A Comment on the Paper 'The Algebraic Structure of the Arbitrary-Order Cone'

Baha Alzalg[†*]

[†]*Department of Mathematics, The University of Jordan, Amman 11942, Jordan*

Abstract. We point out an omission in the paper 'The Algebraic Structure of the Arbitrary-Order Cone' by Alzalg published in J. Optim. Theory Appl. in 2016. In reaction to this omission, some alternative results have been presented.

Keywords. p^{th}-order cones · Semi-inner product · Banach algebras

1 An Omission in Paper [1]

In this section, we point out an omission in [1]. The development in [1] revolves fundamentally around the claim that the p^{th}-order cone (the epigraph of the p-norm) is self-dual, for any $p \geq 1$. By proposing an 'inner product' in [1, Equation (3)], this claim was 'proven' in [1, Lemma 3.1] and a 'Jordan product' was also proposed in [1, Equation (7)]. The bilinearity of each of the products in [1, Equation (3)] and [1, Equation (7)] was considered to satisfy after assuming that all elements operator commute.

The omission in [1] is the assumption that one can assume all elements operator commute (if not, they can be 'scaled' to do so, see page 36 in [1]). In fact, this assumption is a very strong hypothesis and it is not always verified because the only Jordan algebra where all its elements operator commute is $\mathbb{R}^n$. Given this, the products [1, Equation (3)] and [1, Equation (7)] are not linear in the first argument and are not linear in the second argument, whereas the weak and strong duality theorems depend in an essential way on (at least one of) these linearities.

2 Alternative Results

This section almost makes up for the omission in [1]. In this section, we prove that the p^{th}-order cone is "semi-self-dual" for any $p \geq 1$, that is, it is a self-dual cone with respect to a semi-inner product that will be adopted in this short paper. We also consider the algebra associated with this cone and prove that this algebra is a semi-distributive Banach algebra. Finally, we present the application of these results to the duality theory. To be able to state our results, we start by reviewing some definitions related to the semi-inner products and Banach algebras.

*Corresponding author (B. Alzalg). `Email: b.alzalg@ju.edu`

2.1 Preliminary Definitions

For each vector $\boldsymbol{x} \in \mathbb{R}^n$ whose first entry is indexed with 0, we write $\overline{\boldsymbol{x}}$ for the subvector consisting of entries 1 through $n-1$ (therefore $\boldsymbol{x} = (x_0, \overline{\boldsymbol{x}}^{\mathsf{T}})^{\mathsf{T}} \in \mathbb{R} \times \mathbb{R}^n$). We let $\mathcal{E}^n$ denote the n^{th}-dimensional real vector space $\mathbb{R} \times \mathbb{R}^{n-1}$ whose vectors $\boldsymbol{x}$ are indexed from 0.

Let $p \geq 1$ be a real number. The p^{th}*-order cone* (or *arbitrary-order cone*) of dimension n is defined as

$$\mathcal{P}_p := \left\{ \boldsymbol{x} \in \mathcal{E}^n : x_0 \geq \|\overline{\boldsymbol{x}}\|_p \right\}, \quad \text{where} \quad \|\overline{\boldsymbol{x}}\|_p := \left(\sum_{i=1}^{n-1} |x_i|^p \right)^{1/p}.$$

Let $\mathcal{V}$ be a finite dimensional Euclidean vector space over $\mathbb{R}$ with inner product "$[\cdot,\cdot]$". The *dual cone* of a regular cone $\mathcal{K} \subset \mathcal{V}$ is denoted by $\mathcal{K}^\star$ and is defined as

$$\mathcal{K}^\star = \{ \boldsymbol{y} \in \mathcal{V} : [\boldsymbol{x}, \boldsymbol{y}] \geq 0, \forall \boldsymbol{x} \in \mathcal{K} \}. \tag{1}$$

The cone $\mathcal{K}$ is said to be *self-dual* iff it coincides with its dual cone $\mathcal{K}^\star$, i.e., $\mathcal{K} = \mathcal{K}^\star$.

A *semi-inner product* (see for example [2, 3]) for a linear vector space $\mathcal{V}$ over $\mathbb{R}$ is a function (denoted by $[\cdot,\cdot]$) from $\mathcal{V} \times \mathcal{V}$ to $\mathbb{R}$ such that

$$\begin{array}{l} [\alpha \boldsymbol{u}, \boldsymbol{v}] = \alpha[\boldsymbol{u}, \boldsymbol{v}] = [\boldsymbol{u}, \alpha \boldsymbol{v}], \\ [\boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w}] = [\boldsymbol{u}, \boldsymbol{v}] + [\boldsymbol{u}, \boldsymbol{w}], \\ [\boldsymbol{u}, \boldsymbol{u}] \geq 0 \;\text{ and }\; [\boldsymbol{u}, \boldsymbol{u}] = 0 \;\text{ iff }\; \boldsymbol{u} = \boldsymbol{0}, \end{array} \tag{2}$$

for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$ and $\alpha \in \mathbb{R}$. Different from inner products, a semi-inner product is generally not symmetric, i.e., $[\boldsymbol{u}, \boldsymbol{v}] \neq [\boldsymbol{v}, \boldsymbol{u}]$, which is equivalent to saying that $[\boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w}] \neq [\boldsymbol{u}, \boldsymbol{w}] + [\boldsymbol{v}, \boldsymbol{w}]$. So, a semi-inner-product is defined in this paper to be generally nonlinear about its first variable. Our definition presented here might be different from some definitions of the semi-inner product in the literature. For instance, Lumer [2] assumes that the semi-inner-products are generally nonlinear about its second variable instead of its first variable.

The cone $\mathcal{K}^\star$ defined in (1) is said to be *semi-dual cone* of $\mathcal{K}$ the corresponding product is not an inner product but a semi-inner product. In this case, $\mathcal{K}$ is said to be *semi-self-dual* iff $\mathcal{K} = \mathcal{K}^\star$.

A (generally non associative) *Banach algebra* $(\mathcal{B}, \circ)$ is an algebra $\mathcal{B}$ over $\mathbb{R}$ whose underlying space is a Banach space with respect to a norm $\|\cdot\|$ satisfying the inequality $\|\boldsymbol{x} \circ \boldsymbol{y}\| \leq \|\boldsymbol{x}\|\,\|\boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{B}$, where "$\circ$" is a bilinear map from $\mathcal{B} \times \mathcal{B}$ into $\mathcal{B}$. A *semi-distributive Banach algebra* satisfies all axioms of a Banach algebra except of one of the right and left distributive laws [4, Chapter IV].

Now, we introduce the semi-inner product that we deal with throughout this short paper. First, we introduce some notations that will be used in the sequel. Associated with each vector $\boldsymbol{x} \in \mathbb{R}^n$ and each $p \in [1, \infty[$, we define the vector $\boldsymbol{x}^{(p)} \in \mathcal{E}^n$ as

$$\boldsymbol{x}^{(p)} := \begin{bmatrix} x_0 \\ \|\overline{\boldsymbol{x}}\|_p^{2-p} |x_1|^{p-1} \operatorname{sgn}(x_1) \\ \|\overline{\boldsymbol{x}}\|_p^{2-p} |x_2|^{p-1} \operatorname{sgn}(x_2) \\ \vdots \\ \|\overline{\boldsymbol{x}}\|_p^{2-p} |x_{n-1}|^{p-1} \operatorname{sgn}(x_{n-1}) \end{bmatrix},$$

where $\operatorname{sgn}(x_j)$ denotes the sign of x_j for $j = 1, 2, \ldots, n-1$. Note that $\boldsymbol{x}^{(p)}$ can also be written as

$$
\boldsymbol{x}^{(p)} = J_p(\boldsymbol{x})\boldsymbol{x}, \quad \text{where} \quad J_p(\boldsymbol{x}) := \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \frac{|x_1|^{p-2}}{\|\bar{\boldsymbol{x}}\|_p^{p-2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{|x_{n-1}|^{p-2}}{\|\bar{\boldsymbol{x}}\|_p^{p-2}} \end{bmatrix}. \tag{3}
$$

Clearly, the matrix $J_p(\boldsymbol{x})$ reduces to the identity matrix I_n when $p = 2$, and hence the vector $\boldsymbol{x}^{(p)}$ reduces to $\boldsymbol{x}$ when $p = 2$.

The semi-inner product that we deal with throughout this note is the map $[\cdot,\cdot]_p : \mathcal{E}^n \times \mathcal{E}^n \longrightarrow \mathbb{R}$ defined as

$$
[\boldsymbol{x}, \boldsymbol{y}]_p := x_0 y_0 + \overline{\boldsymbol{x}^{(p)}}^{\mathsf{T}} \overline{\boldsymbol{y}} = x_0 y_0 + \|\overline{\boldsymbol{x}}\|_p^{2-p} \sum_{j=1}^{n-1} |x_i|^{p-2} x_j y_j, \tag{4}
$$

for $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{E}^n$ and $p \in [1, \infty[$. Note that $[\cdot,\cdot]_p$ satisfies the properties in (2) for any $p \in [1, \infty[$. Note also that $[\cdot,\cdot]_p$ reduces to the standard inner product when $p = 2$. We point out that this semi-inner product is previously introduced in the literature (see for example, but not limited to: [3]).

2.2 Alternative Results

In this section, we present alternative results to some of those in [1, Sections 3 and 4]. We first prove the semi-self duality of the p^{th}-order cone.

Lemma 2.1. *The p^{th}-order cone $\mathcal{P}_p$ is semi-self dual for any $p \geq 1$.*

Proof. Consider the semi-inner product $[\cdot,\cdot]_p$ defined in (4). Let $\boldsymbol{x} = (x_0; \overline{\boldsymbol{x}}) \in \mathcal{P}_p$, we prove that $\boldsymbol{x} \in \mathcal{P}_p^\star$ by showing that $[\boldsymbol{x}, \boldsymbol{y}]_p \geq 0$ for any $\boldsymbol{y} = (y_0; \overline{\boldsymbol{y}}) \in \mathcal{P}_p$. This is trivial if $\overline{\boldsymbol{x}} = \boldsymbol{0}$ or $\overline{\boldsymbol{y}} = \boldsymbol{0}$. If $\overline{\boldsymbol{x}} \neq \boldsymbol{0}$ or $\overline{\boldsymbol{y}} \neq \boldsymbol{0}$, then

$$
[\boldsymbol{x}, \boldsymbol{y}]_p = x_0 y_0 + \overline{\boldsymbol{x}^{(p)}}^{\mathsf{T}} \overline{\boldsymbol{y}} \geq \|\overline{\boldsymbol{x}}\|_p \|\overline{\boldsymbol{y}}\|_p + \overline{\boldsymbol{x}^{(p)}}^{\mathsf{T}} \overline{\boldsymbol{y}} \geq \left| \overline{\boldsymbol{x}^{(p)}}^{\mathsf{T}} \overline{\boldsymbol{y}} \right| + \overline{\boldsymbol{x}^{(p)}}^{\mathsf{T}} \overline{\boldsymbol{y}} \geq 0,
$$

where the first inequality follows from the fact that $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{P}_p$ and the second inequality follows from the fact that $\left| \overline{\boldsymbol{x}^{(p)}}^{\mathsf{T}} \overline{\boldsymbol{y}} \right| \leq \|\overline{\boldsymbol{x}}\|_p \|\overline{\boldsymbol{y}}\|_p$. Therefore, $\mathcal{P}_p \subseteq \mathcal{P}_p^\star$.

Now, let $\boldsymbol{y} = (y_0; \overline{\boldsymbol{y}}) \in \mathcal{P}_p^\star$, we need to show that $\boldsymbol{y} \in \mathcal{P}_p$. This is trivial if $\overline{\boldsymbol{y}} = \boldsymbol{0}$. If $\overline{\boldsymbol{y}} \neq \boldsymbol{0}$, set $\boldsymbol{x} := \left(\|\overline{\boldsymbol{y}}\|_p; -\overline{\boldsymbol{y}} \right) \in \mathcal{P}_p$, then we have

$$
0 \leq [\boldsymbol{x}, \boldsymbol{y}]_p = x_0 y_0 + \overline{\boldsymbol{x}^{(p)}}^{\mathsf{T}} \overline{\boldsymbol{y}} = \|\overline{\boldsymbol{y}}\|_p y_0 - \overline{\boldsymbol{y}^{(p)}}^{\mathsf{T}} \overline{\boldsymbol{y}} = \|\overline{\boldsymbol{y}}\|_p \left(y_0 - \|\overline{\boldsymbol{y}}\|_p \right),
$$

where the last equality follows from the fact that $\overline{\boldsymbol{y}^{(p)}}^{\mathsf{T}} \overline{\boldsymbol{y}} = \|\overline{\boldsymbol{y}}\|_p^2$. This gives that $y_0 \geq \|\overline{\boldsymbol{y}}\|_p$, and thus implies that $\boldsymbol{y} \in \mathcal{P}_p$. Therefore, $\mathcal{P}_p^\star \subseteq \mathcal{P}_p$. The result is established. $\square$

Now, we consider a semi-distributive Banach algebra associated with the p^{th}-order cone. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{E}^n$ and $p \in [1, \infty[$. We define the multiplication $\square_p : \mathcal{E}^n \times \mathcal{E}^n \longrightarrow \mathcal{E}^n$ as

$$
\boldsymbol{x} \square_p \boldsymbol{y} := \begin{bmatrix} [\boldsymbol{x}, \boldsymbol{y}]_p \\ x_0 \overline{\boldsymbol{y}} + y_0 \overline{\boldsymbol{x}} \end{bmatrix},
$$

where $[\cdot,\cdot]_p$ is the semi-inner product defined in (4).

The vector $\boldsymbol{e} := (1, 0, 0, \ldots, 0)^{\mathsf{T}} \in \mathcal{E}^n$ is clearly the identity vector of $(\mathcal{E}^n, \square_p)$ for any $p \in [1, \infty[$. Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{E}^n$ and $\alpha, \beta \in \mathbb{R}$, it is not hard to see that

$$\boldsymbol{x}\square_p(\alpha\boldsymbol{y} + \beta\boldsymbol{z}) = \alpha\boldsymbol{x}\square_p\boldsymbol{y} + \beta\boldsymbol{x}\square_p\boldsymbol{z}, \quad \text{but generally} \quad (\alpha\boldsymbol{x} + \beta\boldsymbol{y})\square_p\boldsymbol{z} \neq \alpha\boldsymbol{x}\square_p\boldsymbol{z} + \beta\boldsymbol{y}\square_p\boldsymbol{z}.$$

It is also easy to see that $\boldsymbol{x}\square_p\boldsymbol{y} \neq \boldsymbol{y}\square_p\boldsymbol{x}$ in general. We have the following theorem.

Theorem 2.1. *$(\mathcal{E}^n, \square_p)$ is a semi-distributive Banach algebra for any $p \in [1, \infty[$.*

Proof. Let $p \in [1, \infty[$. It is clear that the space $\mathcal{E}^n$ is a Banach space with respect to the p-norm $\|\cdot\|_p$, and that $(\mathcal{E}^n, \square_p)$ is a semi-distributive algebra over $\mathbb{R}$. To establish the result, we need to show that $\|\boldsymbol{x}\square_p\boldsymbol{y}\|_p \leq \|\boldsymbol{x}\|_p\|\boldsymbol{y}\|_p$ for all $\boldsymbol{x}, \boldsymbol{y} \in (\mathcal{E}^n, \square_p)$. The result is implied by the following sequence of equalities and inequalities:

$$\begin{aligned}
\|\boldsymbol{x}\square_p\boldsymbol{y}\|_p^p &= \left\| \begin{bmatrix} x_0 y_0 + \overline{\boldsymbol{x}^{(p)}}^{\mathsf{T}}\overline{\boldsymbol{y}} \\ x_0\overline{\boldsymbol{y}} + y_0\overline{\boldsymbol{x}} \end{bmatrix} \right\|_p^p \\
&= \left| x_0 y_0 + \overline{\boldsymbol{x}^{(p)}}^{\mathsf{T}}\overline{\boldsymbol{y}} \right|^p + \sum_{j=1}^{n-1} \left| x_0 y_j + y_0 x_j \right|^p \\
&\leq \left(|x_0||y_0| + \left| \overline{\boldsymbol{x}^{(p)}}^{\mathsf{T}}\overline{\boldsymbol{y}} \right| \right)^P + \sum_{j=1}^{n-1} \left(|x_0||y_j| + |y_0||x_j| \right)^p \\
&\leq \left(|x_0||y_0| + \|\overline{\boldsymbol{x}}\|_p\|\overline{\boldsymbol{y}}\|_p \right)^P + \sum_{j=1}^{n-1} \left(|x_0||y_j| + |y_0||x_j| \right)^p \\
&= \left(|x_0| + \|\overline{\boldsymbol{x}}\|_p \right)^p \left(\frac{|x_0|}{|x_0| + \|\overline{\boldsymbol{x}}\|_p} |y_0| + \frac{\|\overline{\boldsymbol{x}}\|_p}{|x_0| + \|\overline{\boldsymbol{x}}\|_p} \|\overline{\boldsymbol{y}}\|_p \right)^p \\
&\quad + (|x_0| + |y_0|)^p \sum_{j=1}^{n-1} \left(\frac{|x_0|}{|x_0| + |y_0|} |y_j| + \frac{|y_0|}{|x_0| + |y_0|} |x_j| \right)^p \\
&\leq \left(|x_0| + \|\overline{\boldsymbol{x}}\|_p \right)^p \left(\frac{|x_0|}{|x_0| + \|\overline{\boldsymbol{x}}\|_p} |y_0|^p + \frac{\|\overline{\boldsymbol{x}}\|_p}{|x_0| + \|\overline{\boldsymbol{x}}\|_p} \|\overline{\boldsymbol{y}}\|_p^p \right) \\
&\quad + (|x_0| + |y_0|)^p \sum_{j=1}^{n-1} \left(\frac{|x_0|}{|x_0| + |y_0|} |y_j|^p + \frac{|y_0|}{|x_0| + |y_0|} |x_j|^p \right) \\
&= \left(|x_0| + \|\overline{\boldsymbol{x}}\|_p \right)^{p-1} \left(|x_0||y_0|^p + \|\overline{\boldsymbol{x}}\|_p\|\overline{\boldsymbol{y}}\|_p^p \right) + (|x_0| + |y_0|)^{p-1} \sum_{j=1}^{n-1} \left(|x_0||y_j|^p + |y_0||x_j|^p \right) \\
&= \left(|x_0| + \|\overline{\boldsymbol{x}}\|_p \right)^{p-1} \left(|x_0||y_0|^p + \|\overline{\boldsymbol{x}}\|_p\|\overline{\boldsymbol{y}}\|_p^p \right) + (|x_0| + |y_0|)^{p-1} \left(|x_0|\|\overline{\boldsymbol{y}}\|_p^p + |y_0|\|\overline{\boldsymbol{x}}\|_p^p \right) \\
&\leq |x_0|^p|y_0|^p + |x_0|^p\|\overline{\boldsymbol{y}}\|_p^p + \|\overline{\boldsymbol{x}}\|_p^p|y_0|^p + \|\overline{\boldsymbol{x}}\|_p^p\|\overline{\boldsymbol{y}}\|_p^p \\
&= \left(|x_0|^p + \|\overline{\boldsymbol{x}}\|_p^p \right)\left(|y_0|^p + \|\overline{\boldsymbol{y}}\|_p^p \right) \\
&= \left(\|\boldsymbol{x}\|_p\|\boldsymbol{y}\|_p \right)^p,
\end{aligned}$$

where the first inequality follows from the triangle inequality, the second inequality follows from the inequality $\left| \overline{\boldsymbol{x}^{(p)}}^{\mathsf{T}}\overline{\boldsymbol{y}} \right| \leq \|\overline{\boldsymbol{x}}\|_p\|\overline{\boldsymbol{y}}\|_p$, and the third inequality follows from the fact that the function t^p is convex on $[0, \infty[$ for $p \geq 1$. The proof is complete. □

Note that Theorem 2.1 can be viewed as an alternative result to that in [1, Theorem 4.1].

Looking at the spectral decomposition of $\boldsymbol{x}$ introduced in [1, Section 4], one can see that $\boldsymbol{x}$ is decomposed into eigenvalues ($\lambda_{1,2}(\boldsymbol{x}) := x_0 \pm \|\bar{x}\|_p$) and eigenvectors $\left(\boldsymbol{c}_{1,2}(\boldsymbol{x}) = \frac{1}{2}\left(x_0, \frac{-1}{\|\bar{x}\|_p}\bar{x}^{\mathsf{T}}\right)^{\mathsf{T}}\right)$, and that $f(\boldsymbol{x})$ is defined as $f(\boldsymbol{x}) = f(\lambda_1(\boldsymbol{x}))\boldsymbol{c}_1(\boldsymbol{x}) + f(\lambda_2(\boldsymbol{x}))\boldsymbol{c}_2(\boldsymbol{x})$ for a continuous real-valued function f.

It is easy to see that

$$\boldsymbol{x}\square_p\boldsymbol{x} := \begin{bmatrix} x_0^2 + \|\bar{x}\|_p^2 \\ 2x_0\bar{x} \end{bmatrix} = \lambda_1^2(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}) = x^2.$$

The following theorem is the alternative of [1, Theorem 4.2] and it characterizes the p^{th}-order cones. The proof of this theorem is omitted because it is identical to the proof of [1, Theorem 4.2].

Theorem 2.2. *Let $p \in [1,\infty[$. The p^{th}-order cone equals the cone of squares of $(\mathcal{E}^n, \square_p)$, where the cone of squares of $(\mathcal{E}^n, \square_p)$ is defined as $\mathcal{K}_p := \left\{\boldsymbol{x}^2 : \boldsymbol{x} \in (\mathcal{E}^n, \square_p)\right\}$.*

Now, we can define concepts like the arrow-shaped matrix associated with each vector $\boldsymbol{x} \in (\mathcal{E}^n, \square_p)$ and the quadratic representation of $\boldsymbol{x}$. The *arrow-shaped matrix* associated with $\boldsymbol{x} \in (\mathcal{E}^n, \square_p)$ is defined as

$$\mathrm{Arw}_p(\boldsymbol{x}) := \begin{bmatrix} x_0 & \overline{\boldsymbol{x}^{(p)}}^{\mathsf{T}} \\ \bar{\boldsymbol{x}} & x_0 I_{n-1} \end{bmatrix}.$$

The *quadratic representation* of $\boldsymbol{x} \in (\mathcal{E}^n, \square_p)$ is defined as

$$Q_p(\boldsymbol{x}) := 2\mathrm{Arw}_p^2(\boldsymbol{x}) - \mathrm{Arw}_p(\boldsymbol{x}^2) = \begin{bmatrix} [\boldsymbol{x},\boldsymbol{x}]_p & 2x_0\overline{\boldsymbol{x}^{(p)}}^{\mathsf{T}} \\ 2x_0\bar{\boldsymbol{x}} & \det(\boldsymbol{x})I + 2\bar{\boldsymbol{x}}\overline{\boldsymbol{x}^{(p)}}^{\mathsf{T}} \end{bmatrix}, \tag{5}$$

where, as introduced in [1], $\det(\boldsymbol{x})$ is the *determinant* of $\boldsymbol{x}$ defined as $\det(\boldsymbol{x}) := \lambda_1(\boldsymbol{x})\lambda_2(\boldsymbol{x})$. Similarly, we can also define the quadratic operator for $(\mathcal{E}^n, \square_p)$ as $Q_p(\boldsymbol{x},\boldsymbol{y}) := \mathrm{Arw}_p(\boldsymbol{x})\mathrm{Arw}_p(\boldsymbol{y}) + \mathrm{Arw}_p(\boldsymbol{y})\mathrm{Arw}_p(\boldsymbol{x}) - \mathrm{Arw}_p(\boldsymbol{x}\square_p\boldsymbol{y})$.

Note that $\mathrm{Arw}_p(\boldsymbol{e}) = I_n$, $\mathrm{Arw}_p(\boldsymbol{x})\boldsymbol{e} = \boldsymbol{x}$, $\mathrm{Arw}_p(\boldsymbol{x})\boldsymbol{x} = \boldsymbol{x}\square_p\boldsymbol{x} = \boldsymbol{x}^2$, and, more generally, that $\boldsymbol{x}\square_p\boldsymbol{y} = \mathrm{Arw}_p(\boldsymbol{x})\boldsymbol{y}$. Note also that $Q_p(\boldsymbol{e}) = I_n$ and $Q_p(\boldsymbol{x})\boldsymbol{e} = \boldsymbol{x}^2$.

The quadratic operator for the semi-distributive Banach algebra $(\mathcal{E}^n, \diamond_p)$ associated with the p^{th}-order cone $\mathcal{P}_p$ is defined as

$$\begin{aligned} Q_p(\boldsymbol{x},\boldsymbol{z}) &:= \mathrm{Arw}_p(\boldsymbol{x})\mathrm{Arw}_p(\boldsymbol{z}) + \mathrm{Arw}_p(\boldsymbol{z})\mathrm{Arw}_p(\boldsymbol{x}) - \mathrm{Arw}_p(\boldsymbol{x}\diamond_p\boldsymbol{z}) \\ &= \begin{bmatrix} [\boldsymbol{x},\boldsymbol{z}]_p & (x_0\hat{\boldsymbol{z}}^{\mathsf{T}} + z_0\overline{\boldsymbol{x}^{(p)}}^{\mathsf{T}}) \\ x_0\bar{\boldsymbol{z}} + z_0\bar{\boldsymbol{x}} & (\bar{\boldsymbol{x}}\hat{\boldsymbol{z}}^{\mathsf{T}} + \bar{\boldsymbol{z}}\overline{\boldsymbol{x}^{(p)}}^{\mathsf{T}}) + (x_0z_0 - \overline{\boldsymbol{x}^{(p)}}^{\mathsf{T}}\bar{\boldsymbol{z}})I_{n-1} \end{bmatrix}. \end{aligned}$$

2.3 Application to Duality Theory

The application of the above results to the duality theory is obvious as we will see below. We first introduce the p^{th}-order cone programming problem.

Let n and m be positive integers, $p \geq 1$, $\boldsymbol{b}, \boldsymbol{y} \in \mathbb{R}^m$, $\boldsymbol{c}, \boldsymbol{x}, \boldsymbol{s} \in \mathcal{E}^n$, $A \in \mathbb{R}^{m\times n}$, and A^{T} be its transpose. Let also $[A,\boldsymbol{x}]_p$ be the matrix-vector product defined as

$$[A,\boldsymbol{x}]_p := \left([\boldsymbol{a}_1,\boldsymbol{x}]_p, [\boldsymbol{a}_2,\boldsymbol{x}]_p, \ldots, [\boldsymbol{a}_m,\boldsymbol{x}]_p\right)^{\mathsf{T}} \in \mathbb{R}^m,$$

where $\boldsymbol{a}_j$ is the j^{th} row of A for $j = 1, 2, \ldots, m$. We define the p^{th}*-order cone programming problem and its semi-dual problem* as

$$\begin{array}{llll} & \underline{\textbf{Primal}} & & \underline{\textbf{Semi}-\textbf{Dual}} \\ \min & [\boldsymbol{x}, \boldsymbol{c}]_p & \max & \boldsymbol{b}^\top \boldsymbol{y} \\ \text{s.t.} & [A, \boldsymbol{x}]_p = \boldsymbol{b}, & \text{s.t.} & A^\top \boldsymbol{y} + \boldsymbol{s} = \boldsymbol{c}, \\ & \boldsymbol{x} \in \mathcal{P}_p, & & \boldsymbol{s} \in \mathcal{P}_p. \end{array} \tag{6}$$

Note that the matrix A is defined to map $\mathcal{E}^n$ into $\mathbb{R}^m$, and its transpose $A^\top$ is defined to map $\mathbb{R}^m$ into $\mathcal{E}^n$ such that $[\boldsymbol{x}, A^\top \boldsymbol{y}]_p = [A, \boldsymbol{x}]_p^\top \boldsymbol{y}$. Now, we prove weak and strong semi-duality properties for the pair (6) as justification for referring to it as a semi primal-dual pair.

Theorem 2.3 (Weak semi-duality)**.** *Consider the semi primal-dual pair* (6)*. If $\boldsymbol{x}$ is any primal feasible solution of the primal problem and $(\boldsymbol{y}, \boldsymbol{z})$ is any feasible solution of the semi-dual problem, then the duality gap*

$$[\boldsymbol{x}, \boldsymbol{c}]_p - \boldsymbol{b}^\top \boldsymbol{y} = [\boldsymbol{x}, \boldsymbol{s}]_p \geq 0.$$

Proof. Consider the semi primal-dual pair (6). Because the semi-inner product is linear about its second argument, we have

$$\begin{array}{rcl} [\boldsymbol{x}, \boldsymbol{c}]_p - \boldsymbol{b}^\top \boldsymbol{y} & = & \left[\boldsymbol{x}, A^\top \boldsymbol{y} + \boldsymbol{s}\right]_p - \boldsymbol{b}^\top \boldsymbol{y} \\ & = & \left[\boldsymbol{x}, A^\top \boldsymbol{y}\right]_p + [\boldsymbol{x}, \boldsymbol{s}]_p - \boldsymbol{b}^\top \boldsymbol{y} \\ & = & \boldsymbol{y}^\top [A, \boldsymbol{x}]_p + [\boldsymbol{x}, \boldsymbol{s}]_p - \boldsymbol{y}^\top \boldsymbol{b} \\ & = & [\boldsymbol{x}, \boldsymbol{s}]_p + \boldsymbol{y}^\top ([A, \boldsymbol{x}]_p - \boldsymbol{b}) \\ & = & [\boldsymbol{x}, \boldsymbol{s}]_p. \end{array}$$

Since $\boldsymbol{x} \in \mathcal{P}_p$ and $\boldsymbol{s} \in \mathcal{P}_p = \mathcal{P}_p^\star$, we conclude that $[\boldsymbol{x}, \boldsymbol{s}]_p \geq 0$. □

□

A general form of Farkas' lemma can be used to prove the following theorem. We assume that the m rows of the matrix A are linearly independent.

Theorem 2.4 (Strong semi-duality)**.** *Consider the semi primal-dual pair* (6)*. If both the primal and semi-dual problems have strictly feasible solutions, then they both have optimal solutions $\boldsymbol{x}^\star$ and $(\boldsymbol{y}^\star, \boldsymbol{s}^\star)$, respectively, and $[\boldsymbol{x}^\star, \boldsymbol{c}]_p = \boldsymbol{b}^\top \boldsymbol{y}^\star$ (i.e., $[\boldsymbol{x}^\star, \boldsymbol{s}^\star]_p = 0$ or equivalently $\boldsymbol{x}^\star \square_p \boldsymbol{s}^\star = \boldsymbol{0}$ (Complementary slackness)).*

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