On the Fermat point of a triangle

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Abstract

For a given triangle $\triangle ABC$, Pierre de Fermat posed around 1640 the problem of finding a point P minimizing the sum s_P of the Euclidean distances from P to the vertices A, B, C. Based on geometrical arguments this problem was first solved by Torricelli shortly after, by Simpson in 1750, and by several others. Steeped in modern optimization techniques, notably duality, however, we show that the problem admits a straightforward solution. Using Simpson's construction we furthermore derive a formula expressing s_P in terms of the given triangle. This formula appears to reveal a simple relationship between the area of $\triangle ABC$ and the areas of the two equilateral triangles that occur in the so-called Napoleons Theorem.

1 Introduction

We deal with two problems in planar geometry. Both problems presuppose that three noncollinear points A, B and C are given:

(F) Find a point P minimizing the sum s_P of the Euclidean distances from P to the vertices of $\triangle ABC$.

(D) Find an equilateral triangle T of maximum height h_T, with the points A, B and C on different sides of T.

Fermat's name is associated with various problems and theorems. Fermat posed[1] problem (F) around 1640. In other literature (F) is often referred to as as the 3-point Fermat Problem. For its long history we refer to, e.g., (Krarup and Vajda, 1997; Boltyanski *et al.* , 1999; Bruno *et al.* , 2014).

Using a result of his student Viviani, Torricelli presented a geometrical construction of the Fermat point P, repeated below in Section 3 and accompanied by his correctness proof in Section 5. Viviani's result will be discussed in Section 4.

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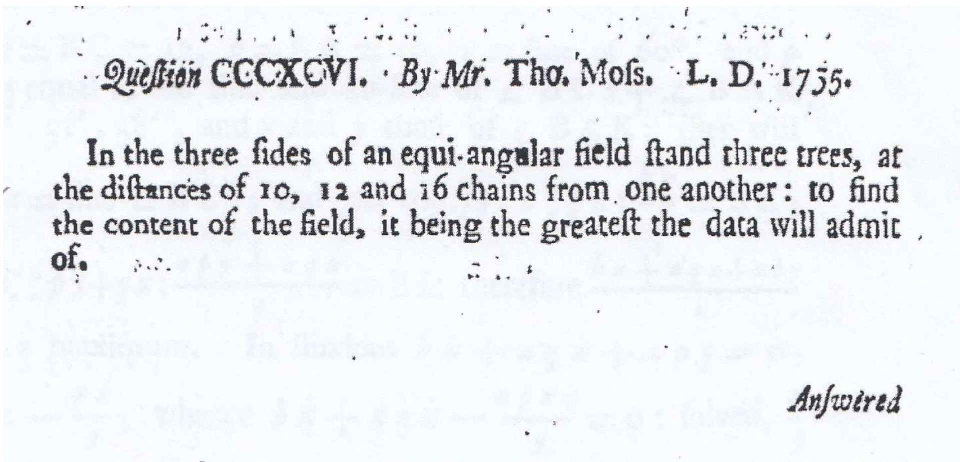
[1]There seems to be uncertainty about the birth year of Fermat, see e.g. (Krarup and Vajda, 1997).

Pierre de Format (1601?-1665) Evangelista Torricelli (1608-1647)

Problem (D) was posed by T. Moss (Moss, 1755) in The Ladies Diary, as shown in Figure 1.[2] As this figure indicates, the problem has been solved but it is not known by whom.



Queſtion CCCXCVI. By Mr. Tho. Moſs. L. D. 1755.

In the three ſides of an equi-angular field ſtand three trees, at the diſtances of 10, 12 and 16 chains from one another: to find the content of the field, it being the greateſt the data will admit of.

Anſwered

Figure 1: Original formulation of problem (D) in the Ladies Diary.

As will be made clear in Section 6, Torricelli's approach already reveals that the problems (F) and (D) are dual to each other.

[2]The Ladies' Diary: or, Woman's Almanack appeared annually in London from 1704 to 1841 after which it was succeeded by The Lady's and Gentleman's Diary. It was a respectable place to pose mathematical problems (brain teasers!) and sustain debate.

Our first aim is to show that application of duality in *conic optimization* yields the solution of (F), straightforwardly. Developed in the 1990s, the field of conic optimization (CO) is a generalization of the more well-known field of linear optimization (LO). Its main source is (Ben-Tal and Nemirovski, 2001). No prior knowledge of LO or CO is needed for reading the paper.

It turns out that two cases need be distinguished depending on the largest angle in the triangle ABC:

Case F1: the largest angle in $\triangle ABC$ is less than 120^o;
Case F2: the largest angle in $\triangle ABC$ is at least 120^o.

Our second aim is to show that in case F1 the problems (F) and (D) are each others dual problem; the meaning of this sentence will be made clear below. The corresponding result can be stated as follows.

Theorem 1.1 *Let X be any point of a triangle ABC of type F1 and T an equilateral triangle such that A, B and C lie on different sides of T. Then one has*

$$h_T \leq s_X. \tag{1}$$

Equality holds if and only if X solves problem (F) and T solves (D).

The inequality (1) expresses so-called *weak duality* for problems (F) and (D), whereas the last statement in Theorem 1.1 says that also *strong duality* holds. The proof of Theorem 1.1 will be given in Section 5.

As far as we know there does not exist a simple formula expressing s_P in terms of the given points A, B and C. Such a formula is derived in Section 8. It uses a simple geometric construction of Simpson that yields a line segment with length s_P, as presented in Section 7. In Section 9 we show that the sides of the so-called Napoleon's outward triangle of $\triangle ABC$ have length $s_P/\sqrt{3}$.

In Section 2 we reformulate (F) as a conic optimization problem and introduce its dual problem. As will be shown in Section 3, the optimality conditions for this pair of problems immediately yield the correctness of Torricelli's construction of the Fermat point.

2 Analytic approach to Fermat's problem

Fermat's problem as stated in (F) is a geometric problem. To begin with we put it in analytic form. To each of the relevant points we associate a vector as follows:

$$x = OP, \quad a_1 = OA, \quad a_2 = OB, \quad a_3 = OC.$$

Here we assume that the plane is equipped with a 2-dimensional coordinate system, with origin O. Moreover, OP denotes the vector that goes from O to P. The distance from P to A is the length of the vector PA. Since $OP + PA = OA$ we have $PA = a_1 - x$. Hence, the distance from P to A is just the Euclidean norm of the vector $a_1 - x$. As a consequence, the sum of the distances from P to A, B and C is equal to $\sum_{i=1}^{3} \|a_i - x\|$. This function will be called our *objective function*. We need to minimize this function when x runs through all vectors in $\mathbf{R}^2$. So (F) can be reformulated as the following optimization problem:

$$\min_x \left\{ \sum_{i=1}^{3} \|a_i - x\| \mid x \in \mathbf{R}^2 \right\}. \tag{2}$$

Each term in the above sum is nonnegative, and since A, B and C are noncollinear, at most one of them can be zero. Hence, the objective function is positive for every $x \in \mathbf{R}^2$. Furthermore, each term is strictly convex in x, as follows from the triangle inequality, and so will be their sum. We conclude that the objective function is strictly convex and positive. This implies that problem (2) has a unique solution, which necessarily is the Fermat point of $\triangle ABC$.

Since the objective function is a sum of Euclidean norms of linear functions of x, and the domain is $\mathbf{R}^2$, (2) is a special case of a convex optimization problem, namely a so-called *conic optimization problem* or a CO problem for short.

Every convex optimization problem has a dual problem. This is a maximization problem with the property that the objective value of any dual feasible solution is less than or equal to the objective value of any 'primal' feasible solution. This is the property called *weak duality*.

Finding a problem with this property can be a tedious task. For the class of CO problems there exists a simple recipe to get a dual problem. For the current purpose it is not necessary to go into further detail. It suffices to mention that the theory of CO yields the following dual problem of (2):

$$\max_{y_i} \left\{ \sum_{i=1}^{3} a_i^T y_i \mid \sum_{i=1}^{3} y_i = 0,\ \|y_i\| \leq 1,\ i \in \{1,\ 2,\ 3\} \right\}. \tag{3}$$

To show that (2) and (3) considered together exhibit the desired property of weak duality, assume that x and the triple $(y_1,\ y_2,\ y_3)$ are feasible for (2) and (3), respectively. We may then write

$$\sum_{i=1}^{3} \|a_i - x\| \geq \sum_{i=1}^{3} \|a_i - x\| \, \|y_i\| \geq \sum_{i=1}^{3} (a_i - x)^T y_i = \sum_{i=1}^{3} a_i^T y_i. \tag{4}$$

The first inequality in (4) is due to the fact that $\|y_i\| \leq 1$ for each i and the second inequality follows from the Cauchy-Schwarz inequality; the equality in (4) follows since $\sum_{i=1}^{3} y_i = 0$, whence also $x^T \sum_{i=1}^{3} y_i = 0$. This shows that we have weak duality.

This is all we need to reveal a property of the Fermat point that enabled Torricelli to geometrically construct this point in a very simple way. As will become clear in the next section the property in question follows by elementary means from (4).

Since (3) maximizes a linear function over a bounded and closed convex domain, an optimal solution certainly exists. This, by the way, is predicted by the duality theory for CO, which also guarantees that the optimal values of (2) and (3) are equal and attained.

3 Solution of Fermat's problem

In this section we show that we have also *strong duality*, i.e., the optimal values of (2) and (3) are equal. Using this we derive the main result of this section, namely that in case F1 the Fermat point P satisfies

$$\angle APB = \angle BPC = \angle CPA = 120^o. \tag{5}$$

As was established in the previous section, (2) has a (unique) optimal solution, denoted by x_P. We have strong duality if and only if there exists a dual feasible y such that we have equalities throughout in (4), with $x = x_P$. This happens if and only if, for each i,

$$\|a_i - x_P\| = \|a_i - x_P\| \, \|y_i\| = (a_i - x_P)^T \, y_i. \tag{6}$$

Let us first consider the case where $x_P \neq a_i$, for each i. Then we have $a_i - x_P \neq 0$ for each i. The first equality in (6) then implies $\|y_i\| = 1$, and then the second equality implies

$$y_i = \frac{a_i - x_P}{\|a_i - x_P\|}. \tag{7}$$

Thus, if x_P is not one of the vertices of $\triangle ABC$ we can conclude that (6) determines each of the vectors y_i uniquely. We need still to verify whether these vectors are dual feasible. This holds if and only if $\sum_{i=1}^{3} y_i = 0$. Since $y_1 + y_2 = -y_3$ it follows that $\|y_1 + y_2\| = \|y_3\| = 1$. Since

$$1 = \|y_1 + y_2\|^2 = \|y_1\|^2 + 2y_1^T y_2 + \|y_2\|^2 = 2y_1^T y_2 + 2,$$

it follows that $y_1^T y_2 = -\frac{1}{2}$. Hence, y_1 and y_2 must make an angle of 120^o. According to (7) the angle between the vectors y_1 and y_2 is the same as the angle between the vectors $a_1 - x$ and $a_2 - x$, which is the angle between PA and PB. Thus, $\angle APB = 120^o$. The same argument applies to any two of the vectors y_i. So we may conclude that the Fermat point P satisfies (5).

The Fermat point must lie in the interior of $\triangle ABC$. This can now be shown by using $\sum_{i=1}^{3} y_i = 0$ once more. This relation is equivalent to

$$\frac{a_1 - x_P}{\|a_1 - x_P\|} + \frac{a_2 - x_P}{\|a_2 - x_P\|} + \frac{a_3 - x_P}{\|a_3 - x_P\|} = 0,$$

which can be rewritten as

$$x_P = \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3, \tag{8}$$

where

$$\lambda_i = \frac{\frac{1}{\|a_i - x_P\|}}{\frac{1}{\|a_1 - x_P\|} + \frac{1}{\|a_2 - x_P\|} + \frac{1}{\|a_3 - x_P\|}}, \qquad i = 1, 2, 3. \tag{9}$$

Note that $\lambda_i > 0$ for each i. The sum of these coefficients being 1, this means that x_P is a convex combination of the vectors a_i. Since each λ_i is positive, it follows that P lies in the interior of $\triangle ABC$.

Since the sum of the angles in the $\triangle APC$ is 180^o, we have $\angle APC + \angle ACP + \angle PAC = 180^o$. Since $\angle APC = 120^o$ and $\angle PAC > 0$ we get $\angle ACP < 60^o$. In the same way we get

$\angle BCP < 60^o$. Hence, $\angle C = \angle ACP + \angle BCP < 120^o$. The same holds for $\angle A$ and $\angle B$. We conclude that if $x_P \neq a_i$ for each i, we are in case F1.

It remains to deal with the case where $x_P = a_i$ for some i. Without loss of generality we assume that this happens for $i = 3$. In that case the optimal value is simply given by $\|a_3 - a_1\| + \|a_3 - a_2\|$. We next show that this happens only in case F2.

Since the points A, B and C are not collinear, we have $x_P \neq a_i$ for $i \in \{1, 2\}$. Hence, y_1 and y_2 are given by (7) and $y_3 = -y_1 - y_2$. Dual feasibility requires $\|y_3\| \leq 1$, which holds if and only if $\|y_1\|^2 + 2y_1^T y_2 + \|y_2\|^2 \leq 1$. Since $\|y_1\| = \|y_2\| = 1$, this holds if and only if $y_1^T y_2 \leq -\frac{1}{2} = \cos 120^o$. Therefore, $\angle APB \geq 120^o$. Since $x_P = a_3$ we have $P = C$ and therefore also $\angle ACB \geq 120^o$, proving that we are in case F2.

At this stage we may conclude that we have completely solved (F) in case F2: then the Fermat point is the vertex with the largest angle, and s_P equals the sum of the distances of that vertex to the other two vertices. As is pointed out in (Boltyanski *et al.* , 1999, p. 235-236) this case has either been overlooked or weakly treated in the early literature. It was explicitly stated for the first time by Bonaventura Cavalieri (1598-1647) (Cavalieri, 1657, p. 504-510). A simple geometric proof of its solution had to wait until 1976 (Sokolowsky, 1976).

It is worthwhile to show Sokolowsky's geometric proof for case F2. Without loss of generality we suppose that $\angle C = \gamma > 120^o$. Now let Q be an arbitrary point in $\mathbf{R}^2$ different from C. In Figure 2 Q lies inside $\triangle ABC$, but the argument below is also valid when Q lies outside the triangle or on its boundary. With $\beta = 180 - \gamma$ we turn the triangle

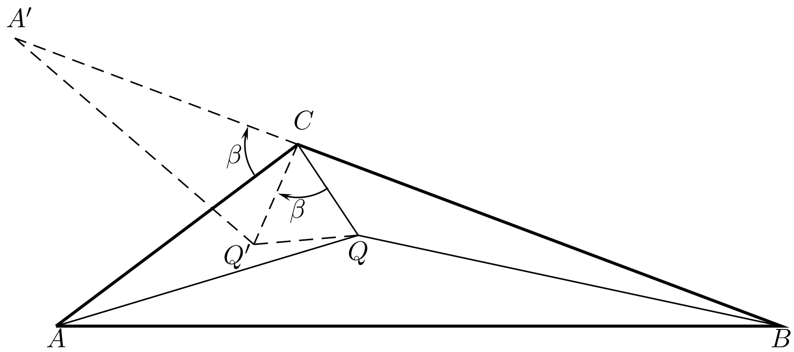


Figure 2: Sokolowsky's proof for case F2.

CQA clockwise around the vertex C over the angle β. This yields a triangle $CQ'A'$ such that the vectors $A'C$ and CB point in the same direction and

$$|CQ| = |CQ'|, \quad |AQ| = |A'Q'|, \quad |CA| = |CA'|.$$

Since $\angle Q'CQ = \beta < 60^o$, the equal base angles in the isosceles triangle $Q'CQ$ exceed β, which implies $|CQ| > |QQ'|$. By using this and $|AQ| = |A'Q'|$ we get

$$s_Q = |QA| + |QB| + |QC| > |Q'A'| + |QB| + |QQ'|.$$

The last expression is the length of the polygonal path $A'Q'QB$, which is certainly larger than the length of the straight path $A'CB$. Since $|A'C| = |AC|$ we obtain

$$s_Q > |AC| + |CB| = s_C.$$

Since this holds for any point Q in $\mathbf{R}^2$ different from C, it follows that C is the Fermat point.

From now on we restrict ourselves to case F1. In that case neither x_P nor s_P is yet known; we only have proved (5). But, as observed by Torricelli[3], this property already enables us to construct P. His surprisingly simple construction is demonstrated in Figure 3. The circumcircles of the outward equilateral triangles on the sides AC and BC intersect at the Fermat point, because the three angles between the dashed lines PA, PB and PC all equal 120^o.

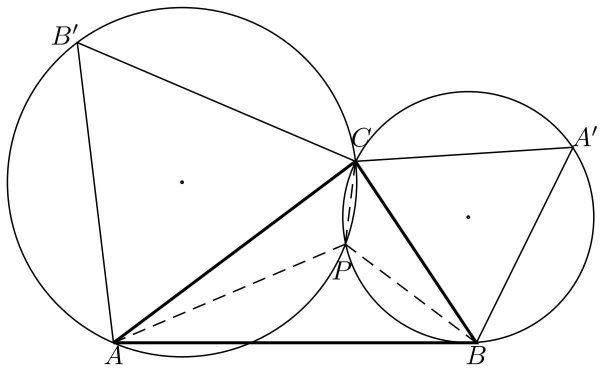


Figure 3: Torricelli's construction of the Fermat point in case F1.

Torricelli also found a geometric proof that (5) must hold. We present this in Section 5. But first Viviani's Theorem is presented in the next section.

4 Viviani's Theorem

In this section we deal with a lemma due to Torricelli's student Vincenzo Viviani (1622-1703) (Viviani, 1659) now known as Viviani's Theorem.

Lemma 4.1 (Viviani's Theorem) *If T is an equilateral triangle then for each point X in T the sum of its distances to the sides of T equals the altitude h_T of T.*

Proof: The proof is easy and demonstrated in Figure 4. The point X divides $\triangle ABC$ into three triangles: $\triangle AXB$, $\triangle BXC$ and $\triangle CXA$. Since the sum of their areas equals the area of $\triangle ABC$ we get

$$\tfrac{1}{2}\,|AB|\,h_1 + \tfrac{1}{2}\,|BC|\,h_2 + \tfrac{1}{2}\,|CA|\,h_3 = \tfrac{1}{2}\,|AB|\,h.$$

Since $|AB| = |BC| = |CA|$, this implies $h = h_1 + h_2 + h_3$, proving the lemma. □

It is worth noting that that if X is on the boundary of T then one or two of the three triangles in the above proof reduce to a single point. But the argument used in the above proof remains valid. This also follows by considering a problem that is quite similar to Fermat's problem but much easier to solve, namely

[3]Torricelli was a famous French number theorist, student of Galileo and discoverer of the barometer.

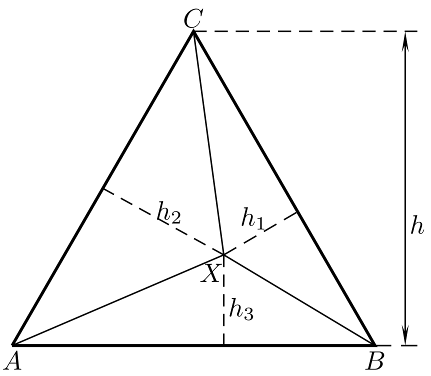


Figure 4: Viviani's Theorem: $h = h_1 + h_2 + h_3$.

(V) Find the point V that minimizes (or maximizes) the sum τ_V of the distances of V to the sides of $\triangle ABC$.

Lemma 4.2 *The minimal (or maximal) value of τ_V is attained at a vertex of $\triangle ABC$, and it equals the height of $\triangle ABC$ seen from that vertex.*

Proof: Let n_A be a unit vector that is orthogonal to BC such that $n_A^T(b-a) > 0$. Then the distance to BC from any point X in $\triangle ABC$ is equal to $n_A^T(b-x)$. Defining n_B and n_C in a similar way, the sum of the distances of X to the sides of $\triangle ABC$ is given by

$$f(x) := n_A^T(b-x) + n_B^T(c-x) + n_C^T(a-x).$$

This function depends linearly on x. Hence, its minimal (or maximal) value is attained at a vertex of $\triangle ABC$ and equals the distance of that vertex to the opposite side of $\triangle ABC$. □

The height value of an equilateral triangle seen from one of its vertices is independent of that vertex. Hence, $f(x)$ is constant on an equilateral triangle, which yields a second proof of Viviani's Theorem.

5 Torricelli's triangle

We now show how Torricelli used Viviani's Theorem to solve Fermat's problem. We have already seen how Torricelli constructed a point P in $\triangle ABC$ that satisfies (5). How did he know that the Fermat point has this property?[4]

For that purpose he introduced another triangle whose sides contain the vertices A, B and C and are perpendicular to AP, BP and CP, respectively, with P such that the three angles at P are 120^o. In this way he obtained the dashed triangle $A_1B_1C_1$ in Figure 5, now called *Torricelli's triangle*. Since $\angle CPB$ is 120^o and $\angle PCA_1 = PBA_1 = 90^o$ we

[4]Some authors who ask this question leave it unanswered, suggesting that it was based on Torricelli's intuition (Gueron and Tessler, 2002, p. 444), (Bruno *et al.* , 2014, p. 88).

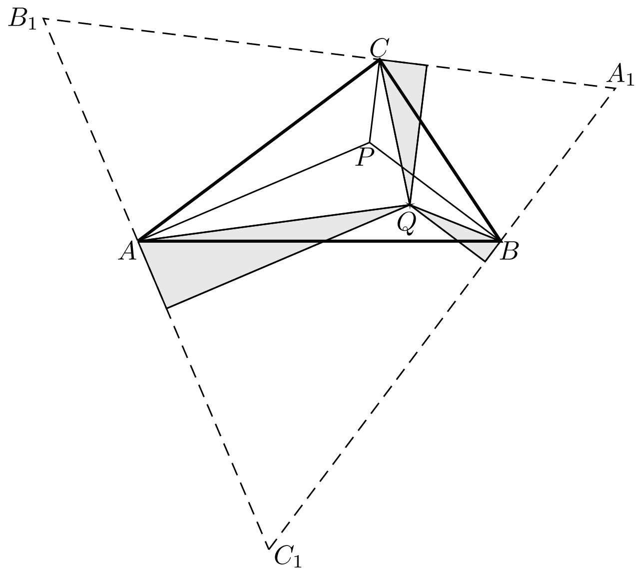


Figure 5: Torricelli's proof of (5).

have $\angle A_1 = 360^o - 120^o - 2 \times 90^o = 60^o$. For similar reasons also $\angle B_1 = \angle C_1 = 60^o$. Hence, $\triangle A_1B_1C_1$ is equilateral.

By the construction of Torricelli's triangle the sum of the distances of P to its sides is equal to $|PA| + |PB| + |PC|$, which is s_P. By Viviani's Theorem this sum is equal to the height of Torricelli's triangle, which we denote by h. Thus, $s_P = h$.

Now let Q be any other point in $\triangle ABC$. Viviani's Theorem tells us that the sum of the distances from Q to the sides of Torricelli's triangle also equals h. The grey triangles in Figure 5 make clear that this sum is less than or equal to $|QA| + |QB| + |QC|$, which is s_Q. So we obtain $s_Q \geq h = s_P$, proving that P minimizes the sum of the distances to A, B and C. It means that P solves (F), which implies that it is the Fermat point.

6 Duality of (F) and (D)

Torricelli's triangle can also be used to prove the duality relation between the problems (F) and (D), as described in Theorem 1.1. In order to prove this theorem we have also drawn another equilateral (dotted) triangle in Figure 6 such that the points A, B and C lie on different sides, as well as line segments from P orthogonal to the sides of this triangle.

Considering the grey triangles in the figure it becomes clear that the sum of the distances of P to the sides of the dotted triangle is less than the sum of the distances of P to the sides of Torricelli's triangle. By Viviani's Theorem these sums are equal to the heights of the respective triangles. The dotted triangle's height is therefore less than the height of the Torricelli triangle. Since the height of Torricelli's triangle equals s_P, this proves Theorem 1.1.

Note that the above proof also makes clear that Torricelli's triangle for $\triangle ABC$ solves problem (D). So, when Torricelli was solving Fermat's problem, i.e., problem (F), he

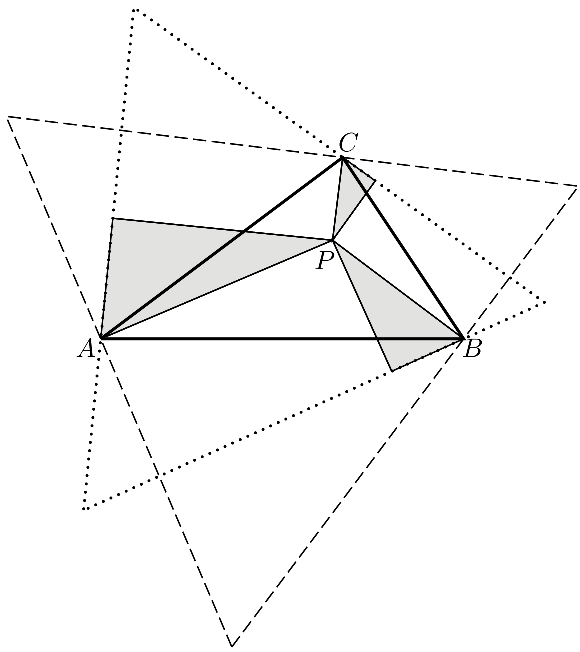


Figure 6: Torricelli's triangle maximizes h_T.

also solved problem (D), although most likely being unaware of this. This is typical for problems that are each others dual. At first sight they seem to be unrelated – though both are based on the same input data, but when solving one of the two problems usually one gets enough information to also solve the other problem. Harold W. Kuhn (Kuhn, 1991, p. 87) states: *Until further evidence is discovered this must stand as the first instance of duality in nonlinear programming* as was established in 1811-1812 by M. Vecten, professor of *mathématiques spéciale* at the Lycée de Nismes.

In the next section we discuss another method to construct the Fermat point. This method will enable us to find a expression for s_P in terms of the given $\triangle ABC$. This will be done in Section 8.

7 Simpson lines

To introduce Simpson's method we use Figure 7, which shows a triangle ABC satisfying F1 and its Fermat point P. So the three angles at P are all equal to 120^o.

We turn $\triangle BPC$ over 60 degrees clockwise around B, yielding $\triangle BP'A'$, as shown in Figure 7. Then $\triangle BCA'$ is isosceles. Since $\angle CBA'$ is 60 degrees it is even equilateral. The same argument yields that also $\triangle BPP'$ is equilateral. Hence, all corners in both triangles are 60^o. Since $\angle APB = 120^o$ it follows that $\angle APP' = 180^o$. This proves that APP' is a straight line segment. The same argument yields that also $PP'A'$ is a straight line segment. It follows that $APP'A'$ is a straight line segment. The length of this segment equals $|AP| + |PP'| + |P'A'| = |AP| + |BP| + |CP|$, which is exactly s_P, the sum of the distances of the Fermat point P to the points A, B and C.

The line AA' is named a *Simpson line*. It can be easily constructed by ruler and compass

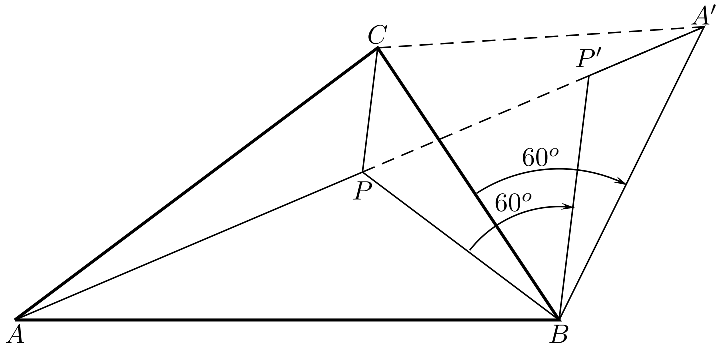


Figure 7: Simpson line AA'.

by first drawing the outward equilateral triangle BCA'. In the same way one may construct Simpson lines BB' and CC'. The Simpson lines intersect at P. This construction of the Fermat point is due to Simpson[5] (1710-1761) (Simpson, 1750). Strangely enough Simpson did not observe that the lengths of the three Simpson lines are the same and equal to s_P. This was first shown by Heinen (Heinen, 1834).

As a byproduct of the fact that $\triangle BPP'$ is equilateral we get $\angle BPP' = 60^o$, which means that the Simpson line PA' is the bisector of $\angle BPC$ as was observed in (Krarup and Vajda, 1997, p. 218). This in turn implies that P is also the Fermat point of $\triangle A'B'C'$, because of (5). Note that $|AA'| + |BB'| + |CC'| = 3s_P$ implies

$$|AP| + |PA'| + |BP| + |PB'| + |CP| + |PC'| = 3s_P.$$

Hence, $|PA'| + |PB'| + |PC'| = 2s_P$, which is the sum of the distances from the Fermat point of $\triangle A'B'C$'to its vertices.

8 Formula for s_P

Using the property of the Simpson lines just found, an expression of s_P in terms of $\triangle ABC$ is immediately at hand, simply by computing the length of the line segment AA' in Figure 7. To compute the point A' we use that it connects the mid point of BC by a line perpendicular to BC. It will be convenient to use below the notations

$$a := OA, \quad b := OB, \quad c := OC.$$

Defining $a' := OA'$, we then have

$$a' = \tfrac{1}{2}(b + c) + d,$$

where d is a vector perpendicular to BC whose length equals the distance of A' to BC. Since $\triangle CBA'$ is equilateral, this distance equals $\frac{1}{2}\sqrt{3}\,\|b - c\|$. Moreover, the vector $[b_2 -$

[5]Thomas Simpson (1710-1761) was an English mathematician. He taught mathematics at the Royal Military Academy and became known for the Simpson rule for numerical integration (Gass and Assad, 2005).

$c_2; c_1 - b_1]$ is orthogonal to BC and has length $\|b - c\|$. Hence,

$$d = \pm\tfrac{1}{2}\sqrt{3}\begin{bmatrix} b_2 - c_2 \\ c_1 - b_1 \end{bmatrix}. \tag{10}$$

One easily understands that d is an outward direction at $\frac{1}{2}(b + c)$ if $d^T(b - a) > 0$ and an inward direction if $d^T(b - a) < 0$. It is worth to have a closer look at the quantity $d^T(b - a)$. We may write

$$\begin{aligned}
\begin{bmatrix} b_2 - c_2 \\ c_1 - b_1 \end{bmatrix}^T \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \end{bmatrix} &= (b_2 - c_2)(b_1 - a_1) + (c_1 - b_1)(b_2 - a_2) \\
&= -b_2 a_1 - c_2 b_1 + c_2 a_1 + c_1 b_2 - c_1 a_2 + b_1 a_2 \\
&= \det[c\, b] - \det[c\, a] + \det[b\, a] = \det\begin{bmatrix} 1 & 1 & 1 \\ c & b & a \end{bmatrix},
\end{aligned}$$

where the last equality sign follows by evaluating the determinant $\det\begin{bmatrix} 1 & 1 & 1 \\ c & b & a \end{bmatrix}$ to its first row. The absolute value of this determinant equals two times the area of $\triangle ABC$. In the sequel we assume that the determinant is positive; if this is not so, interchanging the names of two vertices will make it positive. As a consequence d will be an outward direction at $\frac{1}{2}(b + c)$. So we must use the plus sign in (10). This gives

$$a' = \tfrac{1}{2}(b + c) + \tfrac{1}{2}\sqrt{3}\begin{bmatrix} b_2 - c_2 \\ c_1 - b_1 \end{bmatrix}. \tag{11}$$

As before, we use $|AB|$ to denote the length of line segment AB. So $|AB| = \|a - b\|$. We also use $|ABC|$ to denote the area of $\triangle ABC$.

Lemma 8.1 *In case F1 one has*

$$s_P = \sqrt{\tfrac{1}{2}\left(|AB|^2 + |BC|^2 + |CA|^2\right) + 2\sqrt{3}\,|ABC|}. \tag{12}$$

Proof: From Section 7 we know that $s_P = \|a - a'\|$. Therefore, $s_P^2 = \|a\|^2 + \|a'\|^2 - 2a^T a'$. Since

$$\begin{aligned}
|a'|^2 &= \tfrac{1}{4}\|b + c\|^2 + \tfrac{3}{4}\|b - c\|^2 + \tfrac{1}{2}\sqrt{3}(b + c)^T \begin{bmatrix} b_2 - c_2 \\ c_1 - b_1 \end{bmatrix} \\
&= \|b\|^2 + \|c\|^2 - b^T c + \tfrac{1}{2}\sqrt{3}\left((b_1 + c_1)(b_2 - c_2) + (b_2 + c_2)(c_1 - b_1)\right) \\
&= \|b\|^2 + \|c\|^2 - b^T c + \sqrt{3}\,(b_2 c_1 - b_1 c_2) \\
&= \|b\|^2 + \|c\|^2 - b^T c + \sqrt{3}\det[c\, b].
\end{aligned}$$

and

$$\begin{aligned}
2a^T a' &= a^T\left(b + c + \sqrt{3}\begin{bmatrix} b_2 - c_2 \\ c_1 - b_1 \end{bmatrix}\right) \\
&= a^T b + a^T c + \sqrt{3}\,(a_1 b_2 - a_2 b_1 - a_1 c_2 + a_2 c_1) \\
&= a^T b + a^T c + \sqrt{3}\,(\det[a\, b] - \det[a\, c]),
\end{aligned}$$

it follows that

$$\begin{aligned}
s_P^2 &= \|a\|^2 + \|b\|^2 + \|c\|^2 - a^T b - b^T c - a^T c + \sqrt{3}\,(\det[c\, b] - \det[a\, b] + \det[a\, c]) \\
&= \tfrac{1}{2}\left(\|a - b\|^2 + \|b - c\|^2 + \|c - a\|^2\right) + \sqrt{3}\det\begin{bmatrix} 1 & 1 & 1 \\ c & b & a \end{bmatrix}.
\end{aligned}$$

This gives the lemma. □

Surprisingly enough the formula for s_P just found also appears when dealing with so-called Napoleon triangles. This will be the subject of the next section, where we find a line segment of length $s_P/\sqrt{3}$.

9 Napoleon's triangles

Figure 8 shows a triangle ABC and the three outward equilateral triangles on its sides. The centres of the triangles BCA', CAB' and ABC' are denoted by X, Y and Z, respectively. Napoleon's Theorem[6] states that the triangle XYZ is equilateral. Moreover, the same holds for the triangle $X'Y'Z'$, where X', Y' and Z' are the centres of the inward equilateral triangles on the sides of ABC. The triangle XYZ is called the outward Napoleon triangle and $X'Y'Z'$ the inward Napoleon triangle.

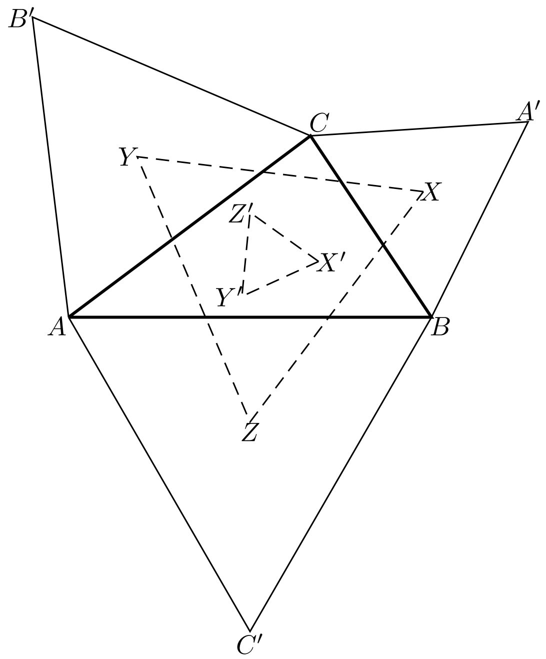


Figure 8: Napoleon's triangles.

The first part of the next theorem is just Napoleon's Theorem. Part of the second statement is also known, namely that XYZ and $X'Y'Z'$ have the same centre (Honsberger, 1973, p. 40, Exercise 8).

Theorem 9.1 *The triangles XYZ and $X'Y'Z'$ are equilateral. The centres of these tri-*

[6]According to (Krarup and Vajda, 1997) and others, any connection between Napoleon's Theorem and the French emperor is highly doubtful. It may amuse the reader to know that Napoleons Theorem is claimed to be one of the most often rediscovered results in triangular geometry. Thus, no less than 32 sources related to it are mentioned in (Wetzel, 1992).

angles coincide with the centre of ABC. Moreover,

$$s_P = \sqrt{3}\,|XY|$$

and

$$|XY|^2 + |X'Y'|^2 = \tfrac{1}{3}\left(|AB|^2 + |BC|^2 + |CA|^2\right) \tag{13}$$
$$|XY|^2 - |X'Y'|^2 = \tfrac{4}{\sqrt{3}}|ABC|. \tag{14}$$

Proof: Let us define $a = OA$, $x = OX$, etc. Then $a' = OA'$ is given by (11). Since $x = \frac{1}{3}(b + c + a')$ it therefore follows that

$$x = \tfrac{1}{3}\left(b + c + \tfrac{1}{2}(b+c) + \tfrac{1}{2}\sqrt{3}\begin{bmatrix} b_2 - c_2 \\ c_1 - b_1 \end{bmatrix}\right) = \tfrac{1}{2}(b+c) + \tfrac{1}{6}\sqrt{3}\begin{bmatrix} b_2 - c_2 \\ c_1 - b_1 \end{bmatrix}.$$

The centre X' of the inward equilateral triangle on side BC is obtained by mirroring X in the line BC, which gives

$$x' = \tfrac{1}{2}(b+c) - \tfrac{1}{6}\sqrt{3}\begin{bmatrix} b_2 - c_2 \\ c_1 - b_1 \end{bmatrix}.$$

In the same way we get the following expressions for y, y', z and z':

$$\begin{aligned} y &= \tfrac{1}{2}(c+a) + \tfrac{1}{6}\sqrt{3}\begin{bmatrix} c_2 - a_2 \\ a_1 - c_1 \end{bmatrix}, \quad & y' &= \tfrac{1}{2}(c+a) - \tfrac{1}{6}\sqrt{3}\begin{bmatrix} c_2 - a_2 \\ a_1 - c_1 \end{bmatrix} \\ z &= \tfrac{1}{2}(a+b) + \tfrac{1}{6}\sqrt{3}\begin{bmatrix} a_2 - b_2 \\ b_1 - a_1 \end{bmatrix}, \quad & z' &= \tfrac{1}{2}(a+b) - \tfrac{1}{6}\sqrt{3}\begin{bmatrix} a_2 - b_2 \\ b_1 - a_1 \end{bmatrix}. \end{aligned}$$

From the above expressions one readily obtains $x + y + z = x' + y' + z' = a + b + c$, which implies that the centres of XYZ, $X'Y'Z'$ and ABC coincide.
One has $|XY| = \|x - y\|$ and $|X'Y'| = \|x' - y'\|$. We start by computing $\|x - y\|$. We may write

$$\begin{aligned} x - y &= \tfrac{1}{2}(b-a) + \tfrac{1}{2\sqrt{3}}\left(\begin{bmatrix} b_2 - c_2 \\ c_1 - b_1 \end{bmatrix} - \begin{bmatrix} c_2 - a_2 \\ a_1 - c_1 \end{bmatrix}\right) \\ &= \tfrac{1}{2}(b-a) + \tfrac{1}{2\sqrt{3}}\begin{bmatrix} a_2 + b_2 - 2c_2 \\ 2c_1 - a_1 - b_1 \end{bmatrix}. \end{aligned}$$

As a consequence we have

$$\begin{aligned} \|x-y\|^2 &= \left(\tfrac{1}{2}(b_1 - a_1) + \tfrac{\sqrt{3}}{6}(a_2 + b_2 - 2c_2)\right)^2 + \left(\tfrac{1}{2}(b_2 - a_2) + \tfrac{\sqrt{3}}{6}(2c_1 - a_1 - b_1)\right)^2 \\ &= \tfrac{1}{4}(b_1 - a_1)^2 + \tfrac{1}{4}(b_2 - a_2)^2 + \tfrac{1}{12}(a_2 + b_2 - 2c_2)^2 + \tfrac{1}{12}(2c_1 - a_1 - b_1)^2 \\ &\qquad + \tfrac{\sqrt{3}}{6}(b_1 - a_1)(a_2 + b_2 - 2c_2) + \tfrac{\sqrt{3}}{6}(b_2 - a_2)(2c_1 - a_1 - b_1) \\ &= \tfrac{1}{12}\left(3(b_1 - a_1)^2 + 3(b_2 - a_2)^2 + (a_2 + b_2 - 2c_2)^2 + (2c_1 - a_1 - b_1)^2\right) \\ &\qquad + \tfrac{\sqrt{3}}{6}\left((b_1 - a_1)(a_2 + b_2 - 2c_2) + (b_2 - a_2)(2c_1 - a_1 - b_1)\right) \\ &= \tfrac{1}{3}\left(a_1^2 + a_2^2 + b_1^2 + b_2^2 + c_1^2 + c_2^2 - a_1c_1 - a_2c_2 - a_1b_1 - b_1c_1 - a_2b_2 - b_2c_2\right) \\ &\qquad + \tfrac{\sqrt{3}}{3}\left(a_2(b_1 - c_1) + b_2c_1 - b_1c_2 + a_1(-b_2 + c_2)\right) \\ &= \tfrac{1}{3}\left(\|a\|^2 + \|b\|^2 + \|c\|^2 - c^Tb - b^Ta - a^Tc\right) \\ &\qquad + \tfrac{1}{\sqrt{3}}\left(\det[b\,a] - \det[c\,a] + \det[c\,b]\right) \\ &= \tfrac{1}{6}\left(\|a-b\|^2 + \|b-c\|^2 + \|c-a\|^2\right) + \tfrac{1}{\sqrt{3}}\det\begin{bmatrix} 1 & 1 & 1 \\ c & b & a \end{bmatrix}. \end{aligned}$$

Comparison of the last expression with the expression for s_P^2 in (12) yields $s_P^2 = 3\,\|x-y\|^2$, which gives the value for $|XY|$ in the theorem. Since the expression is invariant under cyclic permutations of a, b and c it follows that $\|x-y\| = \|y-z\| = \|z-x\|$, which implies that $\triangle XYZ$ is equilateral.

Now consider the triangle $X'Y'Z'$. We have

$$\begin{aligned} x'-y' &= \tfrac{1}{2}(b-a) - \tfrac{1}{2\sqrt{3}}\left(\begin{bmatrix} b_2-c_2 \\ c_1-b_1 \end{bmatrix} + \begin{bmatrix} c_2-a_2 \\ a_1-c_1 \end{bmatrix}\right) \\ &= \tfrac{1}{2}(b-a) - \tfrac{1}{2\sqrt{3}}\begin{bmatrix} a_2+b_2-2c_2 \\ 2c_1-a_1-b_1 \end{bmatrix}. \end{aligned}$$

The only difference with the expression for $x-y$ is the sign of the second term. It is easy to check that a similar derivation as for $\|x-y\|^2$ leads to

$$\|x'-y'\|^2 = \tfrac{1}{6}\left(\|a-b\|^2 + \|b-c\|^2 + \|c-a\|^2\right) - \tfrac{1}{\sqrt{3}}\det\left(\begin{bmatrix} 1 & 1 & 1 \\ c & b & a \end{bmatrix}\right).$$

From this and the expression for $\|x-y\|^2$ we readily obtain (13) and (14). □

The area of an equilateral triangle with side length s equals $\frac{\sqrt{3}}{4}s^2$. Moreover, the area $|ABC|$ of $\triangle ABC$ is equal to $\frac{1}{2}\det\begin{bmatrix} 1 & 1 & 1 \\ c & b & a \end{bmatrix}$. As a consequence of (14) we therefore get a surprisingly simple relation between the areas of Napoleon's triangles and the area of $\triangle ABC$, namely (cf. (Coxeter and Greitzer, 1967, p. 64) and (Honsberger, 1973, p. 40, Exercise 7))

$$|ABC| = |XYZ| - |X'Y'Z'|.$$

10 Concluding remarks

As was made clear the key to the solution of Fermat's problem is property (5), which we obtained straightforwardly by using the conic dual of Fermat's problem. It may be worth pointing out that at first (and second!) sight there is no obvious relation between the 'analytic' dual (3) and the 'geometric' dual (D) of Fermat's problem. Whereas (2) easily is identified as a mathematical model for Fermat's problem (F), it seems harder to recognize (3) as a mathematical model for (D). This is remarkable and seems to deserve further investigation.

Different variants of Fermats original problem have been investigated by several authors. Among the more obvious extensions are weighted versions where the distances are multiplied by positive or negative weights associated with the vertices of $\triangle ABC$ as in e.g. (Gueron and Tessler, 2002) and (Jalal and Krarup, 2003). Other lines of research have focused upon the number of given points and/or the dimension of the space. For an overview the interested reader is referred to (Bruno *et al.* , 2014).

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