On Dantzig figures from graded lexicographic orders

Akshay Gupte Svetlana Poznanović

Abstract

We construct two families of Dantzig figures, which are d-dimensional polytopes with $2d$ facets and an antipodal vertex pair, from convex hulls of initial subsets for the graded lexicographic (grlex) and graded reverse lexicographic (grevlex) orders on $\mathbb{Z}^d_{\geq 0}$. These two polytopes have the same number of vertices, $\mathcal{O}(d^2)$, and the same number of edges, $\mathcal{O}(d^3)$, but are not combinatorially equivalent. We provide an explicit description of the vertices and the facets for both families and describe their graphs along with analyzing their basic properties such as the radius, diameter, existence of Hamiltonian circuits, and chromatic number. Moreover, we also analyze the edge expansions of these graphs.

Keywords. grlex, grevlex, polytope, Dantzig figure

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1 Introduction

A d-polytope is a bounded convex polyhedron whose affine dimension is equal to d. Equivalently, a d-polytope is the convex hull of finitely many points, exactly $d + 1$ of which are affinely independent. It is simple if every vertex is defined by exactly d facets, or equivalently, has exactly d neighboring vertices; otherwise it is non-simple. A d-polytope with n facets is referred to as a (d, n)-polytope. When $n = 2d$, we have a $(d, 2d)$-polytope. A $(d, 2d)$-polytope X is said to be a *Dantzig figure* generated by distinct vertices u and v if u and v do not share a common facet. In this case we say that X is a (u, v)-Dantzig figure. Thus for a Dantzig figure, exactly d distinct facets are incident to each of u and v, and every facet contains exactly one of u or v. This also means that both u and v have exactly d neighboring vertices. Since u and v do not lie on the same facet, they are called an antipodal vertex pair, and a figure may have multiple such pairs. Trivial examples include the hypercube and the simplicial bipyramid.

Dantzig figures were introduced by Dantzig [Dan64] in the context of the Hirsch conjecture on combinatorial diameter of (d, n)-polytopes, and gained prominence after it was shown [KW67] that this conjecture would be true for all polytopes if and only if it was true for simple Dantzig figures. Although the Hirsch conjecture was disproved recently [San12], diameters of special-structured polytopes have always been, and continue to be, the topic of study in literature [PR74; RC98; Bre+13; DLK14; BDLF17]. Besides the connection to combinatorial diameter, Dantzig figures are also important from the perspective of them being polytopes with not too many facets, i.e., belonging to the family of (d, kd)-polytopes for some small constant k. Polytopes with few facets, where few facets generally means $(d, d + k)$-polytopes, have been studied recently for their combinatorial properties [Pad16a; Pad16b]. An important question in polyhedral combinatorics is the identification of different combinatorial types of a particular family of polytopes. This has been answered for $(d, d + k)$-polytopes for small k (typically $k \leq 6$) [Grü03; BS11]. On the other hand, this question has gone largely unanswered for (d, kd)-polytopes, and their explicit construction has received limited attention. There are

Department of Mathematical Sciences, Clemson University
Email addresses: {agupte,spoznan}@clemson.edu

results, though, showing how some (d, kd)-polytopes arise from a term order. Given $\theta, u \in \mathbb{Z}^d$ with $\mathbf{0} \leq \theta \leq u$ and the lexicographic (lex) order $\leq_{\text{lex}}$ on $\mathbb{Z}^d$, the lex polytope is

$$P^{\text{lex}} := \text{conv}\{x \in \mathbb{Z}^d : \mathbf{0} \leq x \leq_{\text{lex}} \theta,\ x \leq u\}.$$

Note that the upper bound $x \leq u$ is necessary to obtain a polytope because the lex constraint $x \leq_{\text{lex}} y$ over the reals defines a neither open nor closed convex cone. The $0\backslash 1$ polytope P^{lex} (i.e., with $u = \mathbb{1}$) was shown to be a $(d, 3d)$-polytope separately by [LS92; GK06]. This was later generalized to arbitrary integral u by [Gup16], who also showed that the polytope $\text{conv}\{x \in [\mathbf{0}, u] \cap \mathbb{Z}^d : \gamma \leq_{\text{lex}} x \leq_{\text{lex}} \theta, x \leq u\}$ defined by one $\leq_{\text{lex}}$ and one $\geq_{\text{lex}}$ order is a $(d, 4d)$-polytope. Thus, lex polytopes are (d, kd)-polytopes[1] for $k \in \{3, 4\}$. To the best of our knowledge, explicit ways of constructing nontrivial Dantzig figures, either simple or non-simple, for arbitrary d are unknown.

Besides identifying families of (d, kd)-polytopes, term orders are also helpful in solving mixed-integer optimization problems. The lex order has been used for breaking symmetry in integer programs [Mar10], which has subsequently led to polyhedral studies of associated polytopes [KP08; HP17]. Another place were lex-ordered sets and the inequalities defining their convex hull appear is in reformulations of mixed-integer problems [Gup+13; EG16]. A third application of term orders is their use in strengthening cutting planes for separating a fractional point in branch-and-cut algorithms [Gup17]. This can be explained briefly as follows. Let $X_I = \{(x, y) \in \mathbb{Z}^n \times \mathbb{R}^m : Ax + By \leq b\}$ be a mixed-integer feasible region and let (x^*, y^*), with $x^* \notin \mathbb{Z}^n$, be optimal to the linear programming relaxation X. There are many well-known techniques [cf. CCZ14] for finding a hyperplane $\alpha x + \beta y \leq \alpha_0$ that separates (x^*, y^*) from X_I, one of the most powerful of them being split cuts. A split cut is obtained by first finding $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that $\pi_0 < \pi x^* < \pi_0 + 1$ and then solving a cut-generating linear program to find a valid inequality to the disjunction $(X \cap \{x : \pi x \leq \pi_0\}) \cup (X \cap \{x : \pi x \geq \pi_0 + 1\})$. This inequality can be further strengthened using a term order $\preccurlyeq$. Let $\tilde{x}$ be the largest (under $\preccurlyeq$) point in $\{x \in \mathbb{Z}^n : \pi x \leq \pi_0\}$ such that $(\tilde{x}, y) \in X_I$ for some y. Similarly, let $\hat{x}$ be the smallest such point in $\{x \in \mathbb{Z}^n : \pi x \geq \pi_0 + 1\}$. If $\mathcal{C}$ (resp. $\mathcal{D}$) is the polytope defined as the convex hull of all integral points less (resp. greater) than or equal to $\tilde{x}$ (resp. $\hat{x}$), then the disjunction $(X \cap \mathcal{C}) \cup (X \cap \mathcal{D})$ can be used to separate a cutting plane that is at least as strong as the one obtained from the split disjunction $\{\pi x \leq \pi_0\} \cup \{\pi x \geq \pi_0 + 1\}$. This approach relies on having a complete facet description of polytopes $\mathcal{C}$ and $\mathcal{D}$ that arise from term orders.

Our Results. We construct two combinatorial types of non-simple d-dimensional Dantzig figures, for any $d \geq 3$, using two term orders related to the lex order. Thus, we not only advance the study of polytopes arising from term orders but also provide a constructive characterization for some Dantzig figures. Furthermore, our polytopes fit in a small grid ($\mathcal{O}(d^2)$ vertices fit in a grid of size $\mathcal{O}(d)$), a class of polytopes in which interesting examples are relatively scarce.

The polytopes we construct are defined by the graded lex (grlex) order ($\preccurlyeq_{\text{gr}}$) and the graded reverse lex (grevlex) order ($\preccurlyeq_{\text{grev}}$). Given $d \geq 3$ and $\theta \in \mathbb{Z}^d_+$ with $\theta \geq \mathbb{1}$, the grlex and grevlex polytopes are, respectively,

$$\mathcal{P} := \text{conv}\{x \in \mathbb{Z}^d \colon \mathbf{0} \leq x \preccurlyeq_{\text{gr}} \theta\}, \qquad \mathcal{Q} := \text{conv}\{x \in \mathbb{Z}^d \colon \mathbf{0} \leq x \preccurlyeq_{\text{grev}} \theta\}. \tag{1}$$

From now we assume that $\theta \geq \mathbb{1}$ and $d \geq 3$. We don't consider the case $d = 2$ because in this case $\mathcal{P}$ and $\mathcal{Q}$ are just quadrilaterals. Our consideration of these lattice polytopes is motivated by lex polytopes being (d, kd)-polytopes for $k \in \{3, 4\}$ and polytopes from term orders being useful for mixed-integer optimization, as mentioned earlier. Also, note that a projection of $\mathcal{P}$ (or $\mathcal{Q}$) yields the lex polytope over a integral simplex (see Remark 1).

[1] Strictly speaking, lex polytopes are, in general, $(d, kd - \epsilon)$-polytopes for $\epsilon \in \{1, 2\}$.

We find the $\mathcal{V}$- and $\mathcal{H}$-representations of these polytopes. The $\mathcal{H}$-representations are obtained using a conic characterization that we develop for arbitrary polytopes. We then characterize the facet-vertex incidence for these polytopes. Based on this, we find that the face lattices of these polytopes are independent of θ: for any $\mathcal{P}$ and $\mathcal{P}'$ corresponding to $\theta, \theta' > \mathbb{1}$, we have $\mathcal{P} \cong \mathcal{P}'$ and for any $\mathcal{Q}$ and $\mathcal{Q}'$ corresponding to $\theta, \theta' \geq \mathbb{1}$, we have $\mathcal{Q} \cong \mathcal{Q}'$.

The facet-vertex incidence then reveals that $\mathcal{P}$ and $\mathcal{Q}$ are Dantzig figures for all d and θ. Specifically, we show that $\mathcal{P} = \mathcal{P}$ is a $(\mathbf{0}, \theta)$-Dantzig figure and $(\mathbf{0}, \theta)$ is the only antipodal vertex pair of $\mathcal{P}$ (Theorem 3.1). Also, $\mathcal{Q} = \mathcal{Q}$ is a $(\mathbf{0}, \theta)$-Dantzig figure and $(\mathbf{0}, \theta)$ is the only antipodal vertex pair of $\mathcal{Q}$ when $d \geq 4$ (Theorem 4.1).

As the polytopes under study are Dantzig figures, which were introduced by Dantzig in relation to the problem of bounding the diameter of polytopes, it is natural to ask whether our polytopes have a large diameter. This is also interesting from the aspect of the question what is the largest diameter of lattice polytopes whose vertex coordinates are integers between 0 and k. The upper bound for d-polytopes was first shown to be kd [KO92] and was recently improved in [DPM16; DP16]. A lower bound was given in [DMO17]. We give a complete description of the graphs of the polytopes, $G(\mathcal{P})$ and $G(\mathcal{Q})$, and show that they have constant and small diameters. Interestingly, $G(\mathcal{P})$ and $G(\mathcal{Q})$, have not only the same number of vertices, but also the same number of edges $\mathcal{O}(d^3)$. However, the graphs are not isomorphic for $\theta > \mathbb{1}$, meaning that $\mathcal{P} \ncong \mathcal{Q}$ in general. We have computationally verified that $\mathcal{P}$ and $\mathcal{Q}$ have the same f-vectors for $d \leq 10$ and we conjecture this to be true for all d. We leave the proof of this conjecture for future work.

A graph is said to have good expansion properties if, roughly speaking, it is sparse but has high connectivity, which is quantified in terms of edge expansion of at least 1. Bounding the edge expansion of graphs of polytopes is of significant interest due to its importance in studying random walks on such graphs and therefore has received much attention [Kai01]. It was shown in [GK06], that $0\backslash1\ P^{\text{lex}}$, even though sparse, has edge expansion at least 1. Since the graphs $G(\mathcal{P})$ and $G(\mathcal{Q})$ are sparse with average degree $\mathcal{O}(d)$, we analyzed their edge expansion. We show that $G(\mathcal{P})$ lies on the threshold for polytopes with good and poor expansion properties with edge expansion $h(G(\mathcal{P})) = 1$ (in general, computing the edge expansion for general graphs is NP-hard [Theorem 2, Kai01]). Numerical results show that $h(G(\mathcal{Q}))$ depends on d but we believe that it is also at least 1.

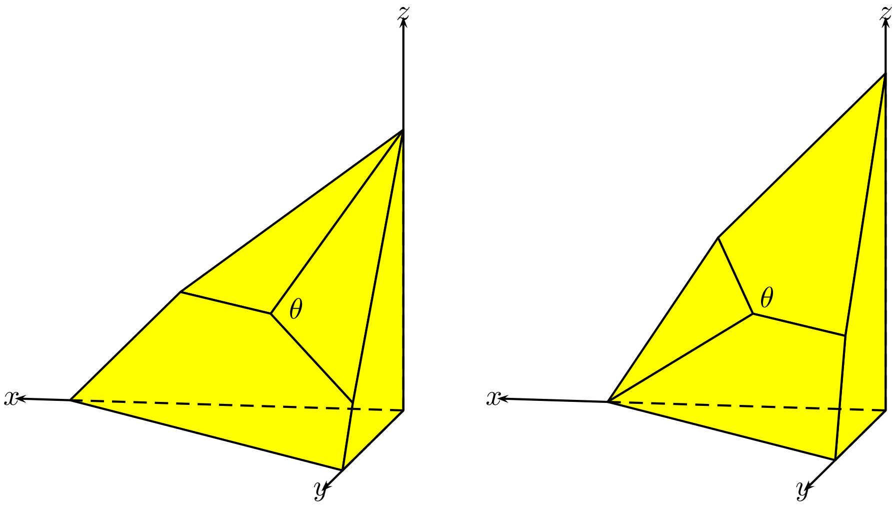
Notation. The vector of all zeros is $\mathbf{0}$, the vector of all ones is $\mathbb{1}$, and the i^{th} unit coordinate vector is $\mathbf{e}_i$. Let $\leq_{\text{lex}}$ denote the lexicographic monomial order. For $x, y \in \mathbb{Z}^d$, we say $x \leq_{\text{lex}} y$ if either $x = y$ or there exists some i with $x_i < y_i$ and $x_k = y_k$ for all $k > i$.[2] The graded lex (grlex) and graded reverse lex (grevlex) monomial orders are denoted as $\preccurlyeq_{\text{gr}}$ and $\preccurlyeq_{\text{grev}}$, respectively, and defined as follows:

1. $x \preccurlyeq_{\text{gr}} y$ if either $\sum_{i=1}^d x_i < \sum_{i=1}^d y_i$, or $\sum_{i=1}^d x_i = \sum_{i=1}^d y_i$ and $x \leq_{\text{lex}} y$,
2. $x \preccurlyeq_{\text{grev}} y$ if either $\sum_{i=1}^d x_i < \sum_{i=1}^d y_i$, or $\sum_{i=1}^d x_i = \sum_{i=1}^d y_i$ and $x \geq_{\text{lex}} y$.

Denoting

$$b := \sum_{i=1}^{d} \theta_i, \qquad \tilde{b}_k := \sum_{i=1}^{k} \theta_i = b - \sum_{i=k+1}^{d} \theta_i \quad 1 \leq k \leq d, \qquad H_0 := \left\{ x \in \mathbb{R}^d \colon \sum_{i=1}^{d} x_i = b \right\}, \tag{2a}$$

[2]Our right-to-left order of coordinate comparison here is opposite to the left-to-right order generally used in literature, but this is immaterial up to permuting the variables.



(a) $\mathcal{P}$ (b) $\mathcal{Q}$

Figure 1: The grlex and grevlex polytopes in $d = 3$ defined by $\theta = (2, 2, 2)$.

as the total and partial sums of θ and the grading hyperplane, it is clear that each of them is built out of two polytopes in the following way:

$$\mathcal{P} = \operatorname{conv}\left(\left\{x \in \mathbb{R}^d_+ \colon \sum_{i=1}^{d} x_i \le b-1\right\} \bigcup \operatorname{conv}\left\{x \in \mathbb{Z}^d_+ \colon \sum_{i=1}^{d} x_i = b, x \le_{\text{lex}} \theta\right\}\right) \tag{2b}$$

$$\mathcal{Q} = \operatorname{conv}\left(\left\{x \in \mathbb{R}^d_+ \colon \sum_{i=1}^{d} x_i \le b-1\right\} \bigcup \operatorname{conv}\left\{x \in \mathbb{Z}^d_+ \colon \sum_{i=1}^{d} x_i = b, x \ge_{\text{lex}} \theta\right\}\right). \tag{2c}$$

Both $\mathcal{P}$ and $\mathcal{Q}$ are d-polytopes since they contain the standard simplex. It is easy to verify that $\mathcal{P} \cap H_0 \subsetneq H_0 \cap \operatorname{conv}\{x \in \mathbb{Z}^d : \mathbf{0} \le x \le_{\text{lex}} \theta\}$. Similarly for $\mathcal{Q}$. Thus $\mathcal{H}$-representations of $\mathcal{P}$ and $\mathcal{Q}$ are not a trivial implication of the known results for lex polytopes.

Figure 1 illustrates these polytopes for $d = 3$ for $\theta = (2, 2, 2)$. Both have 7 vertices, 11 edges, and 6 facets, but they are not isomorphic because $\mathcal{P}$ has one pentagonal, two quadrilateral, and three triangular facets whereas $\mathcal{Q}$ has two triangular and four quadrilateral facets. As we will see, the face lattices of $\mathcal{P}$ and $\mathcal{Q}$ are independent of the actual value of θ when $\theta > 1$.

Remark 1. $\mathcal{P}$ (resp. $\mathcal{Q}$) yields the convex hull of all the integral vectors that belong to a standard integral simplex and are lexicographically smaller (resp. greater) than a fixed integer vector. In particular, denoting $\tilde{x} = (x_2, \ldots, x_d)$ and $\tilde{\theta} = (\theta_2, \ldots, \theta_d)$, we have

$$\operatorname{Proj}_{\tilde{x}}(\mathcal{P} \cap H_0) = \operatorname{conv}\left\{\tilde{x} \in \mathbb{Z}^{d-1}_+ \colon \sum_{i=2}^{d} x_i \le b, \tilde{x} \le_{\text{lex}} \tilde{\theta}\right\}$$

and

$$\operatorname{Proj}_{\tilde{x}}(\mathcal{Q} \cap H_0) = \operatorname{conv}\left\{\tilde{x} \in \mathbb{Z}^{d-1}_+ \colon \sum_{i=2}^{d} x_i \le b, \tilde{x} \ge_{\text{lex}} \tilde{\theta}\right\}.$$

Outline. We begin by providing in Section 2 a conic representation of arbitrary polytopes. This result implies that any Dantzig figure is equal to the intersection of two polyhedral cones, which is obviously the minimum number of cones required to represent any polytope. We use this later to obtain the $\mathcal{H}$-representation of the polytopes $\mathcal{P}$ and $\mathcal{Q}$. The rest of the paper is divided into two parts. Section 3 analyzes the grlex polytope $\mathcal{P}$ and Section 4 analyzes the grevlex polytope $\mathcal{Q}$. For each polytope, we show it is a non-simple Dantzig figure generated by $\mathbf{0}$ and θ and identify all its $\mathcal{O}(d^2)$ vertices and $2d$ facet-defining inequalities. Although $\mathcal{P}$ and $\mathcal{Q}$ appear closely related by definition, they are combinatorially not equivalent, as seen in Figure 1. This necessitates separate proofs, especially for showing the Dantzig figure property in Theorems 3.1 and 4.1, but we condense our arguments whenever possible. We describe $G(\mathcal{P})$ and $G(\mathcal{Q})$, the graphs of these polytopes, and their basic properties, including diameter, in Section 3.3 and Section 4.3.

2 Conic characterization of polytopes

Let $X \subseteq \mathbb{R}^n$ be a d-polytope with set of vertices $\text{vert}(X)$. For every $v \in \text{vert}(X)$, $\mathcal{N}_X(v)$ denotes the set of vertices adjacent to v. Recall that two vertices of a d-polytope are adjacent if and only if there are at least $d-1$ facets that contain both the vertices. The tangent cone at a vertex v (also referred to as a vertex cone) is defined as

$$\mathcal{C}_X(v) := v + \left\{ \sum_{x \in \mathcal{N}_X(v)} \alpha_x (x - v) \colon \alpha \geq \mathbf{0} \right\} = v + \text{cone}\{x - v\}_{x \in \mathcal{N}_X(v)}. \tag{3}$$

By construction, the dimension of this cone cannot be greater than the dimension of X. Observe that

$$X \subseteq \mathcal{C}_X(v) \quad \forall v \in \text{vert}(X). \tag{4}$$

This can be argued as follows.[3] Let $X = \{x \in \mathbb{R}^n : Ax \geq b\}$ be an $\mathcal{H}$-representation of X for some $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$. A basis of X is a n-subset of $[m]$ such that the rows of A indexed by this subset are linearly independent. Consider any $v \in \text{vert}(X)$. By the equivalence of vertices and basic feasible solutions of a polyhedron, there exists some basis B such that v is the unique solution to the linear system $a_i x = b_i$ for $i \in B$, where a_i is the i^{th} row of A. Let $B' := \{i : a_i v = b_i\}$; clearly $B' \supseteq B$ with the inclusion being strict if and only if v is a degenerate vertex. It is easy to argue then that the tangent cone at v can be represented as

$$\mathcal{C}_X(v) = v + \{y \colon a_i y \geq 0, \ i \in B'\}. \tag{5}$$

Now for any x with $Ax \geq b$, we have $x - v$ satisfying $a_i(x - v) \geq 0$ for all $i \in B'$, and therefore $x \in \mathcal{C}_X(v)$.

Equation (4) implies two things. First that the affine dimension of $\mathcal{C}_X(v)$ is equal to d. Secondly, it leads to the inclusion $X \subseteq \cap_{v \in \text{vert}(X)} \mathcal{C}_X(v)$. In fact, equality holds, i.e., every polytope is equal to the intersection of its vertex cones. We will use a stronger version of this statement given in the following lemma. The result, we believe, is folklore, but since we couldn't find a reference, we give the proof for completeness.

Lemma 2.1. *For any $\emptyset \neq S \subseteq \text{vert}(X)$, we have $X = \cap_{v \in S} \mathcal{C}_X(v)$ if and only if there exists a $0 \leq k \leq d-1$ such that every k-dimensional face of X contains some $v \in S$.*

A special case of the above result arises by considering $S = \text{vert}(X)$, which leads to

$$X = \bigcap_{v \in \text{vert}(X)} \mathcal{C}_X(v). \tag{6}$$

[3] A different proof is given in Ziegler [Zie95, Lemma 3.6].

Proof. Since a facet contains a k-dimensional face for every $k = 0, \dots, d-1$, it is clear that it suffices to show the above equivalence for $k = d-1$. Suppose that $X = \cap_{v \in S} \mathcal{C}_X(v)$. By (5), $\mathcal{C}_X(v) = \{x \colon a_i x \geq a_i v,\ i \in B'(v)\}$, where $B'(v)$ is the set of tight inequalities at v. This implies $X = \{x : a_i x \geq a_i v,\ i \in B'(v), v \in S\}$. Note that for every facet of a polyhedron there exists some defining inequality of the polyhedron that represents this facet. Hence if F is a facet of X, then $F = \{x \in X : a_i x = a_i v\}$ for some $i \in B'(v), v \in S$. Then it is clear that $v \in F$.

For the reverse direction, we will need the following.

Claim 2.1. *Let H be a supporting hyperplane of X. Then $X \cap H$ is a facet of X if and only if $\mathcal{C}_X(v) \cap H$ is a facet of $\mathcal{C}_X(v)$ for every $v \in \mathrm{vert}(X) \cap H$.*

Proof. We have that H defines a proper face of X, i.e. the dimension of $X \cap H$ is at least 0 and at most $d-1$. ($\Longleftarrow$) Since $\mathcal{C}_X(v)$ is a d-dimensional polyhedral cone, any $(d-1)$ of its generators are linearly independent, meaning that v and any $(d-1)$-subset of $\mathcal{N}_X(v)$ are affinely independent. Suppose H defines a facet of $\mathcal{C}_X(v)$ for every $v \in \mathrm{vert}(X) \cap H$. Then H contains v and at least $d-1$ vertices in $\mathcal{N}_X(v)$. Therefore H contains d affinely independent vertices of X, making $X \cap H$ a facet of X.

($\Longrightarrow$) Suppose $X \cap H$ is a facet of X. The cone $\mathcal{C}_X(v)$ being d-dimensional for every v, we need to argue that H defines a $(d-1)$-dimensional face of $\mathcal{C}_X(v)$ for every $v \in \mathrm{vert}(X) \cap H$. For every $v \in \mathrm{vert}(X)$, $X \subseteq \mathcal{C}_X(v)$ tells us that $\mathcal{C}_X(v) \nsubseteq H$ and that the points in $\mathcal{N}_X(v) \setminus H$ are all on one side of H. Thus for every $v \in \mathrm{vert}(X) \cap H$, the generators $\{u - v\}_{u \in \mathcal{N}_X(v)}$ of $\mathcal{C}_X(v)$ belong to one of the halfspaces defined by H. Hence H defines a face of $\mathcal{C}_X(v)$. Due to $\mathcal{C}_X(v) \nsubseteq H$, the dimension of this face is at most $d-1$. Since H defines a facet of X, we have $\mathrm{vert}(X \cap H) = \mathrm{vert}(X) \cap H$ and so H contains d affinely independent vertices of X. Now $\mathrm{vert}(X) \cap H \subseteq \mathcal{C}_X(v) \cap H$ tells us that the dimension of the face $\mathcal{C}_X(v) \cap H$ is at least $d-1$, thereby implying that H defines a facet of $\mathcal{C}_X(v)$ for every $v \in \mathrm{vert}(X) \cap H$. ☐

Now suppose every facet of X contains some $v \in S$. It suffices to prove that $\cap_{v \in S} \mathcal{C}_X(v) \subseteq X$ because $X \subseteq \cap_{v \in S} \mathcal{C}_X(v)$ is obvious from (4) and $S \subseteq \mathrm{vert}(X)$. For sake of contradiction, let $x \in \cap_{v \in S} \mathcal{C}_X(v) \setminus X$. Then $cx > c_0$ for some facet-defining inequality $cx \leq c_0$ of X. By assumption, there exists some $\bar{v} \in S$ such that $c\bar{v} = c_0$. By Claim 2.1, we have that $cx \leq c_0$ is a facet-defining inequality of $\mathcal{C}_X(\bar{v})$. But then $cx > c_0$ leads to the contradiction $x \notin \mathcal{C}_X(\bar{v})$. ☐

Remark 2. Lemma 2.1 also holds for pointed d-polyhedra. Let X be a d-polyhedron with $\mathrm{vert}(X) \neq \emptyset$ and the recession cone $\mathrm{rec}(X) = \mathrm{cone}\{r^1, \dots, r^l\}$. For $v \in \mathrm{vert}(X)$, let $R_X(v) := \{r^i : v + r^i \text{ is an edge of } X\}$. The tangent cone at each vertex v is

$$\mathcal{C}_X(v) = v + \mathrm{cone}\{x - v\}_{x \in \mathcal{N}_X(v)} + \mathrm{cone}\{r^i\}_{r^i \in R_X(v)}.$$

Then the above proof naturally extends to give us the same characterization for $X = \cap_{v \in S} \mathcal{C}_X(v)$.

Lemma 2.1 poses an interesting question: for a (d, kd)-polytope X, is there a good lower bound (in terms of d and k) on how many vertex cones are required to describe X? The answer does not seem obvious even for $k = 2$. Even a simpler question does not seem obvious: is there a characterization of $(d, 2d)$-polytopes, or (d, n)-polytopes, that are equal to the intersection of two vertex cones, which is the minimal number required for any polytope? For $(3, 6)$-polytopes, which are called hexahedra and have seven distinct combinatorial types as enumerated in [Num16], one can graphically verify that every $(3, 6)$-polytope is equal to the intersection of two of its vertex cones. For $d = 4$, for the dual of the simplicial 4-polytope wtih 8 vertices P_1^8 in [GS67, pp. 454], it is easy to verify that there does not exist any vertex pair (u, v) such that every facet of $P_1^{8^\circ}$ contains either u or v, meaning that $P_1^{8^\circ}$ requires at least three vertex cones for its description. For general d, the answer to the second question is clearly yes for Dantzig figures, due to Lemma 2.1.

Corollary 2.1. *When X is a (u, v)-Dantzig figure, we have $X = \mathcal{C}_X(u) \cap \mathcal{C}_X(v)$.*

The converse of Corollary 2.1 is not true — not every $(d,2d)$-polytope that is equal to the intersection of two vertex cones is a Dantzig figure; for example in $\mathbb{R}^3$, a pyramid with a pentagonal base is not a Dantzig figure since every pair of vertices shares a common facet. Therefore, the Dantzig figure property is not necessary for a $(d,2d)$-polytope to be described by two cones.

Corollary 2.1 can be used to derive an explicit $\mathcal{H}$-representation for a Dantzig figure (and also for any simple polytope). Since u and v have exactly d neighboring vertices, we may denote $\mathcal{N}_X(u) = \{y^1,\ldots,y^d\}$ and $\mathcal{N}_X(v) = \{z^1,\ldots,z^d\}$. If we let

$$M_u := \begin{bmatrix} y^1-u & y^2-u & \cdots & y^d-u \end{bmatrix}, \qquad M_v := \begin{bmatrix} z^1-v & z^2-v & \cdots & z^d-v \end{bmatrix}$$

denote the $d\times d$ matrices defined by neighbors of u and v, respectively, then the vertex cones $\mathcal{C}_X(u)$ and $\mathcal{C}_X(v)$ are given by $\mathcal{C}_X(u) = u + \{M_u\alpha\colon \alpha \geq \mathbf{0}\}$ and $\mathcal{C}_X(v) = v + \{M_v\alpha\colon \alpha \geq \mathbf{0}\}$. Since the above matrices are nonsingular, these cones are simplicial and we have $\mathcal{C}_X(u) = \{x\colon M_u^{-1}(x-u) \geq \mathbf{0}\}$ and $\mathcal{C}_X(v) = \{x\colon M_v^{-1}(x-v) \geq \mathbf{0}\}$. This combined with Corollary 2.1 yields the following.

Proposition 2.1. *When X is a (u,v)-Dantzig figure, we have the following minimal inequality representation:*

$$X = \{x : M_u^{-1}(x-u) \geq \mathbf{0},\ M_v^{-1}(x-v) \geq \mathbf{0}\}.$$

We will apply this method of deriving the $\mathcal{H}$-representation to our polytopes $\mathcal{P}$ and $\mathcal{Q}$. We remark that the matrix inverses M_u^{-1} and M_v^{-1} may lead to highly ill-conditioned coefficients for facet-defining inequalities of a Dantzig figure, as will be the case for $\mathcal{P}$ and $\mathcal{Q}$. This would not make the $\mathcal{H}$-representation of Proposition 2.1 suitable for computational implementation. In that case, one would seek an extension of the Dantzig figure, where, as is customary in literature, an extension of a polytope $X \subset \mathbb{R}^d$ is a polyhedron $Y \subset \mathbb{R}^{d'}$ and an affine map $\pi\colon \mathbb{R}^{d'} \mapsto \mathbb{R}^d$ such that $X = \pi(Y)$. The size of an extension (Y,π) is counted by the number of facet-defining inequalities in Y. Corollary 2.1 gives us an extension of size $2d$ and the coefficients of the inequalities of this extension are more well-conditioned than those in Proposition 2.1 describing the Dantzig figure in the x-space.

3 The grlex polytope $\mathcal{P}$

In this section we will describe the main properties of the polytope $\mathcal{P}$. To simplify the notation, throughout, we will use $\preccurlyeq$ to denote the grlex order.

3.1 $\mathcal{V}$-polytope

Consider the following integral points:

$$\begin{aligned} w &:= (b-1)\mathbf{e}_d = (0,0,\ldots,0,b-1) && (7a)\\ u^k &:= ((\tilde{b}_{k-1}+1)\mathbf{e}_{k-1}, \theta_k - 1, \theta_{k+1},\ldots,\theta_d) \qquad 3 \leq k \leq d && (7b)\\ v^{j,k} &:= (\tilde{b}_k \mathbf{e}_j, 0,\ldots,0,\theta_{k+1},\ldots,\theta_d) \qquad 1 \leq j < k \leq d && (7c)\end{aligned}$$

where $\tilde{b}_k$ is given by (2a). By construction, we have

Observation 3.1. *$u^k \in H_0$ for all k, $v^{j,k} \in H_0$ for all j,k, $\theta \in H_0$, $\mathbf{0}, w \notin H_0$.*

Since $\theta \geq \mathbb{1}$, we have $w, u^k, v^{j,k} \geq \mathbf{0}$. Also, every u^k and $v^{j,k}$ is $\leq_{\text{lex}}$-less than θ. Thus $w, u^k, v^{j,k} \in \mathcal{P}$. Observe that $\theta \geq \mathbb{1}$ implies that v^{j,k_1} and u^{k_2} coincide if and only if $k_1 = k_2$, $j = k_1 - 1$, and $\theta_{k_1} = 1$.

Our first result shows that the points defined in (7), along with $\mathbf{0}$ and θ, provide a vertex characterization of $\mathcal{P}$.

Proposition 3.1. *Let $\theta \geq \mathbb{1}$ and $d \geq 3$. The vertices of $\mathcal{P}$ are*

$$\mathrm{vert}(\mathcal{P}) = \{\mathbf{0}, \theta, w\} \bigcup \{u^k : 3 \leq k \leq d\} \bigcup \{v^{j,k} : 1 \leq j < k \leq d\}.$$

Proof. It is clear from the definition of $\mathcal{P}$ that $\mathbf{0}$ and w cannot be written as a nontrivial convex combination of integral points in $\mathcal{P}$. Suppose $\theta = \sum_{i=1}^{s} \lambda_i x^i$ is a nontrivial convex combination of some $x^i \in \mathcal{P} \cap \mathbb{Z}^d$. Since $\theta \in \mathcal{P} \cap H_0$ and H_0 defines a face of $\mathcal{P}$, we have $x^i \in \mathcal{P} \cap H_0$ for all i. Let

$$m = \max\{j : x^i_j \neq \theta_j \text{ for some } i\}.$$

Then $x^i \leq_{\text{lex}} \theta$ implies $x^i_m \leq \theta_m$, leading to the contradiction $\sum_{i=1}^{s} \lambda_i x^i_m < \theta_m$. Next, suppose $u^k = \sum_{i=1}^{s} \lambda_i x^i$ is a nontrivial convex combination of some $x^i \in \mathcal{P} \cap \mathbb{Z}^d$. Since $u^k \in H_0$, $x^i \in H_0$ as well. By the same reasoning as for θ we get $x^i_j = \theta_j$ for all i and $j > k$. Also, $x^i_j = 0$ for all i and $j < k-1$. Now, if $x^i_k = \theta_k$ for some i then $x^i_{k-1} = \tilde{b}_{k-1}$ which contradicts $x^i_{k-1} \preccurlyeq \theta$ because $\theta \geq \mathbb{1}$ and $k \geq 3$. So, the only possibility is $x^i = u^k$ for all i. Similarly, one can conclude that all $v^{j,k}$ have to be vertices of $\mathcal{P}$ as well.

Now we argue that if $v \in \mathrm{vert}(\mathcal{P}) \setminus \{\mathbf{0}, w, \theta\}$, then v must be equal to some u^k or $v^{j,k}$. Equation (2b) gives us

$$\mathrm{vert}(\mathcal{P}) \subseteq \mathrm{vert}\{x \in \mathbb{R}^d : \sum_i x_i \leq b-1\} \cup \mathrm{vert}(\mathcal{P} \cap H_0).$$

The vertices of the simplex $\{x \in \mathbb{R}^d : \sum_i x_i \leq b-1\}$ are $\mathbf{0}$ and $(b-1)\mathbf{e}_i$ for $i = 1, \ldots, d$. The point $(b-1)\mathbf{e}_i$ is a convex combination of $\mathbf{0}$ and $b\mathbf{e}_i$ and for $i \leq d-1$, $\theta_d \geq 1$ implies $b\mathbf{e}_i \leq_{\text{lex}} \theta$, and hence $b\mathbf{e}_i \in \mathcal{P} \cap H_0$. Therefore $(b-1)\mathbf{e}_i \notin \mathrm{vert}(\mathcal{P})$ for $i \leq d-1$ and we have

$$\mathrm{vert}(\mathcal{P}) \subseteq \{\mathbf{0}, w\} \cup \mathrm{vert}(\mathcal{P} \cap H_0).$$

Since H_0 defines a face of $\mathcal{P}$, we have $\mathrm{vert}(\mathcal{P} \cap H_0) = \mathrm{vert}(\mathcal{P}) \cap H_0 = \mathrm{vert}(\mathcal{P}) \cap H_0$ and then $\mathbf{0}, w \in \mathrm{vert}(\mathcal{P})$ leads to the equality

$$\mathrm{vert}(\mathcal{P}) = \{\mathbf{0}, w\} \cup \mathrm{vert}(\mathcal{P} \cap H_0).$$

We argued in the first paragraph that $\theta \in \mathrm{vert}(\mathcal{P}) \cap H_0$ and because $\mathrm{vert}(\mathcal{P}) \cap H_0 = \mathrm{vert}(\mathcal{P} \cap H_0)$, we have $\theta \in \mathrm{vert}(\mathcal{P} \cap H_0)$. Let $\bar{x} \neq \theta$ be an arbitrary vertex of $\mathcal{P} \cap H_0$ and define $k := \max\{i : \bar{x}_i \neq \theta_i\}$. Since $\mathrm{vert}(\mathcal{P}) \subseteq \mathbb{Z}^d$, we have $\bar{x} \in \mathbb{Z}^d$ and $\bar{x}_k \in \{0, 1, \ldots, \theta_k - 1\}$. Suppose $1 \leq \bar{x}_k \leq \theta_k - 2$. Then $\bar{x} \in H_0$ implies there exists a $i < k$ such that $1 \leq \bar{x}_i \leq b-1$. For $x', x'' \in H_0 \cap \mathbb{Z}^d$ defined as $x' = \bar{x} + \mathbf{e}_i - \mathbf{e}_k$ and $x'' = \bar{x} - \mathbf{e}_i + \mathbf{e}_k$, note that $1 \leq \bar{x}_k \leq \theta_k - 2$ implies that $\mathbf{0} \leq x', x'' \leq_{\text{lex}} \theta$. Hence $x', x'' \in \mathcal{P} \cap H_0$ and since $\bar{x} = (x' + x'')/2$, we have a contradiction to $\bar{x} \in \mathrm{vert}(\mathcal{P} \cap H_0)$, thereby implying that $\bar{x}_k \in \{0, \theta_k - 1\}$. The assumption $\theta \geq \mathbb{1}$ allows us to make similar arguments when $\bar{x}_k \in \{0, \theta_k - 1\}$ and $\bar{x}_i, \bar{x}_j \geq 1$ for distinct $i, j \leq k-1$. Thus if $\bar{x}_k = 0$, then $\bar{x} \in \mathrm{vert}(\mathcal{P} \cap H_0)$ only if $\bar{x} = v^{j,k}$ for some $j \leq k-1$. Finally let $\bar{x}_k = \theta_k - 1$, $\bar{x}_i = \tilde{b}_{k-1} + 1$ for some $i \leq k-2$ and $\bar{x}_t = 0$ for $t \neq i, k$. In this case, $\bar{x} = \frac{\theta_k - 1}{\theta_k}\sigma^1 + \frac{1}{\theta_k}\sigma^2$, where

$$\sigma^1 := (\tilde{b}_{k-1}\mathbf{e}_i, \theta_k, \theta_{k+1}, \ldots, \theta_d), \quad \sigma^2 := (\tilde{b}_k \mathbf{e}_i, 0, \theta_{k+1}, \theta_{k+2}, \ldots, \theta_d),$$

and $\sigma^1, \sigma^2 \in \mathcal{P} \cap H_0$ due to $\theta \geq \mathbb{1}, k \geq 3, i \leq k-2$. Hence $\bar{x}$ must be equal to u^k when $\bar{x}_k = \theta_k - 1$. □

Observation 3.2. *The d coordinate planes are facets of $\mathcal{P}$, which we call trivial facets.*

Proof. We know $\mathbf{0} \in \mathcal{P}$. The assumption $\theta \geq \mathbb{1}$ implies that $\mathbf{e}_i \preccurlyeq \theta$ for $1 \leq i \leq d$. □

The remaining facets are defined by supporting hyperplanes that have a monotone coefficient property, which we prove next.

Lemma 3.1. *Suppose $\mathcal{P} \subseteq \{x : cx \le c_0\}$ and let $F = \mathcal{P} \cap \{x : cx = c_0\}$ be a face of $\mathcal{P}$. If F is not contained in $x_{i+1} = 0$ for some $i \in \{1, \dots, d-1\}$ then $c_{i+1} \ge \max\{c_i, 0\}$. Consequently, if F is a nontrivial facet then $0 \le c_i \le c_{i+1}$ for all $1 \le i \le d-1$.*

Proof. Let $\bar{x}$ be a vertex on F with $\bar{x}_{i+1} \ge 1$. Consider first $x' = \bar{x} - \mathbf{e}_{i+1}$. Since $\mathbf{0} \le x' \preccurlyeq \bar{x}$, we have $cx' \le c_0 = c\bar{x}$, which implies $c_{i+1} \ge 0$. Similarly, the point $x'' = \bar{x} + \mathbf{e}_i - \mathbf{e}_{i+1}$ also has the property $\mathbf{0} \le x'' \preccurlyeq \bar{x}$. Therefore, we have $cx'' \le c_0 = c\bar{x}$, which yields $c_i \le c_{i+1}$. □

We also note that H_0 defines a facet of $\mathcal{P}$.

Observation 3.3. *$\sum_{i=1}^{d} x_i \le b$ is a facet-defining inequality for $\mathcal{P}$.*

Proof. The face $\mathcal{P} \cap H_0$ contains θ and the $d-1$ coordinate vectors $v^{1,d}, v^{2,d}, \dots, v^{d-1,d}$, and these vertices are affinely independent because of $\theta_d \ge 1$. □

In proving our first main result Theorem 3.1 and deriving the $\mathcal{H}$-representation of $\mathcal{P}$, the adjacancies of $\mathbf{0}$ and θ will be useful.

Proposition 3.2. *Let $\theta \ge \mathbb{1}$ and $d \ge 3$. The neighbors of θ and $\mathbf{0}$ are*

$$\mathcal{N}_{\mathcal{P}}(\theta) = \{w, v^{1,2}\} \cup \{u^k : 3 \le k \le d\}$$
$$\mathcal{N}_{\mathcal{P}}(\mathbf{0}) = \{w\} \cup \{v^{j,d} : 1 \le j \le d-1\}.$$

Proof. Since θ has at least d neighbors, it suffices to show that the other vertices are not neighbors of θ. Now, suppose $\mathcal{P} \subseteq \{x : cx \le c_0\}$ and $F = \mathcal{P} \cap \{x : cx = c_0\}$ is an edge through θ and $v^{j,k}$ for some $k \ge 3$ and $j < k$. Lemma 3.1 implies

$$(c_{k-1} - c_j)(\theta_1 + \cdots + \theta_{k-1} + 1) + (c_k - c_j)(\theta_k - 1) \ge 0$$

which, in turn, implies $cu^k \ge cv^{j,k} = c_0$. Therefore $u^k \in F$, which contradicts the assumption that F is an edge.

Consider any vertex v from the list $\{w, b\mathbf{e}_1, \dots, b\mathbf{e}_{d-1}\}$. Recall that $b\mathbf{e}_j = v^{j,d}$. Note that for $j = 1, \dots, d-1$, $\sum_{i \ne j} x_i = 0$ is a supporting hyperplane for $\mathcal{P}$ that contains only the vertices $\mathbf{0}$ and $b\mathbf{e}_j$. Similarly, $\sum_{i \ne d} x_i = 0$ is a supporting hyperplane that contains only the vertices $\mathbf{0}$ and w. Therefore, $\{w, b\mathbf{e}_1, \dots, b\mathbf{e}_{d-1}\} \subseteq \mathcal{N}_{\mathcal{P}}(\mathbf{0})$. The conic hull of all these neighbors is $\mathbb{R}^d_+$. Hence $\mathcal{C}_{\mathcal{P}}(\mathbf{0})$, the vertex cone at $\mathbf{0}$, contains $\mathbb{R}^d_+$ but since $\mathcal{P} \subseteq \mathbb{R}^d_+$, we in fact have $\mathcal{C}_{\mathcal{P}}(\mathbf{0}) = \mathbb{R}^d_+$. Therefore there does not exist another vertex of $\mathcal{P}$ which is a neighbor of $\mathbf{0}$. □

Theorem 3.1. *Let $\theta \ge \mathbb{1}$ and $d \ge 3$. $\mathcal{P}$ is a $(\mathbf{0}, \theta)$-Dantzig figure and $(\mathbf{0}, \theta)$ is the only antipodal vertex pair of $\mathcal{P}$.*

Proof. Proposition 3.2 gives us that each of $\mathbf{0}$ and θ has exactly d neighboring vertices. For any vertex v of a d-polytope, every facet containing v also contains at least $d-1$ neighbors of v. Hence each of $\mathbf{0}$ and θ lies on exactly d facets. The nonnegativity of the coefficients of the supporting hyperplanes from Lemma 3.1 and the assumption $\theta \ge \mathbb{1}$ imply that there is no nontrivial face containing both $\mathbf{0}$ and θ.

Now we need to prove that every facet of $\mathcal{P}$ contains either $\mathbf{0}$ or θ. Suppose F is a facet of $\mathcal{P}$ given by $cx \le c_0$. If F doesn't contain θ nor any of the vertices $v^{1,k}$, then it is contained in the subspace $x_1 = 0$ and hence be equal to the facet defined by $x_1 \ge 0$ and therefore contain $\mathbf{0}$. So, suppose F contains a vertex $v^{1,k}$ for some k and suppose $c\mathbf{0} < c_0$ and $c\theta < c_0$. Then $v^{1,k} \in F$ implies

$$c\theta < c_0 = c_1 \sum_{i=1}^{k} \theta_i + \sum_{i=k+1}^{d} c_i \theta_i$$

and using Lemma 3.1 we get

$$0 \leq \sum_{i=2}^{k}(c_i - c_1)\theta_i < 0$$

which is a contradiction.

The vertex w is a neighbor of both $\mathbf{0}$ and θ and the vertices θ, u^k and $v^{j,k}$ all belong to the facet defined by H_0. Hence any other antipodal pair of vertices must be of the form $(\mathbf{0}, v)$ or (w, v), where $v = u^k$ for some $3 \leq k \leq d$, or $v = v^{j,k}$ for some $1 \leq j < k \leq d$. However each of these pairs has a common facet defined by some coordinate plane: each of the two pairs $(\mathbf{0}, u^k)$ and (w, u^k) shares the facet $x_1 = 0$; $(\mathbf{0}, v^{j,k})$ share the facet $x_k = 0$; $(w, v^{j,k})$ share the facet $x_k = 0$ if $k < d$ and otherwise, $x_i = 0$ for some $i \leq d-1, i \neq j$. $\square$

3.2 $\mathcal{H}$-polytope

Proposition 3.2 tells us that the vertex cone at $\mathbf{0}$ is $\mathcal{C}_{\mathcal{P}}(\mathbf{0}) = \mathbb{R}^d_+$. Then Proposition 2.1 gives us $\mathcal{P} = \{x \geq \mathbf{0} : M^{-1}(x - \theta) \geq \mathbf{0}\}$, where

$$\begin{aligned} M &:= \begin{bmatrix} v^{1,2}-\theta & u^3-\theta & u^4-\theta & \cdots & u^d-\theta & w-\theta \end{bmatrix} \\ &= \begin{bmatrix} \theta_2 & -\theta_1 & -\theta_1 & -\theta_1 & \cdots & -\theta_1 & -\theta_1 \\ -\theta_2 & \tilde{b}_1+1 & -\theta_2 & -\theta_2 & \cdots & -\theta_2 & -\theta_2 \\ 0 & -1 & \tilde{b}_2+1 & -\theta_3 & \cdots & -\theta_3 & -\theta_3 \\ \vdots & 0 & -1 & \tilde{b}_3+1 & \cdots & -\theta_4 & -\theta_4 \\ \vdots & \vdots & 0 & -1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \tilde{b}_{d-2}+1 & -\theta_{d-1} \\ 0 & \cdots & \cdots & \cdots & \cdots & -1 & \tilde{b}_{d-1}-1 \end{bmatrix} \end{aligned}$$

To describe the inverse of M, denote

$$p_i^j := \begin{cases} \tilde{b}_i \prod_{k=i+1}^{j}(\tilde{b}_k+1) & j > i \\ \tilde{b}_i & j = i \\ 1 & j < i. \end{cases}$$

Proposition 3.3. *Let $\theta \geq \mathbb{1}$ and $d \geq 3$. $\mathcal{P} = \{x \geq \mathbf{0} : Nx \geq N\theta\}$ where $N = M^{-1}$ with*

$$N_{d,i} = -1,\ 1 \leq i \leq d, \quad N_{i,d} = -p_i^{d-1},\ 2 \leq i \leq d, \quad N_{1,d} = \frac{-p_1^{d-1}}{\theta_2},$$

$$N_{i,j} = \begin{cases} N_{i,j+1} + \dfrac{p_1^{j-1}}{\theta_2} & i = 1, 1 \leq j \leq d-1 \\ N_{i,j+1} + p_i^{j-1} & 2 \leq i \leq j \leq d-1 \\ N_{i,j+1} & 1 \leq j < i \leq d-1. \end{cases}$$

Proof. We need to consider several cases when computing $N_{i\cdot}M_{\cdot j}$. First note that

$$N_{1\cdot}M_{\cdot 1} = \theta_2(N_{1,1} - N_{1,2}) = \theta_2/\theta_2 = 1$$

and for $i \geq 2$,

$$
\begin{aligned}
N_{i\cdot}M_{\cdot i} &= -\sum_{l=1}^{i-1} N_{i,l}\theta_i + (\tilde{b}_{i-1}+1)N_{i,i} - N_{i,i+1} \\
&= -N_{i,i}\sum_{l=1}^{i-1}\theta_l + (\tilde{b}_{i-1}+1)N_{i,i} - N_{i,i+1} \\
&= N_{i,i} - N_{i,i+1} \\
&= 1.
\end{aligned}
$$

Consider now the case $1 \leq j < i \leq d$. It is readily seen that

$$
N_{d\cdot}M_{\cdot j} = -\sum_{l=1}^{d} M_{l,j} = 0
$$

and for $i \leq d-1$,

$$
\begin{aligned}
N_{i\cdot}M_{\cdot j} &= -N_{i,i}\sum_{l=1}^{j-1}\theta_l + (\tilde{b}_{j-1}+1)N_{i,j} - N_{i,j+1} \\
&= -N_{i,i}\sum_{l=1}^{j-1}\theta_l + (\tilde{b}_{j-1}+1)N_{i,i} - N_{i,i} \\
&= 0.
\end{aligned}
$$

For $1 \leq i < j \leq d-1$,

$$
\begin{aligned}
N_{i\cdot}M_{\cdot j} - N_{i\cdot}M_{\cdot j-1} &= N_{i\cdot}(M_{\cdot j} - M_{\cdot j-1}) \\
&= N_{i,j-1}(\theta_{j-1} - \tilde{b}_{j-2} - 1) + N_j(\tilde{b}_{j-1}+2) + N_{i,j+1}(-1) \\
&= -N_{i,j-1}(b_{j-1}+1) + N_{i,j}(b_{j-1}+2) - N_{i,j+1} \\
&= (b_{j-1}+1)(N_{i,j} - N_{i,j-1}) + (N_{i,j} - N_{i,j+1}) \\
&= \begin{cases} (b_{j-1}+1)(\frac{-p_i^{j-2}}{\theta_2}) + \frac{p_i^{j-1}}{\theta_2}, & 1 = i < j \leq d-1 \\ (b_{j-1}+1)(-p_i^{j-2}) + p_i^{j-1}, & 1 < i < j \leq d-1 \end{cases} \\
&= \begin{cases} 0, & i+1 = j \leq d-1 \\ -1, & i+1 < j \leq d-1. \end{cases}
\end{aligned}
$$

Therefore, $N_{i\cdot}M_{\cdot j} = 0$ for $1 \leq i < j \leq d-1$.

$$
\begin{aligned}
N_{i\cdot}M_{\cdot d} - N_{i\cdot}M_{\cdot d-1} &= N_{i\cdot}(M_{\cdot d} - M_{\cdot d-1}) \\
&= N_{i,d-1}(-\tilde{b}_{d-2} - 1 - \theta_{d-1}) + N_{i,d}(1 + \tilde{b}_{d-1} - 1) \\
&= \begin{cases} \left(N_{i,d} + \frac{p_i^{d-2}}{\theta_2}\right)(\tilde{b}_{d-1}+1) - \tilde{b}_{d-1}N_{i,d}, & i = 1 \\ (N_{i,d} + p_i^{d-2})(\tilde{b}_{d-1}+1) - \tilde{b}_{d-1}N_{i,d}, & 1 < i \leq d-1 \end{cases} \\
&= \begin{cases} N_{i,d} + \frac{p_i^{d-2}}{\theta_2}(\tilde{b}_{d-1}+1), & i = 1 \\ N_{i,d} + p_i^{d-2}(\tilde{b}_{d-1}+1), & 1 < i \leq d-1 \end{cases} \\
&= \begin{cases} 0, & 1 \leq i \leq d-2 \\ -1, & i = d-1. \end{cases}
\end{aligned}
$$

Therefore, $N_{i\cdot}M_{\cdot d} = 0$ for $1 \leq i \leq d-1$. □

3.3 Graph of the polytope

Let $G(\mathcal{P})$ denote the graph of $\mathcal{P}$. Based on Proposition 3.1, it is clear that if $\theta > 1$, $G(\mathcal{P})$ has $\frac{d^2+d+2}{2}$ vertices.

We next characterize the facet-vertex incidence for $\mathcal{P}$. This information encodes all the faces of a polytope [KP02]. Namely, a subset S of vertices is the vertex set of a face if and only if $S = V(F(S))$, where $F(S)$ is the set of facets incident to all the vertices in S, and $V(T)$ is the set of vertices incident to all facets in T. While the cardinality of S or $F(S)$ doesn't determine the dimension of the face, a k-dimensional face must contain at least $k+1$ vertices and be contained in at least $d-k$ facets [Bro83, Theorem 10.4]. The edges are special, because they contain exactly 2 vertices. Thus, (v, v') is an edge of a polytope if and only if there is a subset of facets incident to both v and v' and all these common facets are not incident to another vertex v''. Any such subset will automatically contain at least $d-1$ facets and this fact can be used to discard pairs of vertices as possible edges. This allows us to enumerate all the edges of $G(\mathcal{P})$.

As can be seen from Corollary 3.1, $G(\mathcal{P})$ depends only on which entries of θ are 1, and for a fixed d, these graphs are isomorphic for all $\theta > \mathbb{1}$ (Figure 2). We then derive some of the basic properties of $G(\mathcal{P})$ such as radius, diameter, coloring number, etc. We also show that when $\theta > \mathbb{1}$, the edge expansion of this graph is equal to one.

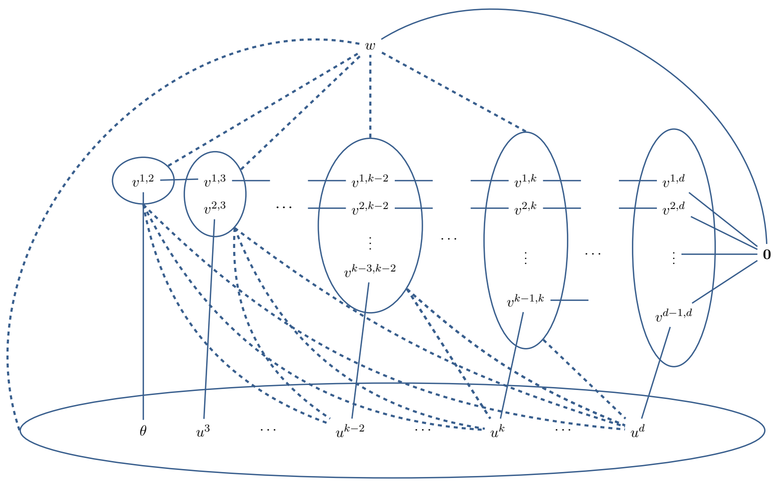


Figure 2: The graph $G(\mathcal{P})$ for a vertex $\theta > \mathbb{1}$ in $\mathbb{R}^d$. The circled vertices form cliques. The dashed edges represent connections between a vertex and a clique.

Since $\mathcal{P}$ is a $(\mathbf{0}, \theta)$-Dantzig figure as per Theorem 3.1, the vertex cones $\mathcal{C}_{\mathcal{P}}(\mathbf{0})$ and $\mathcal{C}_{\mathcal{P}}(\theta)$ are simplicial. Thus the d facet-defining hyperplanes incident to θ (resp. $\mathbf{0}$) are in a one-to-one correspondence with the d neighbors of θ (resp. $\mathbf{0}$). Furthermore, we know that $\mathbf{0}$ and θ do not belong to a common facet. Hence we denote the facet-defining hyperplanes incident to θ as:

$$\begin{aligned} H_w &= \{x \colon N_{d\cdot}(x-\theta) = 0\} = H_0, \quad H_{v^{1,2}} = \{x \colon N_{1\cdot}(x-\theta) = 0\}, \\ H_{u^k} &= \{x \colon N_{(k-1)\cdot}(x-\theta) = 0\} \quad 3 \le k \le d, \end{aligned}$$

where N is the inverse of M from Proposition 3.3 and H_v, for $v \in \mathcal{N}_{\mathcal{P}}(\theta)$, signifies the only facet-defining hyperplane that contains θ but not v. Let $\mathcal{H}_{\mathcal{P}}^{\theta}$ denote the collection of these hyperplanes, i.e.,

$$\mathcal{H}_{\mathcal{P}}^{\theta} := \{H_{v^{1,2}}, H_{u^3}, \dots, H_{u^d}, H_w\}. \tag{8a}$$

The d facet-defining hyperplanes incident to $\mathbf{0}$ are the coordinate planes $H_i := \{x \colon x_i = 0\}$ for $1 \le i \le d$, and we denote this collection by

$$\mathcal{H}^{\mathbf{0}} := \{H_1, H_2, \dots, H_d\}, \quad \mathcal{H}^{\mathbf{0},i} := \{H_1, H_2, \dots, H_i\} \quad 1 \le i \le d. \tag{8b}$$

The vertex-facet incidence for $\mathcal{P}$ is stated in the following result. For $v \in \operatorname{vert}(\mathcal{P})$, let $\psi_{\mathcal{P}}(v)$ denote the subset of facet-defining hyperplanes of $\mathcal{P}$ that contain v.

Proposition 3.4. *Let $\theta \geq \mathbb{1}$ and $d \geq 3$. We have*

$$\begin{aligned}
&\psi_{\mathcal{P}}(\mathbf{0}) = \mathcal{H}^{\mathbf{0}}, \quad \psi_{\mathcal{P}}(\theta) = \mathcal{H}^{\theta}_{\mathcal{P}},\\
&\psi_{\mathcal{P}}(w) = (\mathcal{H}^{\theta}_{\mathcal{P}} \setminus \{H_w\}) \cup \mathcal{H}^{\mathbf{0},d-1},\\
&\psi_{\mathcal{P}}(u^k) = (\mathcal{H}^{\theta}_{\mathcal{P}} \setminus \{H_{u^k}\}) \cup \mathcal{H}^{\mathbf{0},k-2} \cup \begin{cases} H_k & \text{if } \theta_k = 1 \\ \emptyset & \text{if } \theta_k \geq 2 \end{cases}, \qquad 3 \leq k \leq d\\
&\psi_{\mathcal{P}}(v^{j,k}) = (\mathcal{H}^{\theta}_{\mathcal{P}} \setminus \{H_{v^{1,2}}, H_{u^3}, \dots, H_{u^k}\}) \cup (\mathcal{H}^{\mathbf{0},k} \setminus \{H_j\}), \quad 1 \leq j < k \leq d, (j,k) \neq (2,3)\\
&\psi_{\mathcal{P}}(v^{2,3}) = \{H_1, H_3\} \cup \begin{cases} (\mathcal{H}^{\theta}_{\mathcal{P}} \setminus \{H_{u^3}\}) & \text{if } \theta_3 = 1 \\ (\mathcal{H}^{\theta}_{\mathcal{P}} \setminus \{H_{v^{1,2}}, H_{u^3}\}) & \text{if } \theta_3 \geq 2. \end{cases}
\end{aligned}$$

Proof. The expressions for $\psi_{\mathcal{P}}(\mathbf{0})$ and $\psi_{\mathcal{P}}(\theta)$ are obvious. For the other vertices, because it is trivial to check containment in a coordinate plane using our assumption $\theta \geq \mathbb{1}$, we only argue the incidence of the elements of $\mathcal{H}^{\theta}_{\mathcal{P}}$. Since $w \in \mathcal{N}_{\mathcal{P}}(\theta)$, we have $\psi_{\mathcal{P}}(w) \supset \mathcal{H}^{\theta}_{\mathcal{P}} \setminus \{H_w\}$ by construction of $\mathcal{H}^{\theta}_{\mathcal{P}}$. The value of $\psi_{\mathcal{P}}(u^k)$ follows by a similar reasoning. Now fix $k \geq 2$ and $1 \leq j \leq k-1$. Define $\xi := M^{-1}(v^{j,k} - \theta)$. Proposition 3.3 gives us $\xi \geq \mathbf{0}$. The rows of M^{-1} correspond to the hyperplanes $H_{v^{1,2}}, H_{u^3}, \dots, H_{u^d}, H_w$, respectively. Then to prove the claimed expression for $\psi_{\mathcal{P}}(v^{j,k})$, we need to show that $\xi_k = \xi_{k+1} = \cdots = \xi_d = 0$ and $\xi_i > 0$ for $1 \leq i \leq k-1$, except that $\xi_1 = 0$ when $j = 2, k = 3, \theta_3 = 1$.

The last row in M^{-1}, denoted by $M^{-1}_{d\cdot}$, is a vector of -1's, meaning that the facet-defining inequality $M^{-1}_{d\cdot} x \geq M^{-1}_{d\cdot}\theta$ corresponds to the hyperplane H_0, which we know contains $v^{j,k}$. Thus

$$\xi_d = M^{-1}_{d\cdot}(v^{j,k} - \theta) = 0. \tag{9a}$$

Now consider the linear system $M\xi = v^{j,k} - \theta$. The upper Hessenberg structure of M gives us the following recursion:

$$\xi_{i-1} = (\tilde{b}_{i-1} + 1)\xi_i - \theta_i \sum_{t=i+1}^{d} \xi_t - v^{j,k}_i + \theta_i, \qquad 3 \leq i \leq d \tag{9b}$$

$$\xi_1 = \frac{\tilde{b}_1 + 1}{\theta_2}\xi_2 - \sum_{t=3}^{d} \xi_t - \frac{v^{j,k}_2}{\theta_2} + 1 = \frac{\theta_1}{\theta_2}\sum_{t=2}^{d} \xi_t + \frac{v^{j,k}_1 - \theta_1}{\theta_2}. \tag{9c}$$

Note that

$$v^{j,k}_i - \theta_i = \begin{cases} 0 & k+1 \leq i \leq d \\ \tilde{b}_k - \theta_j & i = j \\ -\theta_i & \text{otherwise.} \end{cases} \tag{10}$$

First let us apply (9b) with $i = d$. Invoking $\xi_d = 0$ from (9a) gives us $\xi_{d-1} = \theta_d - v^{j,k}_d$, which is equal to $\theta_d \geq 1$ if $k = d$, otherwise it is zero. Equation (10) and a backward induction on i in (9b) then lead us to

$$\xi_k = \xi_{k+1} = \cdots = \xi_d = 0.$$

The expressions for the remaining ξ's can be obtained from (9b) and (9c) as

$$\xi_{i-1} = (\tilde{b}_{i-1} + 1)\xi_i - \theta_i \sum_{t=i+1}^{k-1} \xi_t - v^{j,k}_i + \theta_i, \qquad 3 \leq i \leq k \tag{11a}$$

$$\xi_1 = \frac{\tilde{b}_1 + 1}{\theta_2}\xi_2 - \sum_{t=3}^{k-1} \xi_t - \frac{v^{j,k}_2}{\theta_2} + 1 = \frac{\theta_1}{\theta_2}\sum_{t=2}^{k-1} \xi_t + \frac{v^{j,k}_1 - \theta_1}{\theta_2}. \tag{11b}$$

For $\xi_2, \ldots, , \xi_{k-1}$, we claim the following:

$$\xi_i = \tilde{b}_i \left(\sum_{t=i+1}^{k-1} \xi_t - 1 \right) + \sum_{t=i+2}^{j-1} \theta_t \qquad i = 2, \ldots, j-1 \tag{12a}$$

$$\xi_i = \tilde{b}_i \sum_{t=i+1}^{k-1} \xi_t + \sum_{t=i+1}^{k} \theta_t \qquad i = j, \ldots, k-1. \tag{12b}$$

Proving this claim implies $\xi_2 > \xi_3 > \cdots > \xi_{k-1} \geq 1$ since $\theta \geq \mathbb{1}$. Equation (11b) gives us $\xi_1 = (\theta_1/\theta_2)(\sum_{t=2}^{k-1} \xi_t - 1)$ if $j > 1$ or $\xi_1 = (\theta_1/\theta_2) \sum_{t=2}^{k-1} \xi_t + (\sum_{t=2}^{k} \theta_t)/\theta_2$ if $j = 1$. For $(j,k) \neq (2,3)$, we then have $\xi_1 > 0$. For $(j,k) = (2,3)$, $\xi_1 = \theta_1(\theta_3 - 1)/\theta_2$, which is equal to zero if and only if $\theta_3 = 1$. Thus, proving equations (12) finishes our proof for $\psi_{\mathcal{P}}(v^{j,k})$.

We prove (12a) and (12b) separately by backward induction on i. For (12b), the base case $\xi_{k-1} = \theta_k$ follows by using $\xi_k = 0$ and (10) in (11a). Assume (12b) to be true for $j < l \leq k-1$ and consider ξ_{l-1}. Applying (11a) and using the induction hypothesis yields

$$\begin{aligned}
\xi_{l-1} &= \tilde{b}_{l-1}\xi_l + \xi_l - \theta_l \sum_{t=l+1}^{k-1} \xi_t + \theta_l \\
&= \tilde{b}_{l-1}\xi_l + \tilde{b}_l \sum_{t=l+1}^{k-1} \xi_t + \sum_{t=l+1}^{k} \theta_t - \theta_l \sum_{t=l+1}^{k-1} \xi_t + \theta_l \\
&= \tilde{b}_{l-1}\xi_l + (\tilde{b}_l - \theta_l) \sum_{t=l+1}^{k-1} \xi_t + \sum_{t=l}^{k} \theta_t \\
&= \tilde{b}_{l-1} \sum_{t=l}^{k-1} \xi_t + \sum_{t=l}^{k} \theta_t.
\end{aligned}$$

For (12a), the base case formula for ξ_{j-1} can be obtained as follows: from (11a) we have

$$\begin{aligned}
\xi_{j-1} &= \tilde{b}_{j-1}\xi_j + \xi_j - \theta_j \sum_{t=j+1}^{k-1} \xi_t - \tilde{b}_k + \theta_j \\
&= \tilde{b}_{j-1}\xi_j + \tilde{b}_j \sum_{t=j+1}^{k-1} \xi_t + \sum_{t=j+1}^{k} \theta_t - \theta_j \sum_{t=j+1}^{k-1} \xi_t - \tilde{b}_k + \theta_j \\
&= \tilde{b}_{j-1}\xi_j + (\tilde{b}_j - \theta_j) \sum_{t=j+1}^{k-1} \xi_t + \sum_{t=j}^{k} \theta_t - \tilde{b}_k \\
&= \tilde{b}_{j-1}\xi_j + \tilde{b}_{j-1} \sum_{t=j+1}^{k-1} \xi_t - \tilde{b}_{j-1} \\
&= \tilde{b}_{j-1} \left(\sum_{t=j}^{k-1} \xi_t - 1 \right).
\end{aligned}$$

The inductive step is similar to that for (12b). $\square$

$\mathcal{P}$ is a d-polytope and hence for any $v, v' \in \operatorname{vert}(\mathcal{P})$, (v, v') is an edge in $G(\mathcal{P})$ if and only if $|\psi_{\mathcal{P}}(v) \cap \psi_{\mathcal{P}}(v')| \geq d-1$ and there does not exist a $v'' \in \operatorname{vert}(\mathcal{P})$ with $\psi_{\mathcal{P}}(v'') \supseteq \psi_{\mathcal{P}}(v) \cap \psi_{\mathcal{P}}(v')$. The formulas for $\psi_{\mathcal{P}}(\cdot)$ in Proposition 3.4 imply a complete list of edges and thereby the degree of each vertex.

Corollary 3.1. *Let $d \geq 3$. If $\theta > \mathbb{1}$, $G(\mathcal{P})$ has $\frac{1}{2}(d^2+d+2)$ vertices, the edges between which are as follows:*

1. $(\theta, w), (\theta, v^{1,2})$, *and* (θ, u^k) *for* $k \geq 3$,
2. $(\mathbf{0}, w)$ *and* $(\mathbf{0}, v^{j,d})$ *for* $1 \leq j \leq d-1$,
3. $(w, v^{2,3})$ *if* $d \geq 4$
4. (w, v) *for* $v \in \mathrm{vert}(\mathcal{P}) \setminus \{v^{2,3}, v^{1,d}, \ldots, v^{d-1,d}\}$,
5. (u^{k_1}, u^{k_2}) *for* $k_1, k_2 \geq 3$,
6. $(v^{k-1,k}, u^k)$ *for* $k \geq 3$,
7. (v^{j,k_1}, u^{k_2}) *for* $2 \leq k_1 \leq k_2 - 2$, $1 \leq j \leq k_1 - 1$,
8. $(v^{j_1,k}, v^{j_2,k})$ *for* $1 \leq j_1, j_2 \leq k-1$,
9. $(v^{j,k}, v^{j,k+1})$ *for* $1 \leq j < k \leq d-1$.

If $\theta_k = 1$, $k \geq 3$, then $u^k = v^{k-1,k}$, and $G(\mathcal{P})$ is a minor of the above-described graph obtained by contracting the edges $\{(u^k, v^{k-1,k}) \colon \theta_k = 1\}$.

Proof. The neighbors of $\mathbf{0}$ and θ are from Proposition 3.2. We will count the number of common facets for each pair of vertices of $\mathcal{P}$. For the pairs that are not on the list above, we will show that there are fewer than $d-1$ facets containing both vertices, and therefore cannot be edges. For the remaining pairs (v, v'), it is then straightforward to verify that no other vertex lies on the common facets for v and v' and, therefore, (v, v') is an edge.

Let us first argue neighbors of w. For $d \geq 4$, w and $v^{2,3}$ share H_1 and H_3 and at least $d-3$ planes from $\mathcal{H}^\theta_{\mathcal{P}}$, implying a possible edge between the two. For $d = 3$, the only coordinate plane they share is H_1 and so an edge exists only if they share $d-2$ planes from $\mathcal{H}^\theta_{\mathcal{P}}$, which happens only when $\theta_3 = 1$. Since $\mathcal{H}^\theta_{\mathcal{P}} \setminus \{H_{v^{1,2}}, H_{u^3}, \ldots, H_{u^d}\} = H_w$, the coordinate vertex $v^{j,d}$ does not share any plane from $\mathcal{H}^\theta_{\mathcal{P}}$ with w and shares all the coordinate planes except H_j and H_d. Now $j \leq d-1$ implies that $|\psi_{\mathcal{P}}(w) \cap \psi_{\mathcal{P}}(v^{j,d})| \leq d-2$ and so there cannot be an edge between w and $v^{j,d}$. For $k \leq d-1$, edge $(w, v^{j,k})$ may exist because $|\psi_{\mathcal{P}}(w) \cap \psi_{\mathcal{P}}(v^{j,k})| \geq d-1$ due to $\psi_{\mathcal{P}}(w) \cap \psi_{\mathcal{P}}(v^{j,k}) \supseteq \{H_{u^{k+1}}, \ldots, H_{u^d}\} \cup (\mathcal{H}^{\mathbf{0},k} \setminus \{H_j\})$. Arguments for the edge (w, u^k) are similar. The u^k's may form a clique since for any $3 \leq k_1 < k_2 \leq d$, $\psi_{\mathcal{P}}(u^{k_1}) \cap \psi_{\mathcal{P}}(u^{k_2}) \supseteq \{H_1\} \cup (\mathcal{H}^\theta_{\mathcal{P}} \setminus \{H_{u^{k_1}}, H_{u^{k_2}}\})$.

Consider v^{j,k_1} and u^{k_2} for $k_2 \geq 3, 1 \leq j < k_1 \leq d$. We use the following cases.

$2 \leq k_1 \leq k_2 - 1$: Here

$$\psi_{\mathcal{P}}(v^{j,k_1}) \cap \psi_{\mathcal{P}}(u^{k_2}) \cap \mathcal{H}^\theta_{\mathcal{P}} = \begin{cases} \mathcal{H}^\theta_{\mathcal{P}} \setminus \{H_{u^3}, H_{u^{k_2}}\} & \text{if } k_1 = 3, \theta_3 = 1 \\ \mathcal{H}^\theta_{\mathcal{P}} \setminus \{H_{v^{1,2}}, H_{u^3}, \ldots, H_{u^{k_1}}, H_{u^{k_2}}\} & k_1 = 3, \theta_3 \geq 2 \text{ or } k_1 \neq 3. \end{cases}$$

The cardinality of this set is $d-2$ for $k_1 = 3, \theta_3 = 1$ or $d - k_1$ otherwise. Also

$$\psi_{\mathcal{P}}(v^{j,k_1}) \cap \psi_{\mathcal{P}}(u^{k_2}) \cap \mathcal{H}^{\mathbf{0}} = \mathcal{H}^{\mathbf{0},k_1-1} \setminus \{H_j\} \cup \begin{cases} H_{k_1} & \text{if } k_1 \leq k_2 - 2 \\ \emptyset & \text{if } k_1 \leq k_2 - 1. \end{cases}$$

Therefore $|\psi_{\mathcal{P}}(v^{j,k_1}) \cap \psi_{\mathcal{P}}(u^{k_2})| \geq d-1$ if and only if $3 \neq k_1 \leq k_2 - 2$ or $k_1 = 3, \theta_3 = 1, k_2 \geq 4$ or $k_1 = 3, \theta_3 \geq 2, k_2 \geq 5$.

$3 \le k_2 \le k_1 - 1$: Here $k_1 \ge 4$. We have $\psi_{\mathcal{P}}(v^{j,k_1}) \cap \psi_{\mathcal{P}}(u^{k_2}) \cap \mathcal{H}^{\theta}_{\mathcal{P}} = \mathcal{H}^{\theta}_{\mathcal{P}} \setminus \{H_{v^{1,2}}, H_{u^3}, \ldots, H_{u^{k_1}}\}$, which are exactly $d - (k_1 - 1)$ common planes from $\mathcal{H}^{\theta}_{\mathcal{P}}$. So an edge exists only if there are at least $k_1 - 2$ common coordinate planes. Note that $\psi_{\mathcal{P}}(v^{j,k_1}) \cap \psi_{\mathcal{P}}(u^{k_2}) \cap \mathcal{H}^{\mathbf{0}} \subseteq \{H_1, \ldots, H_{k_2-2}, H_{k_2}\}$ and so we can have at most $k_2 - 1$ common coordinate planes. Hence, for $k_2 \le k_1 - 2$, there is no edge, and for $k_1 = k_2 + 1$, an edge exists only if $j = k_2 - 1$ and $\theta_{k_2} = 1$.

$3 \le k_1 = k_2 = k$: We have $\psi_{\mathcal{P}}(v^{j,k}) \cap \psi_{\mathcal{P}}(u^{k}) \cap \mathcal{H}^{\theta}_{\mathcal{P}} = \psi_{\mathcal{P}}(v^{j,k}) \cap \mathcal{H}^{\theta}_{\mathcal{P}}$ and so the number of common planes from $\mathcal{H}^{\theta}_{\mathcal{P}}$ is $d-1$ when $k = 3, \theta_3 = 1$ or $d - (k-1)$ otherwise. For $j = k - 1$, the first $k - 2$ coordinate planes are common, giving us a potential edge between $v^{k-1,k}$ and u^k for all k. For $1 \le j \le k - 2$, we get $k - 2$ common coordinate planes if and only if $\theta_k = 1$.

Finally, we argue edges between the v vertices. Each $v^{\cdot,k}$ component may be a clique because $v^{j_1,k}$ and $v^{j_2,k}$ belong to the same $d - (k-1)$ planes from $\mathcal{H}^{\theta}_{\mathcal{P}}$ and share the $k - 2$ coordinate planes in $\mathcal{H}^{\mathbf{0},k} \setminus \{H_{j_1}, H_{j_2}\}$. Now consider v^{j_1,k_1} and v^{j_2,k_2} with $k_1 < k_2$. At most $k_1 - 1$ coordinate planes are shared and exactly $d - (k_2 - 1)$ planes from $\mathcal{H}^{\theta}_{\mathcal{P}}$ are shared, making the total number at most $d + k_1 - k_2$. This upper bound is less than $d - 1$ if $k_2 \ge k_1 + 2$, meaning that in this case, no edges exist between the cliques $v^{\cdot,k_1}$ and $v^{\cdot,k_2}$. If $k_2 = k_1 + 1$, then the upper bound is equal to $d - 1$ and is attained if and only if the first $k_1 - 1$ coordinate planes are shared, which happens only when $j_1 = j_2$.

□

Corollary 3.2. *For $d \ge 3$ and $\theta > \mathbb{1}$, the degrees of the vertices of $G(\mathcal{P})$ are*

$$\deg(\theta) = \deg(\mathbf{0}) = d, \quad \deg(w) = \frac{d^2 - d + 2}{2}, \quad \deg(u^k) = d + \frac{(k-2)(k-3)}{2}, \quad k \ge 3$$
$$\deg(v^{j,k}) = d, \quad 2 \le k \le d, 1 \le j \le k - 1.$$

The total number of edges is $\frac{1}{3}(d^3 + 2d)$ and the average degree is $\frac{2}{3}(d - 1 + \frac{d+2}{d^2+d+2})$.

Proof. The degree of each vertex follows from the list of edges in Corollary 3.1. The number of edges is half the sum of all the degrees, making it equal to

$$\begin{aligned}
&\frac{1}{2}\left[2d + \frac{d^2 - d + 2}{2} + d(d-2) + \sum_{k=3}^{d} \frac{(k-2)(k-3)}{2} + d\sum_{k=2}^{d}(k-1)\right] \\
=&\frac{1}{2}\left[2d + \frac{d^2 - d + 2}{2} + d^2 - 2d + \frac{1}{2}\sum_{k=1}^{d-3} k(k+1) + \frac{d^2(d-1)}{2}\right] \\
=&\frac{1}{2}\left[\frac{d^3 + 2d^2 - d + 2}{2} + \frac{1}{2}\sum_{k=1}^{d-3} k(k+1)\right] \\
=&\frac{1}{2}\left[\frac{d^3 + 2d^2 - d + 2}{2} + \frac{1}{2}\left(\frac{(d-3)(d-2)(2d-5)}{6} + \frac{(d-3)(d-2)}{2}\right)\right] \\
=&\frac{1}{2}\left[\frac{d^3 + 2d^2 - d + 2}{2} + \frac{(d-3)(d-2)(d-1)}{6}\right] \\
=&\frac{1}{12}\left[3d^3 + 6d^2 - 3d + 6 + d^3 - 6d^2 + 11d - 6\right] \\
=&\frac{d^3 + 2d}{3}.
\end{aligned}$$

The average degree is obtained by dividing twice the above number with the number of vertices $(d^2 + d + 2)/2$. □

Corollary 3.3. *Let $\theta \geq \mathbb{1}$ and $d \geq 3$. The graph of $\mathcal{P}$ has the following properties.*

(a) The radius of $G(\mathcal{P})$ is $r(G(\mathcal{P})) = 2$.

(b) The diameter of $G(\mathcal{P})$ is

$$d(G(\mathcal{P})) = \begin{cases} 3 & d \geq 4 \\ 2 & d = 3 \end{cases}$$

(c) $G(\mathcal{P})$ is Hamiltonian.

(d) If $\theta > \mathbb{1}$, the chromatic number of $G(\mathcal{P})$ is $\chi(G(\mathcal{P})) = d$.

Proof. (a) Since the only common neighbor of θ and $\mathbf{0}$ is w, $r(G(\mathcal{P})) \geq 2$. The equality follows from the fact that w can be chosen as a center: for every non-neighbor of w we have $d(w, v^{j,d}) = 2$, because there is a path $w - v^{j,d-1} - v^{j,d}$.

(b) The distance between the non-neighbors of w is $d(v^{j_1,d}, v^{j_2,d}) = 1$. Therefore, $d(G(\mathcal{P})) \leq 3$. The equality follows from the fact that $v^{1,d}$ and θ have no common neighbors.

(c) Suppose first that $\theta > \mathbb{1}$. For each k, $3 \leq k \leq d-1$, let p_k be a Hamiltonian path in the clique $\{v^{j,k} : 1 \leq j < k\}$ between $v^{k-2,k}$ and $v^{k-1,k}$. Then

$$\mathbf{0} - v^{1,d} - v^{2,d} - \cdots - v^{d-1,d} - u^d - u^{d-1} - \cdots - u^3 - \theta - v^{1,2} - p_3 - p_4 - \cdots - p_{d-1} - w - \mathbf{0}$$

is a Hamiltonian cycle in $G(\mathcal{P})$. If $\theta_k = 1$, the same construction gives a Hamiltonian cycle if we take p_k be a Hamiltonian path in the clique $\{v^{j,k} : 1 \leq j < k-1\}$.

(d) Since $\{v^{j,d} : 1 \leq j \leq d-1\} \cup \{\mathbf{0}\}$ is a d-clique, $\chi(G(\mathcal{P})) \geq d$. On the other hand, one can readily check that $\varphi : V \to \{1, 2, \ldots, d\}$, given by $\varphi(0) = \varphi(\theta) = 1, \varphi(w) = 2, \varphi(u^k) = k$, $\varphi(v^{j,k}) = k - j + 1$, is a proper coloring of $G(\mathcal{P})$. $\square$

Finally, we compute the edge expansion of $G(\mathcal{P})$. Recall that the edge expansion of a graph G on n vertices is defined as

$$h(G) = \min_{\substack{S \subseteq V(G): \\ 0 < |S| \leq \frac{n}{2}}} \frac{|\partial S|}{|S|},$$

where $\partial S := \{(u, v) \in E(G) \ : \ u \in S, v \in V(G) \setminus S\}$. Notice that $G(\mathcal{P})$, having $\mathcal{O}(d^2)$ vertices, is a relatively sparse graph with average vertex degree $\mathcal{O}(d)$.

Theorem 3.2. *Let $\theta \geq \mathbb{1}$ and $d \geq 3$. The edge expansion of the graph $G(\mathcal{P})$ is $h(G(\mathcal{P})) = 1$.*

Proof. For the set $S = \{v^{j,d} : 1 \leq j \leq d-1\} \cup \{\mathbf{0}\}$,

$$\partial S = \{(v^{j,d}, v^{j,d-1}) : 1 \leq j \leq d-2\} \cup \{(v^{d-1,d}, u^d), (\mathbf{0}, w)\}$$

and hence

$$\frac{|\partial S|}{|S|} = \frac{d-2+2}{d-1+1} = 1.$$

Therefore,

$$h(G(\mathcal{P})) \leq 1.$$

Suppose now $|S| \leq n/2$. We will consider two cases.

Case 1: $w \in S$. The set ∂S contains $|S^c \cap \mathcal{N}_{\mathcal{P}}(w)|$ edges incident with w. Let $|S^c \cap \mathcal{N}_{\mathcal{P}}(w)^c| = a$. Then $0 \leq a \leq d-1$ and ∂S contains $a(d-1-a)$ edges from the $(d-1)$-clique $\mathcal{N}_{\mathcal{P}}(w)^c$. If $0 \leq a \leq d-2$, then $a(d-1-a) \geq a$ and

$$\frac{|\partial S|}{|S|} \geq \frac{|S^c \cap \mathcal{N}_{\mathcal{P}}(w)| + a}{|S|} = \frac{|S^c \cap \mathcal{N}_{\mathcal{P}}(w)| + |S^c \cap \mathcal{N}_{\mathcal{P}}(w)^c|}{|S|} = \frac{|S^c|}{|S|} \geq 1. \tag{13}$$

If $a = d-1$, then $\{v^{j,d} \colon 1 \le j \le d-1\} \subseteq S^c$. Let $k \ge 2$ be the maximal so that

$$\{v^{j,k} \colon 1 \le j \le k-1\} \cup \{u^{k+2}, \dots, u^d\} \not\subseteq S^c.$$

If such a k doesn't exist then $S \subseteq \{w, \theta, u^3, \mathbf{0}\}$ and

$$\frac{|\partial S|}{|S|} \ge \frac{|S^c \cap \mathcal{N}_{\mathcal{P}}(w)|}{|S|} = \frac{|S^c| - (d-1)}{|S|} \ge \frac{n - 4 - (d-1)}{4} \ge 1.$$

So, assume such a k does exist and let

$$\left| \left(\{v^{j,k} \colon 1 \le j \le k-1\} \cup \{u^{k+2}, \dots, u^d\} \right) \cap S \right| = b \ge 1.$$

Then ∂S contains $b(d-2-b)$ edges from the $(d-2)$-clique $\{v^{j,k} \colon 1 \le j \le k-1\} \cup \{u^{k+2}, \dots, u^d\}$ and, because of the maximality of k, b edges of the type $(v^{j,k}, v^{j,k+1})$ and $(u^{k'}, v^{k'-1,k'})$ for some $j < k$ and $k' \ge k+2$. For $1 < b < d-2$,

$$b(d-2-b) + b \ge d - 1 = a$$

and (13) holds. If $b = d-2$ then $v^{k-1,k} \in S$, $v^{k,k+1} \in S^c$ and ∂S additionally contains one of the edges $(v^{k-1,k}, u^k)$, $(v^{k,k+1}, u^{k+1})$, (u^k, u^{k+1}). Then

$$b(d-2-b) + b + 1 = d - 1 = a$$

and (13) holds. Finally, let $b = 1$. Then either $\{u^2 := \theta, u^3, u^4, \dots, u^d\} \subseteq S^c$ or ∂S contains one of the edges $(v^{d-1,d}, u^d)$, (u^{k_1}, u^{k_2}) for some $k_1 \ne k_2$. So, suppose $\{u^2 := \theta, u^3, u^4, \dots, u^d\} \subseteq S^c$ and let $v^{j,k} \in S$. If $j = k-1$ then $(v^{j,k}, u^k) \in \partial S$. If $j < k-1$, then one of the edges $(v^{j,k}, v^{j,k-1})$, $(u^{k+1}, v^{j,k-1})$ is in ∂S. Either way, we have established that

$$|\partial S| \ge |S^c \cap \mathcal{N}_{\mathcal{P}}(w)| + b(d-2-b) + b + 1 = |S^c \cap \mathcal{N}_{\mathcal{P}}(w)| + d - 1 = |S^c \cap \mathcal{N}_{\mathcal{P}}(w)| + a$$

and (13) holds.

Case 2: $w \notin S$. If $S \subseteq \mathcal{N}_{\mathcal{P}}(w)$ then clearly $|\partial S| \ge |S|$. Suppose $|S \cap \mathcal{N}_{\mathcal{P}}(w)^c| = a \ge 1$. The $a \le d-1$ and ∂S contains $a(d-1-a)$ edges from the $(d-1)$-clique $\mathcal{N}_{\mathcal{P}}(w)^c$. If $1 \le a \le d-2$, then $a(d-1-a) \ge a$ and

$$\frac{|\partial S|}{|S|} \ge \frac{|S \cap \mathcal{N}_{\mathcal{P}}(w)| + a}{|S|} = \frac{|S \cap \mathcal{N}_{\mathcal{P}}(w)| + |S \cap \mathcal{N}_{\mathcal{P}}(w)^c|}{|S|} = \frac{|S|}{|S|} = 1. \tag{14}$$

So, let $a = d-1$. Then $\{v^{j,d} \colon 1 \le j \le d-1\} \subseteq S$. Let $k \ge 2$ be the maximal so that

$$\{v^{j,k} \colon 1 \le j \le k-1\} \cup \{u^{k+2}, \dots, u^d\} \not\subseteq S.$$

Note that such a k exists because otherwise $|S| \ge n/2$. Let

$$\left| \left(\{v^{j,k} \colon 1 \le j \le k-1\} \cup \{u^{k+2}, \dots, u^d\} \right) \cap S^c \right| = b \ge 1.$$

Reasoning as in Case 1, where the role of S and ∂S are swapped, we conclude that ∂S contains at least a more edges and therefore (14) holds. □

4 The grevlex polytope $\mathcal{Q}$

In this section we will describe the main properties of the polytope $\mathcal{Q}$. Throughout, $\preccurlyeq$ will denote the grevlex order.

4.1 $\mathcal{V}$-polytope

Consider the following integral points:

$$\begin{aligned} \bar{u}^k &:= (\tilde{b}_{k-1}\mathbf{e}_{k-1}, \theta_k, \theta_{k+1}, \dots, \theta_d) \qquad 2 \le k \le d+1 & \text{(15a)} \\ \bar{v}^{j,k} &:= ((\tilde{b}_{k-1}-1)\mathbf{e}_j, 0, \dots, 0, \theta_k+1, \theta_{k+1}, \dots, \theta_d) \qquad 1 \le j < k-1 \le d, & \text{(15b)} \end{aligned}$$

where $\tilde{b}_k$ is given by (2a). In particular, $\bar{u}^2 = \theta$, $\bar{u}^{d+1} = b\mathbf{e}_d$, and $\bar{v}^{j,d+1} = (b-1)\mathbf{e}_j$. By construction, we have

Observation 4.1. *$\bar{u}^k \in H_0$ for all k, $\bar{v}^{j,k} \in H_0$ for $k \le d$, $\bar{v}^{j,d+1} \notin H_0$ for $1 \le j \le d-1$.*

Proposition 4.1. *Let $\theta \ge \mathbb{1}$ and $d \ge 3$. The vertices of $\mathcal{Q}$ are*

$$\operatorname{vert}(\mathcal{Q}) = \{\mathbf{0}\} \bigcup \{\bar{u}^k : 2 \le k \le d+1\} \bigcup \{\bar{v}^{j,k} : 1 \le j < k-1 \le d\}.$$

Proof. It is clear that $\mathbf{0}$ cannot be written as a nontrivial convex combination of integral points in $\mathcal{Q}$. Suppose $\bar{u}^k = \sum_{i=1}^s \lambda_i x^i$ is a nontrivial convex combination of some $x^i \in \mathcal{Q} \cap \mathbb{Z}^d$. Since H_0 is a facet of $\mathcal{Q}$, $x^i \in H_0$. Let

$$m = \max\{j : x^i_j \neq \theta_j \text{ for some } i\}.$$

Then $\theta \le_{\text{lex}} x^i$ implies $x^i_m \ge \theta_m$, leading to $\sum_{i=1}^s \lambda_i x^i_m > \theta_m$. Therefore, $m = k-1$. Also, $x^i_j = 0$ for all i and $j < k-2$. So, the only possibility is $x^i = \bar{u}^k$ for all i.

Now suppose $\bar{v}^{j,k} = \sum_{i=1}^s \lambda_i x^i$, $k \le d$ is a nontrivial convex combination of some $x^i \in \mathcal{Q} \cap \mathbb{Z}^d$. As before, we conclude $x^i \in H_0$. Also, $x^r_l = 0$ for $r \in \{1, 2, \dots, j-1, j+1, \dots, k-1\}$. Reasoning the same way as in the case of $\bar{u}^k$, we conclude that $x^i_r = \theta_r$ for $r > k$ and, therefore, $x^i_k \ge \theta_k + 1$. This in turn implies that $x^i_k = \theta_k + 1$ and thus $x^i = \bar{v}^{j,k}$ for all i. The points $\bar{v}^{j,d+1}$, being coordinate vectors, are trivially vertices.

Now we argue that if $v \in \operatorname{vert}(\mathcal{Q}) \setminus \{\mathbf{0}\}$, then v must be equal to some $\bar{u}^k$ or $\bar{v}^{j,k}$. Since

$$\mathcal{Q} = \operatorname{conv}\left(\left\{x \in \mathbb{R}^d : \sum_{i=1}^d x_i \le b-1\right\} \bigcup \operatorname{conv}\left\{x \in \mathbb{Z}^d : \sum_{i=1}^d x_i = b, \theta \le_{\text{lex}} x\right\}\right)$$

we have

$$\operatorname{vert}(\mathcal{Q}) \subseteq \operatorname{vert}\{x \in \mathbb{R}^d : \sum_i x_i \le b-1\} \cup \operatorname{vert}(\mathcal{Q} \cap H_0).$$

Note that $(b-1)\mathbf{e}_d$ is not a vertex of $\mathcal{Q}$ because $\mathbf{0}, b\mathbf{e}_d \in \mathcal{Q}$ and we have already shown that all the other vertices of the simplex $\{x \in \mathbb{R}^d : \sum_i x_i \le b-1\}$ are also vertices of $\mathcal{Q}$. Suppose now that $x \in \mathcal{Q} \cap H_0$ is a vertex of Q. Let $k = \max\{i : x_i \neq \theta_i\}$. Then $k \ge 2$ and $x_k \ge \theta_k + 1$.

- Suppose first $x_k \ge \theta_k + 2$ and there is $j < k$ such that $x_j > 0$. Let $x' = x + \mathbf{e}_j - \mathbf{e}_k$ and $x'' = x - \mathbf{e}_j + \mathbf{e}_k$. Then $x = (x' + x'')/2$ and since $x', x'' \in \mathcal{Q}$, we conclude x is not a vertex of $\mathcal{Q}$. If there is no $j < k$ such that $x_j > 0$ then $x_k = \tilde{b}_k$ and $x = \bar{u}^{k+1}$.

- Suppose now $x_k = \theta_k + 1$. If there exist $i < j < k$ such that $x_i, x_j > 0$ then $x = \left((x + \mathbf{e}_i - \mathbf{e}_j) + (x - \mathbf{e}_i + \mathbf{e}_j)\right)/2$ and, therefore, x is not a vertex of $\mathcal{Q}$. Otherwise either $x = \bar{v}^{j,k}$ for some $j < k-1$ or $x = ((\tilde{b}_{k-1}-1)\mathbf{e}_{k-1}, \theta_k + 1, \theta_{k+1}, \dots, \theta_d)$. But in the latter case

$$x = \frac{\tilde{b}_{k-1}-1}{\tilde{b}_{k-1}}(\tilde{b}_{k-1}\mathbf{e}_{k-1}, \theta_k, \theta_{k+1}, \dots, \theta_d) + \frac{1}{\tilde{b}_{k-1}}(\tilde{b}_k\mathbf{e}_k, \theta_{k+1}, \dots, \theta_d)$$

and, therefore, x is not a vertex of $\mathcal{Q}$. $\square$

As in Observation 3.2, the coordinate planes define facets of $\mathcal{Q}$. We refer to all other facets of $\mathcal{Q}$ as *nontrivial facets*. The grading plane H_0 defines a nontrivial facet since the face $\mathcal{Q} \cap H_0$ contains d affinely independent vertices $\theta = \bar{u}^2, \bar{u}^3, \ldots, \bar{u}^{d+1}$. We show that the coefficients of any nontrivial facet-defining inequality are nonnegative and nonincreasing.

Lemma 4.1. *Suppose $\mathcal{Q} \subseteq \{x : cx \leq c_0\}$ and let $F = \mathcal{Q} \cap \{x : cx = c_0\}$ be a nontrivial face of $\mathcal{Q}$. If F is not contained in $x_i = 0$ for some $i \in \{1, \ldots, d-1\}$ then $0 \leq c_{i+1} \leq c_i$. Consequently, if F is a nontrivial facet then $0 \leq c_{i+1} \leq c_i$ for all $1 \leq i \leq d-1$.*

Proof. Let $\bar{x}$ be a vertex on F with $\bar{x}_i \geq 1$. Consider first $x' = \bar{x} - \mathbf{e}_i$. Since $0 \leq x' \preccurlyeq \bar{x}$, we have $cx' \leq c_0 = c\bar{x}$, which implies $c_{i+1} \geq 0$. The point $x'' = \bar{x} - \mathbf{e}_i + \mathbf{e}_{i+1}$ has the property $0 \leq x'' \preccurlyeq \bar{x}$. Therefore, we have $cx'' \leq c_0 = c\bar{x}$, which yields $c_{i+1} \leq c_i$. $\square$

The above property is useful for characterizing the neighbors of θ.

Proposition 4.2. *Let $\theta \geq \mathbb{1}$ and $d \geq 3$. The neighbors of θ and $\mathbf{0}$ are*

$$\mathcal{N}_{\mathcal{Q}}(\theta) = \{\bar{u}^3\} \cup \{\bar{v}^{1,k} \colon 3 \leq k \leq d+1\}, \quad \mathcal{N}_{\mathcal{Q}}(\mathbf{0}) = \{\bar{u}^{d+1}\} \cup \{\bar{v}^{j,d+1} \colon 1 \leq j \leq d-1\}.$$

Proof. Let $F = \mathcal{Q} \cap \{x \colon cx = c_0\}$ be an edge of $\mathcal{Q}$ defined by $cx \leq c_0$ such that $\theta \in F$. We argue by contradiction that this edge contains exactly one of the proposed neighboring vertices in $\mathcal{N}_{\mathcal{P}}(\mathcal{Q})$, or equivalently that none of the other vertices belong to this edge. First suppose that $\bar{v}^{j,k} \in F$ for some $2 \leq j < k-1 \leq d$. Now $c\bar{v}^{j-1,k} - c\bar{v}^{j,k} = (\tilde{b}_{k-1} - 1)(c_{j-1} - c_j)$, which is nonnegative because of Lemma 4.1. Therefore $\bar{v}^{j-1,k} \in F$, a contradiction to the edge property of F. For $\bar{u}^k$ with $k \geq 4$, similar reasoning carries through by considering $\bar{u}^{k-1}$. For $\mathcal{N}_{\mathcal{Q}}(\mathbf{0})$, noting that the proposed neighbors are coordinate vectors, the proof is exactly the same as that in Proposition 3.2. $\square$

Thus $\mathcal{Q}$ has at least $2d$ facets, d of which are coordinate planes that contain $\mathbf{0}$ and the other d contain θ. We show in Theorem 4.1 that there are no other facets. The following properties about nontrivial facets will be useful.

Lemma 4.2. *Let F be a nontrivial facet of $\mathcal{Q}$ defined by $cx \leq c_0$.*

1. *$\bar{v}^{j,k} \in F$ implies $\bar{v}^{j',k} \in F$ for all $1 \leq j' \leq j$ and $c_1 = c_2 = \cdots = c_j$.*

2. *If F is not defined by H_0, then F contains some coordinate vertex, i.e., there exists some $1 \leq j \leq d-1$ such that $\bar{v}^{j,d+1} \in F$.*

3. *$\bar{u}^k \notin F$ implies $\bar{u}^{k'} \notin F$ for $k+1 \leq k' \leq d+1$.*

Proof. (1) This is because $c\bar{v}^{j,t} - c\bar{v}^{j',t} = (c_j - c_{j'})(\tilde{b}_{t-1} - 1) \leq 0$ and so $c\bar{v}^{j,t} = c_0$ implies $c\bar{v}^{j',t} = c_0$. Consequently, we also get $c_1 = c_2 = \cdots = c_j$.

(2) To argue this, recall from Observation 4.1 that $\operatorname{vert}(\mathcal{Q}) \setminus \{\mathbf{0}, \bar{v}^{1,d+1}, \ldots, \bar{v}^{d,d+1}\} \subseteq H_0$. So if $\bar{v}^{j,d+1} \notin F$ for all j, the nontriviality of F (i.e., $\mathbf{0} \notin F$) then implies that $F = \mathcal{Q} \cap H_0$.

(3) We know from Lemma 4.1 that $F = \{x \in \mathcal{Q} \colon \alpha x = \alpha_0\}$ with $0 \leq \alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_1$. Then for $2 \leq t \leq d$, we have

$$\alpha\bar{u}^t - \alpha\bar{u}^{t+1} = \alpha_{t-1}\tilde{b}_{t-1} + \alpha_t(\theta_t - \tilde{b}_t) = (\alpha_{t-1} - \alpha_t)\tilde{b}_{t-1} \geq 0.$$

Therefore, $\alpha\bar{u}^{d+1} \leq \alpha\bar{u}^d \leq \cdots \leq \alpha\bar{u}^2 = \alpha\theta$. If $\bar{u}^k \notin F$, then $\alpha\bar{u}^k < \alpha_0$ and the claim follows. $\square$

Theorem 4.1. *Let $\theta \geq \mathbb{1}$ and $d \geq 3$. $\mathcal{Q}$ is a $(\mathbf{0}, \theta)$-Dantzig figure.*

Proof. To show that $\mathcal{Q}$ is a $(\mathbf{0}, \theta)$-Dantzig figure, we need to prove that every facet of $\mathcal{Q}$ contains either $\mathbf{0}$ or θ. Let F be a facet of $\mathcal{Q}$ induced by the valid inequality $cx \le c_0$. If F doesn't contain θ nor any of the vertices $\bar{v}^{1,k}$, then it is contained in the subspace $x_1 = 0$ and hence is equal to the trivial facet defined by $x_1 \ge 0$ and therefore contains $\mathbf{0}$. For an arbitrary nontrivial facet F, Lemma 4.1 tells us

$$0 \le c_n \le c_{n-1} \le \cdots \le c_1, \tag{16a}$$

Now assume $\bar{v}^{1,k} \in F$ for some $3 \le k \le d-1$ and $\mathbf{0} \notin F$. Then it must be that F is a nontrivial facet. Suppose for contradiction that $\theta \notin F$. This means $F \neq \mathcal{Q} \cap H_0$ and also has two other implications. First, we have $c_1 > c_{k-1}$. Suppose this is not true, which by (16a) means that $c_1 = c_2 = \cdots = c_{k-1}$. Then

$$0 < c_0 - c\theta = c\bar{v}^{1,k} - c\theta = c_k + c_1(\tilde{b}_{k-1} - 1) - \sum_{j=1}^{k-1} c_1\theta_j = c_k - c_1,$$

which contradicts $c_1 \ge c_k$ from (16a). Second, we have $\bar{u}^k \notin F$ for all k due to $\theta = \bar{u}^2$, the assumption $c\theta < c_0$ and the third item in Lemma 4.2.

Now let j be the maximal index such that $\bar{v}^{j,d+1} \in F$; we know such a j exists because of $F \neq \mathcal{Q} \cap H_0$ and the second item in Lemma 4.2. If $j \ge k-1$, then applying the first item in Lemma 4.2 to $\bar{v}^{j,d+1}$ would imply $c_1 = c_{k-1}$, a contradiction to $c_1 > c_{k-1}$. Hence $1 \le j \le k-2$ and (16a) and maximality of j lead us to

$$c_1 = \cdots = c_j, \qquad c_1 > c_i, \quad i = j+1, \ldots, d. \tag{16b}$$

Using above and $c\theta < c\bar{v}^{1,k}$ by assumption, gives us

$$c_k - c_1 > \sum_{i=j+1}^{k-1} (c_i - c_1)\theta_i. \tag{16c}$$

Since F is not contained in any coordinate plane and $\bar{u}^k \notin F\ \forall k$, we know that for every $t = j+1, \ldots, k-1$, there exist some (i_t, k_t) such that $\bar{v}^{i_t,k_t} \in F$ and either $k_t \le t$ or $i_t = t$. The second possibility $i_t = t$ can be ruled out since applying the first item in Lemma 4.2 to $\bar{v}^{t,k_t}$ would imply $c_1 = c_t$, a contradiction to (16b) due to $t \ge j+1$. Therefore $1 \le i_t < k_t - 1 \le t-1$. Now since $\bar{v}^{i_t,k_t} \in F$, the first item in Lemma 4.2 implies $\bar{v}^{1,k_t} \in F$. Therefore we have $c\bar{v}^{1,k_t} = c\bar{v}^{1,k}$, which upon simplification yields $c_k - c_{k_t} = \sum_{i=k_t}^{k-1}(c_i - c_1)\theta_i$ for $j+1 \le t \le k-1$. Choosing $t = j+1$ gives us

$$c_k - c_{k_{j+1}} = \sum_{i=k_{j+1}}^{k-1} (c_i - c_1)\theta_i = \sum_{i=j+1}^{k-1} (c_i - c_1)\theta_i, \tag{16d}$$

where the second equality is due to $k_{j+1} \le j+1$ by construction, and $c_1 = \cdots = c_j$ from (16b). Since $k_{j+1} \le j+1$, (16b) tells us $c_{k_{j+1}} \le c_1$. Substituting this into (16d) leads to $c_k - c_1 \le \sum_{i=j+1}^{k-1}(c_i - c_1)\theta_i$, but this is a contradiction to (16c). $\square$

Similar to $\mathcal{P}$, for $d \ge 4$ the only antipodal vertex pair of $\mathcal{Q}$ is $(\mathbf{0}, \theta)$. We will show this in Corollary 4.4.

4.2 $\mathcal{H}$-polytope

By Proposition 2.1, we need to invert

$$\bar{M} = \begin{bmatrix} \bar{u}^3-\theta & \bar{v}^{1,3}-\theta & \bar{v}^{1,4}-\theta & \cdots & \bar{v}^{1,d}-\theta & \bar{v}^{1,d+1}-\theta \end{bmatrix}$$

$$= \begin{bmatrix} -\theta_1 & \theta_2-1 & \theta_2+\theta_3-1 & \theta_2+\theta_3+\theta_4-1 & \cdots & \theta_2+\cdots+\theta_{d-1}-1 & \theta_2+\cdots+\theta_d-1 \\ \theta_1 & -\theta_2 & -\theta_2 & -\theta_2 & \cdots & -\theta_2 & -\theta_2 \\ 0 & 1 & -\theta_3 & -\theta_3 & \cdots & -\theta_3 & -\theta_3 \\ \vdots & 0 & 1 & -\theta_4 & \cdots & -\theta_4 & -\theta_4 \\ \vdots & \vdots & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & -\theta_{d-1} & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 1 & -\theta_d \end{bmatrix}$$

Let

$$q_i^j = \begin{cases} \theta_i\theta_j\prod_{k=i+1}^{j-1}(\theta_k+1) & j>i \\ \theta_i & j=i \\ 1 & j<i. \end{cases}$$

Proposition 4.3. *Let $\theta \geq \mathbb{1}$ and $d \geq 3$. $\mathcal{Q} = \left\{x \geq \mathbf{0} : \bar{N}x \geq \bar{N}\theta\right\}$ where $\bar{N} = \bar{M}^{-1}$ with*

$$\bar{N}_{i,d} = \begin{cases} -1 & i=d \\ -\theta_d+1 & i=d-1 \\ -q_{i+1}^d & 2\leq i\leq d-2 \\ -\frac{q_2^d}{\theta_1} & i=1 \end{cases} \qquad \bar{N}_{1,j} = \bar{N}_{1,j+1} + \begin{cases} -\dfrac{1}{\theta_1} & j=1 \\ -\frac{\theta_2-1}{\theta_1} & j=2 \\ -\frac{q_2^j}{\theta_1} & 3\leq j\leq d-1 \end{cases}$$

and for $i \geq 2$

$$\bar{N}_{i,j} = \bar{N}_{i,j+1} + \begin{cases} 0 & j<i \\ -1 & j=i \\ -\theta_j+1 & j=i+1 \\ -q_{i+1}^j & i+2\leq j\leq d-1. \end{cases}$$

Proof. Suppose $i \neq 1, j \neq 1$. Then

$$\begin{aligned} \bar{N}_{i\cdot}\bar{M}_{\cdot j} &= \sum_{l=1}^{j+1}\bar{N}_{i,l}\bar{M}_{l,j+1} \\ &= \bar{N}_{i,1}(\theta_2+\cdots+\theta_j-1) - \sum_{l=2}^{j}\bar{N}_{i,l}\theta_j + \bar{N}_{i,j+1} \\ &= \sum_{l=2}^{j}(\bar{N}_{i,1}-\bar{N}_{i,l})\theta_l + (\bar{N}_{i,j+1}-\bar{N}_{i,1}) \end{aligned} \tag{17}$$

So, if $j < i$, then since $\bar{N}_{i,1} = \bar{N}_{i,2} = \cdots = \bar{N}_{i,i}$ we have $\bar{N}_{i\cdot}\bar{M}_{\cdot j} = 0$. If $j = i$, then $\bar{N}_{i\cdot}\bar{M}_{\cdot j} = \bar{N}_{i,i+1} - \bar{N}_{i,i} = 1$. For $j > i$,

$$\bar{N}_{i\cdot}\bar{M}_{\cdot j} = \bar{N}_{i\cdot}\bar{M}_{\cdot j-1} + (\bar{N}_{i,1}-\bar{N}_{i,j})\theta_j + (\bar{N}_{i,j+1}-\bar{N}_{i,j}).$$

Therefore, if $j = 1 + 1$, then

$$\bar{N}_{i\cdot}\bar{M}_{\cdot i+1} = 1 + (-1)\theta_{i+1} + (\theta_{i+1} - 1) = 0.$$

For, $j > i + 1$, inductively we get

$$\bar{N}_{i\cdot}\bar{M}_{\cdot j} = 0 + (-\sum_{k=i+1}^{j-1} q^k_{i+1})\theta_j + q^j_{i+1} = 0.$$

The entries $\bar{N}_{1\cdot}\bar{M}_{\cdot j}$ and $\bar{N}_{i\cdot}\bar{M}_{\cdot 1}$ can be computed in a similar way. $\square$

4.3 Graph of the polytope

We derive some basic properties of the graph of $\mathcal{Q}$, denoted by $G(\mathcal{Q})$. This graph has $\frac{d^2+d+2}{2}$ vertices enumerated in Proposition 4.1. To find all the edges of $G(\mathcal{Q})$, we characterize the vertex-facet incidence for $\mathcal{Q}$ in Proposition 4.4. Adopting the same approach as in §3.3 to denote H_v, for $v \in \mathcal{N}_{\mathcal{Q}}(\theta)$, as the only facet-defining hyperplane that contains θ but not v, we have

$$\begin{aligned}
&\mathcal{H}^\theta_{\mathcal{Q}} := \{H_{\bar{u}^3}, H_{\bar{v}^{1,3}}, \ldots, H_{\bar{v}^{1,d+1}}\}, \text{ with} \\
&H_{\bar{u}^3} = \{x : \bar{N}_{1\cdot}(x-\theta) = 0\}, \qquad H_{\bar{v}^{1,k}} = \{x : \bar{N}_{(k-1)\cdot}(x-\theta) = 0\} \quad 3 \le k \le d+1,
\end{aligned}$$

where N is the matrix inverse from Proposition 4.3. The hyperplanes incident to $\mathbf{0}$ are the coordinate planes denoted in (8b). As before, for any $v \in \mathrm{vert}(\mathcal{Q})$, let $\psi_{\mathcal{Q}}(v)$ denote the subset of facet-defining hyperplanes of $\mathcal{Q}$ that contain v.

Proposition 4.4. *Let $\theta \ge \mathbb{1}$ and $d \ge 3$. We have*

$$\begin{aligned}
&\psi_{\mathcal{Q}}(\mathbf{0}) = \mathcal{H}^{\mathbf{0}}, \quad \psi_{\mathcal{Q}}(\theta) = \mathcal{H}^\theta_{\mathcal{Q}}, \\
&\psi_{\mathcal{Q}}(\bar{u}^k) = (\mathcal{H}^\theta_{\mathcal{Q}} \setminus \{H_{\bar{u}^3}, H_{\bar{v}^{1,3}}, \ldots, H_{\bar{v}^{1,k-1}}\}) \cup \mathcal{H}^{\mathbf{0},k-2}, \qquad 3 \le k \le d+1 \\
&\psi_{\mathcal{Q}}(\bar{v}^{j,k}) = (\mathcal{H}^\theta_{\mathcal{Q}} \setminus \{H_{\bar{u}^3}, H_{\bar{v}^{1,3}}, \ldots, H_{\bar{v}^{1,j}}, H_{\bar{v}^{1,k}}\}) \cup (\mathcal{H}^{\mathbf{0},k-1} \setminus \{H_j\}), \qquad 2 \le j < k-1 \le d \\
&\psi_{\mathcal{Q}}(\bar{v}^{1,k}) = (\mathcal{H}^\theta_{\mathcal{Q}} \setminus \{H_{\bar{v}^{1,k}}\}) \cup (\mathcal{H}^{\mathbf{0},k-1} \setminus \{H_1\}), \qquad 3 \le k \le d+1.
\end{aligned}$$

Proof. The coordinate planes are trivial to check due to $\theta \ge \mathbb{1}$, whereas $\psi_{\mathcal{Q}}(\mathbf{0})$ and $\psi_{\mathcal{Q}}(\theta)$ follow from the construction of $\mathcal{H}^\theta_{\mathcal{Q}}$ and $\mathcal{H}^{\mathbf{0}}$. It remains to argue the incidence of a hyperplane in $\mathcal{H}^\theta_{\mathcal{Q}}$ onto a $\bar{u}^k$, for $k \ge 3$, or a $\bar{v}^{j,k}$. The formula for $\bar{N}$ in Proposition 4.3 and the monotone property of facet coefficients in Lemma 4.1 tells us that

$$-\bar{N}_{t-1,1} = \cdots = -\bar{N}_{t-1,t-1} > -\bar{N}_{t-1,t} \ge -\bar{N}_{t-1,t+1} \ge \cdots \ge -\bar{N}_{t-1,d} \quad 2 \le t \le d. \tag{18}$$

Consider $\bar{u}^k$ for $k \ge 3$. Since $\bar{u}^3 \notin H_{\bar{u}^3}$ by construction, the third item in Lemma 4.2 implies that $\bar{u}^k \notin H_{\bar{u}^3}$. For any $3 \le j \le k-1$, since $\theta \in H_{\bar{v}^{1,j}}$, we have $\bar{u}^k \notin H_{\bar{v}^{1,j}}$ if and only if $\bar{N}_{(j-1)\cdot}\bar{u}^k < \bar{N}_{(j-1)\cdot}\theta$. Now

$$\begin{aligned}
\bar{N}_{(j-1)\cdot}\bar{u}^k &= \bar{N}_{j-1,k-1}\tilde{b}_{k-1} + \sum_{i=k}^{d} \bar{N}_{j-1,i}\theta_i \\
&= \underbrace{\bar{N}_{j-1,k-1}\tilde{b}_{j-1}}_{<\bar{N}_{j-1,1}\tilde{b}_{j-1}} + \underbrace{\bar{N}_{j-1,k-1}\sum_{i=j}^{k-1}\theta_i}_{\le \sum_{i=j}^{k-1}\bar{N}_{j-1,i}\theta_i} + \sum_{i=k}^{d}\bar{N}_{j-1,i}\theta_i \\
&< \bar{N}_{(j-1)\cdot}\theta,
\end{aligned}$$

where the inequalities are due to equation (18). Therefore $\bar{u}^k \notin H_{\bar{v}^{1,j}}$ for $3 \le j \le k-1$, and so $\bar{u}^k \in H$ for some $H \in \mathcal{H}^\theta_\mathcal{Q}$ only if H is one of the $d-k+2$ hyperplanes $H_{\bar{v}^{1,k}}, H_{\bar{v}^{1,k+1}}, \dots, H_{\bar{v}^{1,d+1}}$. Since exactly $k-2$ coordinate planes contain $\bar{u}^k$ and we know that $|\psi_\mathcal{Q}(\bar{u}^k)| \ge d$ due to $\bar{u}^k$ being a vertex of the d-polytope $\mathcal{Q}$, it follows that $\bar{u}^k \in H_{\bar{v}^{1,t}}$ for $k \le t \le d+1$.

Now we derive $\psi_\mathcal{Q}(\bar{v}^{j,k})$. Note that $k \ge 3$. The construction of $\mathcal{H}^\theta_\mathcal{Q}$ and $\bar{v}^{1,k} \in \mathcal{N}_\mathcal{P}(\theta)$ implies that $\psi_\mathcal{Q}(\bar{v}^{1,k}) \cap \mathcal{H}^\theta_\mathcal{Q} = \mathcal{H}^\theta_\mathcal{Q} \setminus \{H_{\bar{v}^{1,k}}\}$. This leads to $\bar{v}^{j,k} \notin H_{\bar{v}^{1,k}}$ because otherwise the first item in Lemma 4.2 gives the contradiction $\bar{v}^{1,k} \in H_{\bar{v}^{1,k}}$. Consider the hyperplane $H_{\bar{v}^{1,t}} := \{x : \bar{N}_{(t-1)}.x = c_0\}$ for $3 \le t \le d+1, t \ne k$, which contains $\bar{v}^{1,k}$. Then $\bar{v}^{j,k} \in H_{\bar{v}^{1,t}}$ if and only if $\bar{N}_{(t-1)}.\bar{v}^{j,k} = \bar{N}_{(t-1)}.\bar{v}^{1,k}$. Now, $\bar{N}_{(t-1)}.\bar{v}^{j,k} - \bar{N}_{(t-1)}.\bar{v}^{1,k} = (\bar{N}_{t-1,j} - \bar{N}_{t-1,1})(\tilde{b}_{k-1} - 1)$ and since $\tilde{b}_{k-1} > 1$ for $k \ge 3$ due to $\theta \ge \mathbb{1}$, we have $\bar{v}^{j,k} \in H_{\bar{v}^{1,t}}$ if and only if $\bar{N}_{t-1,j} = \bar{N}_{t-1,1}$. Equation (18) tells us that $\bar{N}_{t-1,1} = \bar{N}_{t-1,j}$ if and only if $j \le t-1$, which, along with $t \ne k$, is equivalent to $t \in \{j+1, \dots, k-1, k+1, \dots, d+1\}$. The claim for $\psi_\mathcal{Q}(\bar{v}^{j,k})$ follows. The arguments for $\bar{v}^{j,k} \notin H_{\bar{u}^3}$, for $j \ge 2$, are similar. □

Since (v, v') is an edge in $G(\mathcal{Q})$ if and only if $|\psi_\mathcal{Q}(v) \cap \psi_\mathcal{Q}(v')| \ge d-1$, Proposition 4.4 implies a complete list of edges (Figure 3) and thereby the degree of each vertex.

Corollary 4.1. *Let $\theta \ge \mathbb{1}$ and $d \ge 3$. $G(\mathcal{Q})$ has $\frac{1}{2}(d^2+d+2)$ vertices, the edges between which are as follows:*

1. $(\mathbf{0}, \bar{u}^{d+1})$ *and* $(\mathbf{0}, \bar{v}^{j,d+1})$ *for* $1 \le j \le d-1$,
2. $(\bar{u}^k, \bar{u}^{k+1})$ *for* $2 \le k \le d$,
3. $(\bar{u}^k, \bar{v}^{j,k-1})$ *for* $4 \le k \le d+1$, $1 \le j \le k-3$,
4. $(\bar{u}^j, \bar{v}^{j-1,k})$ *for* $2 \le j \le d$, $j+1 \le k \le d+1$,
5. $(\bar{v}^{j,k_1}, \bar{v}^{j,k_2})$ *for* $3 \le j+2 \le k_1 < k_2 \le d+1$,
6. $(\bar{v}^{j_1,k}, \bar{v}^{j_2,k})$ *for* $1 \le j_1 < j_2 \le k-2 \le d-1$.

Proof. Based on Proposition 4.4, one can check that $|\psi_\mathcal{Q}(v) \cap \psi_\mathcal{Q}(v')| \ge d-1$ only for the pairs of vertices (v, v') given in items 1-6 as well as the pairs $(\bar{v}^{j_1,k_1}, \bar{v}^{j_2,k_2})$ in which ($j_1 < j_2 < k_1 < k_2$ and $k_1 \ge j_2+3$) or ($j_2 < j_1 < k_1 < k_2$ and $k_1 \ge j_1+3$). Therefore, all the other pairs are not edges. For the pairs (v, v') from 1-6, it is straightforward to verify that no other vertex lies on the common facets for v and v', thereby showing that these are edges. However, in the last case, when $j_1 < j_2 < k_1 < k_2$ and $k_1 \ge j_2+3$, we have also $\psi_\mathcal{Q}(\bar{v}^{j_1,k_1}) \cap \psi_\mathcal{Q}(\bar{v}^{j_2,k_2}) \subseteq \psi_\mathcal{Q}(\bar{v}^{j_2,k_1})$. Similarly, when $j_2 < j_1 < k_1 < k_2$ and $k_1 \ge j_1+3$, we have $\psi_\mathcal{Q}(\bar{v}^{j_1,k_1}) \cap \psi_\mathcal{Q}(\bar{v}^{j_2,k_2}) \subseteq \psi_\mathcal{Q}(\bar{v}^{j_1,k_2})$. Therefore, none of these pairs determines an edge of $\mathcal{Q}$. □

Corollary 4.2. *Let $\theta \ge \mathbb{1}$ and $d \ge 3$. The degrees of the vertices of $G(\mathcal{Q})$ are*

$$\deg(\mathbf{0}) = \deg(\bar{u}^k) = d, \quad \deg(\bar{v}^{j,k}) = d+k-j-2.$$

The total number of edges is $\frac{1}{3}(d^3+2d)$ and the average degree is $\frac{2}{3}(d-1+\frac{d+2}{d^2+d+2})$.

As a consequence we see that $\mathcal{P}$ and $\mathcal{Q}$ define two different families of Dantzig figures.

Corollary 4.3. *Let $d \ge 3$ be fixed.*

1. *For any $\mathcal{P}$ and $\mathcal{P}'$ corresponding to $\theta, \theta' > \mathbb{1}$, we have $\mathcal{P} \cong \mathcal{P}'$.*
2. *For any $\mathcal{Q}$ and $\mathcal{Q}'$ corresponding to $\theta, \theta' \ge \mathbb{1}$, we have $\mathcal{Q} \cong \mathcal{Q}'$.*
3. *For $\theta > \mathbb{1}$, $\mathcal{P} \ncong \mathcal{Q}$.*

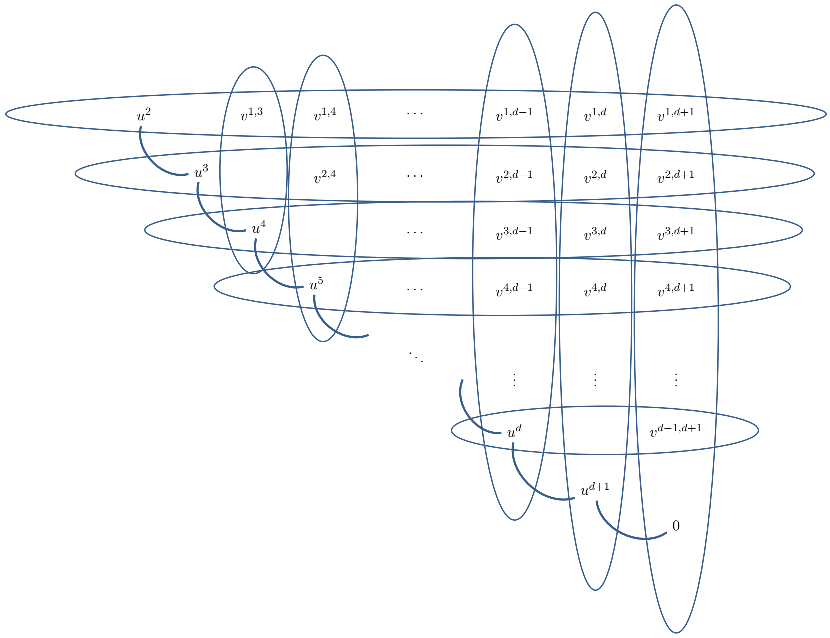


Figure 3: The graph $G(\mathcal{Q})$ for a vertex $\theta \geq \mathbb{1}$ in $\mathbb{R}^d$. The circled vertices form cliques.

Proof. The first two claims follow from Propositions 3.4 and 4.4. The fact that $\mathcal{P}$ and $\mathcal{Q}$ are not combinatorially equivalent is also not hard to see from the properties we have proved so far. For $d = 3$, as can be seen from Figure 1, $G(\mathcal{P})$ has a pentagonal facet, while $G(\mathcal{Q})$ doesn't. For $d \geq 4$, the highest degree vertex in $\mathcal{P}$ is w with $\deg(w) = \frac{d^2-d+2}{2}$, while in $\mathcal{Q}$, the highest degree vertex is $\bar{v}^{1,d+1}$ with $\deg(\bar{v}^{1,d+1}) = 2d - 2 < \frac{d^2-d+2}{2}$. $\square$

In $d = 3$, $\mathcal{Q}$ is a $(\bar{v}^{1,3}, \bar{v}^{2,4})$-Dantzig figure. But, for $d \geq 4$ $(\mathbf{0}, \theta)$ are the only antipodal vertices of $\mathcal{Q}$.

Corollary 4.4. *Let $\theta \geq \mathbb{1}$. For $d \geq 4$, $(\mathbf{0}, \theta)$ is the only antipodal vertex pair that generates the Dantzig figure $\mathcal{Q}$. For $d = 3$, $(\bar{v}^{1,3}, \bar{v}^{2,4})$ is the other antipodal vertex pair.*

Proof. The case $d = 3$ can easily be analyzed from Figure 1. Let $d \geq 4$. Since any antipodal vertex pair (x, y) of a d-dimensional Dantzig figure must have $\deg(x) = \deg(y) = d$, Corollary 4.2 tells us that the only candidate vertices for forming an antipodal pair of $\mathcal{Q}$ are $\mathbf{0}, \theta, \{\bar{u}^k \colon 3 \leq k \leq d+1\}, \{\bar{v}^{j,j+2} \colon 1 \leq j \leq d-1\}$. We also need $\psi_{\mathcal{Q}}(x) \cap \psi_{\mathcal{Q}}(y) = \emptyset$ for any antipodal pair. Proposition 4.4 gives us $H_1 \in \psi_{\mathcal{Q}}(\bar{u}^k) \cap \psi_{\mathcal{Q}}(\bar{v}^{j,j+2})$ for $k \geq 3, j \geq 2$, $\psi_{\mathcal{Q}}(\bar{u}^k) \cap \mathcal{H}^{\theta}_{\mathcal{Q}} \neq \emptyset$ for $k \geq 3$, and $\psi_{\mathcal{Q}}(\bar{v}^{j,j+2}) \cap \mathcal{H}^{\theta}_{\mathcal{Q}} \neq \emptyset$ for $j \geq 1$. The only remaining possibility is $\bar{v}^{1,3}$ but this is also easy to discard with similar arguments. $\square$

Corollary 4.5. *Let $\theta \geq \mathbb{1}$ and $d \geq 3$. The graph of $\mathcal{Q}$ has the following properties.*

(a) The radius of $G(\mathcal{Q})$ is $r(G(\mathcal{Q})) = 2$.

(b) The diameter of $G(\mathcal{Q})$ is $d(G(\mathcal{Q})) = 2$.

(c) $G(\mathcal{Q})$ is Hamiltonian.

(d) The chromatic number of $G(\mathcal{Q})$ is $\chi(G(\mathcal{Q})) = d$.

Proof. Due to the grid-like structure of $G(\mathcal{Q})$ illustrated in Figure 3, one can easily see that the distance between any two vertices is at most 2. Therefore, $r(G(\mathcal{Q})) = d(G(\mathcal{Q})) = 2$. This also allows for an easy construction of a Hamiltonian cycle, for example, one can start with $0 - u^{d+1} - u^d - \cdots - u^2$ and then start traversing the cliques depicted with vertical ellipses in Figure 3 from left to right, before coming back to 0. Finally, $\chi(G(\mathcal{Q})) = d$ because $G(\mathcal{Q})$ has a d-clique and one possible proper coloring with d colors is given by: $\varphi(0) = 1$, $\varphi(\bar{v}^{j,k}) = k + j(\text{mod } d)$, $1 \leq j \leq k - 2 \leq d - 1$, $\varphi(\bar{u}^k) = 2k - 1(\text{mod } d)$, $2 \leq k \leq d$, $\varphi(\bar{u}^{d+1}) = 0$. $\square$

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