Semidefinite Programming Approach to Russell Measure Model

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Abstract

Throughout its evolution, data envelopment analysis (DEA) has mostly relied on linear programming, particularly because of simple primal-dual relations and the existence of standard software for solving linear programs. Although also nonlinear models, such as Russell measure or hyperbolic measure models, have been introduced, their use in applications has been limited mainly because of their computational inconvenience. The common feature of these nonlinear models is that some unknown variables appear in the form of reciprocal values. In this paper, we introduce a novel method for dealing with this type of nonlinearity in DEA. We show how to reformulate the nonlinear model as a semidefinite programming (SDP) problem and describe how to derive the corresponding dual counterpart of the model. Two benefits of our approach are: (1) the SDP reformulated model can be solved efficiently using some standard SDP solvers and, (2) the derived dual program is comparable with the dual counterparts of linear DEA models and allows the common economic interpretations of dual variables as shadow prices. Our approach is applied to the Russell measure model for that also an overview of attempts to overcome the nonlinearities is presented.

Key words: Data envelopment analysis; Semidefinite programming; Second-order cone programming; Russell measure model; Hyperbolic model

1 Introduction

While most DEA models are formulated as linear programming (LP) problems, there are also nonlinear models. One of them, the so-called Russell measure (RM) model, was introduced by Färe and Lovell (1978)‡ as an alternative to

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‡The formulation in Färe and Lovell (1978) used only the inputs in its measure of efficiency. This measure was completed in the subsequent works by Färe, Grosskopf and Lovell (1985), p. 162.

input/output-oriented radial models. Although almost forgotten due to its nonlinearity, it is still worthy of interest. Indeed, the Russell measure possesses most of the properties that, according to (Russell (1985)), a measure of technical efficiency should satisfy. Aggregating both the input and output efficiencies, the RM model is monotone, units invariant and economically well interpretable. Moreover, the RM model is very flexible. It handles particular inputs and outputs efficiencies uniformly which allows various modifications fulfilling environmental modeling purposes. On the other hand, the RM model lacks linearity as reciprocals of some variables enter its objective function - a feature that was considered a serious obstacle both from the computational and from the theoretical point of view (see Cooper et al. (1999), Cooper et al. (2007b) for some basic properties).

The RM model was introduced in the early years of DEA. Since then, a number of models have been developed with an emphasis on their linearity. Even the RM model itself was modified into the enhanced Russell measure (ERM) model (Pastor et al. (1999)), allowing an LP reformulation. Linear models have been preferred in DEA mainly because of their computational convenience and widely available software packages for solving them routinely. Moreover, DEA has benefited from simple primal-dual relations in LP where both the primal and the dual version of each model has its interpretation and deep justification.

Concurrently with the birth of DEA, the fields of linear and nonlinear programming started to change rapidly. The revolution in mathematical programming induced by the publication of Karmarkar's seminal paper in 1984 has significantly influenced all areas of optimization. It has changed the methods, approaches, and finally resulted in the formation of new classes of optimization problems such as semidefinite programming (SDP) and second-order cone programming (SOCP). Both of them belong to a more general class of structured problems called convex conic programming. Since then, various nonlinear convex problems can be solved routinely using interior point methods, which have been implemented in numerous software packages. Nowadays, the challenge in optimization is not as much in designing algorithms as in formulating nonlinear problems as convex (conic) programs. Furthermore, strong duality (known from LP) is available for strictly feasible primal-dual pairs of convex conic programs (see Nemirovski (2007) or Boyd and Vandenberghe (2004) for more details). Due to these advances in mathematical programming, there is no longer reason to restrict oneself to linear models in DEA. Namely, if the nonlinear model allows an SDP or SOCP reformulation, then the corresponding dual problem can be derived with ease and they both can be solved directly using widely available software packages.

Sueyoshi and Sekitani (2007) were the first to show that DEA can benefit from the latest development in convex optimization. They formulated the RM model as an SOCP program, derived the corresponding dual problem and used standard software to solve the primal and dual RM models. The results of the authors are inspiring for a detailed study and practical application of the model.

In this paper, we provide an alternative approach to reformulation of the RM model - not only as a second-order cone program but also as a semidefinite one. Unlike Sueyoshi and Sekitani (2007), where the model is directly rewritten into an SOCP form, we relax the problem using additional variables and reformulate it in an SDP or SOCP form equivalent to the original model. The advantage of this novel approach is that the inequalities for input contractions and output expansions remain linear and hence the corresponding dual variables preserve the interpretation of shadow prices known from linear models. As a result we obtain a dual model that in the case of the RM model is directly comparable with the dual to the slack based measure (SBM) model by Tone (1997). Moreover, our approach is presented in a rather general setting to be applicable also to some other models with a similar type of nonlinearity as in the RM model.

This paper is organized as follows: Section 2 reviews the RM model and specifies its connection with ERM, SBM and some other models. Section 3 provides a methodological basis for dealing with reciprocal variables in conic programming and for the derivation of corresponding dual programs. In Section 4 the dual of the RM model is formulated and its interpretation is discussed. An illustrative example for a comparison of RM with SBM solutions is given in Section 5. Conclusions and possible extensions are summarized in Section 6.

2 Russell measure model

Formulation of the RM model: Consider a set of n decision making units DMU_j $(j = 1, ...n)$, each consuming given amounts of m inputs x_{ij} $(i = 1, ..., m)$ to produce s outputs y_{rj} $(r = 1, ..., s)$. Assume that all inputs and outputs amounts are positive. Let x_j and y_j denote the m and s dimensional column vectors of inputs and outputs of DMU_j $(j = 1, ..., n)$, respectively. By $o \in \{1, ..., n\}$ we denote the index of DMU to be currently evaluated. In order to evaluate $\mathrm{DMU}_o = (x_o, y_o)$, the RM model formulates the following nonlinear program

$$\min_{\theta_i,\phi_r,\lambda_j} \quad \frac{1}{m+s}\left(\sum_{i=1}^{m}\theta_i + \sum_{r=1}^{s}\frac{1}{\phi_r}\right) \tag{1}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij}\lambda_j \le \theta_i x_{io}, \quad \theta_i \le 1, \quad i = 1, \ldots, m, \tag{2}$$

$$\sum_{j=1}^{n} y_{rj}\lambda_j \ge \phi_r y_{ro}, \quad \phi_r \ge 1, \quad r = 1, \ldots, s, \tag{3}$$

$$\lambda_j \ge 0, \quad j = 1, ..., n. \tag{4}$$

This model, as described by (1)–(4), is associated with the production possibility set

$$M = \left\{ (x, y) \in \mathbb{R}^m \times \mathbb{R}^s \mid \sum_{j=1}^{n} x_j\lambda_j \le x,\ \sum_{j=1}^{n} y_j\lambda_j \ge y,\ \lambda \ge 0 \right\},$$

corresponding to constant returns to scale. If the RM model corresponds to variable returns to scale, then the condition $\sum_{j=1}^{n} \lambda_j = 1$ is included both to the constraints of the model and the set M.

Interpretation and properties: Denote θ^*, ϕ^* and λ^* the optimal solution vectors of (1)–(4) and define

$$\hat{x}_i := \sum_{j=1}^{n} x_{ij}\lambda_j^* \ (i = 1, \dots, m) \ \text{ and } \ \hat{y}_r := \sum_{j=1}^{n} y_{rj}\lambda_j^* \ (r = 1, \dots, s). \qquad (5)$$

The unit $(\hat{x}, \hat{y})$ defined in (5) is a projection of (x_o, y_o) on the efficient boundary of M and represents an efficient virtual unit (benchmark) to which (x_o, y_o) is compared. Since the efficiency value should be a number from $(0, 1]$ (regardless of an input/output orientation), it makes sense to interpret the values $\frac{\hat{x}_i}{x_{io}} \leq 1$ and $\frac{y_{ro}}{\hat{y}_r} \leq 1$ as the partial efficiencies of the input i and the output r, respectively. Finally, since the inequalities in (2) and (3) are binding (i.e. they hold with equalities at the optimal solution), we have

$$\theta_i^* = \frac{\hat{x}_i}{x_{io}} \ \text{ and } \ \frac{1}{\phi_r^*} = \frac{y_{ro}}{\hat{y}_r}$$

and hence the optimal value of (1) is the arithmetic mean of the partial efficiencies of the inputs and the outputs.

A certain misinterpretation of the RM objective function (see Cooper *et al.* (1999) and Cooper *et al.* (2007c)) is rectified and some properties of the RM model are summarized in Cooper *et al.* (2007b) as follows: *The objective is to minimize the average value of input plus output efficiencies as measured by the optimal* θ_i *and* $\frac{1}{\phi_r}$. *The resulting measure is complete as well as units invariant and has other attractive properties. However, it is computationally difficult, so the authors turned to alternatives.* In the following we first present three approaches that could be understood as approximations to the RM model. Afterwards we describe the method by Sueyoshi and Sekitani (2007) which allows to solve the RM model itself.

The first approach makes use of the fact that it is not necessary to tie the Russell measure to the model. One can replace the unit $(\hat{x}, \hat{y})$ with some other efficient unit to which (x_o, y_o) can be compared. Such a unit is provided by any model to be linked with the same production possibility set as the RM model. For example, the CCR model provides its max-slack projection and the additive model by Charnes *et.al* (1985) directly yields its projection. Based on such a projection $(\hat{x}, \hat{y})$, the Russell measure can be expressed by

$$\rho_o = \frac{1}{m+s} \left(\sum_{i=1}^{m} \frac{\hat{x}_i}{x_{io}} + \sum_{r=1}^{s} \frac{y_{ro}}{\hat{y}_r} \right). \qquad (6)$$

Obviously, we obtain merely an upper bound of the value provided by the RM model. Nevertheless, the value ρ_o can be interpreted as the Russell based measure of (x_o, y_o) with respect to an efficient $(\hat{x}, \hat{y})$. This approach is used e.g. in

Ševčovič *et al.* (2001) where an efficient virtual unit is obtained by an additive model. In Cooper, Tone (1997) the so-called measure of efficiency proportions (MEP) is introduced - one can easily see that it is equivalent to (6) with $(\hat{x}, \hat{y})$ obtained by the CCR input model as the max-slack projection of (x_o, y_o).

The second approach consists in a linear approximation of nonlinear terms in (1). If we denote

$$s_{io}^- := x_{io} - \theta_i x_{io} \ (i = 1, ..., m) \quad \text{and} \quad s_{ro}^+ = \phi_r y_{ro} - y_{ro} \ (r = 1, ..., s),$$

then

$$\frac{1}{m+s}\left[\sum_{i=1}^{m}\theta_i + \sum_{r=1}^{s}\frac{1}{\phi_r}\right] = \frac{1}{m+s}\left[\sum_{i=1}^{m}\left(1 - \frac{s_{io}^-}{x_{io}}\right) + \sum_{r=1}^{s}\frac{y_{ro}}{y_{ro}+s_{ro}^+}\right].$$

By Cooper *et al.* (1999), the linear approximation

$$\frac{y_{ro}}{y_{ro}+s_{ro}^+} \approx 1 - \frac{s_{ro}^+}{y_{ro}}, \quad \text{for small } s_{ro}^+,$$

can be used in this case to obtain a linear approximation of the objective function (1). Let us mention, however, that due to $s_{ro}^+ \geq 0$ we can obtain even more:

$$\frac{y_{ro}}{y_{ro}+s_{ro}^+} = 1 - \frac{s_{ro}^+}{y_{ro}+s_{ro}^+} \geq 1 - \frac{s_{ro}^+}{y_{ro}}$$

and hence

$$\frac{1}{m+s}\left[\sum_{i=1}^{m}\theta_i + \sum_{r=1}^{s}\frac{1}{\phi_r}\right] \geq 1 - \frac{1}{m+s}\left[\sum_{i=1}^{m}\frac{s_{io}^-}{x_{io}} + \sum_{r=1}^{s}\frac{s_{ro}^+}{y_{ro}}\right]. \qquad (7)$$

We have obtained a lower bound of the value provided by the RM model that is tight enough only for small values of s_{ro}^+. *

The third approach to the RM model is to replace its objective function by another one. This idea is used in Pastor *et al.* (1999), where the arithmetic mean of the input and the output efficiencies (1) is replaced by the product of the arithmetic mean of the input efficiencies and the harmonic mean of the output efficiencies:

$$\min_{\theta_i, \phi_r, \lambda_j} \quad \left(\frac{1}{m}\sum_{i=1}^{m}\theta_i\right) \Big/ \left(\frac{1}{s}\sum_{r=1}^{s}\phi_r\right),$$
$$\text{s.t.} \quad (2) - (4).$$

The model is called Enhanced Russell Graph Measure of Efficiency and we refer to it as the ERM (Enhanced Russell Measure) model. It differs from the RM

*The expression on the right-hand side of (7) is a unit invariant modification of the objective function of the additive model proposed by Charnes (1985). Chang and Sueyoshi (1991) show that the optimal objective value may be negative.

model solely in the objective function. Although it is nonlinear as well, it allows an application of the standard technique of fractional programming to obtain an equivalent linear program. As proved for example in Cooper et al. (2007a), the ERM model is equivalent with the slack based measure (SBM) model by Tone (1997). Moreover, Sueyoshi and Sekitani (2007) prove that the ERM efficiency is less or equal to the RM one.

The approach of Sueyoshi and Sekitani: Unlike all the above approaches yield mere approximations of the exact value of the RM model, Sueyoshi and Sekitani (2007) show that the RM model itself can be solved and its dual model can be derived via an SOCP reformulation of (1)–(4). Actually, they use the substitutions $\frac{1}{\phi_r} =: \psi_r$ in order to move the nonlinearities from (1) to (3). They obtain the RM model in the form:

$$\min_{\theta_i,\psi_r,\lambda_j} \quad \frac{1}{m+s}\left(\sum_{i=1}^{m}\theta_i + \sum_{r=1}^{s}\psi_r\right) \tag{8}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij}\lambda_j \leq \theta_i x_{io}, \quad \theta_i \leq 1, \quad i = 1,\ldots,m, \tag{9}$$

$$\sum_{j=1}^{n} y_{rj}\lambda_j \geq \frac{y_{ro}}{\psi_r}, \quad \psi_r \leq 1, \quad r = 1,\ldots,s, \tag{10}$$

$$\lambda_j \geq 0, \quad j = 1,...,n. \tag{11}$$

Next, they rewrite the nonlinear inequalities in (10) as second-order cone ones:

$$\left(\frac{\sum_{j=1}^{n} y_{rj}\lambda_j + \psi_r}{2}\right)^2 \geq \left(\frac{\sum_{j=1}^{n} y_{rj}\lambda_j - \psi_r}{2}\right)^2 + \left(\sqrt{y_{rk}}\right)^2, \quad r = 1,\ldots,s, \tag{12}$$

and derive the following dual model (Sueyoshi and Sekitani (2007), formulas (13)–(13.6)):

$$\max_{\bar{\mu}_r,\hat{\mu}_r,v_i,\rho_r,\delta_r} \quad \sum_{r=1}^{s}\rho_r\sqrt{y_{ro}} - \sum_{i=1}^{m} v_i x_{io} - \sum_{r=1}^{s}\delta_r \tag{13}$$

$$\text{s.t.} \quad \sum_{r=1}^{s}\left(\frac{\bar{\mu}_r + \hat{\mu}_r}{2}\right) y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1,...,n, \tag{14}$$

$$v_i \geq \frac{1}{m+s}\frac{1}{x_{io}}, \; i = 1,...,m, \tag{15}$$

$$\bar{\mu}_r - \hat{\mu}_r \leq \frac{2}{m+s} + 2\delta_r, \; \delta_r \geq 0, \; r = 1,...,s, \tag{16}$$

$$\bar{\mu}_r^2 \geq \hat{\mu}_r^2 + \rho_r^2, \tag{17}$$

where $\bar{\mu}_r$, $\hat{\mu}_r$ and ρ_r are the dual multipliers corresponding to the r-th second-order cone inequality in (12). Finally, they solve both the primal and dual models on the data set of an illustrative example by means of an SOCP solver.

A comment to the approach by Sueyoshi and Sekitani: The complicated form of the dual model (13–17) allows neither interpretation nor comparison with other multiplicative models. In fact, the common features of all multiplicative models are: (1) they define vectors of decision variables u and v interpreted as the output and input shadow prices; (2) the virtual profit of (x_o, y_o) given as $u^T y_o - v^T x_o$ is maximized under the condition that $u^T y_j - v^T x_j \le 0$ for all $j = 1, \dots, n$. We can see, however, that model (13)–(17) lacks variables that could be interpreted as output shadow prices. It is a consequence of the fact that the second-order cone inequalities for outputs (12) together with their dual multipliers $\bar{\mu}$, $\hat{\mu}$ and ρ satisfy more complex complementarity conditions than in the linear case. Whence, no virtual profit is recognized by the model and strange square roots of output data appear in the objective function instead. Nevertheless, we will see in this paper that by a more sophisticated reformulation of the RM model one can obtain a dual model exhibiting both the above mentioned features.

Main goal of the paper: In the next section we describe a novel general scheme for dealing with reciprocals in DEA. Its basic idea is to keep both variables ϕ and ψ and relax the substitution equality $\frac{1}{\phi} =: \psi$ with the inequality $\frac{1}{\phi} \le \psi$, which is equivalent with a semidefinite constraint. The advantage is that, contrary to Sueyoshi and Sekitani, the constraints (2) and (3) remain linear and hence the corresponding dual multipliers exhibit the usual interpretation of shadow prices. Then, in Section 4 we apply the scheme on the RM model and formulate its dual problem. It allows a direct comparison with the SBM dual.

3 Dealing with the reciprocal value of a variable

General schema: Some of the DEA models yield mathematical programming problems with the minimized objective function containing reciprocal values of some of the variables. As a prototype of such problems consider the following one:

$$\min_{\theta,\phi} \left(\theta + \frac{1}{\phi} \right) \quad \text{s.t.} \quad \phi > 0 \quad \& \quad \text{linear constraints.} \tag{18}$$

By using a new variable ψ such that $\psi \ge \frac{1}{\phi}$, the problem (18) can be equivalently reformulated as

$$\min_{\theta,\phi,\psi} (\theta + \psi) \quad \text{s.t.} \quad \phi \ge 0,\ \psi\phi \ge 1 \quad \& \quad \text{linear constraints.} \tag{19}$$

Here the inequality $\psi\phi \ge 1$ is binding, i.e. it holds with equality at the optimal solution. Now, the problem (19) has a linear objective function, the linear constraints from (18) and a new nonlinear convex constraint $\psi\phi \ge 1$.

Semidefinite programming approach: The inequalities $\psi\phi \ge 1$ and $\phi \ge 0$ are equivalent to the condition

$$P := \begin{pmatrix} \psi & 1 \\ 1 & \phi \end{pmatrix} \succeq 0, \tag{20}$$

reading that the matrix P is positive semidefinite. Hence, the problem (19), where the constraints $\psi\phi \geq 1$ and $\phi \geq 0$ are replaced with the semidefinite constraint $P \succeq 0$ from (20), obtains the form of an SDP problem. Its dual can be constructed by the standard LP procedure with the following difference in dealing with the semidefinite constraint: the Lagrange dual variable corresponding to the primal semidefinite constraint (20) is a 2×2 symmetric, positive semidefinite matrix $Z \succeq 0$ so that the trace of the product of the matrices P and Z is nonnegative, i.e.

$$Z := \begin{pmatrix} z_1 & z_3 \\ z_3 & z_2 \end{pmatrix} \succeq 0, \quad tr(PZ) := \psi z_1 + \phi z_2 + 2z_3 \geq 0. \tag{21}$$

Hence, the corresponding complementarity condition between the optimal solutions P^* and Z^* reads:

$$tr(P^* Z^*) = \psi^* z_1^* + \phi^* z_2^* + 2z_3^* = 0. \tag{22}$$

Since $\psi^* > 0$, $\phi^* > 0$, $z_1^* \geq 0$, $z_2^* \geq 0$, (22) implies $z_3^* \leq 0$. Moreover, due to the binding inequality in (19), also the semidefinite constraint $P \succeq 0$ is binding, i.e., $det(P^*) = \phi^*\psi^* - 1 = 0$. Then it can be deduced from (22) that even the dual semidefinite constraint $Z \succeq 0$ is binding which implies $det(Z^*) = z_1^* z_2^* - z_3^{*2} = 0$. From this and (22) we obtain that either $z_1^* = z_2^* = z_3^* = 0$ or $z_1^* > 0$, $z_2^* > 0$, $z_3^* < 0$. The latter case implies

$$\phi^* = -\frac{z_3^*}{z_2^*} = \frac{\sqrt{z_1^*}}{\sqrt{z_2^*}} \quad \text{and} \quad \psi^* = -\frac{z_3^*}{z_1^*} = \frac{\sqrt{z_2^*}}{\sqrt{z_1^*}} \tag{23}$$

which reviels an interesting correspondence between the primal and the dual optimal solutions. If, in addition $\psi^* = \phi^* = 1$, we obtain $z_1^* = z_2^* = -z_3^*$. In the next section we will deal with the expression $F^* := z_1^* + z_2^* + 2z_3^*$ at the non-zero optimal solution. From (23) we obtain $z_3^* = -\sqrt{z_1^* z_2^*}$. Substituting this into F^* and then using the formula for ψ^* from (23) we obtain

$$F^* = z_1^* + z_2^* + 2z_3^* = (\sqrt{z_1^*} - \sqrt{z_2^*})^2 = z_1^*(1 - \psi^*)^2. \tag{24}$$

Second-order programming approach: Let us note that (19) can be processed also differently to yield an SOCP formulation. Obviously, the inequalities $\psi\phi \geq 1$ and $\phi \geq 0$ are equivalent to the condition

$$\frac{\psi + \phi}{2} \geq \sqrt{\left(\frac{\psi - \phi}{2}\right)^2 + 1} \tag{25}$$

reading that the vector $(\frac{\psi+\phi}{2}, \frac{\psi-\phi}{2}, 1)$ belongs to the second-order cone. Hence, (19), where the nonlinear constraint is replaced with the second-order cone constraint (25), forms an SOCP problem. Its dual problem can be constructed similarly as in the SDP case, but the dual variable corresponding to the second-order cone constraint (25) is a vector (z_1, z_2, z_3) satisfying the dual second-order cone constraint $z_1 \geq \sqrt{z_1^2 + z_3^2}$. The complementarity condition (satisfied by the optimal solution) reads: $\frac{\psi+\phi}{2} z_1 + \frac{\psi-\phi}{2} z_2 + z_3 = 0$.

4 SDP formulation for Russell measure model

The RM model in SDP primal and dual forms: We now apply the results of the previous section to the model (1)–(4). The semidefinite programming formulation of the RM model is

$$\min_{\theta_i,\phi_r,\psi_r\lambda_j} \quad \frac{1}{m+s}\left(\sum_{i=1}^{m}\theta_i + \sum_{r=1}^{s}\psi_r\right) \tag{26}$$

$$\text{s.t.} \qquad (2)-(4), \tag{27}$$

$$\begin{pmatrix}\psi_r & 1\\ 1 & \phi_r\end{pmatrix} \succeq 0,\ r=1,...,s. \tag{28}$$

The corresponding dual problem (where some redundant variables are eliminated) is:

$$\max_{u_r,v_i,Z_r} \quad \sum_{r=1}^{s}u_r y_{ro} - \sum_{i=1}^{m}v_i x_{io} + 1 - \sum_{r=1}^{s}(z_{1r}+z_{2r}+2z_{3r}), \tag{29}$$

$$\text{s.t.} \quad \sum_{r=1}^{s}u_r y_{rj} - \sum_{i=1}^{m}v_i x_{ij} \le 0, \quad j=1,...,n, \tag{30}$$

$$v_i \ge \frac{1}{m+s}\frac{1}{x_{io}},\ i=1,...,m, \tag{31}$$

$$u_r \ge \frac{z_{2r}}{y_{ro}},\ r=1,...,s, \tag{32}$$

$$z_{1r} = \frac{1}{m+s}, \quad Z_r := \begin{pmatrix} z_{1r} & z_{3r}\\ z_{3r} & z_{2r}\end{pmatrix} \succeq 0,\ r=1,...,s. \tag{33}$$

For details of the derivation see Appendix A. We only note here that u_r is the dual variable related to the r-th linear constraint in (3), v_i is the dual variable related to the i-th linear constraint in (3) and $Z_r \succeq 0$ is the dual variable related to the r-th semidefinite constraint in (28). Note that the dual problem (29)–(33) is again an SDP problem.

Formulation of the dual RM model as a nonlinear problem: For purposes of easier interpretation we now rewrite the SDP problem (29)–(33) as a problem with a nonlinear objective function and linear constraints. Indeed, the constraints in (33) are equivalent to

$$z_{3r}^2 \le \frac{1}{m+s}z_{2r} \ \text{ and hence } \ -z_{3r} \le \sqrt{\frac{1}{m+s}z_{2r}}. \tag{34}$$

Since $-z_{3r}$ is maximized in (29) and it does not enter any other constraints of the model, we can replace it by its upper bound $\sqrt{\frac{1}{m+s}z_{2r}}$ to obtain an equivalent model

$$\max_{u_r, v_i, z_r} \quad \sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} + 1 - \sum_{r=1}^{s} F(z_r) \tag{35}$$

$$\text{s.t.} \quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, ..., n, \tag{36}$$

$$v_i \geq \frac{1}{m+s} \frac{1}{x_{io}}, \ i = 1, ..., m, \tag{37}$$

$$u_r \geq \frac{z_r}{y_{ro}}, \ r = 1, ..., s, \tag{38}$$

where

$$z_r := z_{2r} \text{ and } F(z_r) := \frac{1}{m+s} + z_r - 2\sqrt{\frac{1}{m+s} z_r} = \left(\sqrt{\frac{1}{m+s}} - \sqrt{z_r} \right)^2.$$

Let us note that the nonlinear problem (35)–(38) is also the Lagrange dual of the original RM model (1)–(4). It can be derived using the standard techniques and using the concept of conjugate functions.

Interpretation of the dual RM model: It is easy to see that the dual RM model (35)–(38) possesses the main features of linear multiplicative models: it defines the output and input decision variables $u = (u_1, \ldots, u_s)^T$ and $v = (v_1, \ldots, v_m)^T$ that represent output and input shadow prices, respectively. The virtual profit $u^T y_j - v^T x_j$ of any $(x_j, y_j), j = 1, \ldots, n$, does not exceed zero (see (36)) and the virtual profit for (x_o, y_o) enters the objective function which is maximized. For the prices $v_i, i = 1, \ldots, m$, the lower bounds are given in (37) as the reciprocal values of $(m+s)x_{io}$. For the prices $u_r, r = 1, ..., s$, the lower bounds are defined in (38) by the means of the dual variables $z_{2r}, r = 1, ..., s$ that also enter the objective function (35). In comparison with the SBM model (for the reader's convenience we provide the formulation of the SBM model in Appendix B), where the lower bound for u is proportional to the vector of reciprocal values of y_o (see (46) in Appendix B), the Russell dual model offers more freedom for the choice of u, since the lower bound (38) may not be proportional to the vector of reciprocal values of y_o (the reciprocal values of y_{ro} are multiplied by z_{2r} depending on r in (38)). However, we pay for this freedom in the objective function, where the value of the virtual profit for (x_o, y_o) is lowered by the sum of the deviations of z_{2r} from $\frac{1}{m+s}$ measured by the nonlinear square root function $F(z)$.

Note on the variable returns to scale: The RM model described by (1)–(4), or by (26)–(28) corresponds to constant returns to scale. Its modification for variable returns to scale contains the additional condition $\sum_{j=1}^{n} \lambda_j = 1$ in the primal model, and the corresponding dual free variable σ in the dual model that enters the objective function (29) and the left-hand side of each of the inequalities (30) as an additive term.

Note on SOCP reformulation of RM model: An SOCP reformulation of the RM model (different to that provided by Sueyoshi and Sekitani) could be carried out in the framework suggested in the previous section. The corresponding dual is equivalent to the SDP one, although it is expressed in other, somewhat more complicated terms. Thus, we omit it here.

5 An illustrative example

To compare some features of the RM model with the ones of the ERM model, we present an illustrative example. In this simple case the solutions of the corresponding programs can be derived in a closed form. Moreover, the derivation of the dual solutions for the RM model shows how the complementarity conditions mentioned in Section 3 work.

Example: Consider two DMUs, called A and B with a single input and a single output. The data are $A = (1, 1)$ and $B = (2, 1)$.

Solutions to the primal RM and ERM models: It is easy to see that A is efficient and hence its optimal solution in model (1)–(4) is $\theta = \psi = 1$, $\lambda_A = 1$ and $\lambda_B = 0$. On the other hand, B is inefficient and its program (1)- (4) reads:

$$\begin{aligned} \min \quad & \frac{1}{2}\left(\theta + \frac{1}{\phi}\right) \\ \text{s.t.} \quad & \lambda_A + 2\lambda_B \leq 2\theta, \\ & \lambda_A + \lambda_B \geq \phi, \\ & \lambda_A \geq 0,\ \lambda_B \geq 0,\ \theta \leq 1,\ \phi \geq 1. \end{aligned}$$

This program can be solved directly (without a semidefinite formulation) in a closed form. The optimal solution is given in Table 1. We see that the optimal objective value is $\frac{\sqrt{2}}{2}$ and the point $R = (\sqrt{2}, \sqrt{2})$ is the efficient target for B (see Figure 1).

Comparison of the primal solutions: The optimal solutions in the ERM (or the equivalent SBM model) are provided in Table 2. We see that the optimal objective value for the inefficient B in the ERM model is smaller than that in the RM model (this property corresponds to the general result derived by Sueyoshi and Sekitani (2007)). Moreover, we see that the benchmark for B is not uniquely defined in the ERM model (the whole line segment AQ in the Figure 1 represents the set of benchmarks for B), while it is unique in the RM model due to its nonlinearity. Let us note that in real applications it is generally not clear, whether or not the ERM model has alternative benchmarks. By solving the problem numerically (using standard methods and solvers) we obtain one of the possible optimal solution points, which is usually (when e. g. the simpelx method is applied) one of the extreme points of the optimal solution set of benchmarks. For example, the DEA-Solver (Cooper, Seifort, Tone, 2007a) provides for B the benchmark A in the ERM model, which is also the (unique) benchmark for B in the CCR input model.

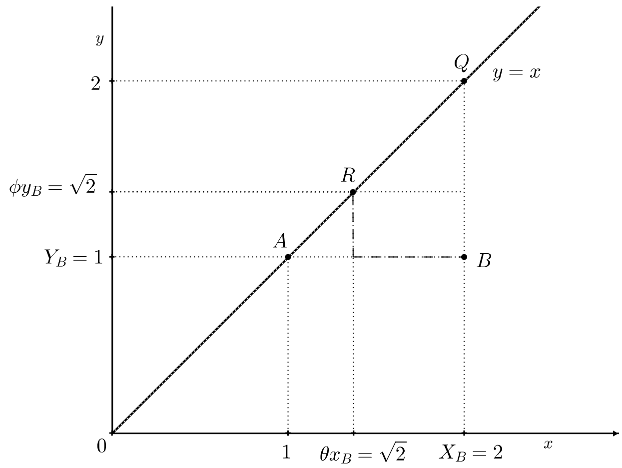

Figure 1: Projection of B into the virtual efficient point R by the RM model in Example 1.

DMU	x	y	eff	θ	ϕ	ψ	λ_A	λ_B	u	v	z	$F(z)$
A	1	1	1	1	1	1	1	0	$= v$	$\geq \frac{1}{2}$	$\frac{1}{2}$	0
B	2	1	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4} - \frac{\sqrt{2}}{2}$

Table 1: Data and optimal solutions in the RM model (1)-(4) and its semidefinite dual (29)-(33) for Example 1.

DMU	x	y	eff	θ	ϕ	λ_A	λ_B	u	v
A	1	1	1	1	1	1	0	$= v$	≥ 1
B	2	1	$\frac{1}{2}$	$\in [\frac{1}{2}, 1]$	$= 2\theta$	$\in [1, 2]$	0	$\frac{1}{2}$	$\frac{1}{2}$

Table 2: Data and optimal solutions in the ERM model (39)-(42) and (43)-(47) for Example 1.

Solutions to dual RM and ERM models: We now compute optimal solutions of the dual RM model, firstly for A. As mentioned in Section 3, for efficient A we have $F(z) = 0$ and hence $z = \frac{1}{2}$. From (36) it follows $u \leq v$, from (37) we have $v \geq \frac{1}{2}$ and from (38) $u \geq z$. Thus the set of optimal solutions is unbounded and consists of all (u, v) such that $u = v \geq \frac{1}{2}$.

For the inefficient B the inequality $v \leq u$ follows from (35). Since $\theta < 1$ and $\phi > 1$, complementarity implies that (36) and (37) are satisfied with equalities. Hence $v = \frac{1}{4}$ and $u = z$. Since $\lambda_A > 0$, from complementarity we obtain $u = v$. Finally from (23), where $z_1 = \frac{1}{2}$ and $z_2 = z$, it follows that $z = \frac{1}{4}$ which implies the solution presented in Table 1.

6 Discussion and conclusions

Remark on strong duality: In Section 4 we have formulated the primal-dual pair of semidefinite programs (26)–(28) and (29)–(33), where both of them are feasible. Let us note that in linear programming the primal-dual feasibility implies strong duality, which means that the optimal solutions to both the primal and the dual program exist and the optimal objective values are equal. Yet, in semidefinite programming duality results are more complex and for strong duality we generally need to presume that both the primal and the dual SDPs are strictly feasible (e.g. Todd (2001), Vandenberghe, Balakhrishnan (2003)). Unfortunately, although our dual problem is strictly feasible, the primal is generally not (for example, the primal problem for an efficient DMU has only one feasible solution). So, what does the duality theory in SDP imply for such a case? If the dual program is strictly feasible, but the primal is only feasible, then their optimal values are equal, though the optimal solution of the dual may fail to exist. As an example for such a phenomenon, a problem with the constraint of the form (28) is often used, where 0 is the infimum of ψ, but it is not attained since ϕ escapes to infinity (Vandenberghe and Boyd (1996)). However, although such problematic semidefinite constraints occur in the formulations of our programs, they do not cause any trouble since the corresponding variables are bounded due to other linear constraints. Hence, in our case, both the primal and the dual problem have optimal solutions and the optimal objective values are equal. Moreover, duality theory (for a reference see e.g. Trnovská (2005)) implies that if the primal problem fails to have a strictly feasible solution, then the nonempty set of optimal solutions of the dual one is unbounded (this property was observed in the illustrative example presented in Section 5).

Application to other non-linear models: In this paper, we have presented a method of dealing with reciprocal values in DEA models. The general scheme has been described in Section 3 and applied to the RM model in Section 4. However, the scheme can be applied also to some other models with a similar type of nonlinearity as in the RM model. Such models can be derived from the RM model, when imposing e.g. $\theta_i = \theta$, for all i and $\phi_r = \phi$, for all r, or even $\theta = \frac{1}{\phi}$. In the latter case the model is known as the hyperbolic efficiency model introduced by Färe *et al.* (1985). It has been often used in environmental applications (see Zhou *et al.* (2008)) but only in the case of constant returns to scale. The reason is that solely this type of returns admits a linear reformulation. Still, the approach described in this paper can be applied to this model regardless of its type of returns.

Computational aspects: To our knowledge, the RM model is not implemented in any standard DEA software package. Nevertheless, its primal (26)–(28) or dual (29)–(33) semidefinite formulation can be efficiently solved using interior point methods. These methods are implemented in cone programming solvers like Sedumi (see Sturm (1999)) or SDPT3 (see Toh et al. (1999) and Tutuncu et al. (2003)). The solvers are supported by CVX, a Matlab-based modeling system for convex optimization (see Grant and Boyd (2008),(2013)). To demonstrate the simplicity of coding semidefinite problems in CVX, an illustrative code is added in Appendix C.

Concluding remarks: This paper provides instructions for a theoretical and practical treatment of the RM model and all of its modifications that may arise from the specific needs of particular applications. It offers a general schema for the reformulation of a model containing reciprocal variables as a semidefinite programming problem and for the derivation of the corresponding dual program. The main advantage of our approach is that it provides the common interpretation of the dual multipliers in the form of shadow prices and that the virtual profit is inherent both in the constraints and the objective function of the dual model.

We hope that the results of this article will encourage further study and practical use of the RM model.

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A Derivation of the dual program in the semidefinite setting

Consider the semidefinite programming formulation for the RM model (16)-(18) in a more compact form:

$$\begin{array}{ll} \min & \frac{1}{m+s}(\mathbf{e}_m^T\theta + \mathbf{e}_s^T\psi), \\ \text{s.t.} & X\lambda - diag(x_o)\theta \leq 0_m, \\ & Y\lambda - diag(y_o)\phi \geq 0_s, \\ & \theta \leq \mathbf{e}_m,\ \mathbf{e}_s \leq \phi,\ \lambda \geq 0_n, \\ & \left(\begin{array}{cc} \psi_r & 1 \\ 1 & \phi_r \end{array} \right) \succeq 0, r = 1, \ldots, s, \end{array}$$

where $X = (x_{ij})$ is an $m \times n$ data matrix and $Y = (y_{rj})$ is an $s \times n$ data matrix, $\mathbf{e}_m, \mathbf{e}_s$ are $m-$ and $s-$dimensional column vectors of ones respectively, $diag(x_o), diag(y_o)$ are $m \times m$ and $s \times s$ diagonal matrices with the entries of the vectors x_o and y_o on the diagonal. Finally, $\lambda = (\lambda_1, \ldots, \lambda_n)^T$, $\theta = (\theta_1, \ldots, \theta_m)^T$, $\phi = (\phi_1, \ldots, \phi_s)^T$, $\psi = (\psi_1, \ldots, \psi_s)^T$ are the variables of the problem and ^{T} is the transposition operator.

Following the idea described e.g. in Boyd, Vandenberghe (2004) we derive the corresponding Lagrange dual program. The related Lagrangian is defined as follows:

$$L(\theta, \phi, \psi, \lambda; v, u, w_1, w_2, Z_1, \ldots, Z_s)$$

$$= \frac{1}{m+s}(\mathbf{e}_m^T\theta + \mathbf{e}_s^T\psi) + v^T(X\lambda - diag(x_o)\theta) + u^T(-Y\lambda + diag(y_o)\phi)$$

$$+w_1^T(\theta - \mathbf{e}_m) + w_2^T(\mathbf{e}_s - \phi) - \sum_{r=1}^{s} tr\left[Z_r\begin{pmatrix} \psi_r & 1 \\ 1 & \phi_r \end{pmatrix}\right],$$

where $v, u, w_1, w_2, Z_1, \ldots, Z_s$ are the Lagrange multipliers corresponding to the constraints and the Lagrangian domain is as follows:

$$(\mathbb{R}^m \times \mathbb{R}^s \times \mathbb{R}^s \times \mathbb{R}^n_+) \times (\mathbb{R}^m_+ \times \mathbb{R}^s_+ \times \mathbb{R}^m_+ \times \mathbb{R}^s_+ \times (\mathcal{S}^2_+)^s).$$

In order to derive the Lagrange dual it is useful to notice that the Lagrangian is a linear function of the primal variables and can be rewritten as follows:

$$L(\theta, \phi, \psi, \lambda; v, u, w_1, w_2, Z_1, \ldots, Z_s)$$

$$= \left[\frac{1}{m+s}\mathbf{e}_m - diag(x_o)v + w_1\right]^T \theta + \left[diag(y_o)u - w_2 - z_2\right]^T \phi + \left[\frac{1}{m+s}\mathbf{e}_s - z_1\right]^T \psi$$

$$+(X^T v - Y^T u)^T \lambda - w_1^T \mathbf{e}_m + w_2^T \mathbf{e}_s - 2z_3^T \mathbf{e}_s,$$

where

$$z_t = (z_{t1}, \ldots, z_{ts})^T,\ t = 1, 2, 3, \quad Z_r = \begin{pmatrix} z_{1r} & z_{3r} \\ z_{3r} & z_{2r} \end{pmatrix}.$$

The dual problem can be formulated directly as

$$\begin{array}{rl} \max & -w_1^T \mathbf{e}_m + w_2^T \mathbf{e}_s - 2z_3^T \mathbf{e}_s, \\ \text{s.t.} & \frac{1}{m+s}\mathbf{e}_m - diag(x_o)v + w_1 = 0, \\ & diag(y_o)u - w_2 - z_2 = 0, \\ & \frac{1}{m+s}\mathbf{e}_s - z_1 = 0, \\ & X^T v - Y^T u \geq 0, \\ & u, v, w_1, w_2 \geq 0, \\ & Z_r = \begin{pmatrix} z_{1r} & z_{3r} \\ z_{3r} & z_{2r} \end{pmatrix} \succeq 0, r = 1, \ldots, s. \end{array}$$

By eliminating the variables w_1, w_2 we obtain its simplified form

$$\begin{array}{rl} \max & y_o^T u - x_o^T v + 1 - \sum_{r=1}^{s} tr\left[Z_r\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right], \\ \text{s.t.} & diag(x_o)v - \frac{1}{m+s}\mathbf{e}_m \geq 0, \\ & diag(y_o)u - z_2 \geq 0, \\ & \frac{1}{m+s}\mathbf{e}_s - z_1 = 0, \\ & X^T v - Y^T u \geq 0, \\ & u, v \geq 0, \\ & Z_r = \begin{pmatrix} z_{1r} & z_{3r} \\ z_{3r} & z_{2r} \end{pmatrix} \succeq 0,\ r = 1, \ldots, s \end{array}$$

which can be reformulated as

$$\begin{array}{ll} \max & y_o^T u - x_o^T v + 1 - \sum_{r=1}^s tr\left[Z_r \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right], \\ \text{s.t.} & Y^T u - X^T v \leq 0, \\ & v \geq \frac{1}{m+s} diag(x_o)^{-1} \mathbf{e}_m, \\ & u \geq diag(y_o)^{-1} z_2, \\ & z_1 = \frac{1}{m+s} \mathbf{e}_s, \\ & Z_r = \begin{pmatrix} z_{1r} & z_{3r} \\ z_{3r} & z_{2r} \end{pmatrix} \succeq 0, \; r = 1, \ldots, s. \end{array}$$

B The SBM model

The SBM model for constant returns to scale reads

$$\min_{\lambda, s^x, s^y} \quad \rho := \frac{1 - \frac{1}{m}\sum_{i=1}^m \frac{s_i^-}{x_{io}}}{1 + \frac{1}{s}\sum_{r=1}^s \frac{s_r^+}{y_{ro}}} \tag{39}$$

$$\text{s.t.} \quad \sum_{j=1}^n x_{ij}\lambda_j + s_i^- = x_{io}, \quad i = 1, ..., m, \tag{40}$$

$$\sum_{j=1}^n y_{rj}\lambda_j - s_r^+ = y_{ro}, \quad r = 1, ..., s, \tag{41}$$

$$s^-, s^+, \lambda \geq 0. \tag{42}$$

The following derivation shows its equivalence with the ERM model:

$$\rho := \frac{1 - \frac{1}{m}\sum_{i=1}^m \frac{s_i^-}{x_{io}}}{1 + \frac{1}{s}\sum_{r=1}^s \frac{s_r^+}{y_{ro}}} = \frac{\frac{1}{m}\sum_{i=1}^m \frac{x_{io} - s_i^-}{x_{io}}}{\frac{1}{s}\sum_{r=1}^s \frac{s_r^+ + y_{ro}}{y_{ro}}} = \frac{\frac{1}{m}\sum_{i=1}^m \theta_i}{\frac{1}{s}\sum_{r=1}^s \phi_r}.$$

The dual to the linearized SBM model reads:

$$\max_{v,u} \quad y_o^T u - x_o^T v + 1 \tag{43}$$

$$\text{s.t.} \quad Y^T u - X^T v \leq 0, \tag{44}$$

$$v \geq \frac{1}{m} diag(x_o)^{-1} \mathbf{e}_m, \tag{45}$$

$$u \geq w \frac{1}{s} diag(y_o)^{-1} \mathbf{e}_s, \tag{46}$$

$$u^T y_o - v^T x_o + 1 = w. \tag{47}$$

C An illustrative CVX code

The following CVX code solves the primal semidefinite programming formulation of the RM model (26)–(28). The problem formulation in the code corresponds to the compact form of the problem (26)–(28) as given in Appendix A.

Note that the code allows to remember the dual variables corresponding with certain type of constraints.

```
load inputs.txt;
load outputs.txt;

X=inputs;
Y=outputs;
n=size(X,2);
m=size(X,1); % number of inputs
s=size(Y,1); % number of outputs

for k=1:n
cvx_begin sdp
   variable phi(s,1)
   variable pssi(s,1)
   variable theta(m,1)
   variable lambda(n,1)
   dual variable v;
   dual variable u;
   dual variable w;

   minimize (ones(m,1)'*theta+ones(s,1)'*pssi)
   v: -X*lambda+diag(X(:,k))*theta>=0;
   u: Y*lambda-diag(Y(:,k))*phi>=0;
    theta-ones(m,1)<=0;
    ones(s,1)-phi<=0;
    lambda>=0;
    w: sum(lambda)==1;
    for i=1:s
        [pssi(i) 1; 1 phi(i)]>=0;
    end
cvx_end

val=cvx_optval/(m+s);
end
```

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