

Reliable single allocation hub location problem under hub breakdowns

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Abstract. The design of hub-and-spoke transport networks is a strategic planning problem, as the choice of hub locations has to remain unchanged for long time periods. However, strikes, disasters or traffic breakdown can lead to the unavailability of a hub for a short period of time. Therefore it is important to consider such events already in the planning phase, so that a proper reaction is possible; once a hub breaks down, an emergency plan has to be applied to handle the flows that were scheduled to be served by this hub. In this paper, we develop a two-stage formulation for the single allocation hub location problem which includes the reallocation of sources to a backup hub in case the hub breaks down. In contrast to related problem formulations from the literature, we keep the non-linear structure of the problem in our model. A branch-and-cut framework based on Benders decomposition is designed to solve large scale instances to proven optimality. Thanks to our decomposition strategy, we keep the structure of the resulting formulation similar to the classical single allocation hub location problem, which in turn allows to use classical linearization techniques from the literature. Our computational experiments show that this approach leads to a significant improvement in the performance when embedded into a standard mixed-integer programming solver. We report optimal solutions for instances much bigger than those solved so far in the literature.

Keywords: Hub location, Hub breakdown, Benders decomposition

1 Introduction

Hub location problems are strategic planning problems which have been studied for almost 30 years [29, 19, 1, 7]. The problem consists of organizing the mutual exchange of flows between a large set of depots by choosing a set of hubs out of the set of possible locations and assigning each flow to a path from source to sink being processed at a small number of hubs in between. The aim is to utilize economies of scale in transportation: Although the transport routes are longer and additional costs for hubs apply, the savings from bundled transport usually outweigh these costs. The economies of scale are usually modeled as

being proportional to the transport volume, defined by multiplication with a discount factor $\alpha \in [0, 1]$. The resulting trade-off has to be optimized. Typical applications of hub-based networks arise in airline [17], postal [13], cargo [34], telecommunication [20] and public transportation [28] services.

There are numerous variants of hub location problems. Depending on the way in which nodes may be assigned to hubs, hub location problems can be classified as either multiple allocation [14] or single allocation [29, 13] hub location problems. In multiple allocation problems, the flow of the same node can be routed through different hubs, while in single allocation problems, each node is assigned to exactly one hub. Thus, all flows originating or destined to a node have to be routed via the respective hub to whom it is assigned. Moreover, each of these problems can be classified as capacitated or uncapacitated depending on various types of capacity restrictions. In particular, there can be limitations on the total flow routed on a hub-hub link [21] or on the volume of flow into the hub nodes [15]. As an extension of these models, in [11, 9, 31] the authors consider a version where individual capacity levels can be installed for each hub location. Accordingly, not only the hub nodes have to be chosen but also the capacity level at which each of them will operate.

O’Kelly [29] proposed the first quadratic integer programming formulation for classic uncapacitated, single allocation p -hub median problem where the number of hubs, denoted by p , is given (the so-called p -Hub Median Problem). Since then, many exact and heuristic algorithms have been proposed in the literature, dealing with locating both a fixed and a variable number of hubs, e.g., by Campbell [6], Ernst and Krishnamoorthy [13], Skorin-Kapov et al. [32], Contreras et al. [10], Meier et al. [27], and Ilić et al. [16]. Due to the quadratic nature of the problem, many attempts have been made to linearize the objective function so that the resulting lower bound is strong enough to be used in a branch-and-bound algorithm. Skorin-Kapov et al. [32] and Ernst and Krishnamoorthy [13] proposed “path based” and “flow based” Mixed Integer Linear Programming (MILP) formulations, respectively. For recent surveys on hub location problems we refer the reader to [1, 7].

In this paper we focus on reliable single allocation hub location problem under hub breakdowns. The traditional hub location models lack one important property: The inclusion of uncertainty in hub operation. Earth quakes, flooding, strikes or accidents might be reasons for a breakdown of a hub. Although those deviations from the standard scenario are rare events, they have a major effect on the entire transportation network as flows get stuck on their normal routes. Since in practice it is of major importance to maintain a high service level, the network has to be recovered in order to maintain all flows.

The concept of reliability under breakdown uncertainty has received considerable attention in the context of the facility location problem in which locations in an one-directional transportation network have to be found in order to minimize the transportation cost. In the model of Snyder and Daskin [33], the breakdown of multiple facilities is considered. Each customer is assigned a primary facility and a set of ordered backup facilities to be chosen if the prior facilities fail. The

authors solve their formulation by using Lagrangian relaxation within a branch-and-bound scheme. Cui et al. [12] extend the model of [33] by location specific instead of homogeneous failure probabilities. A continuous approximation heuristic omitting location and assignment details is developed to find near-optimal solutions. Li et al. [24] only consider scenarios where a single facility may fail, but they introduce fortification costs which can be payed for facilities in order to reduce their failure probabilities. Again a Lagrangian relaxation is used to solve this problem. Furthermore, Álvarez-Miranda et al. [2] introduce the concept of robust recovery proposed by Liebchen et al. [25] to facility location with breakdowns. The authors apply successfully a Benders decomposition approach to solve their recoverable robust problem formulation.

Even though the deterministic single allocation hub location problems have been well-studied over the years, the literature addressing hub breakdowns is rather limited. Strategies to recover transportation networks in case of partially or completely unavailable hubs were studied by Løve et al. [26] and O’Kelly et al. [30] in the context of air transportation and telecommunication systems, respectively. However, these policies have to respond to an initially planned transportation networks, typically without consideration of hub breakdowns. Hence, flows have to be rerouted via other hubs by potentially significantly larger costs than in the initial scenario.

Further models of hub breakdowns in hub location problems were studied for the multiple allocation variant. Chaharsooghi et al. [8] either reassign flows to other hubs or penalize them for not being in service in case of a hub breakdown. They solve instances on up to 80 nodes by a neighborhood search heuristic. The model of Kim [18] represents a problem variant, where the backup hubs are distinct from the original hub and are only used in case a original hub fails. An 18-node instance is solved to optimality. Lei [23] examines the impact of hub breakdown in an existing transportation network. The goal is to identify the subset of r breakdown hubs that leads to the most severe degradation in transportation cost. Instances on up to 40 nodes are solved to optimality.

For the single allocation hub location problem, the impact of hub breakdowns to a transportation network is considered within the strategic planning phase by An et al. [3], Azizi et al. [4], and Tran et al. [35]. An et al. [3] hedge against hub breakdown by rerouting each shipment affected by a closed hub. Note that nodes then could potentially be allocated to more than one hub, in contrast to the usual setting of single allocation hub location problems. The authors could solve instances on up to 25 nodes using Lagrangian relaxation. Azizi et al. [4] determine for each hub a backup hub which completely takes over the flows of the closed hub. Instances on up to 10 nodes are solved exactly and instances on up to 80 nodes heuristically by a genetic algorithm. Tran et al. [35] assume that multiple hub can simultaneously fail. For each hub a sequence of backup hubs is determined, where a backup hub takes over all flow from the original hub, when the original hub and all previous backup hubs break down. Their model is linearized with help of a probability lattice calculating the probability that a path is used in the transportation network. Instances up to 20 nodes are solved

to optimality with a commercial solver. Further the author present a tabu search to solve the problem heuristically.

However, from a mathematical modeling point of view, all three approaches by An et al. [3], Azizi et al. [4], and Tran et al. [35] neglect the quadratic structure of the problem by considering a path based formulation [32] for single allocation hub location problems. This blows up the problem under loss of valuable information and leads to intractable models for large scale instances.

1.1 Our contribution

Our aim is to include the cost of recovering a hub breakdown in a transportation network in its initial design by considering both the transportation cost in the classical, i.e., breakdown-free scenario, and the expected rerouting cost in the breakdown scenarios. Both types of costs are explicitly included in the objective function of the hub location problem. To incorporate the reaction on hub breakdown in a mathematical model, we make use of a two-stage model. In the first stage a transportation network is planned assuming that all hubs are in operation. A reaction to hub breakdown is included in the second stage, where a limited amount of recovery options can be executed to reconstitute a feasible solution. We first define the set of breakdown scenarios, then we specify how to react on the revealed scenarios. More precisely, we determine for each affected hub in the first stage solution a backup hub which takes over all the flow from the closed hub. The aim of our two-stage model is to include the choice of backup hub already in the strategic planning phase. In our model, the first stage decisions set up the transportation network under consideration of the costs of rerouted flow due to hub breakdowns which are computed in the second stage.

In order to solve the problem, we use Benders cuts for dealing with the non-linear structure of the problem. In the resulting formulation, which is a Mixed Integer Quadratically Constrained Quadratic Program (MIQCQP), the reallocation variables along with the dependent non-linear terms of the objective function are replaced by a linear number of continuous variables that model the reallocation cost directly. The resulting second-stage subproblems are reduced to efficiently solvable 1-median problems. For further enhancements, we show that these subproblems have the integral property. Thus, we can take use of their dual optimal solution and provide strong Benders cuts which are consecutively added to a modern branch and cut algorithm for solving the first-stage problem. Thanks to our decomposition strategies, we can still use the well-known flow linearization [13] and solve instances much bigger than those solved in [4, 3] in reasonable time.

Considering the concept of hub breakdowns in the initial design, it is not only helpful to reroute each shipment affected by a closed hub, but also to improve the reliability of the model. More precisely, from the reliability point of view, our computational results show that the choice of hubs potentially differs when taking hub breakdowns into account within the strategic planning phase. For example consider Figure 1 representing the solution for an instance of the well

known CAB dataset with 20 nodes under the classical (Figure 1a) and the reliable (Figure 1b) model. As can be observed hub Chicago (4) remains, but hubs Denver (8) and New York (17) are replaced by Dallas (7) and Philadelphia (18) in the reliable solution.

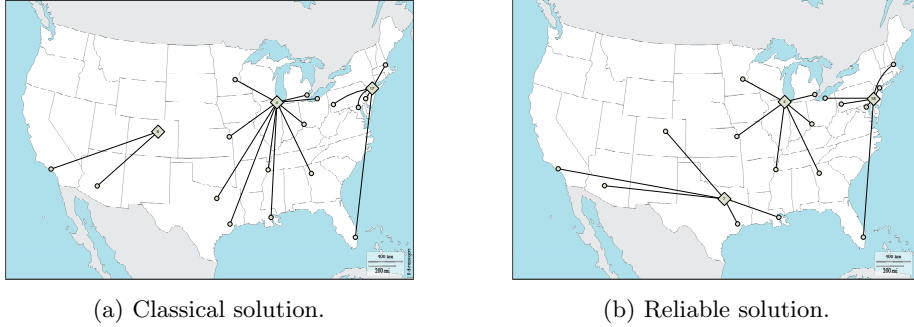


Fig. 1: The choice of hubs differs when taking hub breakdown into account within the strategic planning phase of a transportation network.

1.2 Overview

This paper is organized as follows. The classical Single Allocation Hub Location Problems are described in Section 2. Our two-stage formulation to hedge against hub breakdowns is introduced in Section 3. In Section 4 we describe our reformulation scheme using the Benders cuts and outline our overall solution approach. The computational results are presented in Section 5, and, finally, concluding remarks and possible future works are addressed in Section 6.

2 Classical Single Allocation Hub Location Problems

To model single allocation hub location problem, let us first introduce some notation used in the sequel. We are given a complete directed graph $G = (N, A)$, where $N = \{1, 2, \dots, n\}$ correspond to the origins, destinations and possible hub locations, and A is the arc set. Let w_{ij} be the amount of flow to be transported from node i to node j . We denote by $O_i = \sum_{j \in N} w_{ij}$ and $D_i = \sum_{j \in N} w_{ji}$ the total outgoing flow from node i and the total incoming flow to node i , respectively. For each $l \in N$, let f_l represent the fixed set-up cost of a hub located at node l . The cost per unit of flow for each path $i - l - m - j$ from an origin node i to a destination node j which passes hubs l and m , respectively, is $\chi d_{il} + \alpha d_{lm} + \delta d_{mj}$, where χ , α , and δ are the nonnegative collection, transfer and distribution costs, respectively, and d_{ij} represents the distance between nodes i

and j . For simplification of our notation, we define

$$c_{il} := d_{il} (\chi O_i + \delta D_i) \quad (1)$$

$$q_{iljm} := \alpha w_{ij} d_{lm} \quad (2)$$

for all $i, l, j, m \in N$. Equation (1) sums up all collection and distribution costs between a node i and a hub l . Equation (2) represents the inter-hub transportation cost on the path $i - l - m - j$.

The single allocation hub location problems now consists of selecting a subset of nodes as hubs and assigning the remaining nodes to these hubs such that each nonhub node is assigned to exactly one hub node. Therefore we introduce allocation variables

$$x_{il} = \begin{cases} 1 & \text{if node } i \text{ is allocated to a hub located at } l \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the variables x_{ll} are used to indicate whether l becomes a hub. The objective is to minimize the total cost of the transportation network. The Single Allocation Hub Location Problem (SAHLP) can thus be stated as the following quadratic binary program:

SAHLP :

$$\begin{aligned} \min \quad & \sum_{l \in N} f_l x_{ll} + \sum_{i, l \in N} c_{il} x_{il} + \sum_{i, l, j, m \in N} q_{iljm} x_{il} x_{jm} \\ \text{s.t.} \quad & \sum_{l \in N} x_{il} = 1 \quad i \in N \end{aligned} \quad (3)$$

$$x_{il} \leq x_{ll} \quad i, l \in N \quad (4)$$

$$x_{il} \in \{0, 1\} \quad i, l \in N, \quad (5)$$

where Constraints (3), (4), and (5) are the classical restrictions for the SAHLP. Constraints (3) indicate that node i is allocated to precisely one hub node while Constraints (4) enforce that node i is allocated to a hub node at l only if a hub is located at node l .

Note that we can extend the above model to the Single Allocation p -Hub Median Problem (SAPhMP) where the fixed set-up costs of opening the hubs are replaced by the requirement of opening exactly p hubs, i.e.,

SAPhMP :

$$\begin{aligned} \min \quad & \sum_{i, l \in N} c_{il} x_{il} + \sum_{i, l, j, m \in N} q_{iljm} x_{il} x_{jm} \\ \text{s.t.} \quad & \sum_{l \in N} x_{ll} = p \end{aligned} \quad (6)$$

$$(3), (4), (5),$$

where Constraint (6) enforces the number of open hubs to be p .

3 Single Allocation Hub Location Problems with possible hub breakdown

In order to take the hub breakdown uncertainty into account, we define a finite set \mathcal{S} representing possible scenarios of hub breakdowns. Since hub breakdowns are improbable and also hubs have usually large distances between them, we assume that at most one hub breaks down at once. This leads to a finite set $\mathcal{S} = \{S_0, S_1, \dots, S_n\}$ of $n+1$ scenarios where S_0 represents the classical scenario in which all hubs are in operation and S_1, \dots, S_n describe the case in which the named hub is out of order.

We assume that for each realized breakdown scenario a backup hub is used instead of the closed hub. That means that all flows assigned to the closed hub are rerouted via the backup hub. As we already consider the decision which hub is taken as backup in the strategic planning phase, we have to make two different types of decisions: First, we select for the classical scenario S_0 a subset of nodes as hubs and allocate all non-hub nodes to exactly one hub as the decision variables in SAHLP do. Second, for each scenario S_k , $k \geq 1$, a decision has to be made to react to a breakdown of hub k . This reaction consists in reassigning all flows from the closed hub to the backup hub. Figure 2 illustrates the changes of the transportation network when taking our reactions on hub breakdowns into account. On the left, the standard scenario, where hubs A, B, and C are in operation is depicted. Then hub B breaks down. On the right the reaction is shown, all flows via hub B are now reassigned to hub A.

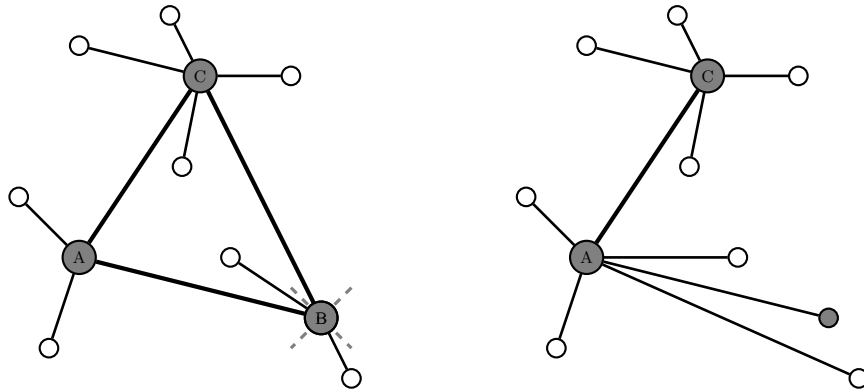


Fig. 2: A transportation network which reacts on the breakdown of hub B.

In order to model the second decision, we define backup and reallocation decision variables for each $k \geq 1$:

$$y_{kl} = \begin{cases} 1 & \text{if hub } l \text{ is a backup for the hub } k \\ 0 & \text{otherwise;} \end{cases}$$

$$z_{il}^k = \begin{cases} 1 & \text{if node } i \text{ is reassigned from hub } k \text{ to the backup hub } l \\ 0 & \text{otherwise.} \end{cases}$$

Before developing the model we need to construct the transportation costs for each realized scenario explicitly. First, since the second-stage decision is a short-term reaction, it is natural to think that whichever decision we take in the future it will be more expensive than if it would have been taken at present. Therefore, we assume that the transportation costs per unit of flow under each scenario S_k with $k \geq 1$ are above the costs for the classical scenario S_0 . For a given scenario S_k , $k \geq 1$, we denote the collection, the transfer and the distribution costs by χ^k , α^k , and δ^k , respectively, where $\chi^k \geq \chi$, $\alpha^k \geq \alpha$, and $\delta^k \geq \delta$. For simplification, we again define the linear and the quadratic cost coefficients for each scenario S_k as $c_{il}^k := d_{il}(\chi^k O_i + \delta^k D_i)$ and $q_{iljm}^k := \alpha^k w_{ij} d_{lm}$ for all $i, l, j, m \in N$ with $l \neq k, m \neq k$.

For given binary variables x and a realized scenario S_k , $k \geq 1$, we introduce transportation cost functions $F_{il}^k(x)$ and $g_k(x)$. The cost function $F_{il}^k(x)$ represents the additional transportation costs which arise by reallocating node i from the closed hub k to backup hub l . This can be computed as the sum of the distribution and collection cost between node i to hub l and the inter-hub transportation cost for paths starting or ending in node i , i.e.,

$$F_{il}^k(x) = c_{il}^k + \sum_{j, m \in N} (q_{iljm}^k + q_{jmil}^k) x_{jm}. \quad (7)$$

However, some of the transportation costs are saved in scenario S_k due to the breakdown of hub k . These are composed of two terms: The first term consists of all distribution and transfer costs that are non-existent in scenario S_k , while the second term computes the inter-hub transportation costs that are canceled in scenario S_k due to rerouting, i.e.,

$$g_k(x) = \sum_{i \in N} c_{ik} x_{ik} + \sum_{i, j, m \in N} \sum_{j, m} (q_{ikjm} + q_{jmik}) x_{ik} x_{jm}. \quad (8)$$

We now define the following real-valued function $Q(x)$ taking a binary allocation vector x as argument:

$$Q(x) = \sum_{k=1}^n p_k Q_k(x) = \sum_{k=1}^n p_k \left(\sum_{i, l} F_{il}^k(x) z_{il}^k - g_k(x) \right) \quad (9)$$

Here, p_k denotes the probability of scenario S_k to occur, for $k \geq 1$.

The single allocation hub location problem with possible hub breakdown is then formulated as:

SAHLP-B :

$$\min \sum_{l \in N} f_l x_{ll} + \sum_{i, l \in N} c_{il} x_{il} + \sum_{i, l \in N} \sum_{j, m \in N} q_{iljm} x_{il} x_{jm} + Q(x)$$

$$\sum_{l \in N: l \neq k} z_{il}^k = x_{ik} \quad i, k \in N \quad (10)$$

$$\sum_{l \in N: l \neq k} y_{kl} = x_{kk} \quad k \in N \quad (11)$$

$$z_{il}^k \leq y_{kl} \quad i, l \in N; l \neq k \quad (12)$$

$$y_{kl} \leq x_{ll} \quad l \in N; l \neq k \quad (13)$$

$$z_{il}^k \in \{0, 1\} \quad i, l \in N; l \neq k \quad (14)$$

$$y_{kl} \in \{0, 1\} \quad l \in N; l \neq k \quad (15)$$

$$(3), (4), (5),$$

where Constraints (10) to (13) connect the location-allocation variables to the reallocation and backup variables. Constraints (10) assure that for every allocation x_{ik} to the breakdown hub k an emergency reallocation z_{il}^k is chosen. Constraints (11) assure that for a breakdown hub k exactly one backup is chosen. Constraints (12) allow the reallocation variables z_{il}^k to a hub at l only if this hub is a backup for hub k . Finally, Constraints (13) enforce the backup variables to a hub at l only if a hub is located at node l . Note that we can easily extend the above model to SApHMP as follows:

SApHMP-B :

$$\min \sum_{i, l} c_{il} x_{il} + \sum_{i, l} \sum_{j, m} q_{iljm} x_{il} x_{jm} + Q(x)$$

$$\text{s.t. } (3) - (6), (10) - (15).$$

The objective function in both SAHLP-B and SApHMP-B corresponds to the expected cost when considering both the classical and the breakdown scenarios. More precisely, expanding $Q(x)$ results in

$$\sum_{l \in N} f_l x_{ll} + \sum_{i, l \in N} (1 - p_l) c_{il} x_{il} + \sum_{i, l \in N} \sum_{j, m \in N} (1 - p_l - p_m) q_{iljm} x_{il} x_{jm}$$

$$+ \sum_{k=1}^n p_k \left(\sum_{i, l} F_{il}^k(x) z_{il}^k \right)$$

where the first term represents the set-up cost with probability of one. The second and third terms are the transportation costs for the classical scenario, while the last term measures the overall rerouting costs of the remaining scenarios.

4 Benders decomposition for SAHLP-B

Due to the nonlinear structure of the SAHLP-B, the standard approach for solving the problem is to linearize it in order to obtain a mixed integer linear program (MILP). The continuous relaxation of the resulting MILP is then solved to optimality to compute a lower bound [6]. However, because of the size of the resulting problems even solving the continuous relaxation is challenging for instances with moderate sizes. To overcome such difficulties, An et al. [3] develop a Lagrangian relaxation approach to solve the continuous relaxation of the resulting MILP. But since their model includes a huge number of variables and constraints, they could solve only instances with up to 25 nodes.

Instead we propose a decomposition based solution method which takes the nonlinear structure of the problem into account. Our solution method is based on Benders decomposition and decomposes the SAHLP-B formulation into a first-stage master problem (classical scenario S_0) and a collection of second-stage subproblems (one subproblem for each scenario S_k with $k \geq 1$). This is possible since the scenario subproblems S_k with $k \geq 1$ can be solved independently for a given set of the first-stage decision variables. In our decomposition, the variables x_{il} are incorporated into a master problem, and then for a given \hat{x} , we have to solve n subproblems to obtain the optimal values of the second-stage reallocation and backup variables z and y , respectively. The resulting master problem is then given by

MP :

$$\min \sum_l f_l x_{ll} + \sum_{i,l} c_{il} x_{il} + \sum_{i,l} \sum_{j,m} q_{iljm} x_{il} x_{jm} + \theta$$

$$\text{s.t. } \theta \geq Q(x) \tag{16}$$

$$\sum_l x_{ll} \geq 2 \tag{17}$$

$$(3), (4), (5),$$

where θ represents the total reallocation costs of all second-stage problems. Note that we add constraint (17) to force the opening of at least two hubs, as needed later. This is no restriction, because the single-hub case can easily be handled in a preprocessing phase.

Let X denote the set of all binary vectors associated with the x_{il} variables. For a given vector $\hat{x} \in X$ and the current value of θ , say $\hat{\theta}$, at a given node of the enumeration tree within the branch-and-bound scheme for MP and for a given

scenario S_k with $k \geq 1$, we consider the following primal subproblem:

$$\text{PS}(\hat{x}, k) : \min Q_k(\hat{x}) = \sum_{i,l \in N} F_{il}^k(\hat{x}) z_{il}^k - g_k(\hat{x}) \quad (18)$$

$$\text{s.t.} \quad \sum_{l \in N: l \neq k} z_{il}^k = \hat{x}_{ik} \quad i \in N \quad (19)$$

$$\sum_{l \in N: l \neq k} y_{kl} = \hat{x}_{kk} \quad (20)$$

$$z_{il}^k \leq y_{kl} \quad i, l \in N; l \neq k \quad (21)$$

$$y_{kl} \leq \hat{x}_{ll} \quad l \in N; l \neq k \quad (22)$$

$$z_{il}^k \in \{0, 1\} \quad i, l \in N; l \neq k \quad (23)$$

$$y_{kl} \in \{0, 1\} \quad l \in N; l \neq k, \quad (24)$$

The constraints arise from SAHLP-B by fixing the allocation vector \hat{x} . In other words, $\text{PS}(\hat{x}, k)$ finds the best backup hub for the failure hub k for given choice of nonhub-hub allocation. Observe that the feasibility of $\text{PS}(\hat{x}, k)$ is ensured by constraints (17). The objective represents the additional cost for rerouting all flows from k to the backup hub.

Now, let $Q_k^*(\hat{x})$ denote the optimal value of subproblem $\text{PS}(\hat{x}, k)$ and let

$$Q^*(\hat{x}) := \sum_{k=1}^n p_k Q_k^*(\hat{x})$$

be the optimal value of the entire second-stage problem. If $\hat{\theta} < Q^*(\hat{x})$, then following [22] we find an integer L-shaped cut as follows:

$$\theta \geq (Q^*(\hat{x}) - L) \left(\sum_{i,l \in N(\hat{x})} x_{il} - \sum_{i,l \in N \setminus N(\hat{x})} x_{il} + 1 - |N(\hat{x})| \right) + L \quad (25)$$

Here L is a valid lower bound on $Q(x)$ and $N(\hat{x}) := \{i, l \in N : \hat{x}_{il} = 1\}$.

4.1 Efficient solution of the subproblems

In order to solve subproblem $\text{PS}(\hat{x}, k)$ efficiently, consider an iteration of Benders Algorithm at a given node of the enumeration tree, with binary solution \hat{x} . Let $H = \{l \in N : \hat{x}_{ll} = 1\}$ be the set of open hubs and consider a scenario S_k with $k \geq 1$. If $k \notin H$, then it is easy to verify that the optimal value of subproblem $\text{PS}(\hat{x}, k)$ is equal to zero. Hence, we only need to consider subproblems for which $k \in H$. In this case we can exploit the subproblem structure to find its optimal value $Q_k^*(\hat{x})$. Let $N^k = \{i \in N : \hat{x}_{ik} = 1\}$ be the set of all nodes that have been assigned to a breakdown hub k and $H_k = H \setminus \{k\}$ be the set of all open hubs different from hub k . Since \hat{x} is integral, subproblem $\text{PS}(\hat{x}, k)$ can be

rewritten as follows:

$$\text{PS}'(\hat{x}, k) : \min Q_k(\hat{x}) = \sum_{i,l:l \in H_k} F_{il}^k(\hat{x}) z_{il}^k - g_k(\hat{x}) \quad (26)$$

$$\text{s.t.} \quad \sum_{l \in H_k} z_{il}^k = 1 \quad i \in N^k \quad (27)$$

$$\sum_{l \in H_k} y_{kl} = 1 \quad (28)$$

$$z_{il}^k \leq y_{kl} \quad i, l; l \in H_k \quad (29)$$

$$z_{il}^k \in \{0, 1\} \quad i, l; l \in H_k \quad (30)$$

$$y_{kl} \in \{0, 1\} \quad l \in H_k. \quad (31)$$

This problem is a 1-median problem which can be solved in a very efficient way: for each $l \in H_k$ we compute

$$\text{val}(l) = \sum_i F_{il}^k(\hat{x}).$$

Then we choose the backup hub l^* for hub k as the one minimizing $\text{val}(l)$.

4.2 Improved Benders cuts

Although subproblems $\text{PS}'(\hat{x}, k)$ for each $\hat{x} \in X$ and each scenario S_k with $k \geq 1$ can be solved efficiently, our preliminary computational experiments showed that using the integer L-shaped cuts alone is not enough to solve big instances to optimality; see Table 1 in Section 5. This is due to the fact that the inequality (25) can cut off only one binary vector at a time. In order to generate more efficient Benders cuts, we propose a modification that uses the dual information of the LP relaxation of the subproblems. For doing so, let us consider again the subproblem $\text{PS}(\hat{x}, k)$.

Proposition 1. *For any binary \hat{x} and any $k \geq 1$, the subproblem $\text{PS}(\hat{x}, k)$ has the integrality property.*

Proof. Consider the LP relaxation of the revised form of subproblem $\text{PS}(\hat{x}, k)$ as in (26) to (31). By subtracting (27) from (28) we have

$$\sum_{l \in H_k} (y_{kl} - z_{il}^k) = 0 \quad i \quad (32)$$

Equation (32) together with (29) implies that

$$y_{kl} = z_{il}^k \quad i, l; l \in H_k \quad (33)$$

This simplifies the LP relaxation of the subproblem to

$$\min Q_k(\hat{x}) = \sum_{l \in H_k} y_{kl} \sum_i F_{il}^k(\hat{x}) - g_k(\hat{x}) \quad (34)$$

$$\text{s.t.} \quad \sum_{l \in H_k} y_{kl} = 1 \quad (35)$$

$$0 \leq y_{kl} \leq 1 \quad l \in H_k \quad (36)$$

and the result follows.

As a consequence of Proposition 1, we can remove the integrality requirement on the variables in $\text{PS}(\hat{x}, k)$ and form its dual linear program. For this, let β, γ, λ , and μ be the dual variables corresponding to constraints (19) to (22), respectively. The dual of $\text{PS}(\hat{x}, k)$ can then be stated as

$$\text{DS}(\hat{x}, k) : \max \quad \sum_i \beta_i \hat{x}_{ik} + \gamma \hat{x}_{kk} - \sum_{l: l \neq k} \mu_l \hat{x}_{ll} - g_k(\hat{x}) \quad (37)$$

$$\text{s.t.} \quad \beta_i - \lambda_{il} \leq F_{il}^k(\hat{x}) \quad i, l; l \neq k \quad (38)$$

$$\gamma + \sum_i \lambda_{il} - \mu_l \leq 0 \quad l; l \neq k \quad (39)$$

$$(\lambda, \mu) \geq 0. \quad (40)$$

By (weak) duality, one optimal solution for $\text{DS}(\hat{x}, k)$ is $(\bar{\beta}, \bar{\gamma}, \bar{\lambda}, \bar{\mu}) = (\bar{\beta}, 0, 0, 0)$ with $\bar{\beta}_i = F_{il^*}^k(\hat{x})$ for all i , where l^* is the chosen backup hub for hub k ; see Section 4.1. Note that the computation of the dual variables $\bar{\beta}$ only requires to solve a 1-median problem. The desired Benders cut is then obtained as

$$\theta \geq \sum_k p_k \left(\sum_i \bar{\beta}_i x_{ik} - g_k(x) \right) \geq \sum_k p_k \left(\sum_i F_{il^*}^k(x) x_{ik} - g_k(x) \right). \quad (41)$$

Using equations (7) and (8) the Benders cut becomes

$$\theta \geq \sum_k p_k \left(\sum_i (c_{il^*}^k - c_{ik}) x_{ik} + \sum_{i,j,m} ((q_{il^*jm}^k + q_{jmil^*}^k) - (q_{ikjm} + q_{jmik})) x_{ik} x_{jm} \right). \quad (42)$$

For a given breakdown of hub k , the Benders cut thus imposes the cost difference between the hub k and its backup hub l^* .

4.3 Solving the master problem

The master problem MP is a MIQCQP containing exponentially many Benders cuts. To solve it, we first linearize it using the classical flow-based linearization of Ernst and Krishnamoorthy [13]. To this end, we introduce the following non-negative flow variables

$$f_{ilm} = x_{il} \sum_j w_{ij} x_{jm} \quad (i, l, m)$$

to represent the total amount of flow originated at node i and routed via hubs located at nodes l and m . The MP then becomes

$$\text{MP}' : \tag{43}$$

$$\min \sum_l f_l x_{ll} + \sum_{i,l} c_{il} x_{il} + \sum_{i,l} \sum_{j,m} \alpha d_{lm} f_{ilm} + \theta$$

$$\text{s.t. (3), (4), (5), (17)}$$

$$\sum_m f_{ilm} - \sum_m f_{iml} = O_i x_{il} - \sum_j w_{ij} x_{jl} \quad i, l \tag{44}$$

$$\sum_m f_{ilm} \leq O_i x_{il} \quad i, l \tag{45}$$

$$f_{ilm} \geq 0 \quad i, l, m \tag{46}$$

$$\begin{aligned} \theta \geq \sum_k p_k \left(\sum_i (c_{il^*}^k - c_{ik}) x_{ik} + \sum_{i,m} (\alpha^k d_{l^*m}^k - \alpha d_{km}) f_{ikm} \right. \\ \left. + \sum_{j,m} (\alpha^k d_{ml^*}^k - \alpha d_{mk}) f_{jmk} \right), \end{aligned} \tag{47}$$

where Constraints (44) are flow balance constraints. Note that following Correia et al. [11] we added Constraints (45) to the model proposed by Ernst and Krishnamoorthy [13] to be sure the model is still correct in case the triangle inequality is not strictly satisfied.

To solve the resulting linearized problem MP' we use Gurobi 6.5 as a state-of-the-art MILP solver. In the branch-and-cut solution framework, each Benders cuts is separated on the fly at each incumbent solution. More precisely, at each new incumbent solution \hat{x} , we solve $|H|$ subproblems PS(\hat{x}, k) to generate the Benders cut (42). Then the quadratic Benders cut is linearized as (47), added to the master problem and its LP relaxation is solved again.

5 Computational results

In this section we present our computational experiments on reliable single allocation hub location problems under hub breakdowns. We implemented the algorithms in C++ with the use of the Gurobi 6.5 solver as subroutine. All experiments were carried out on a single Intel Xeon processor with 3.7 GHz. For numerical tests, we use traditional benchmark instances in hub location research: The CAB and AP datasets.

The former was provided by the Civil Aeronautics Board. It was introduced by O'Kelly [29] and can be found in the OR Library [5]. It contains airline passenger interactions between 25 major cities in the United States of America. On these instances, we focus on hub breakdowns in the SApHMP as considered in previous works such as [3]. We vary the number p of hubs from 2 to 5. The transport cost parameters in the classical scenario are chosen as usual for the CAB dataset: $\alpha \in \{0.2, 0.4, 0.6, 0.8\}$, $\chi = 1.0$, $\delta = 1.0$. We solve the classical

p -hub median problem as baseline SApHMP. For the second-stage transport costs we consider two different settings. The first setting SApHMP-B I shows the effect of rerouting the flows originally attached to the closed hub on the solutions. In order to compare the solutions we assume that the transport costs on rerouted links are equal to the costs in the standard scenario, i.e. $\alpha^k = \alpha$, $\chi^k = \chi$, $\delta^k = \delta$. In practice, rerouting flows lead to higher costs on the rerouted arcs due to short-term reactions. Therefore we assume in the second setting SApHMP-B II that the transportation costs rise by 10% on the rerouted arcs, i.e. $\alpha^k = 1.1\alpha$, $\chi^k = 1.1\chi$, and $\delta^k = 1.1\delta$. The probabilities p_k for a failure of hub k are extracted from Table A2 of [3].

In order to point out that our method is able to solve large instances, we further consider the AP dataset for our computational experiments, which were introduced by Ernst and Krishnamoorthy [13] and can also be found in the OR Library [5]. The data is based on mail flows of Australia Post. The transportation cost parameters are chosen as usual: $\alpha = 0.75$, $\chi = 3.0$, and $\delta = 2.0$. We take over the SApHMP-B I and SApHMP-B II setting from the CAB dataset experiments, i.e. the second-stage transportation cost for rerouted flow are increased by 0% and 10%, respectively. Again, these settings are compared with the classical problem formulation SApHMP. The probability of a hub breakdown is uniformly set to 0.03. We consider the 25, 40, 50, 60, 75, 90 and 100 nodes instances.

5.1 Results on CAB instances

We first demonstrate the computational importance of our Benders cuts method by comparing it to the classical L-shaped cut method and to the framework of An et al. [3]. Then our computational results are discussed in detail.

Algorithmic benefit of Benders cuts:

Our aim is to demonstrate the algorithmic benefit of the Benders cut (42) compared to the classical integer L-shaped cuts (25). In Table 1 we report the runtimes in seconds of our Benders decomposition method using either cut (42) or (25) with a time limit of two hours. When using the Benders cut (42) we could solve all instances within 30 seconds. In turn using the classical integer L-shaped cuts (25) takes our algorithm more than 10 minutes to solve some of the 20 nodes instance and even the time limit was reached on some 25 node instances. In particular, for large p the Benders cuts (42) outperform significantly the integer L-shaped cuts (25). Hence, we only consider Benders cuts (25) for the further computational analysis.

We further compare our method with the solution framework of An et al. [3], which integrated a Lagrangian relaxation in a branch and bound scheme. An et al. [3] reported their results for the SApHMP formulation on the 25 node CAB dataset instances in Table 1. For a fair comparison we adapt their parameter setting, i.e. we chose $\alpha \in \{0.3, 0.5, 0.7\}$, $\chi = 1.0$, $\delta = 1.0$, $\alpha^k = 2\alpha$, $\chi^k = 2\chi$, and $\delta^k = 2\delta$. Note that their experiments were carried out on an Intel Core 2 Duo processor with 3.0 GHz.

Table 1: Runtimes for our Benders decomposition method with cut (42) and (25) on CAB dataset.

Instance			SApHMP-B I		SApHMP-B II	
n	p	α	cuts (42)	cuts (25)	cuts (42)	cuts (25)
20	2	0.2	2.7	4.9	2.5	6.4
20	3	0.2	1.2	11.6	1.5	31.4
20	4	0.2	0.7	177.7	1.2	497.7
20	5	0.2	0.3	177.1	0.7	646.8
20	2	0.4	2.5	3.1	2.5	5.2
20	3	0.4	1.7	7.6	1.9	16.9
20	4	0.4	2.5	47.9	2.9	228.0
20	5	0.4	1.3	81.6	1.5	307.6
20	2	0.6	2.4	3.8	2.4	4.8
20	3	0.6	2.4	5.4	2.2	13.1
20	4	0.6	2.2	29.9	3.0	130.0
20	5	0.6	1.9	87.6	3.0	595.9
20	2	0.8	2.1	2.3	2.0	3.7
20	3	0.8	3.9	10.8	4.3	28.9
20	4	0.8	7.5	44.8	8.7	119.9
20	5	0.8	6.7	37.1	9.1	104.4
25	2	0.2	3.1	6.7	2.9	5.3
25	3	0.2	4.6	336.4	4.0	1972.7
25	4	0.2	4.1	TL	4.9	TL
25	5	0.2	3.0	TL	3.8	TL
25	2	0.4	2.4	5.4	3.4	5.9
25	3	0.4	3.4	72.5	3.5	348.9
25	4	0.4	4.0	1389.7	5.6	TL
25	5	0.4	5.7	4940.3	5.9	TL
25	2	0.6	4.8	6.2	5.5	7.5
25	3	0.6	4.7	40.7	6.0	96.7
25	4	0.6	6.0	138.6	6.2	878.1
25	5	0.6	9.3	1754.2	8.8	TL
25	2	0.8	5.1	5.9	5.0	6.6
25	3	0.8	8.0	13.5	9.9	49.3
25	4	0.8	13.9	83.5	20.1	614.6
25	5	0.8	20.9	309.6	30.4	4459.1

In Figure 3 the runtimes of An et al. and of our method are compared. We observe that our algorithm outperforms the method by An et al. significantly on all instances. This emphasizes the effectiveness of our Benders cuts.

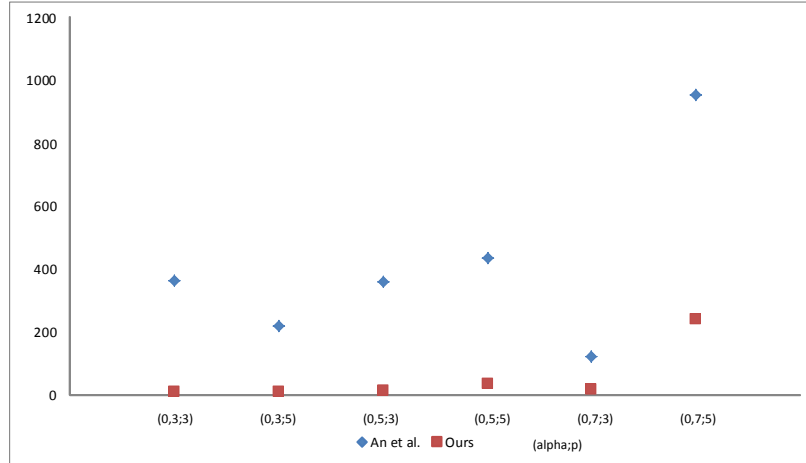


Fig. 3: Comparison of runtimes of An et al. [3] and of our method in seconds on the 25 node CAB instances.

Detailed computational analysis:

For a detailed computational analysis we report our results in Table 2. The first three columns state the number of nodes n , the number of hubs p and the inter-hub discount factor α . Then for each setting SApHMP, SApHMP-B I, and SApHMP-B II the objective and the runtime in seconds are reported. The difference between the objectives of classical solution SApHMP and the stochastic solution SApHMP-B I or SApHMP-B II is the price of robustness, i.e. the amount of extra cost which have to be paid by rerouting the flows affected by a hub breakdown. In Figure 4 the relative prices of robustness on CAB dataset instances are reported. In the SApHMP-B I and SApHMP-B II setting there are 2.6 - 5.7 % and 3.2 - 6.7 % higher costs than in the classical setting, respectively. Note that our backup model guarantees that all flow in the network is maintained in case of a breakdown of a single hub, whereas in the classical formulation only 93.7 - 97.8 % (confer Table 7 in the appendix) of the flow is maintained according to the given hub breakdown probabilities. Thus, our model is appropriate in service-oriented transportation networks.

Table 2 further reports the runtimes for CAB dataset. All instances were solved to optimality within 30 seconds. The efficiency of our solution method further reflects in the fact that the runtime to include hub breakdowns by our

Table 2: Computational results on CAB dataset

Instance			SApHMP		SApHMP-B I		SApHMP-B II	
n	p	α	obj.	time	obj.	time	obj.	time
20	2	0.2	979.09	0.6	1020.23	2.7	1017.64	2.5
20	3	0.2	724.54	0.4	774.46	1.2	783.27	1.5
20	4	0.2	577.62	0.3	626.71	0.7	634.30	1.2
20	5	0.2	467.74	0.3	514.00	0.3	521.02	0.7
20	2	0.4	1042.57	2.1	1077.36	2.5	1080.24	2.5
20	3	0.4	847.77	1.0	889.99	1.7	897.02	1.9
20	4	0.4	727.10	0.8	768.82	2.5	776.92	2.9
20	5	0.4	630.84	0.7	671.61	1.3	679.43	1.5
20	2	0.6	1106.04	2.2	1134.48	2.4	1141.15	2.4
20	3	0.6	971.00	2.4	1003.24	2.4	1010.57	2.2
20	4	0.6	869.16	2.3	904.96	2.2	913.68	3.0
20	5	0.6	793.34	2.3	828.15	1.9	836.70	3.0
20	2	0.8	1169.52	2.0	1190.12	2.1	1196.79	2.0
20	3	0.8	1091.05	5.2	1109.56	3.9	1112.70	4.3
20	4	0.8	1008.49	5.4	1035.12	7.5	1040.68	8.7
20	5	0.8	947.64	6.1	975.65	6.7	980.18	9.1
25	2	0.2	1000.91	0.6	1056.14	3.1	1063.83	2.9
25	3	0.2	767.35	3.6	811.28	4.6	818.73	4.0
25	4	0.2	629.63	0.7	687.01	4.1	695.24	4.9
25	5	0.2	538.37	0.6	587.85	3.0	594.74	3.8
25	2	0.4	1101.63	0.7	1150.72	2.4	1158.41	3.4
25	3	0.4	901.70	2.2	937.39	3.4	945.23	3.5
25	4	0.4	787.52	5.7	830.30	4.0	837.59	5.6
25	5	0.4	707.69	2.1	748.82	5.7	755.69	5.9
25	2	0.6	1201.21	2.9	1244.22	4.8	1251.65	5.5
25	3	0.6	1033.56	5.0	1060.11	4.7	1066.66	6.0
25	4	0.6	939.21	6.3	971.68	6.0	979.49	6.2
25	5	0.6	876.59	12.3	908.29	9.3	915.80	8.8
25	2	0.8	1294.08	7.0	1313.88	5.1	1317.53	5.0
25	3	0.8	1158.83	7.1	1179.16	8.0	1186.75	9.9
25	4	0.8	1087.66	11.5	1113.06	13.9	1120.48	20.1
25	5	0.8	1034.10	15.0	1058.56	20.9	1066.42	30.4

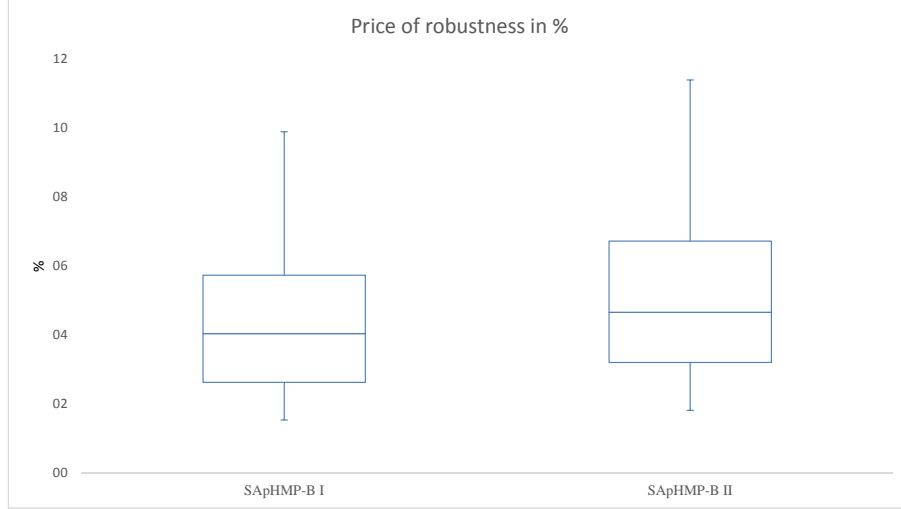


Fig. 4: Price of robustness in % on CAB instances.

Benders decomposition method is not significantly larger than for solving the classical model.

Table 3: Hub locations on CAB dataset

Instance			SApHMP		SApHMP-B I		SApHMP-B II	
n	p	α	obj.	hub loc.	obj.	hub loc.	obj.	hub loc.
20	3	0.2	724.54	4, 12, 17	774.46	4, 12, 17	783.27	4, 12, 17
20	3	0.4	847.77	4, 12, 17	889.99	4, 12, 18	897.02	4, 12, 18
20	3	0.6	971.00	4, 12, 17	1003.24	4, 12, 18	1010.57	4, 12, 18
20	3	0.8	1091.05	4, 8, 17	1109.56	4, 7, 18	1112.70	4, 11, 20

Including hub breakdown in the strategic planning of a transportation network also influences the choice of hubs. In Table 3 we report the chosen hub locations for $n = 20$ and $p = 3$ (hub locations including their backups for further instances can be found in Table 7 in the appendix). First we observe that with $\alpha = 0.2$ in all settings the same set of hub is selected. Then with $\alpha = 0.4, 0.6$ hub 18 is preferred in the backup settings instead of hub 17 in the classical setting. With $\alpha = 0.8$, even two of three hubs are chosen to different. Further higher transportation costs on the rerouted links due to short-term reactions have an explicit impact on the choice of hubs as hubs 8 and 17 in the classical solution are replaced by hubs 7 and 18 in the solution for the SApHMP-B I setting. In Figure 1 both transportation networks are depicted. The solution for the SApHMP-B II setting looks again different. Here, hubs hubs 4, 11 and 20 are chosen. Since the solutions can differ, our claim that hedging against hub break-

down is beneficial to be included in the strategic planning of a transportation network is underlined.

5.2 Results on AP instances

We report our results both for the SApHMP and SAHLP formulation.

Table 4: Computational results on AP dataset (SApHMP)

Instance		SApHMP			SApHMP-B I			SApHMP-B II		
n	p	obj.	time	gap	obj.	time	gap	obj.	time	gap
25	2	175542	1	-	181281	2	-	182433	2	-
25	3	155256	5	-	160278	4	-	161313	4	-
25	4	139197	8	-	143236	4	-	144084	5	-
25	5	123574	4	-	127014	2	-	127816	3	-
40	2	177472	7	-	184153	17	-	185410	20	-
40	3	158831	28	-	163859	27	-	164906	30	-
40	4	143969	27	-	147959	41	-	148865	41	-
40	5	134265	39	-	137814	66	-	138657	78	-
50	2	178484	27	-	184719	37	-	185927	38	-
50	3	158570	27	-	163877	106	-	165795	132	-
50	4	143378	33	-	147190	78	-	148076	109	-
50	5	132367	39	-	135985	107	-	136831	123	-
60	2	179920	64	-	186473	112	-	187656	125	-
60	3	160339	187	-	165429	182	-	166488	257	-
60	4	144720	203	-	148527	387	-	149416	378	-
60	5	132850	454	-	136530	489	-	137365	720	-
75	2	180119	229	-	187008	410	-	188296	520	-
75	3	161057	540	-	166114	1127	-	167173	1328	-
75	4	145734	737	-	149560	1700	-	150341	3120	-
75	5	136011	2050	-	139550	5321	-	140279	7113	-
90	2	179822	878	-	186418	1482	-	187667	1929	-
90	3	160437	1090	-	165576	5078	-	166641	6959	-
90	4	145134	1216	-	148939	TL	1.1 %	149897	TL	1.5 %
90	5	135808	TL	0.3 %	139458	TL	2.7 %	140731	TL	4.1 %
100	2	180225	1006	-	186820	2161	-	188070	2283	-
100	3	160847	1279	-	165977	6611	-	167041	TL	1.7 %
100	4	145897	2073	-	149888	TL	1.1 %	150668	TL	1.1 %
100	5	136929	5133	-	141052	TL	3.9 %	141939	TL	4.2 %

Detailed computational analysis for SApHMP:

For a detailed computational analysis we report our results in Table 4. The first two columns state the number of nodes n and the number of hubs p . Then for each setting SApHMP, SApHMP-B I, and SApHMP-B II the objective and the runtime in seconds are reported. As before, we observe that a price of robustness has to be paid in order to serve all flows in case of a hub breakdown. Figure 5 show that the price of robustness at around 3.1 % and 3.8 % is comparably

small for the SApHMP-B I and SApHMP-B II setting, respectively. In contrast, the classical solutions could only serve about 96.6 % of all flows, where as the backup solution maintains all flows by rerouting them to their backup hubs.

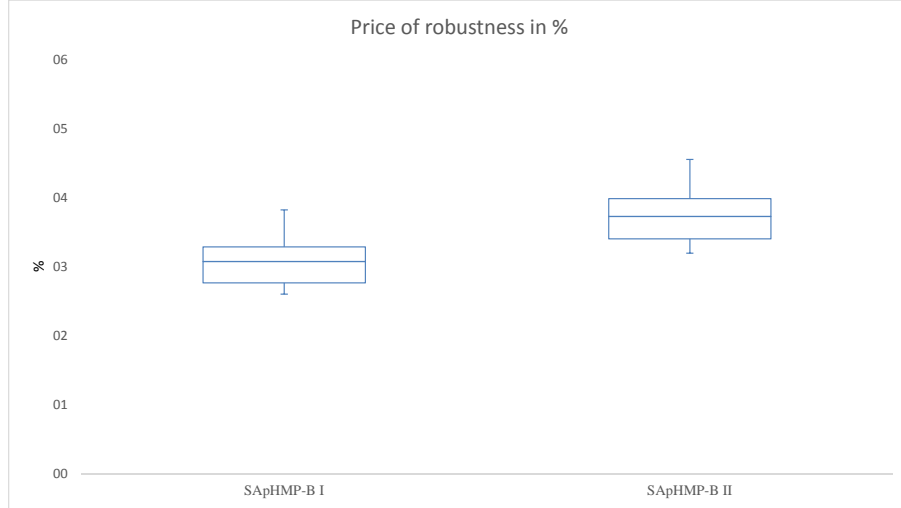


Fig. 5: Price of robustness in % on AP instances.

Table 4 further reports the runtimes for the AP dataset. With increasing instance size, the instances get more involved to solve. All instances on up to 75 nodes and instances on up to 100 with small given number of hubs were solved within two hours to optimality. For the 90 and 100 nodes instances with large given number of hubs we could still provide approximate solutions provably within at most 4.2 % of an optimal solution.

Table 5: Hub locations on AP dataset (SApHMP)

Instance		SApHMP		SApHMP-B I		SApHMP-B II	
n	p	obj.	hub loc.	obj.	hub loc.	obj.	hub loc.
25	4	139197	2, 7, 14, 18	143236	7, 14, 17, 18	144084	7, 14, 17, 18
50	4	143378	14, 28, 33, 35	147190	14, 28, 33, 35	148076	14, 28, 33, 35
75	4	145734	21, 40, 49, 52	149560	21, 40, 49, 52	150341	21, 40, 49, 52
100	4	145897	28, 55, 64, 70	149888 [†]	28, 54, 66, 70	150668 [†]	28, 54, 66, 70

[†] upper bound for the objective function value.

In order to discuss the impact of hedging against hub breakdowns on the solutions we report the selected hub locations for instances $n = 25, 50, 75, 100$ and $p = 4$ in Table 5 (hub locations including their backups for further instances

can be found in Table 8 in the appendix). With $n = 50, 75$ the choice of hubs remains unchanged by including hub breakdown, whereas with $n = 25$ hub 2 is exchanged by hub 17 and with $n = 100$ hubs 55 and 64 are exchanged by hubs 54 and 66. Note that for the 100 nodes instance the optimal solution could be different as we could only prove that the objective of the backup solution is within 1.1 % of an optimal solution. Since we obtain a feasible near-optimal solutions, they are still well suitable in practice.

Table 6: Computational results on AP dataset (SAHLP)

Instance	SAHLP		SAHLP-B I		SAHLP-B II	
n	obj.	time	obj.	time	obj.	time
25	236651	0.9	242389	1.3	243541	1.2
40	240986	16.8	246612	20.8	247790	21.6
50	237422	23.9	243404	74.5	244592	60.6
60	225917	72.9	231746	93.3	232923	113.6
75	235708	253.7	241237	394.6	242389	495.0
90	224196	625.1	230287	1576.6	231497	1553.4
100	238016	1165.6	244184	2523.2	245407	3502.7

Detailed computational analysis for SAHLP:

For a detailed computational analysis we report our results in Table 6. The settings SAHLP, SAHLP-B I, and SAHLP-B II rely on the same parameters as SApHMP, SApHMP-B I, and SApHMP-B II, respectively. For each setting the objective and the runtime in seconds are reported. All instances (up to 100 nodes) were solved to optimality. In Table 9 in the appendix we further report the choice of hub locations. Note that for our instances the choice of hub locations remain unchanged when considering hub breakdowns. This is due to the fact that the rerouting cost are less dominant in the SApHMP-B formulation, since in the objective of the SAHLP the set-up costs are playing an important role.

6 Conclusion

In this paper we introduced two-stage formulations for reliable single allocation hub location problems, which is an important issue in transportation as hub breakdowns have to be managed often in practice. A two-stage reformulation is solved efficiently by reducing the subproblems to 1-median problems. This can be combined with any efficient algorithm to solve the classical problem, in our case a well-known linearization technique. The results show that hedging against hub breakdowns leads to different choices of hub locations. In other words, we prove that it is essential to consider hub breakdowns already in the strategic planning phase of a transportation network. We further showed that taking these uncertainties into account is manageable from the computational point of view.

7 Acknowledgement

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8 Appendix

For sake of completeness we report our results in Table 7, 8, and 9 in detail for all CAB, AP (SAPHMP), and AP (SAHLP) instances, respectively. For the classical solutions, the objective (obj.), the set of chosen hubs (hub loc.), and the relative amount of served flow (serv. flow) are listed. The latter is the quotient between the amount of flow which is still transported in case of hub breakdowns according to the given failure probability and the amount of flow which is transported if all hubs are in operation. For the backup solutions, the objective (obj.), the set of chosen hubs (hub loc.), and the backup hub (bac. hub.) in the order of the hub location are listed. Since for some AP instances the SAPHMP formulation could not be solved to optimally, we report in addition the MIP gap (gap) in Table 8.

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Table 7: Computational results on CAB dataset

n	p	α	obj.	SAPHMP		SAPHMP-B I		SAPHMP-B II			
				hub loc.	serv. flow	obj.	hub loc.	bac. loc.	obj.	hub loc.	bac. loc.
20	2	0.2	979.09	4, 17	95.4 %	1020.23	11, 18	18, 11	1017.64	7, 20	20, 7
20	3	0.2	724.54	4, 12, 17	94.7 %	774.46	4, 12, 17	17, 4, 4	783.27	4, 12, 17	17, 4, 4
20	4	0.2	577.62	4, 12, 16, 17	95.4 %	626.71	4, 12, 16, 17	17, 16, 4, 4	634.30	4, 12, 16, 17	17, 16, 4, 4
20	5	0.2	467.74	4, 7, 12, 14, 17	94.9 %	514.00	4, 7, 12, 14, 17	17, 4, 7, 17, 4	521.02	4, 7, 12, 14, 17	17, 4, 7, 17, 4
20	2	0.4	1042.57	4, 17	95.1 %	1077.36	7, 20	20, 7	1080.24	7, 20	20, 7
20	3	0.4	847.77	4, 12, 17	94.1 %	889.99	4, 12, 18	18, 4, 4	897.02	4, 12, 18	18, 4, 4
20	4	0.4	727.10	1, 4, 12, 17	94.6 %	768.82	1, 4, 12, 17	4, 1, 4, 4	776.92	1, 4, 12, 17	4, 1, 4, 4
20	5	0.4	630.84	4, 7, 12, 14, 17	94.1 %	671.61	4, 7, 12, 14, 17	4, 7, 17, 4	679.43	4, 7, 12, 14, 17	17, 4, 7, 17, 4
20	2	0.6	1106.04	4, 17	94.9 %	1134.48	4, 18	18, 4	1141.15	4, 18	18, 4
20	3	0.6	971.00	4, 12, 17	93.7 %	1003.24	4, 12, 18	18, 4, 4	1010.57	4, 12, 18	18, 4, 4
20	4	0.6	869.16	1, 4, 12, 17	94.1 %	904.96	1, 4, 12, 17	4, 1, 4, 4	913.68	1, 4, 12, 17	4, 1, 4, 4
20	5	0.6	793.34	4, 7, 12, 14, 17	93.6 %	828.19	4, 7, 12, 14, 17	17, 4, 7, 17, 4	836.70	4, 7, 12, 14, 17	17, 4, 7, 17, 4
20	2	0.8	1169.52	4, 17	94.6 %	1190.12	4, 18	18, 4	1196.79	4, 18	18, 4
20	3	0.8	1091.05	4, 8, 17	94.8 %	1109.56	4, 7, 18	18, 4, 4	1112.70	4, 11, 20	20, 4, 4
20	4	0.8	1008.49	1, 4, 8, 17	95.2 %	1035.12	1, 4, 8, 18	4, 18, 4, 4	1040.68	1, 4, 8, 18	4, 18, 4, 4
20	5	0.8	947.64	1, 4, 6, 8, 17	95.2 %	975.65	1, 4, 6, 8, 17	6, 6, 4, 4, 6	980.18	1, 4, 6, 8, 17	6, 6, 4, 4, 6
25	2	0.2	1000.91	12, 20	97.8 %	1056.14	12, 20	20, 12	1063.83	12, 20	20, 12
25	3	0.2	767.35	4, 12, 17	94.6 %	811.28	4, 12, 18	18, 4, 4	818.73	4, 12, 18	18, 4, 4
25	4	0.2	629.63	4, 12, 17, 24	94.5 %	687.01	1, 4, 12, 17	4, 1, 4, 4	695.24	1, 4, 12, 17	4, 1, 4, 4
25	5	0.2	538.37	4, 7, 12, 14, 17	94.7 %	587.85	4, 7, 12, 14, 17	17, 4, 7, 17, 4	594.74	4, 7, 12, 14, 18	18, 4, 7, 18, 4
25	2	0.4	1101.63	12, 20	97.5 %	1150.72	12, 20	20, 12	1158.41	12, 20	20, 12
25	3	0.4	901.70	4, 12, 18	95.3 %	937.39	4, 12, 18	18, 4, 4	945.23	4, 12, 18	18, 4, 4
25	4	0.4	787.52	1, 4, 12, 17	94.4 %	830.30	1, 4, 12, 18	4, 1, 4, 4	837.59	1, 4, 12, 18	4, 1, 4, 4
25	5	0.4	707.69	4, 7, 12, 14, 17	93.9 %	748.82	4, 7, 12, 14, 18	18, 4, 7, 18, 4	755.69	4, 7, 12, 14, 18	18, 4, 7, 18, 4
25	2	0.6	1201.21	12, 20	97.4 %	1244.22	12, 20	20, 12	1251.65	12, 20	20, 12
25	3	0.6	1033.56	2, 4, 12	95.5 %	1060.11	2, 4, 12	4, 2, 4	1066.66	2, 4, 12	4, 2, 4
25	4	0.6	939.21	1, 4, 12, 17	93.9 %	971.68	1, 4, 12, 18	4, 1, 4, 4	979.49	1, 4, 12, 18	4, 1, 4, 4
25	5	0.6	876.59	4, 7, 12, 14, 17	93.5 %	908.29	4, 7, 12, 14, 18	18, 4, 7, 18, 4	915.80	4, 7, 12, 14, 18	18, 4, 7, 18, 4
25	2	0.8	1294.08	12, 20	97.1 %	1313.88	8, 20	20, 8	1317.53	8, 20	20, 8
25	3	0.8	1158.83	2, 4, 12	95.2 %	1179.16	2, 4, 12	4, 2, 4	1186.75	2, 4, 12	4, 2, 4
25	4	0.8	1087.66	1, 4, 12, 18	94.7 %	1113.06	1, 4, 12, 18	4, 1, 4, 4	1120.48	2, 4, 7, 12	4, 2, 4, 7
25	5	0.8	1034.10	1, 4, 7, 12, 18	94.8 %	1058.56	1, 4, 7, 12, 18	18, 1, 4, 7, 4	1066.42	1, 4, 7, 12, 18	18, 1, 4, 7, 4

Table 8: Computational results on AP data set (SApHMP)

n	p	SApHMP				SApHMP-B I				SApHMP-B II			
		obj.	gap	loc.	serv. flow	obj.	gap	hub loc.	bac. loc.	obj.	gap	hub loc.	bac. loc.
25	2	175542	-	8, 18	96.7 %	181281	-	8, 18	18, 8	182433	-	8, 18	18, 8
25	3	155256	-	7, 14, 18	96.6 %	160278	-	7, 14, 18	14, 18, 14	161313	-	7, 14, 18	14, 18, 14
25	4	139197	-	2, 7, 14, 18	96.4 %	143236	-	7, 14, 17, 18	14, 18, 18, 17	144084	-	7, 14, 17, 18	14, 18, 18, 17
25	5	123574	-	2, 7, 14, 17, 18	96.3 %	127014	-	2, 7, 14, 17, 18	7, 14, 18, 18, 17	127816	-	2, 7, 14, 17, 18	7, 14, 18, 18, 17
40	2	177472	-	12, 28	96.7 %	184153	-	12, 28	28, 12	185410	-	12, 28	28, 12
40	3	158831	-	12, 22, 28	96.6 %	163859	-	12, 22, 28	22, 28, 28	164906	-	12, 22, 28	22, 28, 28
40	4	143969	-	12, 22, 26, 28	96.5 %	147959	-	12, 22, 26, 28	22, 28, 28, 26	148865	-	12, 22, 26, 28	22, 28, 28, 26
40	5	134265	-	3, 12, 22, 26, 28	96.4 %	137814	-	3, 12, 22, 26, 28	12, 3, 28, 28, 26	138657	-	3, 12, 22, 26, 28	12, 3, 28, 28, 26
50	2	178484	-	14, 35	96.7 %	184719	-	15, 35	35, 15	185927	-	15, 35	35, 15
50	3	158570	-	14, 28, 35	96.6 %	163877	-	14, 28, 35	28, 35, 28	165795	-	14, 28, 35	28, 35, 28
50	4	143378	-	14, 28, 33, 35	96.5 %	147190	-	14, 28, 33, 35	28, 35, 35, 33	148076	-	14, 28, 33, 35	28, 35, 35, 33
50	5	132367	-	4, 14, 28, 33, 35	96.3 %	135985	-	4, 14, 28, 33, 35	14, 4, 35, 35, 33	136831	-	4, 14, 28, 33, 35	14, 4, 35, 35, 33
60	2	179920	-	16, 42	96.7 %	186473	-	16, 42	42, 16	187656	-	17, 42	42, 17
60	3	160339	-	16, 33, 42	96.6 %	165429	-	16, 33, 42	33, 42, 33	166488	-	16, 33, 42	33, 42, 33
60	4	144720	-	16, 33, 39, 42	96.5 %	148527	-	16, 33, 39, 42	33, 42, 42, 39	149416	-	16, 33, 39, 42	33, 42, 42, 39
60	5	132850	-	4, 17, 34, 39, 42	96.3 %	136530	-	4, 17, 33, 39, 42	17, 33, 17, 42, 39	137365	-	4, 17, 33, 39, 42	17, 33, 17, 42, 39
75	2	180119	-	21, 52	96.7 %	187008	-	21, 52	52, 21	188296	-	21, 52	52, 21
75	3	161057	-	21, 40, 52	96.6 %	166114	-	21, 40, 52	40, 52, 40	167173	-	21, 40, 52	40, 52, 40
75	4	145734	-	21, 40, 49, 52	96.5 %	149560	-	21, 40, 49, 52	40, 52, 52, 49	150341	-	21, 40, 49, 52	40, 52, 52, 49
75	5	136011	-	5, 22, 42, 49, 52	96.3 %	139550	-	5, 22, 41, 49, 52	22, 41, 22, 52, 49	140279	-	5, 22, 41, 49, 52	22, 41, 22, 52, 49
90	2	179822	-	24, 63	96.7 %	186418	-	25, 63	63, 25	187667	-	25, 63	63, 25
90	3	160437	-	24, 49, 62	96.6 %	165576	-	24, 49, 62	49, 62, 49	166641	-	24, 49, 62	49, 62, 49
90	4	145134	-	24, 49, 58, 62	96.5 %	148939	1.1 %	24, 49, 58, 62	49, 62, 62, 58	149897	1.5 %	24, 49, 58, 62	49, 62, 62, 58
90	5	135808	0.3 %	5, 24, 49, 58, 62	96.4 %	139458	2.7 %	5, 24, 49, 57, 62	24, 5, 62, 62, 57	140731	3.4 %	5, 27, 51, 58, 63	27, 5, 27, 63, 58
100	2	180225	-	28, 71	96.7 %	186820	-	29, 71	71, 29	188070	-	29, 71	71, 29
100	3	160847	-	28, 55, 70	96.6 %	165977	-	28, 55, 70	55, 70, 55	167041	1.7 %	28, 55, 70	55, 70, 55
100	4	145897	-	28, 55, 64, 70	96.5 %	149888	1.1 %	28, 54, 66, 70	54, 70, 70, 66	150668	1.1 %	28, 54, 66, 70	54, 70, 70, 66
100	5	136929	-	7, 28, 55, 64, 70	96.4 %	141052	3.9 %	7, 31, 56, 63, 70	31, 56, 31, 70, 63	141939	4.2 %	7, 31, 56, 63, 70	31, 56, 31, 70, 63

Table 9: Computational results on AP data set (SAHLP)

n	SAHLP			SAHLP-B I			SAHLP-B II		
	obj.	hub loc.	serv. flow	obj.	hub loc.	bac. loc.	obj.	hub loc.	bac. loc.
25	236651	8, 18	96.7 %	242389	8, 18	18, 8	243541	8, 18	18, 8
40	240986	14, 28	96.7 %	246612	14, 28	28, 14	247790	14, 28	28, 14
50	237422	15, 36	96.7 %	243404	15, 36	36, 15	244592	15, 36	36, 15
60	225917	17, 42	96.7 %	231746	17, 42	42, 17	232923	17, 42	42, 17
75	235708	22, 52	96.7 %	241237	22, 52	52, 22	242389	22, 52	52, 22
90	224196	23, 63	96.7 %	230287	23, 63	63, 23	231497	23, 63	63, 23
100	238016	29, 73	96.7 %	244184	29, 73	73, 29	245407	29, 73	73, 29

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