. \square

2.3 The Karush-Kuhn-Tucker conditions

First, we define the Lagrangian function $L: \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{E} \rightarrow \mathbb{R}$ associated with problem (P1) as

$$L(x, \mu, \lambda) := f(x) - \langle h(x), \mu \rangle - \langle g(x), \lambda \rangle.$$

We say that $(x, \mu, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{E}$ is a *Karush-Kuhn-Tucker* (KKT) triple of problem (P1) if the following conditions are satisfied:

$$\nabla f(x) - Jh(x)^* \mu - Jg(x)^* \lambda = 0, \quad (\text{P1.1})$$

$$\lambda \in \mathcal{K}, \quad (\text{P1.2})$$

$$g(x) \in \mathcal{K}, \quad (\text{P1.3})$$

$$\lambda \circ g(x) = 0, \quad (\text{P1.4})$$

$$h(x) = 0, \quad (\text{P1.5})$$

where ∇f is the gradient of f , Jg is the Jacobian of g and Jg^* denotes the adjoint of Jg . Usually, instead of (P1.4), we would have $\langle \lambda, g(x) \rangle = 0$, but in view of item (iv) of Proposition 2.4, they are equivalent. Note also that (P1.1) is equivalent to $\nabla L_x(x, \mu, \lambda) = 0$, where ∇L_x denotes the gradient of L with respect to x .

We also have the following definition.

Definition 2.7. *If $(x, \mu, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{E}$ is a KKT triple of (P1) such that*

$$\text{rank } g(x) + \text{rank } \lambda = r,$$

then (x, λ) is said to satisfy the strict complementarity condition.

As for the equality constrained NLP problem (P2), we observe that $(x, y, \mu, \lambda) \in \mathbb{R}^n \times \mathcal{E} \times \mathbb{R}^m \times \mathcal{E}$ is a KKT quadruple if the conditions below are satisfied:

$$\nabla_{(x,y)} \mathcal{L}(x, y, \mu, \lambda) = 0,$$

$$h(x) = 0,$$

$$g(x) - y \circ y = 0,$$

where $\mathcal{L}: \mathbb{R}^n \times \mathcal{E} \times \mathbb{R}^m \times \mathcal{E} \rightarrow \mathbb{R}$ is the Lagrangian function associated with (P2), which is given by

$$\mathcal{L}(x, y, \mu, \lambda) := f(x) - \langle h(x), \mu \rangle - \langle g(x) - y \circ y, \lambda \rangle$$

and $\nabla_{(x,y)} \mathcal{L}$ denotes the gradient of \mathcal{L} with respect to (x, y) .

We can then write the KKT conditions for (P2) as

$$\nabla f(x) - Jh(x)^* \mu - Jg(x)^* \lambda = 0, \quad (\text{P2.1})$$

$$\lambda \circ y = 0, \quad (\text{P2.2})$$

$$g(x) - y \circ y = 0, \quad (\text{P2.3})$$

$$h(x) = 0. \quad (\text{P2.4})$$

Writing the conditions for (P1) and (P2), we see that although they are equivalent problems, the KKT conditions are slightly different. In fact, for (P2), it is not required that λ belongs to \mathcal{K} and this accounts for a great part of the difference between both formulations.

For (P1), we say that the *Mangasarian-Fromovitz constraint qualification* (MFCQ) holds at a point x if $Jh(x)$ is surjective and there exists some $d \in \mathbb{R}^n$ such that

$$\begin{aligned} Jh(x)d &= 0, \\ g(x) + Jg(x)d &\in \text{int } \mathcal{K}, \end{aligned}$$

where $\text{int } \mathcal{K}$ denotes the interior of \mathcal{K} . See, for instance, Equation (2.190) in [8]. If x is a local minimum for (P1), MFCQ ensures the existence of a pair of Lagrange multipliers (μ, λ) and that the set of multipliers is bounded.

We also can define a nondegeneracy condition as follows.

Definition 2.8. *Suppose that $x \in \mathcal{K}$ is such that*

$$\begin{aligned} \mathbb{R}^m &= \text{Im } Jh(x), \\ \mathcal{K} &= \text{lin } \mathcal{T}_{\mathcal{K}}(g(x)) + \text{Im } Jg(x), \end{aligned}$$

where $\text{Im } Jg(x)$ denotes the image of the linear map $Jg(x)$, $\mathcal{T}_{\mathcal{K}}(g(x))$ denotes the tangent cone of \mathcal{K} at $g(x)$, and $\text{lin } \mathcal{T}_{\mathcal{K}}(g(x))$ is the lineality space of $\mathcal{T}_{\mathcal{K}}(g(x))$, i.e., $\text{lin } \mathcal{T}_{\mathcal{K}}(g(x)) = \mathcal{T}_{\mathcal{K}}(g(x)) \cap -\mathcal{T}_{\mathcal{K}}(g(x))$. Then, x is said to be nondegenerate.

This definition is related to the transversality condition introduced by Shapiro in [29], see Definition 4 therein. Furthermore, Definition 2.8 is a special case of a more general definition of nondegeneracy discussed in the work by Bonnans and Shapiro, see Section 4.6.1 of [8].

For (P2), we say that the *linear independence constraint qualification* (LICQ) is satisfied at a point (x, y) if the gradients of the constraints are linearly independent.

2.4 Second order conditions for (P2)

For (P2), we say that the second order sufficient condition (SOSC-NLP) holds if

$$\langle \nabla_{(x,y)}^2 \mathcal{L}(x, y, \mu, \lambda)(v, w), (v, w) \rangle > 0,$$

for every nonzero $(v, w) \in \mathbb{R}^n \times \mathcal{E}$ such that $Jg(x)v - 2y \circ w = 0$ and $Jh(x)v = 0$, where $\nabla_{(x,y)}^2 \mathcal{L}$ denotes the Hessian of \mathcal{L} with respect to (x, y) . See [3, Section 3.3] or [26, Theorem 12.6]. We can also present the SOSC-NLP in terms of the Lagrangian of (P1).

Proposition 2.9. *Let $(x, y, \mu, \lambda) \in \mathbb{R}^n \times \mathcal{E} \times \mathbb{R}^m \times \mathcal{E}$ be a KKT quadruple of problem (P2). The SOSC-NLP holds if*

$$\langle \nabla_x^2 L(x, \mu, \lambda)v, v \rangle + 2\langle w \circ w, \lambda \rangle > 0 \quad (2.9)$$

for every nonzero $(v, w) \in \mathbb{R}^n \times \mathcal{E}$ such that $Jg(x)v - 2y \circ w = 0$ and $Jh(x)v = 0$.

Proof. Note that we have

$$\nabla_{(x,y)}^2 \mathcal{L}(x, y, \mu, \lambda) = \nabla_{(x,y)}^2 [L(x, \mu, \lambda) + \langle y \circ y, \lambda \rangle].$$

Therefore,

$$\langle \nabla_{(x,y)}^2 \mathcal{L}(x, y, \mu, \lambda)(v, w), (v, w) \rangle = \langle \nabla_x^2 L(x, \mu, \lambda)v, v \rangle + \langle \nabla_y^2 \langle y \circ y, \lambda \rangle w, w \rangle.$$

Due to the fact that the underlying algebra is Euclidean, we have $\nabla_y \langle y \circ y, \lambda \rangle = 2y \circ \lambda$ and $\nabla_y^2 \langle y \circ y, \lambda \rangle = 2L_\lambda$, where we recall that L_λ is the linear operator satisfying $L_\lambda(z) = \lambda \circ z$ for every z . We then conclude that

$$\langle \nabla_y^2 \langle y \circ y, \lambda \rangle w, w \rangle = \langle w, 2L_\lambda(w) \rangle = 2\langle w \circ w, \lambda \rangle,$$

which implies that (2.9) holds. \square

Similarly, we have the following second order necessary condition (SONC). Note that we require the LICQ to hold.

Proposition 2.10. *Let (x, y) be a local minimum for (P2) and $(x, y, \mu, \lambda) \in \mathbb{R}^n \times \mathcal{E} \times \mathbb{R}^m \times \mathcal{E}$ be a KKT quadruple such that LICQ holds. Then, the following SONC-NLP holds:*

$$\langle \nabla_x^2 L(x, \mu, \lambda)v, v \rangle + 2\langle w \circ w, \lambda \rangle \geq 0 \quad (2.10)$$

for every $(v, w) \in \mathbb{R}^n \times \mathcal{E}$ such that $Jg(x)v - 2y \circ w = 0$ and $Jh(x)v = 0$.

Proof. See [26, Theorem 12.5] for the basic format of the second order necessary condition for NLPs. In order to express the condition in terms of the Lagrangian of (P1), we proceed as in the proof of Proposition 2.9. \square

Under the assumption that \mathcal{E} is decomposed as $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_\ell$, the term $2\langle w \circ w, \lambda \rangle$ can be written as

$$\sum_{i=1}^{\ell} 2\langle w_i \circ w_i, \lambda_i \rangle,$$

which brings Propositions 2.9 and 2.10 closer to the format described, for instance, in Lemma 3.2 of [16].

Furthermore, we observe that in Propositions 2.9 and 2.10 an extra term appears together with the Lagrangian of (P1). This term is connected with the so-called *sigma-term* that appears in second order optimality conditions for optimization problems over general closed convex cones and plays an important role in the construction of no-gap optimality conditions. Classical references for second order optimality conditions for general cones include the works of Kawasaki [22], Cominetti [11], Bonnans, Cominetti and Shapiro [6], and the book by Bonnans and Shapiro [8]. In particular, we refer to sections 3.2, 3.3, 5.2 and 5.3 in [6].

3 A criterion for membership in \mathcal{K}

We need the following auxiliary result.

Proposition 3.1. *Let (\mathcal{E}, \circ) be a simple Euclidean Jordan algebra and let c_i and c_j be two orthogonal primitive idempotents. Then*

$$V\left(c_i, \frac{1}{2}\right) \cap V\left(c_j, \frac{1}{2}\right) \neq \{0\}.$$

Proof. See Corollary IV.2.4 in [12]. \square

Since \mathcal{K} is self-dual, we have that $\lambda \in \mathcal{K}$ if and only if $\langle \lambda, w \circ w \rangle \geq 0$ for all $w \in \mathcal{E}$. Lemma 3.2 refines this criterion for the case where \mathcal{K} is *simple*.

Lemma 3.2. *Let (\mathcal{E}, \circ) be a simple Euclidean Jordan algebra of rank r and $\lambda \in \mathcal{E}$. The following statements are equivalent:*

(i) $\lambda \in \mathcal{K}$.

(ii) There exists $y \in \mathcal{E}$ such that $y \circ \lambda = 0$ and

$$\langle w \circ w, \lambda \rangle > 0, \quad (3.1)$$

for every $w \in \mathcal{E}$ satisfying $y \circ w = 0$ and $w \neq 0$.

Moreover, any y satisfying (ii) is such that

- (a) $\text{rank } y = r - \text{rank } \lambda$, i.e., y and λ satisfy strict complementarity,
- (b) if σ and σ' are non-zero eigenvalues of y , then $\sigma + \sigma' \neq 0$.

Proof. (i) \Rightarrow (ii) If $\lambda \in \mathcal{K}$, we write its spectral decomposition as

$$\lambda = \sum_{i=1}^r \sigma_i c_i,$$

where we assume that only the first $\text{rank } \lambda$ eigenvalues are positive and the others are zero. If $\text{rank } \lambda = r$, we take $y = 0$. Otherwise, take

$$y = \sum_{i=\text{rank } \lambda + 1}^r c_i.$$

Note that y is an idempotent and that λ lies in the relative interior of the cone of squares of the Jordan algebra $V(y, 0)$. Hence, the condition (3.1) is satisfied.

(ii) \Rightarrow (i), together with (a) and (b) We write

$$y = \sum_{i=1}^r \sigma_i c_i = \sum_{i=1}^{\text{rank } y} \sigma_i c_i,$$

where $\{c_1, \dots, c_r\}$ is a Jordan frame, and we assume that the first $\text{rank } y$ eigenvalues of y are nonzero and the others are zero. Then, following Theorem 2.3, we write

$$\lambda = \sum_{i \leq j} \lambda_{ij} = \sum_{i=1}^r \lambda_{ii} + \sum_{i < j} \lambda_{ij},$$

where $\lambda_{ij} \in V_{ij}$. Using the operation rules in Theorem 2.3, we get

$$c_k \circ \lambda_{ij} = \begin{cases} \lambda_{ij}, & \text{if } i = j = k \\ 0, & \text{if } \{k\} \cap \{i, j\} = \emptyset \\ \frac{\lambda_{ij}}{2}, & \text{if } i < j, \quad \{k\} \cap \{i, j\} = \{k\}. \end{cases}$$

Therefore,

$$y \circ \lambda = \sum_{i=1}^{\text{rank } y} \sigma_i \lambda_{ii} + \sum_{1 \leq i < j \leq \text{rank } y} \left(\frac{\sigma_i + \sigma_j}{2} \right) \lambda_{ij} + \sum_{1 \leq i \leq \text{rank } y < j} \frac{\sigma_i}{2} \lambda_{ij}. \quad (3.2)$$

By hypothesis, we have $y \circ \lambda = 0$. Since the V_{ij} 's are mutually orthogonal subspaces, we conclude that all terms inside the summations in (3.2) must be zero. In particular, we have $\sigma_i \lambda_{ii} = 0$, for every $i \leq \text{rank } y$. As the σ_i 's are nonzero for those indexes, we have $\lambda_{ii} = 0$ so that

$$\lambda = \sum_{\text{rank } y < i \leq r} \lambda_{ii} + \sum_{i < j} \lambda_{ij}, \quad (3.3)$$

We now show that (b) holds. Suppose for the sake of contradiction that $\sigma_i + \sigma_j = 0$ for some $i < j \leq \text{rank } y$. By Proposition 3.1, there is a nonzero $w \in V(c_i, 1/2) \cap V(c_j, 1/2)$. Since $w \circ c_k = 0$ for $k \neq i, k \neq j$, we have

$$y \circ w = (\sigma_i c_i \circ w) + (\sigma_j c_j \circ w) = \left(\frac{\sigma_i + \sigma_j}{2} \right) w = 0.$$

Moreover, $w \circ w \in V(c_i, 1) + V(c_j, 1)$, due to (2.5). By (3.3) and the orthogonality among the V_{ij} ,

$$\langle w \circ w, \lambda \rangle = 0,$$

since λ has no component in neither $V(c_i, 1) = V_{ii}$ nor $V(c_j, 1) = V_{jj}$. This contradicts (3.1), and so it must be the case that (b) holds.

Let $c = c_1 + \cdots + c_{\text{rank } y}$. We will now show that $V(c, 0) = V(y, 0)$. Since $\mathcal{E} = V(c, 0) \oplus V(c, 1/2) \oplus V(c, 1)$ and $y \in V(c, 1)$, we have $V(c, 0) \subseteq V(y, 0)$, since $V(c, 0) \circ V(c, 1) = \{0\}$.

The next step is to prove that $V(y, 0) \subseteq V(c, 0)$. Suppose that $y \circ w = 0$ and write $w = \sum_{i \leq j} w_{ij}$ with $w_{ij} \in V_{ij}$ as in Theorem 2.3. As in (3.2), we have

$$y \circ w = \sum_{i=1}^{\text{rank } y} \sigma_i w_{ii} + \sum_{1 \leq i < j \leq \text{rank } y} \left(\frac{\sigma_i + \sigma_j}{2} \right) w_{ij} + \sum_{1 \leq i \leq \text{rank } y < j} \frac{\sigma_i}{2} w_{ij} = 0.$$

Because σ_i, σ_j and $\sigma_i + \sigma_j$ are all nonzero when $i \leq j \leq \text{rank}$, it follows that $w_{ij} = 0$ for $i \leq j \leq \text{rank } y$ and for $i \leq \text{rank } y < j$, since w_{ij} all lie in mutually orthogonal subspaces. Therefore, $w = \sum_{\text{rank } y < i \leq j} w_{ij}$ and Theorem 2.3 implies that $c \circ w = 0$, which shows that $V(y, 0) \subseteq V(c, 0)$.

We now know that $V(c, 0) = V(y, 0)$. By Theorem 2.2, $V(c, 0)$ is an Euclidean Jordan algebra and its rank is $r - \text{rank } y$, since $c_{\text{rank } y+1} + \cdots + c_r$ is the identity in $V(c, 0)$. Then, the condition (3.1) means that $\langle z, \lambda \rangle > 0$ for all z in the cone of squares of the algebra $V(c, 0)$. By item (iii) of Proposition 2.4, this means that λ belongs to the (relative) interior of $\mathcal{K}' = \{w \circ w \mid w \in V(c, 0)\}$. This shows both that $\lambda \in \mathcal{K}$ and that $\text{rank } \lambda = \text{rank } \mathcal{K}' = r - \text{rank } y$. These are items (i) and (a). \square

Lemma 3.2 does not apply directly when \mathcal{K} is not simple because the complementarity displayed in item (b) only works “inside the same blocks”. That is essentially the only aspect we need to account for. As a reminder, we have

$$\begin{aligned} \mathcal{E} &= \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_\ell, \\ \mathcal{K} &= \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_\ell, \end{aligned}$$

where the \mathcal{E}_i are simple Euclidean Jordan algebras of rank r_i and \mathcal{K}_i is the cone of squares of \mathcal{E}_i . The rank of \mathcal{E} is $r = r_1 + \cdots + r_\ell$.

Theorem 3.3. *Let (\mathcal{E}, \circ) be an Euclidean Jordan algebra of rank r and $\lambda \in \mathcal{E}$. The following statements are equivalent:*

(i) $\lambda \in \mathcal{K}$.

(ii) There exists $y \in \mathcal{E}$ such that $y \circ \lambda = 0$ and

$$\langle w \circ w, \lambda \rangle > 0, \tag{3.4}$$

for every $w \in \mathcal{E}$ satisfying $y \circ w = 0$ and $w \neq 0$.

Moreover, any y satisfying (ii) is such that

- (a) $\text{rank } y = r - \text{rank } \lambda$, i.e., y and λ satisfy strict complementarity,
- (b) if σ and σ' are non-zero eigenvalues of y belonging to the same block, then $\sigma + \sigma' \neq 0$.

Proof. $\boxed{(i) \Rightarrow (ii)}$ Write $\lambda = \lambda_1 + \cdots + \lambda_\ell$, according to block division in \mathcal{E} . Then apply Lemma 3.2 to each λ_i to obtain y_i , and let $y = y_1 + \cdots + y_\ell$.

$\boxed{(ii) \Rightarrow (i)}$ Write $\lambda = \lambda_1 + \cdots + \lambda_\ell$ and $y = y_1 + \cdots + y_\ell$. Then, the inequality (3.4) implies that for every i ,

$$\langle w_i \circ w_i, \lambda_i \rangle > 0, \quad (3.5)$$

for all nonzero w_i with $w_i \circ y_i = 0$. Therefore, Lemma 3.2 applies to each y_i , thus concluding the proof. \square

Theorem 3.3 extends Lemma 1 in [25] for all symmetric cones. For a product of second order cones, no similar result is explicitly stated in [16], but it can be derived from the proof of Proposition 3.2 therein.

4 Comparison of KKT points for (P1) and (P2)

Although (P1) and (P2) share the same local minima, their KKT points are not necessarily the same. However, if (x, μ, λ) is a KKT triple for (P1), it is easy to construct a KKT quadruple for (P1) according to the next proposition.

Proposition 4.1. *Let $(x, \mu, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{E}$ be a KKT triple for (P1), then $(x, \sqrt{g(x)}, \mu, \lambda)$ is a KKT quadruple for (P2).*

Proof. The quadruple $(x, \sqrt{g(x)}, \mu, \lambda)$ satisfies (P2.1), (P2.3) and (P2.4). We will now check that (P2.2) is also satisfied. We can write

$$\lambda = \sum_{i=1}^{\text{rank } \lambda} \sigma_i c_i,$$

where $\{c_1, \dots, c_r\}$ is a Jordan frame for λ such that $\sigma_i > 0$ for $i = 1, \dots, \text{rank } \lambda$. By item (iv) of Proposition 2.4 and (P1.4), we have $\langle \lambda, g(x) \rangle = 0$, which implies that $\langle c_i, g(x) \rangle = 0$ for $i = 1, \dots, \text{rank } \lambda$. Again, by item (iv) of Proposition 2.4, we obtain that $c_i \circ g(x) = 0$ for $i = 1, \dots, \text{rank } \lambda$.

Let $c = c_1 + \cdots + c_{\text{rank } \lambda}$. Using Theorem 2.2, we write

$$\mathcal{E} = V(c, 1) \oplus V(c, 1/2) \oplus V(c, 0).$$

We then have $\lambda \in V(c, 1)$ and $g(x) \in V(c, 0)$. Because $V(c, 0)$ is also an Euclidean Jordan algebra, we have $\sqrt{g(x)} \in V(c, 0)$. Finally, since $V(c, 1) \circ V(c, 0) = \{0\}$, we readily obtain $\lambda \circ \sqrt{g(x)} = 0$, which is (P2.2). \square

It is not true in general that if (x, y, μ, λ) is a KKT quadruple for (P2), then (x, μ, λ) is a KKT triple for (P1). Nevertheless, the only obstacle is that λ might fail to belong to \mathcal{K} .

Proposition 4.2. *Let $(x, y, \mu, \lambda) \in \mathbb{R}^n \times \mathcal{E} \times \mathbb{R}^m \times \mathcal{E}$ be a KKT quadruple for (P2) such that $\lambda \in \mathcal{K}$. Then, (x, μ, λ) is a KKT triple for (P1).*

Proof. Under the current hypothesis, (P1.1), (P1.2), (P1.3) and (P1.5) are satisfied. Due to (P2.3), we have

$$0 = \langle y, y \circ \lambda \rangle = \langle y \circ y, \lambda \rangle = \langle g(x), \lambda \rangle,$$

where the second equality follows from Property 3 of the \mathcal{E} operator. Therefore, by item (iv) of Proposition 2.4, we obtain $g(x) \circ \lambda = 0$, which is (P1.4). \square

We then have the following immediate consequence.

Proposition 4.3. *Let $(x, y, \mu, \lambda) \in \mathbb{R}^n \times \mathcal{E} \times \mathbb{R}^m \times \mathcal{E}$ be a KKT quadruple for (P2).*

- (i) *If y and λ satisfy the assumptions of item (ii) of Theorem 3.3, then (x, μ, λ) is a KKT triple for (P1) satisfying strict complementarity.*
- (ii) *If SOSC-NLP holds at (x, y, μ, λ) , then (x, μ, λ) is a KKT triple for (P1) satisfying strict complementarity.*

Proof. (i) By Theorem 3.3, $\lambda \in \mathcal{K}$. Therefore, by Proposition 4.2, (x, μ, λ) is a KKT triple for (P1). Moreover, due to item (a) of Theorem 3.3, we have $\text{rank } y + \text{rank } \lambda = r$. As $g(x) = y \circ y$, we have $\text{rank } g(x) + \text{rank } \lambda = r$ as well.

- (ii) If SOSC-NLP holds at (x, y, μ, λ) , then taking $v = 0$ in (2.9) in Proposition 2.9, we obtain that y and λ satisfy the assumptions of item (ii) of Theorem 3.3. Therefore, the result follows from the previous item. \square

Note that Propositions 4.2 and 4.3 extend previous results obtained in Section 3 of [16] for NSOCs and in Section 3 of [25] for NSDPs. In comparison, in this work, besides the fact that we are focused on a more general family of cones, we are also considering equality constraints.

5 Comparison of constraint qualifications

In order to understand the differences between constraint qualifications for (P1) and (P2), we first have to understand the shape of the tangent cones of \mathcal{K} . A description can be found in Section 2.3 in the work by Kong, Tunçel and Xiu [24]. Apart from that, we can also use the relations described by Pataki in Lemma 2.7 of [28]. For the sake of self-containment, we will give an account of the theory. In what follows, if $C \subseteq \mathcal{E}$, we define $C^\perp := \{z \in \mathcal{E} \mid \langle z, y \rangle = 0, \forall y \in C\}$.

Let $z \in \mathcal{K}$, where the rank of \mathcal{K} is r . We will now proceed to describe the shape of $\mathcal{T}_{\mathcal{K}}(z)$ and $\text{lin } \mathcal{T}_{\mathcal{K}}(z)$. First, denote by $\mathcal{F}(z, \mathcal{K})$ the *minimal face* of \mathcal{K} which contains z . Denote by $\mathcal{F}(z, \mathcal{K})^\Delta$ the *conjugated face* of $\mathcal{F}(z, \mathcal{K})$, which is defined as $\mathcal{K}^* \cap \mathcal{F}(z, \mathcal{K})^\perp$. Since, \mathcal{K} is self-dual, we have $\mathcal{F}(z, \mathcal{K})^\Delta = \mathcal{K} \cap \mathcal{F}(z, \mathcal{K})^\perp$. Now, the discussion in Section 2 and Lemma 2.7 of [28] shows that

$$\begin{aligned} \mathcal{F}(z, \mathcal{K})^\Delta &= \mathcal{K} \cap \{z\}^\perp, \\ \mathcal{T}_{\mathcal{K}}(z) &= \mathcal{F}(z, \mathcal{K})^{\Delta*}, \\ \text{lin } \mathcal{T}_{\mathcal{K}}(z) &= \mathcal{F}(z, \mathcal{K})^{\Delta\perp}. \end{aligned}$$

Our next task is to describe $\mathcal{F}(z, \mathcal{K})$. Let $\{c_1, \dots, c_r\}$ be a Jordan frame for z and write the spectral decomposition of z as

$$z = \sum_{i=1}^{\text{rank } z} \sigma_i c_i,$$

where $\sigma_1, \dots, \sigma_{\text{rank } z}$ are positive. Now, define $c = c_1 + \dots + c_{\text{rank } z}$. Then, c is an idempotent and Theorem 2.2 implies that

$$\mathcal{E} = V(c, 1) \oplus V\left(c, \frac{1}{2}\right) \oplus V(c, 0).$$

A result by Faybusovich (Theorem 2 in [13]) implies that $\mathcal{F}(z, \mathcal{K})$ is the cone of squares in $V(c, 1)$, that is,

$$\mathcal{F}(z, \mathcal{K}) = \{y \circ y \mid y \in V(c, 1)\}.$$

Then, we can see that $\mathcal{F}(z, \mathcal{K})^\Delta$ is precisely the cone of squares of $V(c, 0)$. We remark this fact as a proposition.

Proposition 5.1. $\mathcal{F}(z, \mathcal{K})^\Delta = \{y \circ y \mid y \in V(c, 0)\}$.

Proof. We first show that $\mathcal{F}(z, \mathcal{K})^\Delta \subseteq \{y \circ y \mid y \in V(c, 0)\}$. If $w \in \mathcal{K}$ and $\langle w, z \rangle = 0$, then we must have $\langle c_i, w \rangle = 0$, for every $i \in \{1, \dots, \text{rank } z\}$. Then Lemma 3.2 implies that $c_i \circ w = 0$ for those i . This shows that $c \circ w = 0$, so that $w \in V(c, 0)$. As $w \in \mathcal{K}$ and $V(c, 0)$ is an Euclidean Jordan algebra, we have $w = y \circ y$ for some $y \in V(c, 0)$.

Now, let $w = y \circ y$ with $y \in V(c, 0)$. As $z \in V(c, 1)$ and $w \in V(c, 0)$ we have $\langle w, z \rangle = 0$, so that $w \in \mathcal{F}(z, \mathcal{K})^\Delta$. \square

If we restrict ourselves to $V(c, 0)$, then $\mathcal{F}(z, \mathcal{K})^\Delta$ is a genuine symmetric cone, since it is a cone of squares induced by an Euclidean Jordan algebra. In particular, $\mathcal{F}(z, \mathcal{K})^\Delta$ is self-dual in the sense that $\mathcal{F}(z, \mathcal{K})^\Delta = \{w \in V(c, 0) \mid \langle w, v \rangle \geq 0, \forall v \in \mathcal{F}(z, \mathcal{K})^\Delta\}$. Following the Peirce decomposition, we conclude that

$$\mathcal{T}_{\mathcal{K}}(z) = \mathcal{F}(z, \mathcal{K})^{\Delta*} = V(c, 1) \oplus V\left(c, \frac{1}{2}\right) \oplus \mathcal{F}(z, \mathcal{K})^\Delta, \quad (5.1)$$

$$\text{lin } \mathcal{T}_{\mathcal{K}}(z) = \mathcal{F}(z, \mathcal{K})^{\Delta\perp} = V(c, 1) \oplus V\left(c, \frac{1}{2}\right) \oplus \{0\}, \quad (5.2)$$

where we recall that $\text{lin } \mathcal{T}_{\mathcal{K}}(z)$ denotes the largest subspace contained in the cone $\mathcal{T}_{\mathcal{K}}(z)$.

We are now prepared to discuss the difference between constraint qualifications for (P1) and (P2). This discussion is analogous to the one in Section 2.8 of [25]. We remark that a similar discussion for the special case of nonlinear programming appears in Section 3 of the work by Jongen and Stein [20]. First, we recall that nondegeneracy for (P1) at a point x is the same as saying that the following condition holds:

$$w \in (\text{lin } \mathcal{T}_{\mathcal{K}}(g(x)))^\perp, Jg(x)^*w + Jh(x)^*v = 0 \quad \Rightarrow \quad w = 0, v = 0. \quad (\text{Nondegeneracy})$$

On the other hand, LICQ holds for (P2) at a point (x, y) if the following condition holds:

$$w \circ y = 0, Jg(x)^*w + Jh(x)^*v = 0 \quad \Rightarrow \quad w = 0, v = 0. \quad (\text{LICQ})$$

We need the following auxiliary result.

Proposition 5.2. *Let $z = y \circ y$. Then $(\text{lin } \mathcal{T}_{\mathcal{K}}(z))^\perp \subseteq \ker L_y$, where $\ker L_y$ is the kernel of L_y . If $y \in \mathcal{K}$, then $\ker L_y \subseteq (\text{lin } \mathcal{T}_{\mathcal{K}}(z))^\perp$ as well.*

Proof. Using (5.2), we have

$$(\text{lin } \mathcal{T}_{\mathcal{K}}(z))^\perp = \{0\} \oplus \{0\} \oplus V(c, 0),$$

where we assume that $z = \sum_{i=1}^{\text{rank } z} \sigma_i c_i$ with $\sigma_i > 0$ for $i \leq \text{rank } z$ and $c = c_1 + \dots + c_{\text{rank } z}$. Let $w \in V(c, 0)$. Recall that y and z share a Jordan frame, so we may assume that $y \in V(c, 1)$. Since $V(c, 0) \circ V(c, 1) = \{0\}$, we see that $y \circ w = 0$, that is, $w \in \ker L_y$. This shows that $(\text{lin } \mathcal{T}_{\mathcal{K}}(z))^\perp \subseteq \ker L_y$.

Now, suppose that $y \in \mathcal{K}$, $w \in \ker L_y$. Since y is the square root of z that belongs to \mathcal{K} , we may assume that $y = \sum_i \sqrt{\sigma_i} c_i$. Then, we decompose w as $w = \sum_{i \leq j} w_{ij}$, as in Theorem 2.3, with $w_{ij} \in V_{ij}$. Then, as in (3.2), we have

$$y \circ w = \sum_{i=1}^{\text{rank } y} \sqrt{\sigma_i} w_{ii} + \sum_{1 \leq i < j \leq \text{rank } y} \left(\frac{\sqrt{\sigma_i} + \sqrt{\sigma_j}}{2} \right) w_{ij} + \sum_{1 \leq i \leq \text{rank } y < j} \frac{\sqrt{\sigma_i} w_{ij}}{2} = 0.$$

The condition $y \circ w = 0$, the fact that the $\sqrt{\sigma_i}$ are positive, and the orthogonality among the w_{ij} imply that $w = \sum_{\text{rank } y < i \leq j} w_{ij}$, so that $w \circ c = 0$. Hence $w \in (\text{lin } \mathcal{T}_{\mathcal{K}}(z))^\perp$. \square

Corollary 5.3. *If $(x, y) \in \mathbb{R}^n \times \mathcal{K}$ satisfies LICQ for problem (P2), then nondegeneracy is satisfied at x for (P1). On the other hand, if x satisfies nondegeneracy and if $y = \sqrt{g(x)}$, then (x, y) satisfies LICQ for (P2).*

Proof. Follows from combining Proposition 5.2 with (LICQ) and (Nondegeneracy). \square

6 Second order conditions for (P1)

Using the connection between (P1) and (P2), we can state the following second order conditions.

Proposition 6.1 (A Sufficient Condition via Slack Variables). *Let $(x, \mu, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{E}$ be a KKT triple of problem (P1). Suppose that*

$$\langle \nabla_x^2 L(x, \mu, \lambda) v, v \rangle + 2 \langle w \circ w, \lambda \rangle > 0, \quad (\text{SOSC-NSCP})$$

for every nonzero $(v, w) \in \mathbb{R}^n \times \mathcal{E}$ such that $Jg(x)v - 2\sqrt{g(x)} \circ w = 0$ and $Jh(x)v = 0$. Then, x is a local minimum for (P1), $\lambda \in \mathcal{K}$, and strict complementarity is satisfied.

Proof. If (x, μ, λ) is a KKT triple for (P1), then $(x, \sqrt{g(x)}, \mu, \lambda)$ is a KKT quadruple for (P2). Then, from Proposition 2.9, we conclude that x must be a local minimum. Taking $v = 0$ in (SOSC-NSCP), we see that

$$\langle w \circ w, \lambda \rangle > 0,$$

for all w such that $\sqrt{g(x)} \circ w = 0$. Due to Theorem 3.3, we have $\lambda \in \mathcal{K}$ and $\text{rank } \sqrt{g(x)} + \text{rank } \lambda = r$. As $\text{rank } \sqrt{g(x)} = \text{rank } g(x)$, we conclude that strict complementarity is satisfied. \square

Interestingly, the condition in Proposition 6.1 is strong enough to ensure strict complementarity. And, in fact, when strict complementarity holds and \mathcal{K} is either the cone of positive semidefinite matrices or a product of Lorentz cones, the condition in Proposition 6.1 is equivalent to the second order sufficient conditions described by Shapiro [29] and Bonnans and Ramírez [7]. See [16], [17] and [25] for more details. We also have the following necessary condition.

Proposition 6.2 (A Necessary Condition via Slack Variables). *Let $x \in \mathbb{R}^n$ be a local minimum of (P1). Assume that $(x, \mu, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{E}$ is a KKT triple for (P1) satisfying nondegeneracy. Then the following condition holds:*

$$\langle \nabla_x^2 L(x, \mu, \lambda) v, v \rangle + 2 \langle w \circ w, \lambda \rangle \geq 0, \quad (\text{SONC-NSCP})$$

for every $(v, w) \in \mathbb{R}^n \times \mathcal{E}$ such that $Jg(x)v - 2\sqrt{g(x)} \circ w = 0$ and $Jh(x)v = 0$.

Proof. If (x, μ, λ) is a KKT triple for (P1), then $(x, \sqrt{g(x)}, \mu, \lambda)$ is a KKT quadruple for (P2). Moreover, if x is a local minimum for (P1), then (x, y) is a local minimum for (P2). As x satisfies nondegeneracy, LICQ is satisfied at (x, y) , so that we are under the hypothesis of Proposition 2.10. \square

7 A simple augmented Lagrangian method and its convergence

In [27], Noll warns against naively extending algorithms for NLPs to nonlinear conic programming. One of the reasons is that those extensions often use unrealistic second order conditions which ignore the extra terms that appear in no-gap SOSCs for nonlinear cones. He then argues that such conditions are unlikely to hold in practice. He goes on to prove convergence results for an augmented Lagrangian method for NSDPs based on the no-gap optimality conditions obtained by Shapiro [29].

We have already shown in [25] that if $\mathcal{K} = \mathcal{S}_+^n$, then Shapiro's SOSC for (P1) and the classical SOSC for (P2) are equivalent, under strict complementarity, see Propositions 10, 11, 13 and 14 therein. This suggests that it is viable to design appropriate algorithms for (P1) by studying the NLP formulation (P2) and importing convergence results from nonlinear programming theory while avoiding the issues described by Noll. Furthermore, in some cases, we can remove the slack variable y altogether from the final algorithm. We will illustrate this approach by describing an augmented Lagrangian method for (P1).

Bertsekas also suggested a similar approach in [2], where he analyzed augmented Lagrangian methods for *inequality* constrained NLPs by first reformulating them as *equality* constrained NLPs with the aid of squared slack variables. Kleinmichel and Schönefeld described a method for NLPs in [23] where squared slack variables were used not only to deal with the inequality constraints but in place of the Lagrange multipliers, as a way to force them to be nonnegative. More recently, Sun, Sun and Zhang [32] showed how to obtain a convergence rate result for an augmented Lagrangian method for NSDPs using slack variables, under the hypothesis of strict complementarity¹, see Theorem 3 therein. Here we take a closer look at this topic and extend their Theorem 3.

7.1 Augmented Lagrangian method for (P1)

Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$, $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be twice differentiable functions and consider the following NLP:

$$\begin{aligned} & \underset{x}{\text{minimize}} && \varphi(x) \\ & \text{subject to} && \psi(x) = 0. \end{aligned} \tag{7.1}$$

Following Section 17.3 in [26], given a multiplier $\lambda \in \mathbb{R}^m$ and a penalty parameter $\rho \in \mathbb{R}$, define the augmented Lagrangian $\mathfrak{L}_\rho : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ for (7.1) by

$$\mathfrak{L}_\rho(x, \lambda) = \varphi(x) - \langle \psi(x), \lambda \rangle + \frac{\rho}{2} \|\psi(x)\|^2.$$

For problem (P2), the augmented Lagrangian is given by

$$\mathfrak{L}_\rho^{\text{Slack}}(x, y, \mu, \lambda) = f(x) - \langle h(x), \mu \rangle + \frac{\rho}{2} \|h(x)\|^2 - \langle g(x) - y \circ y, \lambda \rangle + \frac{\rho}{2} \|g(x) - y \circ y\|^2. \tag{7.2}$$

¹We remark that their main contribution was to show a convergence rate result using the strong second order sufficient condition, nondegeneracy but without strict complementarity.

We have the following basic augmented Lagrangian method.

Algorithm 1: Augmented Lagrangian Method for (P2)

- 1 Choose initial points x_1, y_1 , initial multipliers μ_1, λ_1 and an initial penalty ρ_1 .
 - 2 $k \leftarrow 1$.
 - 3 Let (x_{k+1}, y_{k+1}) be a minimizer of $\mathfrak{L}_{\rho_k}^{\text{Slack}}(\cdot, \cdot, \mu_k, \lambda_k)$.
 - 4 $\mu_{k+1} \leftarrow \mu_k - \rho_k h(x_{k+1})$.
 - 5 $\lambda_{k+1} \leftarrow \lambda_k - \rho_k (g(x_{k+1}) - y_{k+1} \circ y_{k+1})$.
 - 6 Choose a new penalty ρ_{k+1} with $\rho_{k+1} \geq \rho_k$.
 - 7 Let $k \leftarrow k + 1$ and return to Step 3.
-

We will show how to remove the slack variable from the augmented Lagrangian.

Proposition 7.1. *The following equation holds:*

$$\min_y \mathfrak{L}_{\rho}^{\text{Slack}}(x, y, \mu, \lambda) = f(x) - \langle h(x), \mu \rangle + \frac{\rho}{2} \|h(x)\|^2 + \frac{1}{2\rho} \left(-\|\lambda\|^2 + \|[\lambda - \rho g(x)]_+\|^2 \right). \quad (7.3)$$

Moreover if (x^*, y^*) is a minimum of $\mathfrak{L}_{\rho}^{\text{Slack}}(\cdot, \cdot, \mu, \lambda)$, then $y^* \circ y^* = \left[g(x^*) - \frac{\lambda}{\rho} \right]_+$.

Proof. In the partial minimization $\min_y \mathfrak{L}_{\rho}^{\text{Slack}}(x, y, \mu, \lambda)$, we look at the terms that depend on y .

$$\begin{aligned} & \min_y \left(-\langle g(x) - y \circ y, \lambda \rangle + \frac{\rho}{2} \|g(x) - y \circ y\|^2 \right) \\ &= \min_y \left(-\langle g(x) - y \circ y, \lambda \rangle + \frac{\rho}{2} \left\| g(x) - \frac{\lambda}{\rho} - y \circ y + \frac{\lambda}{\rho} \right\|^2 \right) \\ &= \min_y \left(\frac{\rho}{2} \left\| g(x) - \frac{\lambda}{\rho} - y \circ y \right\|^2 - \frac{\|\lambda\|^2}{2\rho} \right) \\ &= -\frac{\|\lambda\|^2}{2\rho} + \frac{\rho}{2} \min_y \left\| g(x) - \frac{\lambda}{\rho} - y \circ y \right\|^2. \end{aligned}$$

Note that

$$\min_y \left\| g(x) - \frac{\lambda}{\rho} - y \circ y \right\|^2 = \min_{z \in \mathcal{K}} \left\| g(x) - \frac{\lambda}{\rho} - z \right\|^2. \quad (7.4)$$

Then, (7.4) together with Lemma 2.6 implies that

$$\begin{aligned} -\frac{\|\lambda\|^2}{2\rho} + \frac{\rho}{2} \min_y \left\| g(x) - \frac{\lambda}{\rho} - y \circ y \right\|^2 &= -\frac{\|\lambda\|^2}{2\rho} + \frac{\rho}{2} \left\| P_{\mathcal{K}} \left(\frac{\lambda}{\rho} - g(x) \right) \right\|^2 \\ &= -\frac{\|\lambda\|^2}{2\rho} + \frac{1}{2\rho} \|[\lambda - \rho g(x)]_+\|^2. \end{aligned}$$

It follows that

$$\min_y \mathfrak{L}_{\rho}^{\text{Slack}}(x, y, \mu, \lambda) = f(x) - \langle h(x), \mu \rangle + \frac{\rho}{2} \|h(x)\|^2 + \frac{1}{2\rho} \left(-\|\lambda\|^2 + \|[\lambda - \rho g(x)]_+\|^2 \right)$$

and hence (7.3) holds.

Finally, note that (7.4) implies that if (x^*, y^*) is a minimum of $\mathfrak{L}_{\rho}^{\text{Slack}}(\cdot, \cdot, \mu, \lambda)$, then $y^* \circ y^* = \left[g(x^*) - \frac{\lambda}{\rho} \right]_+$. \square

Proposition 7.1 suggests the following augmented Lagrangian for (P1):

$$\mathfrak{L}_\rho^{\text{Sym}}(x, \mu, \lambda) = f(x) - \langle h(x), \mu \rangle + \frac{\rho}{2} \|h(x)\|^2 + \frac{1}{2\rho} \left(-\|\lambda\|^2 + \|[\lambda - \rho g(x)]_+\|^2 \right). \quad (7.5)$$

Moreover, due to Lemma 2.6 and Proposition 7.1, we can write the multiplier update in Step 5 of Algorithm 1 as

$$\lambda_{k+1} \leftarrow [\lambda_k - \rho_k g(x_{k+1})]_+.$$

This gives rise to the following augmented Lagrangian method for (P1). Note that the squared slack variable y is absent.

Algorithm 2: Augmented Lagrangian Method for (P1)

- 1 Choose an initial point x_1 , initial multipliers μ_1, λ_1 and an initial penalty ρ_1 .
 - 2 $k \leftarrow 1$.
 - 3 Let x_{k+1} be a minimizer of $\mathfrak{L}_{\rho_k}^{\text{Sym}}(\cdot, \mu_k, \lambda_k)$.
 - 4 $\mu_{k+1} \leftarrow \mu_k - \rho_k h(x_{k+1})$.
 - 5 $\lambda_{k+1} \leftarrow [\lambda_k - \rho_k g(x_{k+1})]_+$.
 - 6 Choose a new penalty ρ_{k+1} with $\rho_{k+1} \geq \rho_k$.
 - 7 Let $k \leftarrow k + 1$ and return to Step 3.
-

Note that Algorithms 1 and 2 are equivalent in the sense that any sequence of iterates $(x_k, y_k, \mu_k, \lambda_k)$ for Algorithm 1 is such that (x_k, μ_k, λ_k) is a valid sequence of iterates for Algorithm 2. Conversely, given a sequence (x_k, μ_k, λ_k) for Algorithm 2, the sequence $(x_k, \sqrt{[g(x_k) - \lambda/\rho_k]_+}, \mu_k, \lambda_k)$ is valid for Algorithm 1. However, there could be computational differences between both algorithms. On the one hand, the subproblem in Algorithm 2 has less variables than the subproblem in Algorithm 1. On the other hand, if f, h, g are twice differentiable, the same is true for $\mathfrak{L}_\rho^{\text{Slack}}(\cdot, \cdot, \mu, \rho)$, while $\mathfrak{L}_\rho^{\text{Sym}}(\cdot, \mu, \rho)$ will not necessarily be twice differentiable in general. In this sense, the subproblem in Algorithm 1 is smoother than the one in Algorithm 2. In the appendix, we describe some preliminary numerical experiments aimed at exploring the difference between both approaches.

Note also that when \mathcal{K} is the cone of positive semidefinite matrices or a product of second order cones, Algorithm 2 gives exactly the same augmented Lagrangian method with quadratic penalty discussed extensively in the literature. This is because, due to Proposition 2.5, the projection $[\lambda_k - \rho_k g(x_{k+1})]_+$ is just the result of zeroing the negative eigenvalues of $\lambda_k - \rho_k g(x_{k+1})$.

7.2 Convergence results

Here, we will reinterpret a result of [2]. We will then use it to prove an analogous theorem for (P1). This extends Theorem 3 in [32] for all nonlinear symmetric cone programs.

Proposition 7.2 (Proposition 2.4 in [2]). *Suppose that $(x^*, y^*, \mu^*, \lambda^*) \in \mathbb{R}^n \times \mathcal{E} \times \mathbb{R}^m \times \mathcal{E}$ is a KKT quadruple for (P2) such that*

- (2.9) is satisfied,
- LICQ is satisfied.

Moreover, let $\hat{\rho}$ be such that $\nabla^2 \mathfrak{L}_{\hat{\rho}}^{\text{Slack}}$ is positive definite². Then there are positive scalars $\hat{\delta}, \hat{\epsilon}, \hat{M}$ such that

²Such a $\hat{\rho}$ always exists, see the remarks before Proposition 2.4 in [2].

1. for all (μ, λ, ρ) in the set $\hat{D} := \{(\mu, \lambda, \rho) \mid |\mu - \mu^*| + |\lambda - \lambda^*| < \hat{\delta}\rho, \hat{\rho} \leq \rho\}$, the following problem has a unique solution:

$$\begin{aligned} & \underset{x, y}{\text{minimize}} \quad \mathfrak{L}_\rho^{\text{Slack}}(x, y, \mu, \lambda) \\ & \text{subject to} \quad (x, y) \in B_{\hat{\epsilon}}(x^*) \times B_{\hat{\epsilon}}(y^*), \end{aligned} \quad (7.6)$$

where $B_{\hat{\epsilon}}(x^*) \subseteq \mathbb{R}^n$ and $B_{\hat{\epsilon}}(y^*) \subseteq \mathcal{E}$ are the spheres with radius $\hat{\epsilon}$ centered at x^* and y^* , respectively. Denote such a solution by $(x(\mu, \lambda, \rho), y(\mu, \lambda, \rho))$. Then, $(x(\cdot, \cdot, \cdot), y(\cdot, \cdot, \cdot))$ is continuously differentiable in the interior of \hat{D} and satisfies

$$|(x(\mu, \lambda, \rho), y(\mu, \lambda, \rho)) - (x^*, y^*)| \leq \frac{\hat{M}}{\rho} |(\mu, \lambda) - (\mu^*, \lambda^*)|, \quad (7.7)$$

for all $(\mu, \lambda, \rho) \in \hat{D}$.

2. For all $(\mu, \lambda, \rho) \in \hat{D}$, we have

$$|(\hat{\mu}(\mu, \lambda, \rho), \hat{\lambda}(\mu, \lambda, \rho)) - (\mu^*, \lambda^*)| \leq \frac{\hat{M}}{\rho} |(\mu, \lambda) - (\mu^*, \lambda^*)|, \quad (7.8)$$

where

$$\begin{aligned} \hat{\mu}(\mu, \lambda, \rho) &= \mu - \rho h(x(\mu, \lambda, \rho)), \\ \hat{\lambda}(\mu, \lambda, \rho) &= \lambda - \rho(g(x(\mu, \lambda, \rho)) - y(\mu, \lambda, \rho)^2). \end{aligned}$$

Our goal is to prove the following result.

Proposition 7.3. *Suppose that $(x^*, \mu^*, \lambda^*) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{E}$ is a KKT triple for (P1) such that*

- (SOSC-NSCP) is satisfied,
- Nondegeneracy (see Definition 2.8) is satisfied.

Then there are positive scalars $\delta, \epsilon, M, \bar{\rho}$ such that

1. For all (μ, λ, ρ) in the set $D := \{(\mu, \lambda, \rho) \mid |\mu - \mu^*| + |\lambda - \lambda^*| < \delta\rho, \bar{\rho} \leq \rho\}$, the following problem has a unique solution:

$$\begin{aligned} & \underset{x}{\text{minimize}} \quad \mathfrak{L}_\rho^{\text{Sym}}(x, \mu, \lambda) \\ & \text{subject to} \quad x \in B_\epsilon(x^*). \end{aligned} \quad (7.9)$$

Denote such a solution by $x(\mu, \lambda, \rho)$. Then, $x(\cdot, \cdot, \cdot)$ is continuously differentiable in the interior of D and satisfies

$$|x(\mu, \lambda, \rho) - x^*| \leq \frac{M}{\rho} |(\mu, \lambda) - (\mu^*, \lambda^*)| \quad (7.10)$$

for all $(\mu, \lambda, \rho) \in D$.

2. For all $(\mu, \lambda, \rho) \in D$, we have

$$|(\bar{\mu}(\mu, \lambda, \rho), \bar{\lambda}(\mu, \lambda, \rho)) - (\mu^*, \lambda^*)| \leq \frac{M}{\rho} |(\mu, \lambda) - (\mu^*, \lambda^*)|, \quad (7.11)$$

where

$$\begin{aligned} \bar{\mu}(\mu, \lambda, \rho) &= \mu - \rho h(x(\mu, \lambda, \rho)), \\ \bar{\lambda}(\mu, \lambda, \rho) &= [\lambda - \rho g(x(\mu, \lambda, \rho))]_+. \end{aligned}$$

Note that the argument in Proposition 7.1 shows that (7.9) is equivalent to

$$\begin{aligned} & \underset{x,y}{\text{minimize}} \quad \mathfrak{L}_\rho^{\text{Slack}}(x,y,\mu,\lambda) \\ & \text{subject to} \quad (x,y) \in B_\epsilon(x^*) \times \mathcal{E}. \end{aligned} \tag{7.12}$$

Moreover, if (\hat{x}, \hat{y}) is an optimal solution for problem (7.12), we have $\hat{y}^2 = [g(\hat{x}) - \lambda/\rho]_+$. Therefore, if $\hat{\delta}$ and $\hat{\epsilon}$ provided by Proposition 7.2 were such that $\sqrt{[g(x) - \lambda/\rho]_+}$ stays in the ball $B_{\hat{\epsilon}}(\sqrt{g(x^*)})$ for all $x \in B_{\hat{\delta}}(x^*)$, then it would be very straightforward to prove Proposition 7.3. As this is not generally the case, in the proof we have to argue that we can adjust $\hat{\delta}, \hat{\epsilon}$ appropriately. The proof is similar to [32], but we need to adjust the proof to make use of the Euclidean Jordan algebra machinery. Before we proceed, we need a few auxiliary lemmas.

Lemma 7.4. *Let \mathcal{E} be an Euclidean Jordan algebra with rank r and let $\psi_{\mathcal{E}}$ denote the function that maps y to $\sqrt{|y|}$, where $|y| = \sqrt{y^2}$. Let $\mathcal{E}^* = \{y \in \mathcal{E} \mid \text{rank } y = r\}$. Then, $\psi_{\mathcal{E}}$ is continuously differentiable over \mathcal{E}^* .*

Proof. Let $Q = \{x \in \mathbb{R}^r \mid x_i \neq 0, \forall i\}$ and $\varphi : \mathbb{R}^r \rightarrow \mathbb{R}^r$ be the function that maps $x \in \mathbb{R}^r$ to $(\sqrt{|x_1|}, \dots, \sqrt{|x_r|})$. Then, following the discussion in Section 6 of [1], $\psi_{\mathcal{E}}$ is the spectral map generated by φ . That is, if $y \in \mathcal{E}$ and its spectral decomposition is given by $y = \sum_{i=1}^r \sigma_i c_i$, then

$$\psi_{\mathcal{E}}(y) = \sum_{i=1}^r \varphi_i(\sigma_i) c_i,$$

where $\varphi_i : \mathbb{R} \rightarrow \mathbb{R}$ are the component functions of φ . Then, Theorem 53 in [1] shows that $\psi_{\mathcal{E}}$ is continuously differentiable over \mathcal{E}^* , because φ is continuously differentiable over Q . A similar conclusion also follows from Theorem 3.2 in [31]. \square

Lemma 7.5. *Let \mathcal{E} be an Euclidean Jordan algebra with rank r , let c be an idempotent and $w \in V(c, 1/2)$. Then there are $w_0 \in V(c, 0)$, $w_1 \in V(c, 1)$ such that*

$$\begin{aligned} w^2 &= w_0^2 + w_1^2, \\ \|w_0^2\| &= \|w_1^2\|. \end{aligned}$$

Proof. According to Theorem 2.2, $w^2 \in V(c, 0) + V(c, 1)$. As $w^2 \in \mathcal{K}$, this implies the existence of $w_0 \in V(c, 0)$ and $w_1 \in V(c, 1)$ such that $w^2 = w_0^2 + w_1^2$. From the proof of Proposition IV.1.1 in [12], we see that, in fact,

$$w_0^2 = w^2 - c \circ w^2, \quad w_1^2 = c \circ w^2.$$

Then, we have

$$\begin{aligned} \|w_0^2\|^2 &= \|w^2\|^2 - 2\langle w^2, c \circ w^2 \rangle + \|w_1^2\|^2 \\ &= \|w^2\|^2 - 2\langle w^3, c \circ w \rangle + \|w_1^2\|^2 \\ &= \|w^2\|^2 - \langle w^3, w \rangle + \|w_1^2\|^2 \\ &= \|w_1^2\|^2, \end{aligned}$$

where the second equality follows from the power associativity of the Jordan product and the fact that the algebra is Euclidean, which implies that $\langle w^2, c \circ w^2 \rangle = \langle w^3 \circ w, c \rangle = \langle w^3, c \circ w \rangle$. Then, the third equality follows from $w \in V(c, 1/2)$. \square

Proof of Proposition 7.3. Note that if (SOSC-NSCP) and nondegeneracy are satisfied at (x^*, μ^*, λ^*) , then $(x^*, \sqrt{g(x^*)}, \mu^*, \lambda^*)$ satisfies (2.9) and LICQ, due to Corollary 5.3. So let $\hat{\rho}, \hat{\delta}, \hat{\epsilon}, \hat{M}$ and \hat{D} be as in Proposition 7.2.

First, we consider the spectral decomposition of $g(x^*)$. Without loss of generality, we may assume that

$$g(x^*) = \sum_{i=1}^{\text{rank } g(x^*)} \sigma_i c_i,$$

where the first $\text{rank } g(x^*)$ eigenvalues are positive and the remaining are zero. Then, we let $c = c_1 + \dots + c_{\text{rank } g(x^*)}$ and consider the Euclidean Jordan algebra $V(c, 1)$ together with its associated symmetric cone $\mathcal{F} = \{w \circ w \mid w \in V(c, 1)\}$. We know that $V(c, 1)$ has rank equal to $\text{rank } g(x^*)$. Moreover, by Proposition 2.4, $g(x^*)$ belongs to the relative interior of \mathcal{F} . In particular, we may select $\hat{\epsilon}_1 \in (0, \hat{\epsilon})$ such that $\|z - g(x^*)\| \leq \hat{\epsilon}_1$ and $z \in V(c, 1)$ implies that z lies in the relative interior of \mathcal{F} as well.

Now, we take the function $\psi_{V(c,1)}$ from Lemma 7.4. Note that if $v \in \mathcal{K}$, then $\psi_{V(c,1)}(v) = \sqrt{v}$. Since $\psi_{V(c,1)}$ is continuously differentiable in $V(c, 1)^*$, the mean value inequality tells us that for $\|v - g(x^*)\| \leq \hat{\epsilon}_1$ and $v \in V(c, 1)$ we have

$$\left\| \sqrt{v} - \sqrt{g(x^*)} \right\| \leq R \|v - g(x^*)\|, \quad (7.13)$$

where R is the supremum of $\|J\psi_{V(c,1)}\|$ over the set $V(c, 1) \cap B_{\hat{\epsilon}_1}(g(x^*))$. We then let $\hat{\epsilon}_2 \in (0, \epsilon_1]$ be such that

$$4R^2 \hat{\epsilon}_2^2 + \hat{\epsilon}_2 \sqrt{2r} + \hat{\epsilon}_2 \sqrt{r} \leq \hat{\epsilon}_1^2. \quad (7.14)$$

Since g is continuously differentiable, again by the mean value inequality, there is l_g such that for every $x \in B_{\hat{\epsilon}}(x^*)$ we have

$$\|g(x) - g(x^*)\| \leq l_g \|x - x^*\|. \quad (7.15)$$

We are now ready to construct the neighborhood D . We select $\epsilon, \delta, \bar{\rho}, M$ such that the following conditions are satisfied:

1. $\epsilon \in (0, \hat{\epsilon}], \delta \in (0, \hat{\delta}], \bar{\rho} \geq \hat{\rho}, M \geq \hat{M}$.
2. $l_g \epsilon + \delta + \left\| \frac{\lambda^*}{\bar{\rho}} \right\| \leq \hat{\epsilon}_2$.
3. ϵ, δ are small enough and $M, \bar{\rho}$ are large enough such that the conclusions of Proposition 7.2 hold for those $\epsilon, \delta, M, \bar{\rho}$ and such that the neighborhood $B_{\hat{\epsilon}}(x^*) \times B_{\hat{\epsilon}}(y^*)$ in (7.6) can be replaced by $B_\epsilon(x^*) \times B_{\hat{\epsilon}_1}(y^*)$ without affecting the conclusion of the theorem.

We then have

$$D = \{(\mu, \lambda, \rho) \mid |\mu - \mu^*| + |\lambda - \lambda^*| < \delta \rho, \bar{\rho} \leq \rho\}.$$

For all $(\mu, \lambda, \rho) \in D$ and x such that $\|x - x^*\| \leq \epsilon$, we have

$$\begin{aligned}
\left\| \left[g(x) - \frac{\lambda}{\rho} \right]_+ - g(x^*) \right\| &= \left\| \left[g(x) - \frac{\lambda}{\rho} \right]_+ - [g(x^*)]_+ \right\| \\
&\leq \left\| g(x) - g(x^*) - \frac{\lambda - \lambda^*}{\rho} - \frac{\lambda^*}{\rho} \right\| \\
&\leq \|g(x) - g(x^*)\| + \left\| \frac{\lambda - \lambda^*}{\rho} \right\| + \left\| \frac{\lambda^*}{\rho} \right\| \\
&\leq l_g \|x - x^*\| + \delta + \left\| \frac{\lambda^*}{\rho} \right\| \\
&\leq \hat{\epsilon}_2,
\end{aligned}$$

where the first inequality follows from the fact that the projections are nonexpansive maps and the fourth inequality follows by the definition of D , (7.15) and the fact that $\bar{\rho} \leq \rho$.

Now we will show that $\left\| \sqrt{[g(x) - \lambda/\rho]_+} - \sqrt{g(x^*)} \right\| \leq \hat{\epsilon}_1$ holds as well, which is our primary goal. More generally, we will show that if $v \in \mathcal{K} \cap B_{\hat{\epsilon}_2}(g(x^*))$ then $\sqrt{v} \in B_{\hat{\epsilon}_1}(\sqrt{g(x^*)})$.

Thus suppose that $v \in \mathcal{K}$ is such that $v \in B_{\hat{\epsilon}_2}(g(x^*))$. We consider the Peirce decomposition of \sqrt{v} with respect to the idempotent c , as in Theorem 2.2. We write $\sqrt{v} = w_1 + w_2 + w_3$, where $w_1 \in V(c, 1)$, $w_2 \in V(c, 1/2)$, $w_3 \in V(c, 0)$. We then have $v = w_1^2 + w_2^2 + w_3^2 + 2w_2 \circ (w_1 + w_3)$. Also by Theorem 2.2, $2w_2 \circ (w_1 + w_3) \in V(c, 1/2)$ and $w_2^2 = w_{2,1}^2 + w_{2,0}^2$, for some $w_{2,1} \in V(c, 1)$ and $w_{2,0} \in V(c, 0)$. We group the terms of $v - g(x^*)$ as follows

$$v - g(x^*) = (w_1^2 + w_{2,1}^2 - g(x^*)) + (w_3^2 + w_{2,0}^2) + (2w_2 \circ (w_1 + w_3)), \quad (7.16)$$

where the terms in parentheses belong, respectively, to the mutually orthogonal subspaces $V(c, 1)$, $V(c, 0)$ and $V(c, 1/2)$. Therefore, $\|v - g(x^*)\|^2 \leq \hat{\epsilon}_2^2$ implies that

$$\|w_1^2 + w_{2,1}^2 - g(x^*)\|^2 \leq \hat{\epsilon}_2^2, \quad (7.17)$$

$$\|w_3^2\|^2 \leq \hat{\epsilon}_2^2, \quad (7.18)$$

$$\|w_{2,0}^2\|^2 \leq \hat{\epsilon}_2^2, \quad (7.19)$$

where the last two inequalities follows from the fact that $\|w_3^2 + w_{2,0}^2\|^2 = \|w_3^2\|^2 + \langle w_3^2, w_{2,0}^2 \rangle + \|w_{2,0}^2\|^2 \leq \hat{\epsilon}_2^2$ and that $\langle w_3^2, w_{2,0}^2 \rangle \geq 0$, since $w_3^2, w_{2,0}^2 \in \mathcal{K}$. From (7.17), (7.19) and Lemma 7.5, we obtain

$$\|w_1^2 - g(x^*)\| \leq \|w_1^2 + w_{2,1}^2 - g(x^*)\| + \|w_{2,1}^2\| \leq 2\hat{\epsilon}_2.$$

We then use (7.13) to conclude that

$$\left\| w_1 - \sqrt{g(x^*)} \right\| \leq 2R\hat{\epsilon}_2, \quad (7.20)$$

since $\sqrt{w_1^2} = w_1$ holds³.

³If we fix $v \in \mathcal{K}$, there might be several elements a satisfying $a \circ a = v$, but only one of those a belongs to \mathcal{K} . With our definition, \sqrt{v} is precisely this a . As $\sqrt{v} \in \mathcal{K}$, the element w_1 appearing in its Pierce decomposition also belongs to \mathcal{K} . To see that, note that if w_1 had a negative eigenvalue σ , we would have $\langle \sqrt{v}, \sigma d \rangle = \langle w_1, \sigma d \rangle < 0$, where d is the idempotent associated to σ . This shows that $\sqrt{w_1^2} = w_1$ indeed.

Recall that given $z \in \mathcal{K}$, we have $\|z\|^2 = \sigma_1^2 + \dots + \sigma_r^2$, where σ_i are the eigenvalues of z . Therefore, $\|z\|^2$ is the 1-norm of the vector $u = (\sigma_1^2, \dots, \sigma_r^2)$, which is majorized by $\|u\|_2 \sqrt{r} = \|z^2\| \sqrt{r}$, i.e., $\|z\|^2 \leq \|z^2\| \sqrt{r}$. This, together with (7.18), (7.19) and Lemma 7.5 imposes the following inequalities on the w_i :

$$\|w_3\|^2 \leq \hat{\epsilon}_2 \sqrt{r}, \quad (7.21)$$

$$\|w_2\|^2 \leq \|w_2^2\| \sqrt{r} = \|w_{2,0}^2\| \sqrt{2r} \leq \hat{\epsilon}_2 \sqrt{2r}. \quad (7.22)$$

From $\sqrt{g(x^*)} \in V(c, 1)$, (7.20), (7.21) and (7.22), we obtain

$$\begin{aligned} \left\| \sqrt{v} - \sqrt{g(x^*)} \right\|^2 &= \left\| w_1 - \sqrt{g(x^*)} \right\|^2 + \|w_2\|^2 + \|w_3\|^2 \\ &\leq 4R^2 \hat{\epsilon}_2^2 + \hat{\epsilon}_2 \sqrt{2r} + \hat{\epsilon}_2 \sqrt{r} \\ &\leq \hat{\epsilon}_1^2, \end{aligned}$$

where the last inequality follows from (7.14). To recap, we have shown that whenever $(\mu, \lambda, \rho) \in D$ and x is such that $\|x - x^*\| \leq \epsilon$, then

$$\sqrt{\left[g(x) - \frac{\lambda}{\rho} \right]_+} \in B_{\hat{\epsilon}_1}(\sqrt{g(x^*)}).$$

Thus, letting $x(\mu, \lambda, \rho)$ be a minimizer of (7.9) and $\hat{y} = \sqrt{\left[g(x(\mu, \lambda, \rho)) - \lambda/\rho \right]_+}$, we have $x(\mu, \lambda, \rho) \in B_{\hat{\epsilon}}(x^*)$, and $\hat{y} \in B_{\hat{\epsilon}_1}(\sqrt{g(x^*)})$. Now, we note that $(x(\mu, \lambda, \rho), \hat{y})$ is a minimizer of (7.6) with $y^* = \sqrt{g(x^*)}$ and $B_{\hat{\epsilon}}(x^*) \times B_{\hat{\epsilon}_1}(y^*)$ in place of $B_{\hat{\epsilon}}(x^*) \times B_{\hat{\epsilon}}(y^*)$. In fact, let (\bar{x}, \bar{y}) be the minimizer of (7.6). By Proposition 7.1, we have

$$\begin{aligned} \mathfrak{L}_\rho^{\text{Slack}}(x(\mu, \lambda, \rho), \hat{y}, \mu, \lambda) &= \mathfrak{L}_\rho^{\text{Sym}}(x(\mu, \lambda, \rho), \mu, \lambda) \\ &\leq \mathfrak{L}_\rho^{\text{Sym}}(\bar{x}, \mu, \lambda) \\ &= \min_y \mathfrak{L}_\rho^{\text{Slack}}(\bar{x}, y, \mu, \lambda) \\ &\leq \mathfrak{L}_\rho^{\text{Slack}}(\bar{x}, \bar{y}, \mu, \lambda) \end{aligned}$$

As $(x(\mu, \lambda, \rho), \hat{y})$ is feasible to (7.6), we conclude that $(x(\mu, \lambda, \rho), \hat{y})$ is indeed a minimizer of (7.6) with $B_{\hat{\epsilon}}(x^*) \times B_{\hat{\epsilon}_1}(y^*)$ in place of $B_{\hat{\epsilon}}(x^*) \times B_{\hat{\epsilon}}(y^*)$. Furthermore, since the minimizer of (7.6) is unique, it follows that $x(\mu, \lambda, \rho)$ is the unique minimizer of (7.9). Due to Proposition 7.2 and the choice of $\epsilon, \delta, M, \bar{\rho}$ (see items 1 to 3 above), $x(\cdot, \cdot, \cdot)$ must be differentiable in the interior of D . Since (7.7) holds, then (7.10) holds as well. This concludes item 1.

Item 2 also follows from the fact that

$$\begin{aligned} \hat{\lambda}(\mu, \lambda, \rho) &= \lambda - \rho(g(x(\mu, \lambda, \rho)) - \hat{y}^2) \\ &= \lambda - \rho \left(g(x(\mu, \lambda, \rho)) - \left[g(x(\mu, \lambda, \rho)) - \frac{\lambda}{\rho} \right]_+ \right) \\ &= [\rho g(x(\mu, \lambda, \rho)) - \lambda]_+ - (\rho g(x(\mu, \lambda, \rho)) - \lambda) \\ &= [\lambda - \rho g(x(\mu, \lambda, \rho))]_+ \\ &= \bar{\lambda}(\mu, \lambda, \rho), \end{aligned}$$

where the second to last equality follows from Lemma 2.6. Hence the estimate in (7.8) also implies the estimate in (7.11). \square

8 Concluding remarks

In this paper we presented a discussion on optimality conditions for nonlinear symmetric cone programs through slack variables. By doing so, we obtain an ordinary nonlinear programming problem, which is more straightforward to analyze. This connection gives some interesting insights, such as Theorem 3.3, and makes it possible to analyze algorithms for (P1) as we did in Section 7.

However, one slightly upsetting part of the usage of slack variables is the fact that when the second order sufficient conditions (SOSCs) are written down, we get that strict complementarity is automatically satisfied. In particular, if we have a KKT tuple that *does not* satisfy strict complementarity and we wish to check whether it is a local minimum, there is no way to apply the theory described in this paper. Of course, it is indeed possible to derive SOSCs without assuming strict complementarity, as it was done in [7, 29]. However, at this point we do not know how to explain why this difference arises. An interesting research topic would be to find out whether (P1) admits another reformulation as a nonlinear programming problem without this deficiency.

Appendix

As we discussed in Section 7.1, Algorithms 1 and 2 are equivalent in terms of generated sequences, but there should be some differences from the computational point of view. To illustrate this, we implemented both algorithms in MATLAB R2016b, under Ubuntu 16.04, with core i7 3.9GHz and 16GB of RAM. The numerical experiments here are simple, so the subproblems were solved simply with the default `fminunc` function, that uses a quasi-Newton method. Noting that the sequences generated by both algorithms are the same only if the subproblems are solved exactly, we set up the optimization tolerance for the subproblems as 10^{-8} . We also considered some ideas from the augmented Lagrangian solver ALGENCAN (see Chapter 12 in [5]), so the initial penalty parameter is dependent of the problem, and its update is done by using an analysis of the infeasibility measure. Moreover, the termination criteria consist in satisfying KKT conditions approximately, with precision 10^{-4} .

We compare the algorithms with three problems, which are special cases of NSCP: nonlinear programming problem (NLP) number 1 from Hock and Schittkowski's list [19] (and originally by Betts [4]), nonlinear second-order cone programming problem (NSOCP) by Kanzow, Ferenczi and Fukushima [21], and nonlinear semidefinite programming problem (NSDP) by Noll [27] (see Section 12 therein). The results are described in Table 1, with “inner iterations” as the total number of iterations for solving the subproblems. All problems were successfully solved, using both algorithms. Also, for each problem, the sequences of iterates $\{x_k\}$ generated by the algorithms were the same. However, the number of inner iterations and the total times were clearly different. As expected, the slack variables approach tends to require more time, because of the increase of the dimension of the problem.

Whether there exist some cases where Algorithm 1 performs better is still a matter of further investigation. In particular, the projection operator used only in Algorithm 2 is computationally expensive for the NSDP case, so depending on the optimization instance, the slack approach may be advantageous. We note that Burer and Monteiro have proposed a very successful variant of Algorithm 1 for linear SDPs [9, 10], where they exploit the existence of optimal solutions with low rank. In [25], for the NSDP case, we also presented a numerical comparison between an augmented Lagrangian method for the formulation (P1) and for (P2). And, indeed, for some small problems, the formulation (P2) had faster running times. However, we used the software PENLAB [15], which uses a different kind of augmented Lagrangian that is not covered by the discussion in Section 7.

Table 1: Comparison of Algorithms 1 (with slack) and 2 (without slack).

Problems	Methods	Outer iterations (k)	Inner iterations	Time (s)
NLP	Algorithm 1	1	50	0.017317
	Algorithm 2	1	51	0.014929
NSOCP	Algorithm 1	5	76	0.069946
	Algorithm 2	5	18	0.018908
NSDP	Algorithm 1	5	230	0.210977
	Algorithm 2	5	10	0.036631

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