The Robust Uncapacitated Lot Sizing Model with Uncertainty Range

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Abstract

We study robust versions of the uncapacitated lot sizing problem, where the demand is subject to uncertainty. The robust models are guided by three parameters, namely, the total scaled uncertainty budget, the minimum number of periods in which one would like the demand to be protected against uncertainty, and the minimum scaled protection level per period. We solve the proposed models in polynomial time and give numerical results to show their effectiveness. Under various problem scenarios, we show that the models provide a good trade-off between the robustness of the solution, i.e., to what extend the solution is feasible, and the quality of the objective value. In addition, we show that in many scenarios it is sufficient to apply a special case of the proposed model in which only the uncertainty budget is needed.

Keywords: inventory, robustness and sensitivity analysis, lot sizing, uncertainty budget

1. Introduction

Data uncertainty is inherent to many optimization problems. In recent years a body of literature has been developed under the name of robust optimization, in which one optimizes against the worst-case instances that might arise by using min-max type objective functions. Soyster [24] proposed the so-called absolute robust model with interval uncertainty, i.e., for each uncertain parameter an upper bound and a lower bound are applied. Very often, there is not necessarily a solid statistical basis to back up the choice of a particular distribution of the uncertain parameter, although such knowledge may be of use to obtain reasonable uncertainty sets [6]. Hence interval based uncertainty sets are widely used in the robust optimization literature, see in [7] and [15] for an overview. By construction, in a minimiza-

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tion problem Soyster's absolute robust model is equivalent to a deterministic model with the uncertain parameters attaining the upper bound of the uncertainty interval. Consequently, the absolute robust solution is feasible in all realizations of uncertain data, at the expense of high objective values. This issue of over-conservatism hindered the adoption of robust optimization techniques until the mid-1990s, when Ben-Tal et al. [4, 5, 6], El Ghaoui and Lebret [11], El Ghaoui et al. [12] and Bertsimas and Sim [8] started investigating models with parametric uncertainty sets. In Ben-Tal and Nemirovski [4, 5, 6] and El Ghaoui et al. [11, 12], the authors allow the uncertainty sets for the data to be ellipsoids, and propose efficient algorithm to solve convex optimization problems. The resulting robust formulations involve conic quadratic problems. In Bertsimas and Sim [8], the authors define a family of polyhedral uncertainty sets that encode a budget of uncertainty in terms of cardinality constraints: the number of parameters of the problem that are allowed to vary from their nominal values. They show that the robust formulation of a linear programming problem is still linear. For a review of the recent development in robust optimization, we refer the reader to [13].

In this paper, we study an n-period lot sizing model and suitable robust models to deal with demand uncertainty. We start by describing the deterministic version in which there is no uncertainty. We let the demand in period i be $d_i \geq 0$ for $i = 1, \ldots, n$. For each period i, there is a setup cost f_i if production takes place, and a unit production cost $p_i > 0$. If inventory is carried to the next period, there is an inventory holding cost h_i per unit. With the aim of minimizing the total setup cost and the variable production and inventory holding costs, the deterministic Uncapacitated Lot Sizing model can be formulated as follows [20],

$$
\begin{aligned}
ULS: \quad & \min_{\mathbf{y},\mathbf{x},\mathbf{I}} \sum_{i=1}^{n} (f_i y_i + p_i x_i + h_i I_i) && (1)\\
\text{s.t.} \quad & I_{i-1} + x_i = I_i + d_i,\ \forall i = 1, \ldots, n && (2)\\
& x_i \leq M\, y_i,\ \forall i = 1, \ldots, n && (3)\\
& y_i \in \{0, 1\},\ \forall i = 1, \ldots, n && (4)\\
& x_i, I_i \geq 0,\ \forall i = 1, \ldots, n, && (5)
\end{aligned}
$$

where y_i is equal to 1 if there is production in period i and 0 otherwise, x_i is the amount produced in period i, I_i is the inventory level at the end of period i, and M is a large constant. Throughout this paper, we use bold characters for vectors, i.e., $\mathbf{y} = (y_i)$ and similarly for others. As it is common in the literature, a period k with zero incoming inventory level,

$I_{k-1} = 0$, is called a regeneration period. Then ULS problem can be solved efficiently by decomposing the planning horizon into a sequence of regeneration intervals $[k, t]$ defined by regeneration periods k and $t + 1$, [19]. To calculate the cost of each subplan, Wagner and Whitin [25] proposed the Zero Inventory Ordering (ZIO) property. The ZIO property implies that production only takes place at the first period of the regeneration interval and covers the total demand of the whole interval, i.e., $I_{i-1}x_i = 0$. The authors show that the ULS problem can be solved in $\mathcal{O}(n^2)$ time. For convenience, we define the notation $d_{ij} = \sum_{k=i}^{j} d_k$, $h_{ij} = \sum_{k=i}^{j} h_k$, and $\alpha_{ij} = p_i + h_{ij}$ for $i \leq j$. If $i > j$, we assume $d_{ij} = 0 = h_{ij} = \alpha_{ij}$. We also denote by m_i the production period for demand in period i. Then $\alpha_{m_i,i}$ is the unit cost of delivering the demand in period i from its production period m_i.

Realizing the uncertain nature of input data, we study a robust version of the lot sizing problem with demand uncertainty. We propose a robust model which adopts the idea of budget of uncertainty. Theoretical studies on robust optimization with uncertainty budget include, but not limited to, Ben-Tal and Nemirovski [4, 5, 6], Bertsimas and Sim [8], El Ghaoui and Lebret [11] and El Ghaoui et al. [12], and it has wide applications in, e.g., portfolio optimization [14, 17], production planning [2, 10], network design [3, 18], and knapsack problems [16]. In this paper, we model d_i for $i \in 1, \ldots, n$ as an uncertain parameter that takes values in the symmetric and bounded interval $[\tilde{d}_i - \hat{d}_i, \tilde{d}_i + \hat{d}_i]$, where the nominal value $\tilde{d}_i$ denotes the mean demand and $\hat{d}_i$ the deviation from the nominal demand. We impose a scaled budget of demand uncertainty Γ^d that takes values in the interval $[0, n]$, such that the total scaled uncertainty level, $\sum_1^n \frac{d_i - \tilde{d}_i}{\hat{d}_i}$, is bounded from above by Γ^d. Studies of robust lot sizing problems are available in the literature. In [26], Zhang studies a minimax regret version of the robust lot sizing problem. Moreover, in [1, 9, 23], authors consider robust lot sizing models with uncertain demand budget. With backlogging, Bertsimas and Thiele [9] examine the relative performance of the lot sizing model with budget of uncertainty compared to the stochastic policy obtained by dynamic programming. Alem and Morabito [1] apply the uncertainty budget technique on lot sizing problems with capacity and storage constraints, overtime and backlogging in the Brazil furniture industry. In Solyah et al. [23], the authors use the facility location reformulation of the lot sizing model with backlogging to formulate robust versions of the problem with various robust optimization methodologies. With demand uncertainty, it is shown that the new robust formulation is polynomially solvable when the initial inventory is zero or negative.

Existing robust lot sizing models rewrite the feasible region of the lot sizing model to account for possible deviations of the demand, thus focusing on the feasibility level of the problem instances. We instead modify both the objective function as well as the feasible region. Using a min-max objective function, the Uncapacitated Lot Sizing Model with Uncertainty Budget (ULSUB) model aims to be protected against demand uncertainty where this harms the most the objective value. Restricted by the budget of demand uncertainty Γ^d, the demand is only allowed to fluctuate in a selected group of periods. In the rest of this study, we refer to the periods in which demand is allowed to fluctuate the worst-case periods.

The ULSUB model can be understood as a "reasonable" worst-case approach. It chooses to allocate worst-case periods in places that lead to the largest increase in the objective value. In worst-case periods, we constrain the level of demand deviation from the nominal value is at least β, thus making the protection meaningful. Then, by adjusting Γ^d and β, one can control the trade-off between robustness, i.e., to what extend the solution is feasible, and quality of the objective value, and therefore arrive at a less conservative solution.

Ideally, we would like to consider a small amount of worst-case periods to ensure a good quality objective value, and at the same time reach a satisfactory robustness level. Nevertheless, we show in the paper later that this may not always be the case. Given an uncertainty budget Γ^d, we introduce an additional parameter $\Theta \in \mathbb{Z}$ that restricts the minimum number of periods in which demand is protected against uncertainty. In those periods, we again constrain the level of demand deviation from the nominal value is at least β. We show that in this way, the ULS model with Uncertainty Range (ULSUR) has the chance to improve further the trade-off between robustness and the quality of the objective value.

In this paper, after showing that both the ULSUR and the ULSUB models are polynomially solvable, we provide comprehensive computational experiments on the factors that affect the performance of the proposed models. By benchmarking with the nominal model, a deterministic ULS model in which the demands are equal to the nominal value, and Soyster's absolute robust model, we investigate how cost and demand parameter settings affect the performance of the proposed models. In addition, we give insights on when controlling the uncertainty budget is sufficient to obtain satisfactory results. The rest of the paper is organized as follows. We formulate the proposed models in Section 2. To solve the problem, we derive theoretical properties in Section 3 and propose a dynamic programming solution approach in Section 4. In Section 5, we present the computational experiment where we benchmark the ULSUB and the ULSUR models against the nominal and Soyster's models in terms of feasibility and objective value. In Section 6, we give a summary of the paper and discuss directions for future research.

2. Formulations

In the following, we propose the ULSUR model. To begin with, we formally introduce some additional notations that will be used throughout this section.

- Γ^d: the overall scaled uncertainty budget for demand, i.e., we enforce that $\sum_{i=1}^{n} \frac{d_i - \tilde{d}_i}{\hat{d}_i} \leq \Gamma^d$;
- Θ: the minimum number of periods we would like the model to take into account the possibility that the demand may deviate from the nominal values and provide protection against the uncertainty;
- $\mathbf{z}$: $z_i = 1$ if i is a worst-case period, and 0 otherwise;
- $\mathscr{Z}$: the set of worst-case periods, i.e., the set of i in which $z_i = 1$;
- $\mathbf{w}$: the scaled deviation level in each period, $w_i = \frac{d_i - \tilde{d}_i}{\hat{d}_i}$;
- β: the minimum scaled deviation level in each period, i.e., $w_i \geq \beta$.

The uncertainty budget Γ^d controls the "depth" of data uncertainty, i.e., to what extent the total data uncertainty needs to be considered in order to achieve a solution that is robust and gives a less conservative objective value than Soyster's model. By introducing Θ, one also has the control over the "width" of data uncertainty, i.e., to what extent the Γ^d degree of total data uncertainty should be spread out over the planning horizon. Note that as the ULS is a cost minimization problem, in worst-case periods the demand will go above the nominal value and consequently the total cost rises. In this period, to make the protection against uncertainty meaningful, we constrain that the scaled deviation of demand, $\frac{d_i - \tilde{d}_i}{\tilde{d}_i}$, is at least $\beta \in (0, 1)$.

Then the ULSUR model can be formulated as follows,

$$(ULSUR) \qquad \min_{\mathbf{y}} \max_{\mathbf{z},\mathbf{w}} \sum_{i=1}^{n} [f_i y_i + \min_{\mathbf{x},\mathbf{I}} (p_i x_i + h_i I_i)] \tag{6}$$

$$\text{s.t.} \qquad I_{i-1} + x_i = I_i + \tilde{d}_i + w_i \hat{d}_i, \; \forall i = 1, \ldots, n \tag{7}$$

$$x_i \leq M\, y_i,\ \forall i = 1, \ldots, n \tag{8}$$

$$\sum_{i=1}^{n} z_i \geq \Theta \tag{9}$$

$$\sum_{i=1}^{n} w_i \leq \Gamma^d \tag{10}$$

$$\beta z_i \leq w_i \leq z_i,\ \forall i = 1, \ldots, n \tag{11}$$

$$y_i \in \{0, 1\},\ \forall i = 1, \ldots, n \tag{12}$$

$$x_i, I_i \geq 0,\ \forall i = 1, \ldots, n \tag{13}$$

$$z_i \in \{0, 1\},\ \forall i = 1, \ldots, n, \tag{14}$$

where $z_i = 1$ if i is a worst-case period, and 0 otherwise, and w_i is the deviation level. We denote by $\mathcal{Z}$ the set of worst-case periods. Constraints (7) are similar to their counterpart in the ULS formulation, where we take into account the demand deviation. Constraint (9) imposes the minimum number of periods to be protected, whereas constraint (10) imposes an upper bound of the total deviation level. Constraints (11) ensure that in the worst-case periods the deviation levels are at least β, and in the remaining periods the deviation levels are equal to zero. Constraints (14) make sure that $\mathbf{z}$ is well defined. The remaining constraints, namely (8), (12) and (13), are as in the ULS formulation.

The problem under consideration is a two-stage one. For more details about two-stage robust optimization models we refer the readers to [15]. The decision $\mathbf{y}$ is made at the the first stage before we know the demand realization. The production and the inventory decisions ($\mathbf{x}$, $\mathbf{I}$) are made at the second stage, based on the realization of demand. Given $\mathbf{y}$ and $\mathbf{d}$ ($\mathbf{z}$ and $\mathbf{w}$), the ZIO property implies that the optimal production quantity and the inventory level are uniquely determined. Hence the objective function of the ULSUR model has the following equivalent form,

$$\min_{\mathbf{y}} \max_{\mathbf{z},\mathbf{w}} \sum_{i=1}^{n} [f_i y_i + p_i x_i(\mathbf{y}, \mathbf{z}, \mathbf{w}) + h_i I_i(\mathbf{y}, \mathbf{z}, \mathbf{w})].$$

In the rest of the paper and where convenient, we drop out the dependence on $\mathbf{y}$, $\mathbf{z}$, and $\mathbf{w}$. Moreover, we use the decision vector $(\mathbf{z}^\star, \mathbf{w}^\star)$ to denote the optimal allocation of uncertainty periods and uncertainty budget repsectively for the ULSUR model, and $\mathcal{Z}^\star$ the corresponding set of worst-case periods.

Below, we propose a relaxation of the ULSUR model, the ULS model with Uncertainty Budget (ULSUB). Unlike the ULSUR model which controls both the "width" of the demand uncertainty via the parameter Θ and the "depth" via the parameter Γ^d, the ULSUB model concentrates on the depth by imposing the uncertainty budget only. Then the two-stage ULSUB model is formulated as follows,

$$(ULSUB) \qquad \min_{\mathbf{y}} \max_{\mathbf{z},\mathbf{w}} \sum_{i=1}^{n} [f_i y_i + \min_{\mathbf{x},\mathbf{I}} (p_i x_i + h_i I_i)] \tag{15}$$

$$\text{s.t.} \qquad I_{i-1} + x_i = I_i + \tilde{d}_i + w_i \hat{d}_i, \ \forall i = 1, \dots, n \tag{16}$$

$$x_i \le M\, y_i, \ \forall i = 1, \dots, n \tag{17}$$

$$\sum_{i=1}^{n} w_i \le \Gamma^d \tag{18}$$

$$\beta z_i \le w_i \le z_i, \ \forall i = 1, \dots, n \tag{19}$$

$$y_i \in \{0, 1\}, \ \forall i = 1, \dots, n \tag{20}$$

$$x_i, I_i \ge 0, \ \forall i = 1, \dots, n. \tag{21}$$

The formulation resembles the one of the ULSUR model in (6)-(14). As the ULSUB model no longer models the "width" of demand uncertainty, constraint (9) is excluded.

Please note that in the ULSUR and the ULSUB model, the worst-case periods are dependent on the setup decision and hence they are considered as a combination of the cost of production and the size of the deviation of demand. This distinguishes our model from other robust models with an uncertainty budget. For instance, a natural way to protect against demand uncertainty in lot sizing problem would be producing more in earlier periods, i.e., setting $z_i = 1$ for $i = 1, \dots, \Gamma^d$, $\Gamma^d \in \mathbb{Z}$. It is clear that this allocation of Γ^d is a feasible solution to our model, but it may not be the optimal one.

It is obvious that the ULSUB model is a speacial case of the ULSUR model when $\Theta = 0$. In the following we show that the ULSUR model is equivalent to the ULSUB model when $\Theta \le \Gamma^d$. In order to show that, we need the following technical result.

Lemma 2.1. *For a given setup decision vector* $\mathbf{y}$*, there exists an optimal* $(\mathbf{z}^\star, \mathbf{w}^\star)$ *of the ULSUB model such that* $\lfloor \Gamma^d \rfloor \le |\mathcal{Z}^\star| \le \lceil \Gamma^d \rceil$.

Proof. We will first show that $\lfloor \Gamma^d \rfloor \le |\mathcal{Z}^\star|$. We will prove the statement by contradiction.

Suppose that $|\mathcal{Z}^\star| < \lfloor \Gamma^d \rfloor$. We will show that we can construct an alternative $(\mathbf{z}', \mathbf{w}')$ to $(\mathbf{z}^\star, \mathbf{w}^\star)$, which leads to an increased objective value for $\mathbf{y}$, and therefore we arrive at a contradiction that $(\mathbf{z}^\star, \mathbf{w}^\star)$ is optimal. We let $\mathcal{Z}' = \mathcal{Z}^\star \cup \mathcal{Z}^{new}$, where $\mathcal{Z}^{new} \subset \{1, \dots, n\} \setminus \mathcal{Z}^\star$ and $|\mathcal{Z}^{new}| = \lfloor \Gamma^d \rfloor - |\mathcal{Z}^\star|$. Hence we have $|\mathcal{Z}'| = \lfloor \Gamma^d \rfloor$. We let $\mathbf{w}' = \mathbf{z}'$. Then, by construction,

$$\begin{aligned}
&\sum_{i=1}^n z'_i = \lfloor \Gamma^d \rfloor \\
&\sum_{i=1}^n w'_i = \lfloor \Gamma^d \rfloor \le \Gamma^d \\
&\beta z'_i \le w'_i \le z'_i, \; \forall i = 1, \dots, n.
\end{aligned}$$

Therefore, $(\mathbf{z}', \mathbf{w}')$ is a feasible solution to the ULSUB model. Then for $\mathbf{y}$,

$$w'_i = z'_i \ge z^\star_i \ge w^*_i,$$

where this inequality is strict for $i \in \mathcal{Z}' \setminus \mathcal{Z}^\star$, which is not empty. Thus, $(\mathbf{z}', \mathbf{w}')$ leads to a higher production quantity and therefore to a higher objective value. We arrive at a contradiction with the fact that $(\mathbf{z}^\star, \mathbf{w}^\star)$ is optimal.

We now show that $|\mathcal{Z}^\star| \le \lceil \Gamma^d \rceil$. Again, we prove the statement by contradiction. Given a setup decision vector $\mathbf{y}$, the production periods for demand in period i, m_i, are known for all $i = 1, \dots, n$, and hence $\alpha_{m_k,i}\hat{d}_i$ are known. By ordering $\alpha_{m_i,i}\hat{d}_i$ in decreasing order for $i \in \mathcal{Z}^\star$, we define $\mathcal{Z}' = \mathcal{Z}^\star \setminus \{k\}$, where $k \in \arg\min\{\alpha_{m_i,i}\hat{d}_i \,:\, i \in \mathcal{Z}^\star\}$. We construct the corresponding $\mathbf{w}'$, such that

$$\begin{aligned}
&w'_i = 0, \; \forall i \notin \mathcal{Z}' \\
&w'_i \ge w^*_i, \; \forall i \in \mathcal{Z}' \\
&\sum_{i=1}^n w'_i = \sum_{i=1}^n w^\star_i,
\end{aligned} \tag{22}$$

where (22) is possible since $|\mathcal{Z}'| \ge \lceil \Gamma^d \rceil$. Moreover,

$$\sum_{i \in \mathcal{Z}^*} \alpha_{m_i,i} w^*_i \hat{d}_i = \sum_{i \in \mathcal{Z}'} \alpha_{m_i,i} w^*_i \hat{d}_i + \alpha_{m_k,k} w^*_k \hat{d}_k \le \sum_{i \in \mathcal{Z}'} \alpha_{m_i,i} w'_i \hat{d}_i,$$

i.e., $(\mathbf{z}', \mathbf{w}')$ leads to a greater or equal total cost for $\mathbf{y}$. If the cost is greater, we arrive at a contradiction that $(\mathbf{z}^\star, \mathbf{w}^\star)$ is optimal. Otherwise, if the cost is equal, we can choose

$(\mathbf{z}', \mathbf{w}')$ over $(\mathbf{z}^\star, \mathbf{w}^\star)$. By repeating the same process until $|\mathbf{z}'| = \lceil \Gamma^d \rceil$ we arrive at the desired result. □

Proposition 2.2. *If $\Theta \leq \lfloor \Gamma^d \rfloor$, the ULSUR model is equivalent to the ULSUB model.*

Proof. Recall that the ULSUB model is equal to the ULSUR model, where constraint (9) has been removed. Therefore, to show the equivalence, it is enough to show that if $\Theta \leq \lfloor \Gamma^d \rfloor$, constraint (9) is satisfied for optimal solution of the ULSUB model. This easily follows from Lemma 2.1. □

In the rest of the paper, we assume that $\Theta > \lfloor \Gamma^d \rfloor$ when discussing the ULSUR model. The ULSUB model can reduce the total cost while maintaining high feasibility. We will illustrate this with the following 4-period example. We assume that inventory holding cost is zero, and let $\mathbf{f} = (f, f, f, f)$ and $\mathbf{p} = (p, p+\epsilon, p+\epsilon, p+\epsilon)$ for some $\epsilon > 0$. We assume that the uncertain demand d_i is a uniform random variable in $[\tilde{d} - \hat{d}, \tilde{d} + \hat{d}]$. For convenience, we let $\Gamma^d \in \mathbb{Z}$, i.e., $\Gamma^d = 0, \ldots, 4$. As $p_i + h_i = p_i \leq p_{i+1}$ for $i = 1, 2, 3$, and there is a setup cost f, the optimal solution of the ULSUB model considers the entire planning horizon as a single regeneration interval. It is easy to see that α_{1i} is the unit contribution to the total cost of demand. Hence, with given values for Γ^d and β, and because the uncertain demand intervals are time-invariant, the ULSUB allocates the worst-case periods according to the size of α_{1i} for $i = 1, \ldots, 4$, i.e., the Γ^d periods with the highest $\alpha_{1,i}$ are assigned as worst-case periods with demand attaining upper bound.

In the following, we list an optimal worst-case period allocation for each value of Γ^d:

- ULSUB model with $\Gamma^d = 0$, which is equivalent to the nominal model, would lead to production in period 1, with $d_i = \tilde{d}$ for $i = 1, \ldots, 4$;
- ULSUB model with $\Gamma^d = 1$ would lead to production in period 1, with $d_4 = \tilde{d} + \hat{d}$ while $d_i = \tilde{d}$ for $i = 1, 2, 3$;
- ULSUB model with $\Gamma^d = 2$ would lead to production in period 1, with $d_4 = d_3 = \tilde{d} + \hat{d}$ while $d_i = \tilde{d}$ for $i = 1, 2$;
- ULSUB model with $\Gamma^d = 3$ would lead to production in period 1, with $d_2 = d_3 = d_4 = \tilde{d} + \hat{d}$ while $d_1 = \tilde{d}$;
- ULSUB model with $\Gamma^d = 4$, which is equivalent to Soyster's worst-case model, would lead to production in period 1, with $d_1 = d_2 = d_3 = d_4 = \tilde{d} + \hat{d}$.

Although the solution of the nominal model gives the least total cost, the probability of it being feasible is only 0.5. On the other hand, the Soyster's worst-case model guarantees the solution is feasible for all realizations of demand, but the model also leads to the highest total cost. The ULSUB model provides a trade-off between feasibility and total cost. With $\Gamma^d = 2$, the total demand being produced is $4\tilde{d} + 2\hat{d}$. As there is only one production period, using a linear transformation of the Irwin-Hall distribution, which models the sum of i.i.d. uniform random variables, we know that the probability of realized total demand being above $4\tilde{d} + 2\hat{d}$ is less than 5%. That is to say the solution of the ULSUB model with $\Gamma^d = 2$ is feasible for more than 95% of the data realizations, whilst reducing the total cost by $2p\hat{d}$ comparing with the Soyster's worst-case model. Letting $\Gamma^d = 3$ leads to an even greater level of feasibility and a $p\hat{d}$ reduction in total cost. Hence in this case, the ULSUB model manages to reduce the total cost while maintaining high feasibility.

However, the ULSUB model could fail to perform in some situations. Consider again this 4-period with similar data settings. This time we assume that there is no setup cost and $h_i = \epsilon + \eta$ for $i = 1, \ldots, 4$, where η is positive. As $p_i + h_i = p + \epsilon + \eta > p_{i+1}$ for $i = 1, 2, 3$, and there is no setup cost, the optimal solution of the ULSUB model considers every period a regeneration interval of its own. In the following, we list an optimal worst-case period allocation for each value of Γ^d:

- ULSUB model with $\Gamma^d = 0$ would lead to production in every period, with $d_i = \tilde{d}$ for $i = 1, \ldots, 4$;
- ULSUB model with $\Gamma^d = 1$ would lead to production in every period, with $d_2 = \tilde{d} + \hat{d}$ while $d_i = \tilde{d}$ for $i = 1, 3, 4$;
- ULSUB model with $\Gamma^d = 2$ would lead to production in every period, with $d_2 = d_3 = \tilde{d} + \hat{d}$ while $d_i = \tilde{d}$ for $i = 1, 4$;
- ULSUB model with $\Gamma^d = 3$ would lead to production in every period, with $d_2 = d_3 = d_4 = \tilde{d} + \hat{d}$ while $d_1 = \tilde{d}$;
- ULSUB model with $\Gamma^d = 4$, which is equivalent to Soyster's worst-case model, would lead to production in every period, with $d_1 = d_2 = d_3 = d_4 = \tilde{d} + \hat{d}$.

In this example, the first regeneration interval (the first period) is not a worst-case period if $\Gamma^d < 4$, as assigning d_1 to upper bound leads to the smallest increase in total cost compared to other periods. However, with $\Gamma^d < 4$, the solutions are infeasible for any demand realization with $d_1 > \tilde{d}$, which could happen with probability 0.5. As d_1 Hence the probability of the solutions of the ULSUB model being feasible in this example is below 0.5. In Section 5, we will show that in similar situations, the ULSUR model provides improved results over the ULSUB model.

3. Properties

In the following, Lemmas 3.1-3.2 are the core in deriving a solution approach for the ULSUR model, and Lemmas 3.3-3.4 are for the ULSUB model.

Lemma 3.1. *Given a setup decision vector* $\mathbf{y}$*, there exists an optimal solution* $(\mathbf{z}^\star, \mathbf{w}^\star)$ *of the ULSUR model, such that* $\sum_{i=1}^n z_i^* = \Theta$ *and* $\sum_{i=1}^n w_i^* = \Gamma^d$*.*

Proof. We will first prove that $\sum_{i=1}^n z_i^* = \Theta$. We prove the statement by contradiction. Suppose that $\sum_{i=1}^n z_i^* > \Theta$. We show that we can construct an alternative $(\mathbf{z}', \mathbf{w}')$ to $(\mathbf{z}^\star, \mathbf{w}^\star)$, with $\sum_{i=1}^n z_i' = \Theta$ and $\sum_{i=1}^n w_i' = \sum_{i=1}^n w_i^*$, which leads to an increased objective value for $\mathbf{y}$ and we arrive at a contradiction that $(\mathbf{z}^\star, \mathbf{w}^\star)$ is optimal. The proof resembles that used in Lemma 2.1.

Since $\mathbf{y}$ is given, m_i are known for all $i = 1, \ldots, n$. With an abuse of notation, we define $\mathbf{z}'$ be the optimal solution of the following optimization problem,

$$\begin{aligned} &\min_{\mathbf{z}} \sum_{i=1}^n \alpha_{m_i,i} \hat{d}_i z_i \\ \text{s.t.} \quad & z_i \le z_i^*, \ \forall i = 1, \ldots, n \\ & \sum_{i=1}^n z_i \le \Theta. \end{aligned}$$

We let $\mathcal{Z}'$ denote the set of i in which $z_i' = 1$. Then $\mathcal{Z}'$ is a subset of $\mathcal{Z}^\star$ consisting of the Θ worst-case periods with the greatest contribution to the objective value.

Using $\Theta > \lfloor \Gamma^d \rfloor$ and (10), we know that

$$\sum_{i=1}^n w_i^\star \le \Gamma^d \ \le \Theta = \sum_{i=1}^n z_i',$$

and therefore, we can construct $\mathbf{w}'$, such that

$$\begin{aligned} & w'_i = 0,\ \forall i \notin \mathcal{Z}' \\ & w'_i \geq w^*_i,\ \forall i \in \mathcal{Z}' \\ & \sum_{i=1}^{n} w'_i = \sum_{i=1}^{n} w^\star_i, \end{aligned} \tag{23}$$

where at least one of the inequalities in (23) is strict. Since the relocation from $\mathbf{w}^*$ to $\mathbf{w}'$ has been done to periods with at least the same cost, we have

$$\sum_{i \in \mathcal{Z}^*} \alpha_{m_i,i} w^*_i \hat{d}_i \ \leq \ \sum_{i \in \mathcal{Z}'} \alpha_{m_i,i} w'_i \hat{d}_i.$$

If the cost is greater, we arrive at a contradiction that $(\mathbf{z}^\star, \mathbf{w}^\star)$ is optimal. Otherwise, if the cost is equal, we can choose $(\mathbf{z}', \mathbf{w}')$ over $(\mathbf{z}^\star, \mathbf{w}^\star)$ and we arrive at the desired result.

Note that in a similar way, we can prove that $\sum_{i=1}^{n} w^*_i = \Gamma^d$. Assuming that $\mathbf{z}^*$ satisfies that $\sum_{i=1}^{n} z^*_i = \Theta$, and $\sum_{i=1}^{n} w^*_i < \Gamma^d$, we show that we can construct an alternative $\mathbf{w}'$ to $\mathbf{w}^\star$, with $\sum_{i=1}^{n} w'_i = \Gamma^d > \sum_{i=1}^{n} w^*_i$, which leads to an increased objective value for $\mathbf{y}$ and we arrive at a contradiction that $(\mathbf{z}^\star, \mathbf{w}^\star)$ is optimal. Using $\Theta > \Gamma^d$, we can construct $\mathbf{w}'$, such that

$$\begin{aligned} & w'_i = 0,\ \forall i \notin \mathcal{Z}^* \\ & w'_i \geq w^*_i,\ \forall i \in \mathcal{Z}^* \\ & \sum_{i=1}^{n} w'_i = \Gamma^d, \end{aligned} \tag{24}$$

where at least one of the inequalities in (23) is strict. Thus, it is easy to see that

$$\sum_{i \in \mathcal{Z}^*} \alpha_{m_i,i} w^*_i \hat{d}_i < \sum_{i \in \mathcal{Z}^*} \alpha_{m_i,i} w'_i \hat{d}_i,$$

i.e., $\mathbf{w}'$ leads to an increased total cost, and we arrive at a contradiction that $(\mathbf{z}^\star, \mathbf{w}^\star)$ is optimal. □

Lemma 3.2. *Given a setup decision vector* $\mathbf{y}$*, there exists an optimal solution* $(\mathbf{z}^\star, \mathbf{w}^\star)$ *of the ULSUR model, such that there is at most one period* $r \in \mathcal{Z}^\star$ *in which* $\beta < w^\star_r < 1$.

Proof. We will prove the statement by contradiction. Suppose that there exist periods l and r, such that $z^\star_l = 1 = z^*_r$, $\beta < w^\star_l < 1$ and $\beta < w^\star_r < 1$. We will distinguish three cases.

If $\alpha_{m_l,l}\hat{d}_l = \alpha_{m_r,r}\hat{d}_r$,

$$\alpha_{m_l,l}(\tilde{d}_l + w_l^\star \hat{d}_l) + \alpha_{m_r,r}(\tilde{d}_r + w_r^\star \hat{d}_r) = \alpha_{m_l,l}[\tilde{d}_l + (w_l^\star + \epsilon)\hat{d}_l] + \alpha_{m_r,r}[\tilde{d}_r + (w_r^\star - \epsilon)\hat{d}_r],$$

with $\epsilon > 0$ a sufficiently small quantity. Then we can always increase w_l^* while reducing w_r^* until $w_l^* = 1$ or $w_r^* = \beta$ without loss of optimality.

Otherwise, if $\alpha_{m_l,l}\hat{d}_l > \alpha_{m_r,r}\hat{d}_r$,

$$\alpha_{m_l,l}(\tilde{d}_l + w_l^\star \hat{d}_l) + \alpha_{m_r,r}(\tilde{d}_r + w_r^\star \hat{d}_r) < \alpha_{m_l,l}[\tilde{d}_l + (w_l^\star + \epsilon)\hat{d}_l] + \alpha_{m_r,r}[\tilde{d}_r + (w_r^\star - \epsilon)\hat{d}_r].$$

Hence increasing w_l^* while reducing w_r^* leads to an increased total cost, and we arrive at a contradiction.

Finally, if $\alpha_{m_l,l}\hat{d}_l < \alpha_{m_r,r}\hat{d}_r$, increasing w_r^* while reducing w_l^* leads to an increased total cost, and we again arrive at a contradiction. □

Using the same arguments as in the proofs of Lemma 3.1 and Lemma 3.2, we have the following lemmas.

Lemma 3.3. *Given a setup decision vector* $\mathbf{y}$*, there exists an optimal solution* $(\mathbf{z}^\star, \mathbf{w}^\star)$ *of the ULSUB model, such that*

(1) *either* $\sum_{i=1}^n z_i^* = \lceil \Gamma^d \rceil$ *and* $\sum_{i=1}^n w_i^* = \Gamma^d$,
(2) *or* $\sum_{i=1}^n z_i^* = \lfloor \Gamma^d \rfloor$ *and* $\sum_{i=1}^n w_i^* = \lfloor \Gamma^d \rfloor$.

Lemma 3.4. *Given a setup decision vector* $\mathbf{y}$*, there exists an optimal solution* $(\mathbf{z}^\star, \mathbf{w}^\star)$ *of the ULSUB model, such that there is at most one period* $r \in \mathcal{Z}^*$ *in which* $\beta < w_r^* < 1$.

4. The solution approach

Using the ZIO property, the vector $\mathbf{y}$ splits the planning horizon into a sequence of regeneration intervals [25], where no inventory enters or leaves the intervals and the inventory levels within the interval are positive. In this section we first propose a Dynamic Programming (DP) algorithm for both the ULSUR models that resembles the one proposed in [25]. We then show that with slight modifications, the same DP can also be used to solve the ULSUB model.

As shown in Lemma 3.2, in worst-case periods the optimal deviation level of demand, $\mathbf{w}^*$, has three forms. To be more specific, in a worst-case period i, $w_i^* = 1$, $w_i^* = \beta$, or $\beta < w_i^* < 1$. We denote by γ^u the number of worst-case periods in which $w_i^* = 1$, by γ^l the number in which $w_i^* = \beta$, by $\gamma^m \in \{0, 1\}$ whether there exists a period r such that $\beta < w_r^* < 1$. Later in this section, we will count how many of each type are for given Γ^d, Θ, and β.

We denote by $\mathbf{\Gamma} = (\gamma^u, \gamma^m, \gamma^l)$ a vector of dimension three, with the number of each type of worst-case periods. We also denote by $\mathbf{i}$ as a vector of dimension three, with each element in the vector equal to i. We let $\mathbf{\Gamma}_i \leq \mathbf{i}$ denote the vector with the number of each type of worst-case periods in $[1, i]$ for $i = 2, \ldots, n+1$. We define $\mathbf{\Gamma}_1 = \mathbf{0}$. We construct a network with nodes $(k, \mathbf{\Gamma}_k)$ in which the DP moves from node $(k, \mathbf{\Gamma}_k)$ to $(t+1, \mathbf{\Gamma}_{t+1})$, defining the subplan $[(k, \mathbf{\Gamma}_k), (t+1, \mathbf{\Gamma}_{t+1})]$, for $1 \leq k \leq t \leq n$, $\mathbf{0} \leq \mathbf{\Gamma}_k \leq \mathbf{\Gamma}_{t+1} \leq \mathbf{\Gamma}$. As incoming inventory into the regeneration interval $[k, t]$ is always zero, none of the production decisions taken in the previous subplan affect the regeneration interval $[k, t]$. Hence, given $\mathbf{\Gamma}_k$ and $\mathbf{\Gamma}_{t+1}$, assigning worst-case periods in one subplan is independent from assigning in the others.

In $[(k, \mathbf{\Gamma}_k), (t+1, \mathbf{\Gamma}_{t+1})]$, we let $\mathbf{\Delta} = (\Delta^u, \Delta^m, \Delta^l) = \mathbf{\Gamma}_{t+1} - \mathbf{\Gamma}_k$. We denote by $\phi(k, t, \mathbf{\Delta})$ the minmax cost of the subplan with $\mathbf{\Delta}$ worst-case periods. Let $G(t+1, \mathbf{\Gamma}_{t+1})$ be the minmax cost over the first t periods with $\mathbf{\Gamma}_{t+1}$ worst-case periods, where $1 \leq t \leq n$ and $\mathbf{0} \leq \mathbf{\Gamma}_{t+1} \leq \mathbf{\Gamma}$, and $S(k, t+1, \mathbf{\Gamma}_{t+1})$ be the minmax cost over the first t periods with $\mathbf{\Gamma}_{t+1}$ worst-case periods, with the additional information that $I_{k-1} = 0$ and $I_i > 0$ for $k \leq i < t$, i.e., $[k, t]$ is the last regeneration interval in $[1, t]$. As in a ULS model, we minimize the total cost in $[1, t]$ by choosing appropriate setup period. Then, we have

$$G(t+1, \mathbf{\Gamma}_{t+1}) \quad = \quad \min_{k:\, 1 \leq k \leq t} S(k, t+1, \mathbf{\Gamma}_{t+1}).$$

We also need to maximize the total cost in $[1, t]$ by allocating the budget of uncertainty. Hence, we also have

$$S(k, t+1, \mathbf{\Gamma}_{t+1}) \quad = \quad \max_{\mathbf{\Delta}:\, \mathbf{0} \leq \mathbf{\Delta} \leq \min\{(t+1-k) \cdot \mathbf{1}, \mathbf{\Gamma} - \mathbf{\Gamma}_k\}} \{G(k, \mathbf{\Gamma}_{t+1} - \mathbf{\Delta}) + \phi(k, t, \mathbf{\Delta})\}.$$

We define $G(1, \mathbf{0}) = S(1, 1, \mathbf{0}) = 0$. Therefore, by construction, the optimal solution to the ULSUR model is equal to $G(n+1, \mathbf{\Gamma})$.

As the models are uncapacitated, in the subplan $[(k, \mathbf{\Gamma}_k), (t+1, \mathbf{\Gamma}_{t+1})]$, there is only production at period k with the total unit cost to deliver one unit in period i equal to α_{ki}. It remains to allocate $\mathbf{\Delta}$ worst-case periods in the subplan. Then we have the min-max cost of the subplan as

$$\phi(k, t, \mathbf{\Delta}) = f_k + \sum_{i=k}^{t} \alpha_{ki} \tilde{d}_i + E(k, t, \mathbf{\Delta}),$$

where

$$E(k, t, \mathbf{\Delta}) := \max_{\substack{\mathbf{z}^u \in \{0,1\}^{t+1-k}, \\ \mathbf{z}^l \in \{0,1\}^{t+1-k}, \\ \mathbf{z}^m \in \{0,1\}^{t+1-k}, \\ \sum_{i=k}^{t} z_i^u \leq \Delta^u, \\ \sum_{i=k}^{t} z_i^l \leq \Delta^l, \\ \sum_{i=k}^{t} z_i^m \leq \Delta^m, \\ z_i^u + z_i^l + z_i^m \leq 1.}} \sum_{i=k}^{t} \alpha_{ki} \hat{d}_i [z_i^u + \beta z_i^l + w^* z_i^m].$$

Function $E(k, t, \mathbf{\Delta})$ gives the maximum cost incurred in $[(k, \mathbf{\Gamma}_k), (t+1, \mathbf{\Gamma}_{t+1})]$ by setting Δ^u demands to upper bounds, Δ^l to the minimum protection level, and Δ^m to $\beta < w^* < 1$, where the value of w^* will come from Lemma 4.1. For given values of k, t, and $\mathbf{\Delta}$, calculating $E(k, t, \mathbf{\Delta})$ is equivalent to a unordered partial sorting problem. Given $\alpha_{ki} \hat{d}_i$ for $i = k, \ldots, t$, first assigning the Δ^u periods with largest values of $\alpha_{ki} \hat{d}_i$ with $z_i^u = 1$, followed by Δ^m and then Δ^l periods, gives the optimal solution.

For given values of k, t, and $\mathbf{\Delta}$, calculating $E(k, t, \mathbf{\Delta})$ is equivalent to a unordered partial sorting problem, which can be done in $\mathcal{O}(n)$ time. By calculating $E(1, n, \mathbf{\Delta})$ first and deleting the lowest value of $\alpha_{ki} \hat{d}_i$, $E(k, t, \mathbf{\Delta})$ can be obtained in $\mathcal{O}(n)$ time for all values of $t \geq k$. Then all values of $E(k, t, \mathbf{\Delta})$ can be calculated in an $\mathcal{O}(\Theta n)$ time preprocessing procedure. Given t and $\mathbf{\Gamma}_{t+1}$, calculating $G(t+1, \mathbf{\Gamma}_{t+1})$ can be done in $\mathcal{O}(n)$ time due to the minimization over k. For each $S(k, t+1, \mathbf{\Gamma}_{t+1})$, we need to consider the maximization over $\mathbf{\Delta}$. Using Lemma 4.1 we know that given γ^u, γ^m and γ^l follows using the fact that $\gamma^u + \gamma^m + \gamma^l = \Theta$, hence solving the maximization takes $\mathcal{O}(\Theta)$ time. Therefore, all values of $S(k, t+1, \mathbf{\Gamma}_{t+1})$ and $G(t+1, \mathbf{\Gamma}_{t+1})$ can be calculated in $\mathcal{O}(\Gamma^d \Theta n^2)$ time, i.e., the DP solves the ULSUR model in $\mathcal{O}(\Gamma^d \Theta n^2)$ time.

Given the DP, it remains to derive $\mathbf{\Gamma}$ and w^* for the ULSUR model.

Lemma 4.1. *Given Γ^d, Θ, and β, we have that*

$$\gamma^u = \lfloor \frac{\Gamma^d - \beta\Theta}{1-\beta} \rfloor,$$
$$\gamma^m = \begin{cases} 1 & \text{if } \Gamma^d - \gamma^u > \beta \cdot (\Theta - \gamma^u), \\ 0 & \text{otherwise}, \end{cases}$$
$$\gamma^l = \Theta - \gamma^u - \gamma^m,$$

and in period r, if $\gamma^m = 1$, the level of deviation is

$$w_r^\star = \Gamma^d - \gamma^u - \beta\gamma^l.$$

Proof. By Lemma 3.1, there exists an optimal solution such that

$$\gamma^u + w_r^* \gamma^m + \beta\gamma^l = \Gamma^d, \tag{25}$$
$$\gamma^u + \gamma^m + \gamma^l = \Theta. \tag{26}$$

Then by (25) and (26),

$$\begin{aligned} \Gamma^d - \gamma^u &= \beta\gamma^l + w_r^\star \gamma^m \\ &\geq \beta\gamma^l + \beta\gamma^m \\ &= \beta(\gamma^l + \gamma^m) \\ &= \beta(\Theta - \gamma^u). \end{aligned}$$

Hence

$$\begin{aligned} (1-\beta)\gamma^u &\leq \Gamma^d - \beta\Theta \\ \gamma^u &\leq \frac{\Gamma^d - \beta\Theta}{1-\beta} \\ \gamma^u &\leq \lceil \frac{\Gamma^d - \beta\Theta}{1-\beta} \rceil. \end{aligned}$$

We now prove that $\gamma^u \geq \lfloor \frac{\Gamma^d - \beta\Theta}{1-\beta} \rfloor$. By Lemma 3.1, we can substitute γ^l with $\Theta - \gamma^u - \gamma^m$ and rewrite (25) as follows,

$$\gamma^u + w_r^* \gamma^m + \beta\gamma^l = \Gamma^d$$
$$\gamma^u + w_r^* \gamma^m + \beta(\Theta - \gamma^u - \gamma^m) = \Gamma^d$$

$$(1-\beta)\gamma^u + (w_r^\star - \beta)\gamma^m = \Gamma^d - \beta\Theta.$$

By definition, $\beta < w_r^\star < 1$, thus

$$\begin{aligned}
(1-\beta) + (1-\beta)\gamma^u &> \Gamma^d - \beta\Theta \\
1 + \gamma^u &> \frac{\Gamma^d - \beta\Theta}{1-\beta} \\
\gamma^u &> \frac{\Gamma^d - \beta\Theta}{1-\beta} - 1 \\
\gamma^u &\geq \lfloor \frac{\Gamma^d - \beta\Theta}{1-\beta} \rfloor.
\end{aligned}$$

The desired result follows using that $\gamma^u \in \mathbb{Z}$.

Given that we have proven the result for γ^u, the results for γ^m and γ^l follow trivially. By Lemma 3.2, we know that in the rest of the worst-case periods, the uncertainty level is exactly β, with the possibility that in one period r, $\beta < w_r^\star < 1$. Then existence of γ^m is determined by the size of $\Gamma^d - \gamma^u - \beta(\Theta - \gamma^u)$, i.e., if $\Gamma^d - \gamma^u - \beta(\Theta - \gamma^u) > 0$, $\gamma^m = 1$ with $w_r^\star = \Gamma^d - \gamma^u - \beta\gamma^l$.

□

Using Lemma 3.3 and Lemma 3.4, we derive a similar result for the ULSUB model.

Lemma 4.2. *Given Γ^d, Θ, and β, we have that*

(1) *if* $\Gamma^d - \lfloor \Gamma^d \rfloor \geq \beta$,

$$\begin{aligned}
\gamma^u &= \lfloor \Gamma^d \rfloor, \\
\gamma^m &= 1, \\
\gamma^l &= 0,
\end{aligned}$$

and in period r, if $\gamma^m = 1$, the level of deviation is

$$w_r^\star = \Gamma^d - \gamma^u.$$

(2) *otherwise, if* $\Gamma^d - \lfloor \Gamma^d \rfloor < \beta$,

(a) *either*

$$\gamma^u = \lfloor \Gamma^d \rfloor,$$
$$\gamma^m = 0,$$
$$\gamma^l = 0,$$

(b) *or*

$$\gamma^u = \lfloor \Gamma^d \rfloor - 1,$$
$$\gamma^m = 1,$$
$$\gamma^l = 1,$$

and in period r, the level of deviation is

$$w_r^\star = \Gamma^d - \gamma^u - \beta.$$

Proof. Using Lemma 3.3 and Lemma 3.4 and constructing a similar argument as in the proof of Lemma 4.2, we can show that the statement holds when $\Gamma^d - \lfloor \Gamma^d \rfloor \geq \beta$. While $\Gamma^d - \lfloor \Gamma^d \rfloor < \beta$, we need to consider two cases. We can either leave $\Gamma^d - \lfloor \Gamma^d \rfloor$ as unused, as in Case (2)(a), or we can reduce γ^u as in Case (2)(b) by one and assign the remaining uncertainty budget of size $1 + \Gamma^d - \lfloor \Gamma^d \rfloor$ so that $\gamma^m = 1 = \gamma^l$.

□

For the ULSUB model, the proposed DP algorithm can be applied directly if $\Gamma^d - \lfloor \Gamma^d \rfloor \geq \beta$ by using the $\mathbf{\Gamma}$ vetor derived in Case (1) in Lemma 4.2. If $\Gamma^d - \lfloor \Gamma^d \rfloor < \beta$, the DP has to run twice for both $\mathbf{\Gamma}$ vectors in Case (2) in Lemma 4.2. The one with the highest cost leads to the optimal solution of the ULSUB model. It is easy to see that the ULSUB model can be solved in $\mathcal{O}((\Gamma^d)^2 n^2)$ time using Lemma 4.2.

5. Numerical analysis

The purpose of this computational study is to investigate the effectiveness in terms of protection against the demand uncertainty of the proposed models. We will show performance results for the ULSUR and the ULSUB models, against the nominal model and Soyster's absolute robust model [24]. In the nominal model, we have a deterministic ULS where demands are equal to $\tilde{d}_i$. Soyster's model is equivalent to the ULSUB model with $\Gamma^d = n$. Throughout this section, we will assume that the demand in period i, D_i, is i.i.d..

As commonly used in the literature, we assume the demands to be uniformly distributed, i.e., $D_i \sim U(\tilde{d}_i - \hat{d}_i, \tilde{d}_i + \hat{d}_i)$. We present a comprehensive set of results, where we test different values for

- the cost parameters $\mathbf{f}$, $\mathbf{p}$, and $\mathbf{h}$; and
- the size of the deviation component of the demand $\hat{\mathbf{d}}$.

In particular, we experiment with $f_i = 50, 100, 200, 300$. For the unit inventory holding costs, we assume that $h_i = \lambda \cdot \sum_{t=1}^{n} p_t/n$, where $\lambda = 0.10, 0.20, 0.30, 0.40$. For the nominal demand, we assume $\tilde{d}_i = 30$, while for the deviation component, $\hat{d}_i = \delta \tilde{d}_i$, with $\delta = 0.25$, 0.5, 0.75, 1.00. The protection of periods is heavily influenced by $\alpha_{m_i,i}\hat{d}_i$, where recall that $\alpha_{m_i,i} = p_{m_i} + h_{m_i,i-1}$. Therefore, we will distinguish three patterns for $\mathbf{p}$. In Sections 5.1 and 5.2, we will consider time-invariant unit production costs, followed by non-increasing ones in Section 5.3, and non-decreasing ones in Section 5.4. In all these experiments, we have $n = 15$.

The goal of this experiment is to test the protection against demand uncertainty of the robust models using two criteria, namely feasibility and objective function. In order to do so, and for each of the models, we generate 5000 realizations of the demand vector, and report feasibility and average objective function costs among the feasible instances. More precisely, for a given robust model we obtain the optimal solution $(\mathbf{y}^*, \mathbf{x}^*)$. For each realization of the demand vector, we first check whether $(\mathbf{y}^*, \mathbf{x}^*)$ is feasible, and in that case we calculate the cost as in the ULS. In what follows, R_f reports the percentage of feasible instances, V the average cost across the feasible instances for that model, and R_n the percentage increase in average cost comparing to the nominal model, i.e., $R_n = \frac{V - V_n}{V_n} \cdot 100$, where V_n is the average cost of the nominal model.

Both the ULSUR and the ULSUB models have parameters. We have tested our models for all possible values of Γ^d and Θ, while fixing $\beta = 0.2$. We have experimented with other values of β, but the results are similar. For each set of the experiments, we present the results of the "best" parameters of the ULSUR model and the ULSUB model. The goodness of the parameters is measured by V, while we impose a minimum feasibility level on R_f. We present results for several values of feasibility threshold, namely, 70%, 80%, 90%.

5.1. The base experiment

We first present results of the base experiment. Then in the following sections we test our models with various parameter settings, altering one at a time. In the base experiment,

Feasible level	Model	Γ^d	Θ	V	R_n	R_f
70%	ULSUB	3	-	2528.8	11.0	71.8
	ULSUR	3	15	2509.3	10.1	78.3
80%	ULSUB	4	-	2619.0	14.9	89.7
	ULSUR	4	13	2604.4	14.3	83.8
90%	ULSUB	5	-	2703.2	18.6	93.1
	ULSUR	5	12	2688.8	18.0	92.0
	Nominal	-	-	2278.5	-	36.6
	Soyster	-	-	3569.4	56.7	100.0

Table 1: The results of the base experiment, with setup cost $f_i = 200$, unit production cost $p_i = 3$, and unit inventory holding cost $h_i = 0.3$. Demand realizations are generated uniformly in $(15, 45)$.

the problem has time-invariant setup cost $f_i = 200$, unit production cost $p_i = 3$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n = 0.3$. For the demand, we let $\hat{d}_i = 15$. The results can be found in Table 1. From the table we can see that the nominal model achieves the lowest average total cost over feasible realizations. However the solution is feasible only for 37% of the realizations. On the other hand, Soyster's model is the most robust in terms of feasibility, at the expense of the highest average cost, i.e., more than 50% higher than the nominal model. There is no big difference between the ULSUR model and the ULSUB model, both models showing their effectiveness by reaching a satisfactory feasibility level while maintaining reasonable cost, i.e., increase in average cost is below 20%.

5.2. Time-invariant unit production cost

We carry on the experiment by altering the parameter settings of the base experiment, one at a time. To begin with, we experiment with f_i and present the results in Table 2, followed by the unit production cost. In addition to $p_i = 3$, we let $p_i = 2$, 4, and 5 and the results are shown in Table 3. Then the experiment is repeated with various values of the unit inventory holding cost. The results are presented in Table 4. The advantage of the ULSUR model and the ULSUB model over the nominal model and Soyster's absolute robust model, as in Section 5.1, seems to be insensitive to changes in cost parameters. Taking 90% feasibility level for example, the proposed models lead to a 30%-40% cost reduction comparing to the absolute robust model. This reduction rises above 50% if the unit inventory holding cost is increased. Lastly, we vary the size of the deviation component of the demand, $\hat{d}$, and the results are presented in Table 5. It can be seen in the table that if the demand deviation is increased, all robust model suffer from relative higher objective values comparing to the nominal model, i.e., the R_n values in Table 5 are higher than those in the Tables 2-4. However, the cost reduction led by the proposed robust models is the most significant. The average cost of Soyster's absolute robust model doubles it of the nominal model, while the proposed models lead to a 30%-40% increase in average cost and reaching at least 90% feasibility level.

5.3. Non-increasing unit production cost

In this section, we first let the unit production costs to be a non-increasing sequence with $p_1 = 4 = 200\% \cdot p_{15}$ and $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$, whilst all other parameters are as in the base experiment. The results are shown in Table 6. We then experiment with different parameter settings as in Section 5.2, and the results are presented in Tables 7-9 respectively. In addition, we test other three sets of unit production costs, all of which are non-increasing, and the results are shown in Table 10.

The results coincide with those for the time-invariant unit production cost in Sections 5.1 and 5.2. The proposed ULSUR and ULSUB models show their usefulness, in particular, when unit inventory holding cost or the deviation component of the demand is high. The models achieve average costs that are around 70% less than the Soyster's model, while attaining at least 90% feasibility level in these scenarios. In all cases, the nominal model is only feasible for around 30% of the demand realizations.

5.4. Non-decreasing unit production cost

In this section, we let the unit production cost to be a non-decreasing sequence with $p_{15} = 4 = 200\% \cdot p_1$, and let $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$, while keep all other parameters as in the base experiment. The result is shown in Table 11. We then experiment with different parameter settings as in Section 5.2, and the results are presented in Tables 12-14 respectively. In addition, we test other three sets of unit production cost, all of which are non-decreasing, and the results are shown in Table 15.

While the results in general agree with those in the previous sets of experiments, we see that the ULSUR model in general outperforms the ULSUB model, particularly with high setup cost, high unit inventory holding cost, and high demand deviation component. For instance, the ULSUR model reduces the total cost by 19% if $f = 300$ and a minimum of 90% feasibility level is required, by 20% if $h = 0.4 \cdot \sum_{i=1}^{n} p_i/n$, and by 16% if $\hat{d} = 30$.

6. Conclusion

Using the concept of uncertainty budget, in this paper we study robust lot sizing models. We assume the uncertainty lies in the demand and propose a polynomial time DP to solve the models. We provide a comprehensive computational experiments on the factors that affect the performance of the ULSUR and ULSUB models. We see that the proposed models outperform the nominal model and Soyster's absolute robust model in general, being cost-efficient while attaining high feasibility level. The advantage of the proposed robust models is quite insensitive to variations of cost parameters and demand deviations.While the proposed models perform similarly in the first two sets of experiments, the ULSUR shows its effectiveness when unit production cost is non-decreasing, especially when the cost parameters, or the size of demand deviation is large. In this case, using the ULSUR leads to a $6-20\%$ average objective value reduction comparing to the ULSUB model.

Note that the theoretical results in this paper can easily be generalized to multiples types of uncertainty, e.g., setup and unit production cost uncertainty. Controlling total demand deviation level while monitoring closely the objective value in a certain model is an interesting area which deserves further attention. The bw-robustness criterion, which is first proposed by Roy [22] for cost coefficient uncertainty in the objective function, allows the decision-maker to identify a solution that guarantees an objective value, in a maximization problem, of at least w in all scenarios, and maximizes the probability of reaching a target value of $b > w$. We feel that modifying the criterion to monitor the total demand deviation level, i.e., imposing a maximum total demand deviation level b while minimizing the objective value in scenarios in which total demand deviation level is closest to some desire level w. In reality, this new criterion has its own practical importance. Forecast of demand may have several levels, e.g., monthly forecast and annually forecast, the criterion links the two levels of demand uncertainty together.

Another stream of research links uncertainty budget with decisions. Poss [21] studies the so-called variable budgeted uncertainty in combinatorial optimization model, an extension of the budgeted uncertainty introduced by Bertsimas and Sim [8], and shows that the model is less conservative for decision vectors with few non-zero components. In our case, it is also interesting to see whether making Γ^d and Θ dependent on the setup decision $\mathbf{y}$ further improves the performance of the proposed models.

f	Feasible level	Model	Γ^d	Θ	V	R_n	R_f
50	70%	ULSUB	3	-	2064.0	11.9	75.6
		ULSUR	3	13	2047.5	11.0	72.3
	80%	ULSUB	6	-	2281.1	23.6	96.2
		ULSUR	5	14	2178.2	18.1	80.4
	90%	ULSUB	6	-	2281.1	23.6	96.2
		ULSUR	6	9	2271.1	23.1	90.5
		Nominal	-	-	1845.0	-	31.3
		Soyster	-	-	3016.9	63.5	100.0
100	70%	ULSUB	3	-	2228.8	10.5	71.8
		ULSUR	3	3	2228.8	10.5	72.3
	80%	ULSUB	4	-	2319.0	15.0	89.7
		ULSUR	4	13	2304.4	14.2	83.8
	90%	ULSUB	8	-	2652.8	31.5	97.8
		ULSUR	8	12	2640.5	30.9	92.0
		Nominal	-	-	2017.1	-	31.3
		Soyster	-	-	3247.9	61.0	100.0
200	70%	ULSUB	3	-	2528.8	11.0	71.8
		ULSUR	3	15	2509.3	10.1	78.3
	80%	ULSUB	4	-	2619.0	14.9	89.7
		ULSUR	4	13	2604.4	14.3	83.8
	90%	ULSUB	5	-	2703.2	18.6	93.1
		ULSUR	5	12	2688.8	18.0	92.0
		Nominal	-	-	2278.5	-	36.6
		Soyster	-	-	3569.4	56.7	100.0
300	70%	ULSUB	3	-	2704.3	9.11	72.0
		ULSUR	3	3	2704.3	9.11	72.0
	80%	ULSUB	4	-	2795.5	12.8	88.8
		ULSUR	4	13	2792.2	12.7	86.5
	90%	ULSUB	5	-	2874.5	16.0	90.1
		ULSUR	5	5	2874.5	16.0	90.5
		Nominal	-	-	2478.5	-	36.6
		Soyster	-	-	3825.9	54.4	100.0

Table 2: The results for time-invariant unit production cost, with various settings of **f**, unit production cost $p_i = 3$, and unit inventory holding cost $h_i = 0.3$. Demand realizations are generated uniformly in $(15, 45)$.

p	Feasible level	Model	Γ^d	Θ	V	R_n	R_f
2	70%	ULSUB	3	-	1802.9	9.1	72.0
		ULSUR	3	3	1802.9	9.1	72.0
	80%	ULSUB	4	-	1863.7	12.8	88.8
		ULSUR	4	8	1859.3	12.5	84.0
	90%	ULSUB	5	-	1916.4	16.0	90.1
		ULSUR	5	5	1916.4	16.0	90.1
		Nominal	-	-	1652.3	-	36.6
		Soyster	-	-	2550.6	54.4	100.0
3	70%	ULSUB	3	-	2528.8	11.0	71.8
		ULSUR	3	15	2509.3	10.1	78.3
	80%	ULSUB	4	-	2619.0	14.9	89.7
		ULSUR	4	13	2604.4	14.3	83.8
	90%	ULSUB	5	-	2703.2	18.6	93.1
		ULSUR	5	12	2688.8	18.0	92.0
		Nominal	-	-	2278.5	-	36.6
		Soyster	-	-	3569.4	56.7	100.0
4	70%	ULSUB	4	-	3171.8	9.8	71.8
		ULSUR	4	4	3171.8	9.8	71.8
	80%	ULSUB	4	-	3292.0	13.9	89.7
		ULSUR	4	14	3269.4	13.1	81.6
	90%	ULSUB	5	-	3404.3	17.8	93.1
		ULSUR	5	12	3385.1	17.2	90.6
		Nominal	-	-	2889.5	-	31.3
		Soyster	-	-	4559.3	57.8	100.0
5	70%	ULSUB	3	-	3814.7	10.2	71.8
		ULSUR	3	8	3812.8	10.1	73.4
	80%	ULSUB	4	-	3965.0	14.5	89.7
		ULSUR	4	14	3937.1	13.7	81.6
	90%	ULSUB	5	-	4105.3	18.6	93.1
		ULSUR	5	12	4081.4	17.9	90.6
		Nominal	-	-	3461.9	-	31.3
		Soyster	-	-	5546.6	60.2	100.0

Table 3: The experiments for various settings of the time-invariant unit production cost $\mathbf{p}$, with setup cost $f_i = 200$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly in $(15, 45)$.

h	Feasible level	Model	Γ^d	Θ	V	R_n	R_f
$0.1 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB	3	-	2528.8	11.0	71.8
		ULSUR	3	15	2509.3	10.1	78.3
	80%	ULSUB	4	-	2619.0	14.9	89.7
		ULSUR	4	13	2604.4	14.3	83.8
	90%	ULSUB	5	-	2703.2	18.6	93.1
		ULSUR	5	12	2688.8	18.0	92.0
		Nominal	-	-	2278.5	-	36.6
		Soyster	-	-	3569.4	56.7	100.0
$0.2 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB	3	-	2972.6	10.7	71.8
		ULSUR	3	8	2970.0	10.6	72.3
	80%	ULSUB	4	-	3108.0	15.8	89.7
		ULSUR	4	14	3074.5	14.5	81.6
	90%	ULSUB	8	-	3595.6	33.9	97.8
		ULSUR	5	13	3202.7	19.3	90.6
		Nominal	-	-	2684.3	-	31.3
		Soyster	-	-	4470.9	66.6	100.0
$0.3 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB	3	-	3472.1	14.4	75.6
		ULSUR	3	9	3384.0	11.5	70.9
	80%	ULSUB	6	-	3853.4	27.0	96.2
		ULSUR	5	14	3688.4	21.5	82.5
	90%	ULSUB	6	-	3853.4	27.0	96.2
		ULSUR	6	8	3840.0	26.5	92.9
		Nominal	-	-	3035.0	-	25.4
		Soyster	-	-	5250.8	73.0	100.0
$0.4 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB	3	-	3801.1	14.1	75.6
		ULSUR	3	13	3735.1	12.2	72.3
	80%	ULSUB	6	-	4264.6	38.1	96.2
		ULSUR	5	8	4025.0	20.9	86.0
	90%	ULSUB	6	-	4264.6	28.1	96.2
		ULSUR	6	11	4214.0	26.5	90.5
		Nominal	-	-	3330.2	-	23.9
		Soyster	-	-	5992.8	80.0	100.0

Table 4: The experiments for time-invariant unit production cost $\mathbf{p}$, with setup cost $f_i = 200$, and various settings of $\mathbf{h}$. Demand realizations are generated uniformly in $(15, 45)$.

$\hat{d}$	Feasible level	Model	Γ^d	Θ	V	R_n	R_f
7.5	70%	ULSUB	3	-	2347.7	3.0	72.0
		ULSUR	3	3	2347.7	3.0	72.0
	80%	ULSUB	4	-	2393.3	5.0	88.8
		ULSUR	4	15	2390.0	4.9	84.0
	90%	ULSUB	5	-	2432.8	6.8	90.1
		ULSUR	5	5	2432.8	6.8	90.1
		Nominal	-	-	2278.5	-	31.3
		Soyster	-	-	2894.7	27.0	100.0
15	70%	ULSUB	3	-	2528.8	11.0	71.8
		ULSUR	3	15	2509.3	10.1	78.3
	80%	ULSUB	4	-	2619.0	14.9	89.7
		ULSUR	4	13	2604.4	14.3	83.8
	90%	ULSUB	5	-	2703.2	18.6	93.1
		ULSUR	5	12	2688.8	18.0	92.0
		Nominal	-	-	2278.5	-	31.3
		Soyster	-	-	3569.4	56.7	100.0
22.5	70%	ULSUB	3	-	2683.2	17.8	71.8
		ULSUR	3	15	2668.4	17.1	78.3
	80%	ULSUB	4	-	2818.5	23.7	89.7
		ULSUR	4	14	2793.4	22.6	81.6
	90%	ULSUB	5	5	2944.8	29.2	93.1
		ULSUR	5	13	2923.3	28.3	90.6
		Nominal	-	-	2278.5	-	31.3
		Soyster	-	-	4442.2	95.0	100.0
30	70%	ULSUB	3	-	2837.6	24.5	71.8
		ULSUR	3	3	2837.6	24.5	71.8
	80%	ULSUB	4	-	3018.0	32.5	89.7
		ULSUR	4	14	2984.5	31.0	81.6
	90%	ULSUB	5	-	3186.4	39.8	93.1
		ULSUR	5	12	3157.7	38.6	90.6
		Nominal	-	-	2278.5	-	31.3
		Soyster	-	-	4918.9	115.9	100.0

Table 5: The experiments for time-invariant unit production cost, with setup cost $f_i = 200$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly using various setting of $\hat{\mathbf{d}}$ as shown in the table.

Feasible level	Model	Γ^d	Θ	V	R_n	R_f
70%	ULSUB	2	-	2642.1	7.9	76.0
	ULSUR	2	8	2626.8	7.3	72.3
80%	ULSUB	3	-	2753.2	12.4	88.9
	ULSUR	3	12	2702.0	10.3	82.2
90%	ULSUB	4	-	2838.2	15.8	95.3
	ULSUR	4	14	2791.0	14.0	91.4
-	Nominal	-	-	2449.2	-	31.3
-	Soyster	-	-	3771.3	54.0	100.0

Table 6: The experiments for non-increasing unit production cost, with setup cost $f_i = 200$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly in $(15, 45)$.

f	Feasible level	Model	Γ^d	Θ	V	R_n	R_f
50	70%	ULSUB	2	-	2095.2	9.6	73.3
		ULSUR	2	6	2082.3	8.9	70.6
	80%	ULSUB	3	-	2200.9	15.1	87.7
		ULSUR	3	11	2153.4	12.6	80.2
	90%	ULSUB	4	-	2278.9	19.2	94.5
		ULSUR	4	11	2261.2	18.3	91.0
		Nominal	-	-	1911.7	-	23.9
		Soyster	-	-	3100.1	62.2	100.0
100	70%	ULSUB	2	-	2314.4	8.4	74.1
		ULSUR	2	7	2307.5	8.1	70.6
	80%	ULSUB	3	-	2409.3	12.8	86.6
		ULSUR	3	11	2388.1	11.8	82.4
	90%	ULSUB	4	-	2506.8	17.4	93.7
		ULSUR	4	14	2462.4	15.3	90.6
		Nominal	-	-	2135.3	-	27.7
		Soyster	-	-	3370.4	57.8	100.0
200	70%	ULSUB	2	-	2642.1	7.9	76.0
		ULSUR	2	8	2626.8	7.3	72.3
	80%	ULSUB	3	-	2753.2	12.4	88.9
		ULSUR	3	12	2702.0	10.3	82.2
	90%	ULSUB	4	-	2838.2	15.9	95.3
		ULSUR	4	14	2791.0	14.0	91.4
		Nominal	-	-	2449.2	-	31.3
		Soyster	-	-	3771.3	54.0	100.0
300	70%	ULSUB	2	-	2893.5	7.4	77.7
		ULSUR	2	10	2865.6	6.4	70.4
	80%	ULSUB	3	-	3005.4	11.6	89.8
		ULSUR	3	14	2952.0	9.6	81.2
	90%	ULSUB	4	-	3079.9	14.3	94.7
		ULSUR	4	15	3056.5	13.5	91.8
		Nominal	-	-	2693.6	-	36.6
		Soyster	-	-	4071.3	51.1	100.0

Table 7: The experiments for non-increasing unit production cost, with various settings of $\mathbf{f}$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly in $(15, 45)$.

h	Feasible level	Model	Γ^d	Θ	V	R_n	R_f
$0.1 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB	2	-	2642.1	7.9	76.0
		ULSUR	2	8	2626.8	7.3	72.3
	80%	ULSUB	3	-	2753.2	12.4	88.9
		ULSUR	3	12	2702.0	10.3	82.2
	90%	ULSUB	4	-	2838.2	15.9	95.3
		ULSUR	4	14	2791.0	14.0	91.4
		Nominal	-	-	2449.2	-	31.3
		Soyster	-	-	3771.3	54.0	100.0
$0.2 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB	3	-	3207.3	14.7	86.6
		ULSUR	2	6	3061.9	9.5	70.6
	80%	ULSUB	3	-	3207.3	14.7	86.6
		ULSUR	3	11	3157.9	12.9	81.5
	90%	ULSUB	4	-	3299.4	18.0	91.6
		ULSUR	4	11	3293.7	17.8	90.2
		Nominal	-	-	2796.5	-	27.7
		Soyster	-	-	4619.4	65.2	100.0
$0.3 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB	3	-	3571.5	18.3	75.6
		ULSUR	3	13	3511.2	16.3	72.3
	80%	ULSUB	4	-	3771.3	24.9	93.4
		ULSUR	3	9	3557.6	17.8	80.9
	90%	ULSUB	4	-	3771.3	24.9	93.4
		ULSUR	4	6	3744.2	24.0	90.3
		Nominal	-	-	3019.8	-	27.7
		Soyster	-	-	5368.4	77.8	100.0
$0.4 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB	3	-	3903.6	21.9	75.6
		ULSUR	3	13	3826.7	19.5	72.3
	80%	ULSUB	4	-	4150.1	29.6	93.4
		ULSUR	3	9	3885.2	21.4	80.9
	90%	ULSUB	4	-	4150.1	29.6	93.4
		ULSUR	4	6	4115.6	28.6	90.3
		Nominal	-	-	3201.3	-	23.9
		Soyster	-	-	6117.4	91.1	100.0

Table 8: The experiments for non-increasing unit production cost, with setup cost $f_i = 200$, and various settings of $\mathbf{h}$. Demand realizations are generated uniformly in $(15, 45)$.

$\hat{d}$	Feasible level	Model	Γ^d	Θ	V	R_n	R_f
7.5	70%	ULSUB	2	-	2496.6	1.9	76.0
		ULSUR	2	8	2489.0	1.6	72.3
	80%	ULSUB	3	-	2552.2	4.2	88.9
		ULSUR	3	12	2526.6	3.2	82.2
	90%	ULSUB	4	-	2594.7	5.9	95.3
		ULSUR	4	14	2571.1	5.0	91.4
		Nominal	-	-	2449.2	-	31.3
		Soyster	-	-	3061.2	25.0	100.0
15	70%	ULSUB	2	-	2642.1	7.9	76.0
		ULSUR	2	8	2626.8	7.3	72.3
	80%	ULSUB	3	-	2753.2	12.4	88.9
		ULSUR	3	12	2702.0	10.3	82.2
	90%	ULSUB	4	-	2838.2	15.9	95.3
		ULSUR	4	14	2791.0	14.0	91.4
		Nominal	-	-	2449.2	-	31.3
		Soyster	-	-	3771.3	54.0	100.0
22.5	70%	ULSUB	2	-	2787.6	13.8	76.0
		ULSUR	2	8	2764.6	12.9	72.3
	80%	ULSUB	3	-	2954.2	20.6	88.9
		ULSUR	3	12	2877.5	17.5	82.2
	90%	ULSUB	4	-	3069.7	25.3	93.1
		ULSUR	4	14	3010.9	22.9	91.4
		Nominal	-	-	2449.2	-	31.3
		Soyster	-	-	4470.8	82.5	100.0
30	70%	ULSUB	3	-	3118.8	27.3	87.4
		ULSUR	2	8	2902.4	18.5	72.3
	80%	ULSUB	3	-	3118.8	27.3	87.4
		ULSUR	3	12	3053.0	24.7	82.2
	90%	ULSUB	4	-	3304.8	24.6	93.1
		ULSUR	4	14	3230.8	34.9	91.4
		Nominal	-	-	2449.2	-	31.3
		Soyster	-	-	5150.8	110.3	100.0

Table 9: The experiments for non-increasing unit production cost, with setup cost $f_i = 200$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly using various setting of $\hat{\mathbf{d}}$ as shown in the table.

$\mathbf{p}$	Feasible level	Model	Γ^d	Θ	V	R_n	R_f
$p_1 = 4.25$ and $p_{15} = 1.75$	70%	ULSUB	2	-	2682.8	6.8	76.0
		ULSUR	2	8	2665.8	6.1	72.3
	80%	ULSUB	3	-	2797.7	11.4	88.9
		ULSUR	3	12	2741.7	9.1	82.2
	90%	ULSUB	4	-	2884.0	14.8	95.3
		ULSUR	4	14	2832.4	12.7	91.4
		Nominal	-	-	2512.5	-	31.3
		Soyster	-	-	3821.8	52.1	100.0
$p_1 = 4$ and $p_{15} = 2$	70%	ULSUB	2	-	2642.1	7.9	76.0
		ULSUR	2	8	2626.8	7.3	72.3
	80%	ULSUB	3	-	2753.2	12.4	88.9
		ULSUR	3	12	2702.0	10.3	82.2
	90%	ULSUB	4	-	2838.2	15.9	95.3
		ULSUR	4	14	2791.0	14.0	91.4
		Nominal	-	-	2449.2	-	31.3
		Soyster	-	-	3771.3	54.0	100.0
$p_1 = 3.75$ and $p_{15} = 2.25$	70%	ULSUB	2	-	2601.4	5.9	76.0
		ULSUR	2	7	2583.2	5.2	71.2
	80%	ULSUB	3	-	2681.3	9.2	86.4
		ULSUR	3	12	2657.4	8.2	81.4
	90%	ULSUB	4	-	2792.5	13.7	95.3
		ULSUR	4	14	2749.6	12.0	91.4
		Nominal	-	-	2455.9	-	31.3
		Soyster	-	-	3720.8	51.5	100.0
$p_1 = 3.5$ and $p_{15} = 2.5$	70%	ULSUB	2	-	2560.7	5.5	76.0
		ULSUR	2	7	2544.7	4.8	71.2
	80%	ULSUB	3	-	2639.5	8.7	86.4
		ULSUR	3	12	2618.3	7.9	81.4
	90%	ULSUB	4	-	2746.8	13.1	95.3
		ULSUR	4	14	2708.3	11.6	91.4
		Nominal	-	-	2427.7	-	31.3
		Soyster	-	-	3670.4	51.2	100.0

Table 10: The experiments for various settings of the non-increasing unit production cost $\mathbf{p}$, with setup cost $f_i = 200$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly in $(15, 45)$.

Feasible level	Model	Γ^d	Θ	V	R_n	R_f
70%	ULSUB	4	-	2347.3	13.8	72.0
	ULSUR	3	14	2270.4	10.1	74.6
80%	ULSUB	6	-	2501.1	21.3	88.6
	ULSUR	4	15	2345.9	13.8	82.2
90%	ULSUB	8	-	2664.9	29.2	96.6
	ULSUR	6	15	2501.5	21.3	91.2
	Nominal	-	-	2061.9	-	37.9
	Soyster	-	-	3299.5	60.0	100.0

Table 11: The experiments for non-decreasing unit production cost, with setup cost $f_i = 200$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly in $(15, 45)$.

f	Feasible level	Model	Γ^d	Θ	V	R_n	R_f
50	70%	ULSUB	10	-	2442.0	37.3	78.0
		ULSUR	7	15	2210.8	24.3	73.7
	80%	ULSUB	11	-	2525.8	42.0	80.2
		ULSUR	11	15	2519.6	41.7	91.0
	90%	ULSUB	12	12	2604.0	46.4	96.2
		ULSUR	11	15	2519.6	41.7	91.0
		Nominal	-	-	1778.2	-	31.6
		Soyster	-	-	2879.4	61.9	100.0
100	70%	ULSUB	6	-	2280.4	19.8	74.9
		ULSUR	3	15	2069.1	8.7	72.0
	80%	ULSUB	9	-	2508.5	31.8	91.7
		ULSUR	7	11	2355.7	23.8	83.4
	90%	ULSUB	9	-	2508.5	31.8	91.7
		ULSUR	8	15	2436.6	28.0	91.7
		Nominal	-	-	1903.2	-	31.6
		Soyster	-	-	3048.8	60.2	100.0
200	70%	ULSUB	4	-	2347.3	13.8	72.0
		ULSUR	3	14	2270.4	10.1	74.6
	80%	ULSUB	6	-	2501.1	21.3	88.6
		ULSUR	4	15	2345.9	13.8	82.2
	90%	ULSUB	8	-	2664.9	29.2	96.6
		ULSUR	6	15	2501.5	21.3	91.2
		Nominal	-	-	2061.9	-	37.9
		Soyster	-	-	3299.5	60.0	100.0
300	70%	ULSUB	2	-	2355.7	7.1	80.1
		ULSUR	2	2	2355.7	7.1	80.1
	80%	ULSUB	2	-	2355.7	7.1	80.1
		ULSUR	2	2	2355.7	7.1	80.1
	90%	ULSUB	8	-	2864.9	30.3	96.6
		ULSUR	3	11	2440.0	11.0	90.9
		Nominal	-	-	2199.0	-	50.1
		Soyster	-	-	3499.5	59.1	100.0

Table 12: The experiments for non-decreasing unit production cost, with various settings of **f**, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly in $(15, 45)$.

h	Feasible level	Model	Γ^d	Θ	V	R_n	R_f
$0.1 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB	4	-	2347.3	13.8	72.0
		ULSUR	3	14	2270.4	10.1	74.6
	80%	ULSUB	6	-	2501.1	21.3	88.6
		ULSUR	4	15	2345.9	13.8	82.2
	90%	ULSUB	8	-	2664.9	29.2	96.6
		ULSUR	6	15	2501.5	21.3	91.2
		Nominal	-	-	2061.9	-	37.9
		Soyster	-	-	3299.5	60.0	100.0
$0.2 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB	3	-	2807.7	16.3	70.5
		ULSUR	3	3	2807.7	16.3	70.5
	80%	ULSUB	6	-	3127.2	29.5	91.2
		ULSUR	5	11	3014.7	24.84	83.1
	90%	ULSUB	6	-	3127.2	29.5	91.2
		ULSUR	6	6	3127.2	29.5	91.2
		Nominal	-	-	2414.9	-	31.6
		Soyster	-	-	4294.5	77.8	100.0
$0.3 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB	5	-	3466.4	32.4	75.8
		ULSUR	5	5	3466.4	32.4	75.8
	80%	ULSUB	9	-	4050.6	54.8	96
		ULSUR	6	10	3608.7	37.9	82.8
	90%	ULSUB	9	-	4050.6	54.8	96
		ULSUR	7	15	3758.8	43.6	90.6
		Nominal	-	-	2617.4	-	27.4
		Soyster	-	-	5109.6	95.2	100.0
$0.4 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB	7	-	4198.8	51.7	80.2
		ULSUR	7	7	4198.8	51.7	80.2
	80%	ULSUB	7	-	4198.8	51.7	80.2
		ULSUR	7	7	4198.8	51.7	80.2
	90%	ULSUB	12	-	5127.9	85.2	98.1
		ULSUR	9	15	4578.9	65.4	91.4
		Nominal	-	-	2768.6	-	25.4
		Soyster	-	-	5868.1	112.0	100.0

Table 13: The experiments for non-decreasing unit production cost, with setup cost $f_i = 200$, and various settings of $\mathbf{h}$. Demand realizations are generated uniformly in $(15, 45)$.

$\hat{d}$	Feasible level	Model	Γ^d	Θ	V	R_n	R_f
7.5	70%	ULSUB	4	-	2162.0	4.9	72.0
		ULSUR	3	13	2123.3	3.0	74.6
	80%	ULSUB	6	-	2239.0	8.6	88.6
		ULSUR	4	15	2161.3	4.8	82.2
	90%	ULSUB	8	-	2320.9	12.6	96.6
		ULSUR	6	15	2239.2	8.6	91.2
		Nominal	-	-	2061.9	-	37.9
		Soyster	-	-	2638.2	27.9	100.0
15	70%	ULSUB	4	-	2347.3	13.8	72.0
		ULSUR	3	14	2270.4	10.1	74.6
	80%	ULSUB	6	-	2501.1	21.3	88.6
		ULSUR	4	15	2345.9	13.8	82.2
	90%	ULSUB	8	-	2664.9	29.2	96.6
		ULSUR	6	15	2501.5	21.3	91.2
		Nominal	-	-	2061.9	-	37.9
		Soyster	-	-	3299.5	60.0	100.0
22.5	70%	ULSUB	4	-	2532.5	22.8	72.0
		ULSUR	3	13	2416.3	17.2	74.6
	80%	ULSUB	6	-	2763.3	34.0	88.6
		ULSUR	4	15	2530.4	22.7	82.2
	90%	ULSUB	8	-	3009.0	45.9	96.6
		ULSUR	6	15	2763.9	34.0	91.2
		Nominal	-	-	2061.9	-	37.9
		Soyster	-	-	3960.9	92.1	100.0
30	70%	ULSUB	4	-	2717.7	31.8	72.0
		ULSUR	3	13	2562.8	24.3	74.6
	80%	ULSUB	6	-	3025.4	46.7	88.6
		ULSUR	4	15	2715.0	31.7	82.2
	90%	ULSUB	8	-	3353.1	63.1	96.6
		ULSUR	6	15	3026.3	46.8	91.2
		Nominal	-	-	2061.9	-	37.9
		Soyster	-	-	4621.3	124.1	100.0

Table 14: The experiments for non-decreasing unit production cost, with setup cost $f_i = 200$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly using various setting of $\hat{\mathbf{d}}$ as shown in the table.

$\mathbf{p}$	Feasible level	Model	Γ^d	Θ	V	R_n	R_f
$p_1 = 1.75$ and $p_{15} = 4.25$	70%	ULSUB	5	-	2369.3	19.4	77.2
		ULSUR	3	14	2212.1	11.5	72.3
	80%	ULSUB	7	-	2520.3	27.0	88.7
		ULSUR	4	15	2287.1	15.3	82.2
	90%	ULSUB	9	-	2680.7	35.1	96.6
		ULSUR	7	14	2519.8	27.0	91.2
		Nominal	-	-	1984.1	-	50.1
		Soyster	-	-	3217.9	62.2	100.0
$p_1 = 2$ and $p_{15} = 4$	70%	ULSUB	4	-	2347.3	13.8	72.0
		ULSUR	3	14	2270.4	10.1	74.6
	80%	ULSUB	6	-	2501.1	21.3	88.6
		ULSUR	4	15	2345.9	13.8	82.2
	90%	ULSUB	8	-	2664.9	29.2	96.6
		ULSUR	6	15	2501.5	21.3	91.2
		Nominal	-	-	2061.9	-	37.9
		Soyster	-	-	3299.5	60.0	100.0
$p_1 = 2.25$ and $p_{15} = 3.75$	70%	ULSUB	3	-	2327.6	11.9	74.6
		ULSUR	3	13	2327.3	11.9	70.7
	80%	ULSUB	5	-	2483.2	19.4	88.4
		ULSUR	4	15	2404.6	15.6	82.2
	90%	ULSUB	7	-	2650.1	27.4	96.6
		ULSUR	6	15	2563.8	23.3	91.2
		Nominal	-	-	2080.1	-	36.6
		Soyster	-	-	3381.1	62.5	100.0
$p_1 = 2.5$ and $p_{15} = 3.5$	70%	ULSUB	3	-	2384.2	12.5	70.7
		ULSUR	3	-	2384.2	12.5	70.7
	80%	ULSUB	4	-	2467.4	16.4	87.4
		ULSUR	4	7	2463.4	16.2	82.2
	90%	ULSUB	6	-	2636.3	24.4	96.6
		ULSUR	5	14	2545.4	20.1	90.9
		Nominal	-	-	2119.4	-	36.6
		Soyster	-	-	3462.7	63.4	100.0

Table 15: The experiments for various settings of the non-decreasing unit production cost $\mathbf{p}$, with setup cost $f_i = 200$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly in $(15, 45)$.

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