A hybrid approach for Bi-Objective Optimization

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Abstract

A large number of the real world planning problems which are today solved using Operations Research methods are actually multi-objective planning problems, but most of them are solved using single-objective methods. The reason for converting, i.e. simplifying, multi-objective problems to single-objective problems is that no standard multi-objective solvers exist and specialized algorithms need to be programmed from scratch.

In this article we will present a hybrid approach, which both operate in decision space and in objective space. The approach enables massive efficient parallelisation and can be used to a vide variety of bi-objecive Mixed Integer Programming models. We test the approach on the biobjejctive extension of the classic traveling salesman problem, on the standard dataset, and determine for the full set of nondominated points. This has only been done once before [1], and in our approach we do it in a fraction of the time.

Keywords: biobjective optimization; mixed integer programming; traveling salesman problem; branch-and-cut algorithm

1. Introduction

Most real-world problems which are optimized using Operations Research (or) methods are actually multi-objective. Only very seldom do the OR practitioners consider their optimization problem as multi-objective optimization problems, i.e. i most cases the objectives are simply summed, possibly with weights thus identifying the more important objectives. While this approach may work in practice, it is problematic since it ignores a multitude other (Pareto) optimal solutions. While there has been research going on in multi-objective optimization for decades, an increased interest in exact solution of multi-objective optimization problems has flourished in the last decade (i.e. since 2006): [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. This research has however not yet lead to general practical solvers of multi-objective Mixed Integer Programming (MOMIP) models.

Generally there have been two different types of approaches: Either Criteria Space Search (CSS) algorithms or Desicion Space Search (DSS) algorithms. We will briefly describe the most important CSS algorithms and DSS algorithms below.

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1.1. Criteria Space Search algorithms

CSS algorithms are iterative algorithms which utilize a series of single-objective optimizations, where extra constraints are added to the MOMIP models criteria space. The two "classic" CSS algorithms are the Epsilon-constraint algorithm [15] and the Two-Phase algorithm [16]. The epslion-constraint algorihtm, starts by finding one of the two lexiographic points. This point is then ruled out by a constraint on one of the criteria space variables and another (lexiographic) optimization leads to the next point of the Pareto front. The Two-Phase algorithm also starts with the two lexiographic points. In the first phase, the efficient point of the pareto front are found by solving a series single-objective MIP's. In the second phase, a series of "triangles" in criteria space are iteratively searched for solutions, again using single-objective MIP's or problem-dependent enumeration methods.

While the CSS algorithms have been known for decades modern versions have been constructed which seems quite promising, e.g. [3, 6].

1.2. Decision Space Search algorithms

DSS algorithms are (basically) Branch & Bound algorithms, tweaked to work with more than one objective. This requires a re-definition of the fathoming rules, new types of branching and new versions of pre-processing, of probing etc. The cutting plane generation is however **not** changed ! Building an efficient Branch & Cut algorithm from scratch is a monumental task, which only very few researchers could possible do and the developed algorithm would most likely still be seriously inferior to standard solvers like CPLEX or Gurobi, if applied to single-objective MIP models. This approach has been attempted a few times in [17, 18]. Recently, there has however been approaches which utilize the so-called call-back procedures in modern solvers as CPLEX or Gurobi [10, 13]. This solution overcome **some** of the issues.

1.3. Which approach is best ? CSS or DSS ?

The question of which approach is the best, i.e. can solve the largest MOMIP models fastest is in our opinion open. More research is required and both approaches has their advantages and disadvantages: The CS algorithms can utilize the fast development of new solver versions, which promise continuous improvement. Their main disadvantage is that they solve a large set of almost alike MIP models, and they will most likely process the same intermediate solutions, i.e. nodes in their Branch & Bound tree, many times. The DSS algorithms on the other hand suffer from reduced strength in the fathoming process and the extra requirement of Integer branching, see Section 3.2. On the other hand, they will ideally only see a node once.

In our opinion the question is not CSS or DSS, but **CSS and DSS** ! We believe that the paradigms should be mixed to improve performance, thus creating hybrid methods. We have already demonstrated this in our previous paper [10], where a special branching method, called Pareto Branching, is introduced. This allows branching in the criteria space, thus creating a hybrid approach. In our opinion this is a viable approach: To "enrich" DSS algorithms, with features from CSS algorithms, to speed up performance.

We test our new hybrid approach on the BI-objective Travelling Salesman Problem (BITSP). Since the first proposal of the TSP problem [19], TSP has formed a testbed for Operations Research methods. The Branch & Cut algorithm was originally developed for solving the TSP problem, starting with the cutting plane approach [20] which was in the 1980s developed to solve large scale TSP problems by [21, 22], ending with the current record holder Concorde [23]. In the 1990s, the cutting plane techniques made their way into general MIP model solvers [24], resulting in spectacular performance gains of MIP solvers [25]. Therefore, it is worth noting that the techniques applied in this paper to the BITSP problem can easily be applied to a large set of Bi-Objective Mixed Integer Programs.

1.4. Parallel Approaches

Parallelization of time-consuming algorithms is a classic way of achieving faster results. This approach has gained more importance in the last decade, where the clock-frequency has not increased significantly, but the number of cores on each chip has. Today, access to a large cluster of e.g. linux computers is relatively easy,cheap and common. This also means that solvers today like CPLEX or Gurobi focus a lot on parallel execution of especially Branch & Bound algorithms. Given that Mulit-Objective Mixed Integer Programming (MOMIP) models are significant harder to solve than standard MIP models, parallelization of MOMIP solvers is a pretty obvious approach to improve the speed of MOMIP solvers. Parallelization of single-objective Branch & Bound solvers, is well understood and implemented, but only for limited parallelization, e.g. less than 100 processors. For massive parallelization, communication between the different processors becomes problematic (they have to exchange new incumbent values). In this article we present an approach where Criteria Space is used to obtain trivial parallelization, i.e. parallelization where no communication is needed between the processors.

1.5. Contributions

The main contributions in this paper to the field of multiobjective optimization can be summarized as follows:

1. The feasible set of nondominated points (in criterion space) are partitioned into a number of independent sets (slices). This is the basis for the development of the parallel method.
2. The slices are constructed in an “optimal” way, given a trial set of points in the feasible set in criterion space. This “optimal” partition is very important, because the solution time depends heavily on how the partition is done. We believe that the OR community can take advantage of this insight when solving other multiobjective optimization problems. One can think on the procedure as a parallel decomposition method.
3. The approach is tested on the biobjective TSP. The BITSP test problems are the full generally acknowledged BITSP test set, used in a number of other articles.
4. The method developed in this paper is generic and can be used on other biobjective (mixed integer) problems. BITSP is used to illustrate the approach. We expect that

this approach can pave the way for general biobjective MIP solvers, just like the branch-and-cut algorithms originally developed for TSP were later turned into general solvers.

It should be mentioned that without partitioning the set of nondominated points in an optimal way into a number of slices, we cannot solve the biobjective TSP instances.

In §2 we give the preliminaries and in §3 we give a short overview of the branch-and-bound algorithm developed in [10]. In §4 we develop the parallel version of the branch-and-bound algorithm described in §3, and in §5 we apply it to the biobjective traveling salesman problem. In §6 we present a few results, and we conclude the paper in §7.

2. Preliminaries

In this section we give some notation and describe the biobjective mixed integer programming model for which a branch-and-bound method was developed in [10]. The branch-and-bound method will be described in Section 3.

We assume that there are only two objectives. The first objective only contains binary variables, whereas the second objective may contain both binary and continuous variables. These restrictions make it possible to represent the set of nondominated points as a discrete set in two-dimensional space.

Let $c^1, c^2 \in \mathcal{R}^n$ be two n-dimensional vectors and $h^2 \in \mathcal{R}^p$ be a p-dimensional vector. Let $z(x, y) = (z_1(x, y), z_2(x, y)) = (c^1x, c^2x + h^2y)$ be two linear objective functions with nonnegative n-dimensional integral variables $x \in \mathcal{Z}^n_+$ and nonnegative p-dimensional continuous variables $y \in \mathcal{R}^p_+$. Let A be an $m \times n$ matrix, G be an $m \times p$ matrix, and $b \in \mathcal{R}^m$.

The resulting biobjective mixed integer programming (BOMIP) model is shown in equation (1).

$$\min\{z(x, y) \mid Ax + Gy \geq b,\ x \in \mathcal{Z}^n_+,\ y \in \mathcal{R}^p_+\} \tag{1}$$

Let $\mathcal{X}$ denote the set of feasible solutions to (1), that is $\mathcal{X} = \{(x, y) \in \mathcal{Z}^n_+ \times \mathcal{R}^p_+ \mid Ax + Gy \geq b\}$. $\mathcal{X}$ is also referred to as the feasible set in *decision space*. Let $\mathcal{Z} = \{z \in \mathcal{R}^2 \mid z_1 = c^1x,\ z_2 = c^2x + h^2y,\ (x, y) \in \mathcal{X}\}$ denote the corresponding feasible set in *criterion space.*

A point $z^2 \in \mathcal{Z}$ is *dominated* by $z^1 \in \mathcal{Z}$ if the condition in (2) is satisfied

$$z^1_i \leq z^2_i \quad i = 1, 2, \quad \text{and} \quad z^1 \neq z^2 \tag{2}$$

If $z^2 \in \mathcal{Z}$ is *dominated* by $z^1 \in \mathcal{Z}$ we write $z^1 \prec z^2$.

The set of *efficient solutions* $\mathcal{X}_E$ in decision space is defined as

$$\mathcal{X}_E = \{(x, y) \in \mathcal{X} \mid \nexists (\bar{x}, \bar{y}) \in \mathcal{X} : z(\bar{x}, \bar{y}) \prec z(x, y)\},$$

and the set of *nondominated points* $\mathcal{Z}_N$ in criterion space is given by

$$\mathcal{Z}_N = \{z \in \mathcal{R}^2 \mid z = z(x, y),\ (x, y) \in \mathcal{X}_E\}.$$

We denote by z^{UL} the *upper/left* point and by z^{LR} the *lower/right* point. It is well-known [26] that these two points are nondominated. The point z^{UL} can be found by solving the lexicographic optimization problem $\text{lexmin}\{(z_1(x,y), z_2(x,y)) \mid (x,y) \in \mathcal{X}\}$, and the point z^{LR} can be found by solving the lexicographic optimization problem $\text{lexmin}\{(z_2(x,y), z_1(x,y)) \mid (x,y) \in \mathcal{X}\}$. Given z^{UL} and z^{LR} we know that if $z = (z_1, z_2)$ is a nondominated point then $z_1 \in [z_1^{UL}, z_1^{LR}]$ and $z_2 \in [z_2^{LR}, z_2^{UL}]$. In Figure 1, the grey rectangle, spanned by the two points z^{UL} and z^{LR}, is the only area where nondominated points may exist.

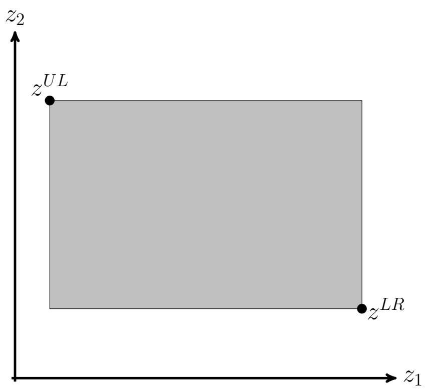


Figure 1: The criterion space

In Section 3, a biobjective branch-and-bound algorithm is presented, which can find all nondominated points to problem 1. The algorithm uses the (relaxation of the) following problem:

$$\min\{z_1 + \gamma z_2 \mid Ax + Gy \geq b,\ x \in \mathcal{Z}_+^n,\ y \in \mathcal{R}_+^p\}, \tag{3}$$

where $\gamma \in]0, \infty[$ is a positive number.

2.1. Division of Criteria Space

In classical single-objective Branch & Bound algorithms, the algorithm only "operate" on the decision variables. In CSS algorithms, the most typical approach is to add extra constraints in criteria space in combination with a single-objective solve. Exactly how this is done, depends on the specific CSS algorithm in question.

The full criteria space is illustrated in Figure 1, spanned by the two lexiographic points z^{UL} and z^{LR}. Since we are only looking for efficient solutions $\mathcal{X}_E$, these can only reside (visually) in the grey area of Figure 1. In the next Section 3 we describe our bi-objective Branch & Cut algorithm. We can use this algorithm to find all the efficient solutions (i.e. the Pareto Front), for the whole criteria space. But this may require an excessive use of time and memory (on the computer).

The basic hypothesis in this article is: The solution time of our Branch & Cut algorithm described in Section 3 is dependent to a certain degree on the size of the objective space to be searched (among other things).

To illustrate the hypothesis, assume that we are somehow able to find some efficient points in the criteria space, say in this case, two points, and assume we already know the lexicographic points z^{UL} and z^{LR}. This case is illustrated in Figure 2.

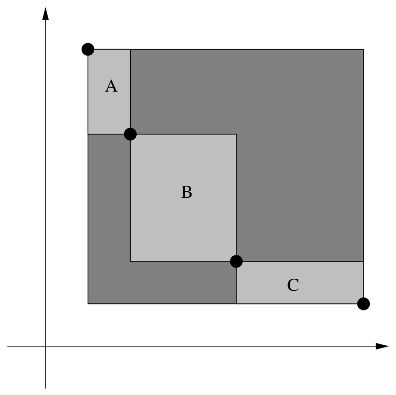


Figure 2: Division of the criterion space, using two intermediate efficient points

In Figure 2 the two intermediate points divides the whole criteria space into 3 separate light-grey areas. It is easily seen that we can only find efficient points in the light-grey areas. Our hypothesis then states that it is faster to execute our Branch & Cut algorithm from Section 3 three times on each of the smaller light-grey rectangles A,B and C, than to execute the Branch & Cut algorithm on the entire dark-grey rectangle between z^{UL} and z^{LR}.

The next question is then how to constrain the light-grey areas, which is not as simple as it seems ! Assuming that all MIP solvers ultimately are single-objective, the question is whether the objectives z_1 and z_2 are both in the objective or the coefficient in front of one of them is zero.

2.1.1. One Zero coefficient

To the best of our knowledge, this case is only valid for CSS algorithms. Most of them will start with the lexiographic points z^{UL} and z^{LR} and generate the points one by one. In this case, we speculate that the criteria space reduction shown in Figure 2, has no effect on the performance. All these CSS algorithms though suffer of an important drawback: To avoid weekly efficient solutions, each time a new point is found, lexiographic optimization has to be performed, i.e. first according to one criteria and then according to the other. Experimenting with this on our BITSP problem gave a surprising result: After fixing one objective, the second round of optimization on average took 3 times longer. This seemed contra-intuitive, since we constrain the feasible space for the second round. The data also showed that the second round used fewer nodes in the Branch & Bound algorithm and fewer dual-simplex iterations. The reason turns out to be that converting one objective to a constraint added a dense row to the constraints, and this lead to much slower solution due to slower row operations in the dual-simplex [27].

2.1.2. Positive weights

When using MIP solvers, with two non-zero weights we think our hypothesis is correct and we will show emperical results in Section 6. This is the case for all DS algorithms, but also at least one CSS algorithm [5, 6]. The in [5, 6] carefull adjustment of the size of the weights on the two objectives is utilized avoid lexiographic optimization, and only using one solve to obtain an efficient solution. With positive weights, there is however a problem with the box-bounds illustrated in Figure 2: The lower-bounding constraint (in case of minimization) creates problems for the Branch & Cut algorithm. Assume that an algorithm utilize weighted objectives for a bi-objective MIP model, see the MIP model 4, where also upper and lower bounds on the criteria variables z_1 and z_2, forming the reduced box area in criteria space.

$$\begin{array}{l}
\min \alpha \cdot z_1 + (1-\alpha) \cdot z_2 \\
\text{s.t.} \\
z_1 = \sum\limits_{j=1}^{n} c_j^1 x_j \\
z_2 = \sum\limits_{j=1}^{n} c_j^2 x_j \\
\sum\limits_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \cdots, m \\
LB_{z_1} \leq z_1 \leq UB_{z_1} \\
LB_{z_2} \leq z_2 \leq UB_{z_2}
\end{array} \tag{4}$$

When a Branch & Cut solver is used to solve a MIP model like the one given in model 4.

For all DS algorithms and CSS algorithms with two non-zero weights, assuming minimization on both objectives, it is always a good idea to add upper-bounds on both objectives. **It is however a bad idea to add lower bounds !** The reason for this is that the lower-bounds in most situations (unless the problem is very easy), makes all reduced costs of the variables zero and all dual variables zero. This "blinds" the Branch & Bound algorithm such that random branching decisions has to be made, until the lower bounds become non-binding. This is the reason we prefer a different way of bounding in criteria space: As (pizza) slices, see Figure 3

In Figure 3 the three areas from the previous Figure 2 are constrained by the upper bounds, **and** two slice constraints between the efficient points and origo. This is illustrated in Figure 3, where the areas A,B and C have actually been enlarged, but notice that the contain the full A,B,C rectangles from Figure 2. This slice division also avoids "blinding" the MIP solver, and the slice constraints can actually be considered to help the MIP solver by "guiding" the algorithm in the right way.

3. Branch-and-bound algorithm

In [10] we developed a branch-and-bound algorithm for solving problem 3. The algorithm is shown in Algorithm 1.

Algorithm 1: Biobjective branch-and-bound algorithm

```
   Input: Problem (1) and $\gamma$
   Output: Set of nondominated points for problem (1)
 1 Formulate a weighted LP (3) for problem (1)
 2 Add the solution of the LP as a node to tree
 3 while tree not empty do
 4     get LP from tree
       /* fathom section                              */
 5     if node infeasible then
 6         fathom
 7         continue
 8     end
 9     else if node bounded then
10         fathom
11         continue
12     end
       /* branching section                           */
13     if node integer then
14         if not dominated: save solution in set of nondominated points
15         perform integer branching
16         add children to tree
17         continue
18     end
19     else if node dominated then
20         perform Pareto branching
21         add children to tree
22         continue
23     end
24     else
25         standard variable branching
26         add children to tree
27     end
28 end
29 Return set of nondominated points $\mathcal{Z}_N$
```

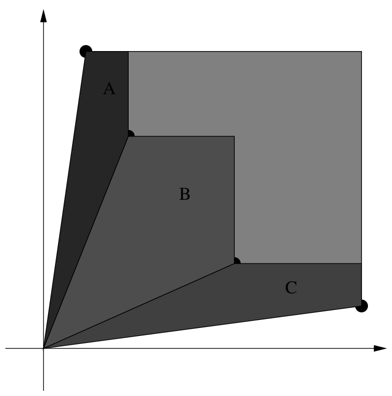


Figure 3: Division of the criterion space, using slices

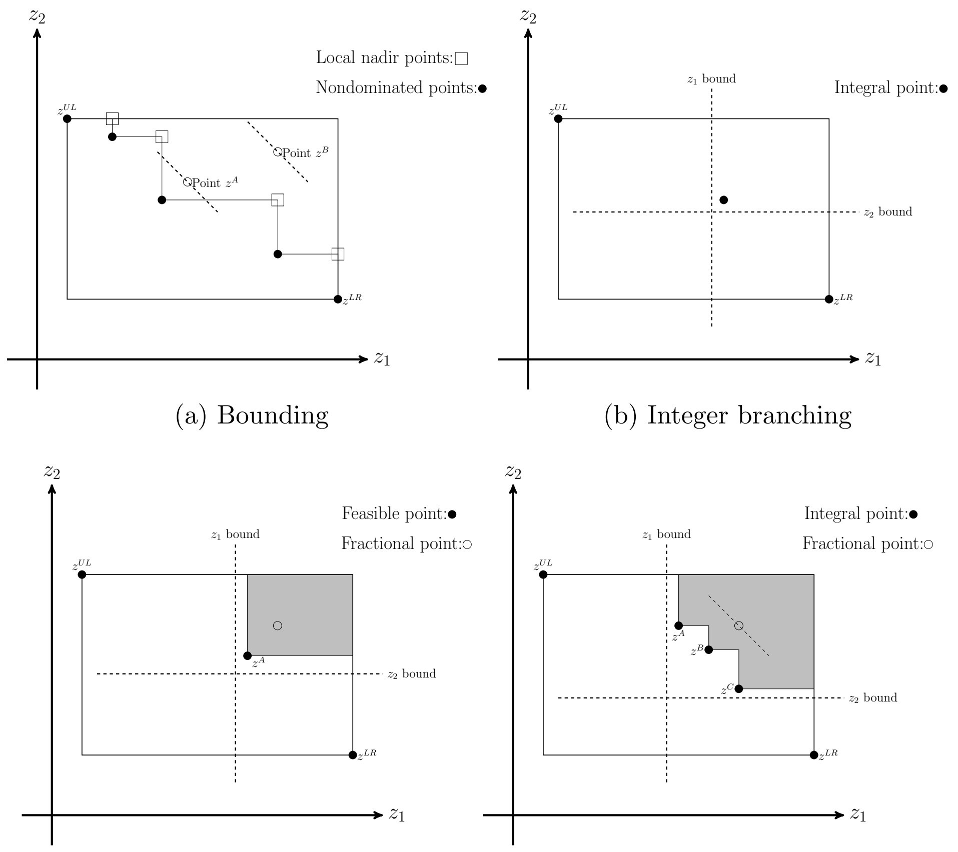
The relaxed model with the new aggregated objective function (3) can be solved with a standard LP solver. Then the main loop is entered in line 3. The algorithm does not terminate until the tree data-structure is empty. In line 4 a new node is retrieved from the tree data-structure for consideration. First we check if the node is infeasible (line 5), in which case it can be fathomed. Then we check if the node can be bounded (line 9), see Section 3.1, in which case the node can also be fathomed. If the node is neither infeasible nor can be bounded we check if the node is integral (line 13), i.e. (integer) variables have integral values, in which case the node *cannot* be fathomed. Instead so-called integer-branching (line 15) is performed, see Section 3.2. If integer branching is not performed, so-called Pareto branching (line 20) is attempted, see Section 3.3. Finally, if none of the above cases are possible, standard variable branching (line 25) is performed.

3.1. Bounding

To bound a (fractional) solution it must be guaranteed that no successor of this node in the branch tree can be part of $\mathcal{Z}_N$. The situation is illustrated in Figure 4a, where the local nadir points are marked with open squares. Two nodes z^A and z^B are shown as open circles and both are dominated. But node z^A cannot be fathomed whereas node z^B can. The reason is that the objective value of z^B is worse than the objective value of the worst local nadir point.

3.2. Integer branching

When a node is integral, it is still possible that later branches of the node can become part of $\mathcal{Z}_N$. Hence so-called integer branching is necessary, see Figure 4b. The integer solution is shown as the filled circle. Now two branches are created, one where a new upper bound is given for z_1 and one where a new upper bound is given for z_2. Hence the branch is set to be lower than the node z_1 value by $z_1 - \delta$. If the coefficients c_i^1 are integers the above cut is correct provided that $\delta < 1$. Also, the z_2 upper bound can be set to $z_2 - \delta$, for a small positive value of δ.



(a) Bounding (b) Integer branching

(c) Pareto branching (d) Improved Pareto branching

Figure 4: **Bounding and branching**

3.3. Pareto branching

The algorithm in Algorithm 1 attempts a specialized type of branching first, if the current node cannot be bounded. If the node is *dominated* by a solution in the current incumbent set of nondominated points, Pareto branching can be performed, see Figure 4c. The non-integral node, shown as an open circle, is dominated by point z^A, so two new branches can be created, one where z_1 is less than the z_1 bound of point z^A and one where z_2 is less than the z_2 bound of point z^A.

The Pareto branching procedure, shown in Figure 4c, is essential for the performance of the biobjective branch-and-bound algorithm, but it can be improved, see Figure 4d. Again the node is dominated by a solution, point z^B, in the current incumbent set of nondominated points, but instead of just using the z_1 and z_2 values of point z^A, we can utilize the value of the objective function in program (3), illustrated by the stapled line in the figure. Because of the objective function, we can be sure that the children of the current point can never improve the set of nondominated points between point z^A and z^B and that it can never improve the set of nondominated points between point z^B and point z^C. Hence the branch

z_1 value is less than the z_1 value for point z^A and the branch z_2 is less than the z_2 value for point z^C.

3.4. Slicing

In our previous paper [10] the two most important algorithmic techniques which improve the performance is Pareto Branching and a technique which is called Slicing. Each particular slice is determined by two lines originating from origo. If nothing is known about the set of nondominated points the angle between the two lines defining a particular slice is the same, no matter what slice it is, as illustrated in Figure 5.

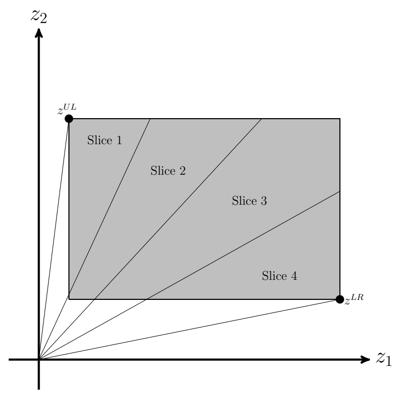


Figure 5: Visualisation of slicing

In [10] slicing was used to improve bounding, see [10] for details. In this paper we will still utilize the improved bounding, but at the same time the slices are solved in parallel, as described in Section 4. In order to have slices, two extra constraints need to be added, see below in program (5).

$$\min\{z_1 + \gamma z_2 \mid Ax + Gy \geq b,\ \alpha^{down} z_1 \leq z_2 \leq \alpha^{up} z_1,\ x \in \mathcal{Z}^n_+,\ y \in \mathcal{R}^p_+\} \tag{5}$$

4. Criterion Space Parallelization

Traditional parallelization of branch-and-bound algorithms typically split the search tree into different sub-trees, each of which is solved using its own processor. The only communication which is necessary for the different processes is to exchange the value of the best incumbent solution. It should, however, be mentioned that the leading solver products, Gurobi, Cplex and Xpress use proprietary code and may hence use very different parallelization schemes than the classic approach.

Sub-tree parallelization for biobjective branch-and-bound algorithms is, however, problematic: *The whole set of nondominated points $\mathcal{Z}_N$ is incumbent* in biobjective branch-and-bound algorithms. Given that $\mathcal{Z}_N$ may consist of a large number of points, and any new point that is not dominated by the local set of nondominated points needs to be communicated to all the other sub-tree algorithms. Hence the communication needs become significantly higher.

Below, we will present an alternative approach: Criterion space parallelization, where each slice is solved separately on different processors.

The problem of communication between the slices arises because each slice does not know the nondominated end points, i.e. the two nondominated points closest to the radial constraints. Through pre-computation we are able to find these points in two steps: First calculate the nondominated points around each radial slice constraint, see Section 4.1. Afterwards, calculate the slice nondominated end point, see Section 4.1.

4.1. Finding radial-constraint pairs of nondominated points

Given the BOMIP model (1) the set of nondominated points $\mathcal{Z}_N$ are located inside the square with extreme points $(z_1^{UL}, z_2^{LR}), (z_1^{UL}, z_2^{UL}), (z_1^{LR}, z_2^{UL})$, and (z_1^{LR}, z_2^{LR}). This square can be partitioned into two parts $\triangle_{up}$ and $\triangle_{dowm}$ by adding a radial constraint $z_2 = \alpha z_1$:

$$\begin{aligned}\triangle_{up} &= \{z \in \mathcal{R}^2 \mid z_2 \geq \alpha z_1,\ z_1^{UL} \leq z_1 \leq z_1^{LR},\ z_2^{LR} \leq z_2 \leq z_2^{UL}\}\\ \triangle_{down} &= \{z \in \mathcal{R}^2 \mid z_2 \leq \alpha z_1,\ z_1^{UL} \leq z_1 \leq z_1^{LR},\ z_2^{LR} \leq z_2 \leq z_2^{UL}\}\end{aligned}$$

We may make two restrictions of problem 1. These are

$$\min\{(z(x,y) \mid c^2x + h^2y \geq \alpha c^1 x,\ (x,y) \in \mathcal{X}\} \tag{6}$$

$$\min\{(z(x,y) \mid c^2x + h^2y \leq \alpha c^1 x,\ (x,y) \in \mathcal{X}\} \tag{7}$$

Let $\mathcal{X}_{up}$ denote the feasible set to problem (6) and let $\mathcal{X}_{down}$ denote the feasible set to problem (7). Furthermore, let $\mathcal{Z}_{N,up}$ denote the set of nondominated points to problem (6) and let $\mathcal{Z}_{N,down}$ denote the set of nondominated points to problem (7). Clearly, $\mathcal{Z}_{N,up} \subseteq \triangle_{up}$ and $\mathcal{Z}_{N,down} \subseteq \triangle_{down}$. Because $\mathcal{X}_{up} \subseteq \mathcal{X}$ it follows that $\mathcal{Z}_N \cap \triangle_{up} \subseteq \mathcal{Z}_{N,up}$. Similarly, $\mathcal{Z}_N \cap \triangle_{down} \subseteq \mathcal{Z}_{N,down}$. Also, $\mathcal{Z}_N \subseteq (\mathcal{Z}_{N,up} \cup \mathcal{Z}_{N,down})$. However, it may be the case that $\mathcal{Z}_N \cap \triangle_{up} \subset \mathcal{Z}_{N,up}$ or $\mathcal{Z}_N \cap \triangle_{down} \subset \mathcal{Z}_{N,down}$.

We want to find two nondominated points z^A and z^B to problem (1). Point z^A should belong to $\triangle_{up}$ and be as close to the radial constraint $z_2 = \alpha z_1$ as possible. Similarly, point z^B should belong to $\triangle_{down}$ and be as close to the radial constraint $z_2 = \alpha z_1$ as possible, see Figure 6. The two points z^A and z^B are a pair of Radial Constraint Nondominated Points (RCNP). To find these points we define two optimization problems, namely one for each of the parts $\triangle_{up}$ and $\triangle_{down}$. For the part $\triangle_{up}$ we can define the model $R(\alpha, \geq)$:

$$R(\alpha, \geq) \qquad \operatorname{lexmin}\{(z_2(x,y), z_1(x,y)) \mid (x,y) \in \mathcal{X}_{up}\}, \tag{8}$$

and for the part $\triangle_{down}$ we can define the model $R(\alpha, \leq)$:

$$R(\alpha, \leq) \qquad \operatorname{lexmin}\{(z_1(x,y), z_2(x,y)) \mid (x,y) \in \mathcal{X}_{down}\} \tag{9}$$

Algorithm 2 shows how the two RCNP points z^A and z^B can be determined.

Algorithm 2: Radial constraint nondominated point pairs algorithm

```
    Input: $R(\alpha, \geq)$ and $R(\alpha, \leq)$
    Output: RCNP
 1  Solve $R(\alpha, \geq)$. Solution $z^A = (z_1^A, z_2^A)$.
 2  Solve $R(\alpha, \leq)$ Solution $z^B = (z_1^B, z_2^B)$.
 3  if $z^A \prec z^B$ then
 4      // We need to re-calculate $z^B$
 5      Add the constraint $\{z \in \mathcal{R}^2 \mid z_2 \leq z_2^A - \epsilon\}$ to problem $R(\alpha, \leq)$ and solve it.
         Solution $z^B = (z_1^B, z_2^B)$
 6      return $z^A, z^B$
 7  end
 8  else if $z^B \prec z^A$ then
 9      // We need to re-calculate $z^A$
10      Add the constraint $\{z \in \mathcal{R}^2 \mid z_1 \leq z_1^B - \epsilon\}$ to problem $R(\alpha, \geq)$ and solve it.
         Solution $z^A = (z_1^A, z_2^A)$
11      return $z^A, z^B$
12  end
13  else
14      // neither $z^A$ nor $z^B$ dominated by each other
15      return $z^A, z^B$
16  end
```

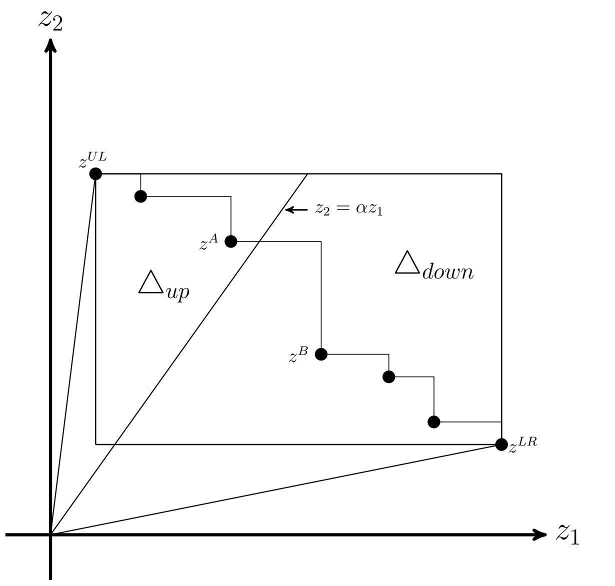


Figure 6: z^A and z^B are a pair of Radial Constraint Nondominated Points (RCNP)

4.1.1. RCNP features

We would like to prove a number of different features.

Theorem 1. *The two RCNP points z^A and z^B found in Algorithm 2 belong to $\mathcal{Z}_N$.*

Proof.
Let z^A and z^B be the two points found in Algorithm 2 by solving $R(\alpha, \geq)$ and $R(\alpha, \leq)$. Notice that $z^A \in \mathcal{Z}_{N,up}$ and $z^B \in \mathcal{Z}_{N,down}$, see Figure 7.

Case 1: $z^A = z^B$.
z^A is not dominated by any points in $\mathcal{Z}_{N,up}$, and z^B is not dominated by any points in $\mathcal{Z}_{N,down}$. We can therefore conclude that z^A is not dominated by any points in $\mathcal{Z}_{N,up} \cup \mathcal{Z}_{N,down}$. It follows that $z^A \in \mathcal{Z}_N$.

In the following we assume that $z^A \neq z^B$.

Case 2: $z^A \prec z^B$.
In this case $z_1^A \leq z_1^B$ and $z_2^A \leq z_2^B$, with at least one strict inequality. In this case B is re-calculated, see Algorithm 2. For a sufficiently small positive value of ϵ we add the constraint $\{z_2 \leq z_2^A - \epsilon\}$ to problem $R(\alpha, \leq)$. Denote the re-calculated point as $z^{B'} \in \mathcal{Z}_{N,down}$. We know that $z_1^A < z_1^{B'}$ and $z_2^A > z_2^{B'}$.

Assume that $z^{B'} \notin \mathcal{Z}_N$. Then there exists a point $z^C \in \mathcal{Z}_N$ such that $z^C \prec z^{B'}$. Because $\mathcal{Z}_N \subseteq (\mathcal{Z}_{N,up} \cup \mathcal{Z}_{N,down})$ it follows that $z^C \in \mathcal{Z}_{N,up}$. Also, $z_1^C \leq z_1^{B'}$ and $z_2^C \leq z_2^{B'}$, with at least one strict inequality. But then $z_2^C \leq z_2^{B'} < z_2^A$. But this contradicts that $z^A \in \mathcal{Z}_{N,up}$. We can therefore conclude, that $z^{B'} \in \mathcal{Z}_N$.

If $z^A \notin \mathcal{Z}_N$ then there is a nondominated point $z^D \in \mathcal{Z}_N$ such that $z_1^D \leq z_1^A$ and $z_2^D \leq z_2^A$, with at least one strict inequality. Because $\mathcal{Z}_N \subseteq (\mathcal{Z}_{N,up} \cup \mathcal{Z}_{N,down})$ it follows that $z^D \in \mathcal{Z}_{N,down}$. But $z_1^D \leq z_1^A < z_1^{B'}$ contradicting that $z_{B'} \in \mathcal{Z}_{N,down}$.

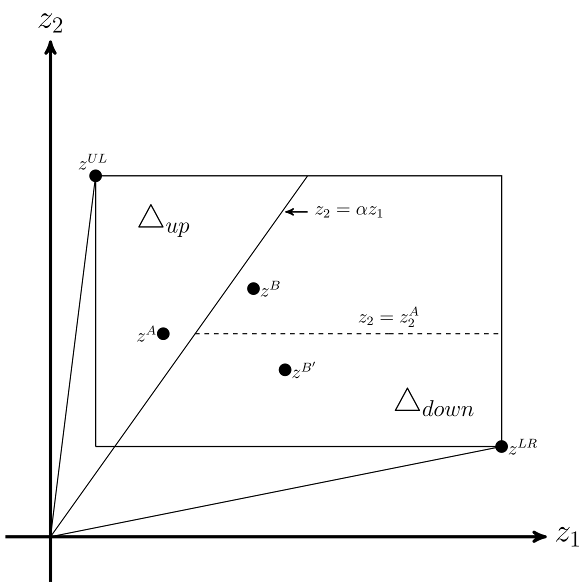


Figure 7: z^A and z^B found in Algorithm 2

Case 3: $z^B \prec z^A$.
Similar to case 2.

Case 4: $z^A \nprec z^B$ and $z^B \nprec z^A$.
If $z_1^B < z_1^A$ it follows that $z_2^B \leq \alpha z_1^B < \alpha z_1^A \leq z_2^A$. This contradicts that $z^A \nprec z^B$. We can conclude that $z_1^B > z_1^A$ and $z_2^B < z_2^A$. Assume that $z^B \notin \mathcal{Z}_N$. Then there exists a point $z^D \in \mathcal{Z}_N$ such that $z^D \prec z^B$. Because $\mathcal{Z}_N \subseteq (\mathcal{Z}_{N,up} \cup \mathcal{Z}_{N,down})$ it follows that $z^D \in \mathcal{Z}_{N,up}$. We know that $z_1^D \leq z_1^B$ and $z_2^D \leq z_2^B$, with at least one strict inequality. This implies $z_2^D < z_2^A$, a contradiction. We conclude that $z^B \in \mathcal{Z}_N$. Similarly it can be seen that $z^A \in \mathcal{Z}_N$. □

Theorem 2. *The two RCNP points z^A and z^B found in Algorithm 2 are consecutive non-dominated points in $\mathcal{Z}_N$.*

Proof.
Let z^A and z^B be the two points found in Algorithm 2 by solving $R(\alpha, \geq)$ and $R(\alpha, \leq)$. Notice that $z^A \in \mathcal{Z}_{N,up}$ and $z^B \in \mathcal{Z}_{N,down}$.

Case 1: $z^A \nprec z^B$ and $z^B \nprec z^A$.
Suppose that z^A and z^B are not consecutive nondominated points in $\mathcal{Z}_N$. Then there exists a point $z^C \in \mathcal{Z}_N$ with the property

$$z_1^B < z_1^C < z_1^A \text{ and } z_2^B > z_2^C > z_2^A$$

Clearly, $z^C \in \mathcal{Z}_{N,up}$ or $z^C \in \mathcal{Z}_{N,down}$. If $z^C \in \mathcal{Z}_{N,up}$ then it contradicts that $z^A \in \mathcal{Z}_{N,up}$, and if $z^C \in \mathcal{Z}_{N,down}$ it contradicts that $z^B \in \mathcal{Z}_{N,down}$.

Case 2: $z^A \prec z^B$**.**
As in Theorem 1 point z^B is re-calculated. The new point $z^{B'} \in \mathcal{Z}_{N,down}$. The remaining part of the proof can be carried out as in case 1. □

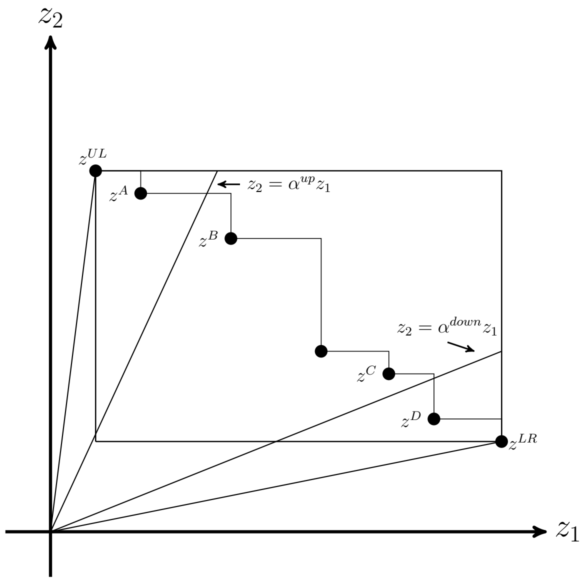


Figure 8: RCNP points for a sliced BOMIP

As mentioned in Subsection 3.4 we use slicing to improve our bounding in the biobjective branch-and-bound algorithm. In particular we want to determine the set of nondominated points within each slice in parallel (each slice is solved separately on different processors). The set of nondominated points in a particular slice $\alpha^{down} z_1 \leq z_2 \leq \alpha^{up} z_1$ is found by solving problem (5). If Algorithm 2 is used with input $R(\alpha^{up}, \geq)$ and $R(\alpha^{up}, \leq)$ we find the nondominated points z^A and z^B, and if Algorithm 2 is used with input $R(\alpha^{down}, \geq)$ and $R(\alpha^{down}, \leq)$ we find the nondominated points z^C and z^D, see Figure 8.

By construction, the nondominated points z^B and z^C are the "outer" points of the part of $\mathcal{Z}_N$ *inside* the slice defined by $\alpha^{down} z_1 \leq z_2 \leq \alpha^{up} z_1$. Hence, we can solve this slice in isolation from the other slices. Because both points are part of $\mathcal{Z}_N$ according to Theorem 1, no point exists which dominates these, and hence, no point found *outside* this slice can dominate them, hence no point outside this slice can affect the inside set of nondominated points, and the slice can be solved completely separated from the other slices.

The following lemmas are easy to verify.

Lemma 3. *If $z^A = z^C$ or $z^B = z^D$, then the interior of the slice $\alpha^{down} z_1 \leq z_2 \leq \alpha^{up} z_1$ contains no nondominated points.*

Lemma 4. *If $z^A \neq z^C$ and $z^B \neq z^D$ and if $z^B = z^C$, then the interior of the slice contains only one nondominated point, namely the point z^B.*

4.2. Heuristic construction of the slices

In Subsection 4.1 we showed how it is possible to find RCNP points which can divide the problem of finding $\mathcal{Z}_N$ into smaller parts, which can be solved independently of each other. If nothing is known about the set of nondominated points we may simply divide the rectangle spanned by z^{UL} and z^{LR}, where nondominated points may exist into, say k, slices as illustrated in Figure 5.

In our experiments with the parallel branch-and-bound algorithm we noticed that the solution time for each specific slice was very dependent on the area of the rectangle spanned by the two "outer" RCNP points within the slice (such as the points z^B and z^C illustrated in Figure 8). The smaller the area of the rectangle spanned by the two RCNP points, the faster is the set of nondominated points within the slice found. Given that we have k slices, we have k such rectangles, one in each slice. Because we are solving the problem using a parallel branch-and-bound algorithm the total solution time is very dependent upon the solution time of the slice containing the rectangle with the largest area. Therefore, we may aim at defining the k slices in such a way that the size of the area of the largest of the k rectangles is as small as possible. Unfortunately, to be able to do that we need to know the full set of nondominated points.

Now suppose that we know a subset of the set of (not necessarily nondominated) points in the feasible set in criterion space that includes the two nondominated points z^{UL} and z^{LR}. Given a partition of the rectangle spanned by z^{UL} and z^{LR} into k slices, we can within each slice determine the area of the rectangle spanned by the subset of the set of nondominated solutions contained in that slice. Then we know the area of the largest of the k rectangles. The problem is then to divide the rectangle spanned by z^{UL} and z^{LR} into k slices such that the area of the largest of the k rectangles calculated in the way described above is as small as possible.

The above method is illustrated in Figure 9. Assume that we know a subset of the feasible set of points in criterion space consisting of 8 points: $z^{UL} = (2,16), z^A = (4,14), z^B = (5,10), z^C = (6,9), z^D = (9,8), z^E = (11,7.2), z^F = (16,5.5), z^{LR} = (20,4)$. We want to divide the rectangle spanned by z^{UL} and z^{LR} into 3 slices. In Figure 9a the angles between the two lines defining a slice are the same for all three slices. The rectangles within each slice determined by the known subset of nondominated points are also illustrated. The three areas, left to right, are 18, 3 and 28.8. In Figure 9b we have illustrated the optimal partitioning of the rectangles spanned by z^{UL} and z^{LR} into 3 slices. The three areas, left to right, are 4, 16.8 and 6. Notice, that the area of the largest rectangle in Figure 9b is much smaller in this particular case than the area of the largest rectangle in Figure 9a. The area of the rectangle spanned by the two points z^{UL} and z^{LR} is 216. In Figure 9a the area of the largest rectangle constitutes 13.3% of the area of the rectangle spanned by the two points z^{UL} and z^{LR}. Similarly, in Figure 9b the area of the largest rectangle constitutes 7.8% of the area of the rectangle spanned by the two points z^{UL} and z^{LR}. The sums of the three areas in the two figures constitute 23.1% and 12.4%, respectively, of the area of the rectangle spanned by the two points z^{UL} and z^{LR}.

Assume, that we have an initial set $N = \{z^{UL}, z^1, z^2, \cdots, z^{K-2}, z^{LR}\}$ of (not necessarily nondominated) points in the feasible set in criterion space, that includes the two nondominated points z^{UL} and z^{LR}. The points are ordered increasingly according to the first coordinate, and there are K points in the initial set. The two points z^{UL} and z^{LR} may be found using lexicographic optimization, whereas the other points may have been found

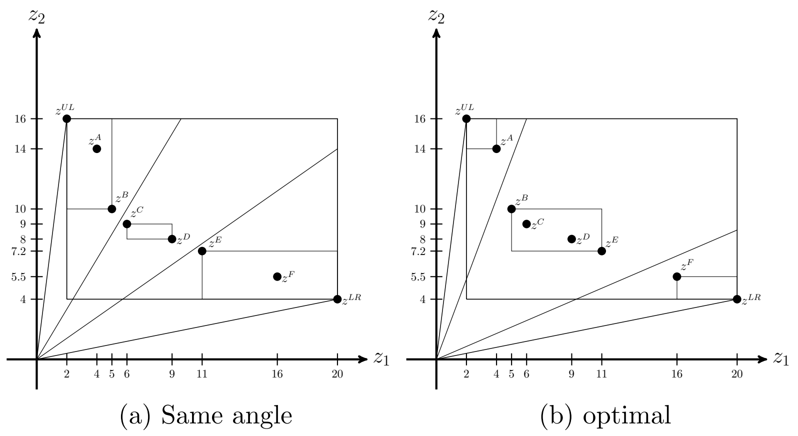


Figure 9: **Partitioning $z^{UL} - z^{LR}$ rectangle into slices**

by using a heuristic. The problem of dividing the rectangle defined by z^{UL} and z^{LR} into k slices, such that the maximal area between the two "outer" points in a slice is as small as possible, can be solved by a flow algorithm. We assume that each slice contains at least two points. We define an acyclic network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with nodeset $\mathcal{V} = \{1, 2, \cdots, n\}$ and edgeset $\mathcal{E} = \{(i, j) \mid i, j \in \mathcal{V}\}$:

- Column 0 contains node z^{UL} and column k contains node z^{LR}.
- Column 1 contains a node corresponding to the points $\{z^1, z^2, \cdots, z^{K-2k+1}\}$.
- Column 2 contains a node corresponding to the points $\{z^3, \cdots, z^{K-2k+3}\}$.
- We continue the construction in this way. Each time we form a new column the two first nodes from the previous column is removed, and the two successor nodes of the last node in the previous column is added. Each of the columns $2, 3, \cdots, k-1$ contains $K - 2k + 1$ points.

  Column $k - 1$ contains the nodes $\{z^{2k-3}, z^{2k-1}, \cdots, z^{K-3}\}$.
- There are arcs from node z^{UL} in column 1 to all nodes in column 2. The weights of the arcs are the value of the area between the corresponding two points. For instance, the value of the arc from z^{UL} in column 1 to node z^1 in column 2 is the value of the area of the rectangle spanned by z^{UL} and z^1.
- The value of an arc from a node z^i in column 2 to node z^j, $i + 2 \leq j \leq K - 2k + 3$ in column 3 is the value of the area of the rectangle spanned by the points z^{i+1} and z^j. Similarly, the values of arcs leaving a node z^i in column a to a node z^j, $i + 2 \leq j \leq K - 2k + 2a - 1$ in the successive column is the area of the rectangle spanned by the points z^{i+1} and z^j.

Now consider a path $\mathcal{P}$ in the above mentioned network. It has this form $\mathcal{P} = \{z^{UL}, \bar{z}^1, \cdots, \bar{z}^{k-1}, z^{LR}\}$. So it starts at node z^{UL}, passes through one node in each of the $k-1$ columns and ends at node z^{LR}. The interpretation of the path is that $\bar{z}^1$ is the end point of slice 1, $\bar{z}^2$ is the end point of slice 2, etc.

In Figure 10 we have shown how the construction of the network is done using the 8 points $z^{UL}, z^A, \cdots, z^F, z^{LR}$ mentioned earlier in this subsection. One path is $\mathcal{P} = \{z^{UL}, z^A, z^E, z^{LR}\}$ This means that z^A is the ending point of slice 1, z^E is the ending point of slice 2, and that z^{LR} is the ending point of slice 3. Therefore, slice 1 consists of the two points $\{z^{UL}, z^A\}$, slice 2 consists of the four points $\{z^B, z^C, z^D, z^E\}$, and slice 3 consists of the two points $\{z^F, z^{LR}\}$. The values of the three arcs are $(z^{UL}, z^A) = 6, (z^A, z^E) = 16.8$ and $(z^E, z^{LR}) = 6$.

A Bellman-Ford style flow algorithm can applied to find the path which has the smallest maximal weight attached to it. In practice the algorithm is very fast. The quality of the slices depends, however, on the quality of the initial set of points.

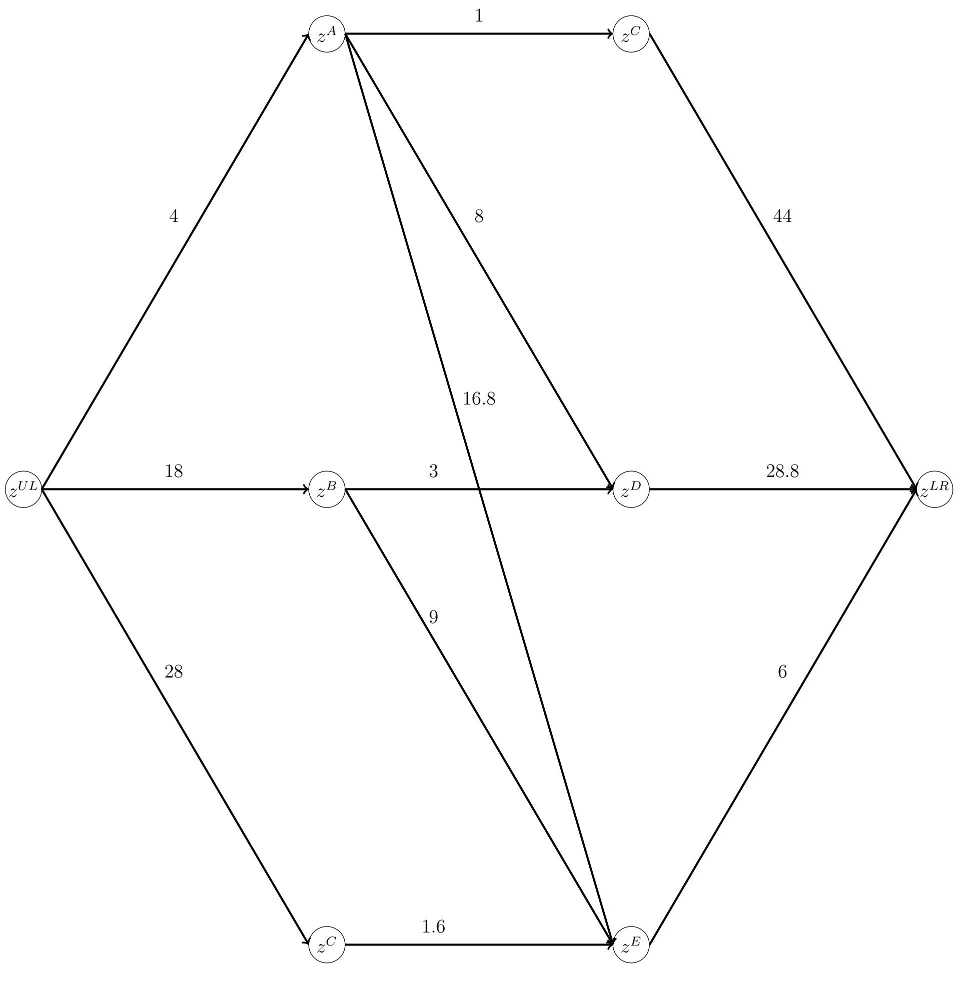


Figure 10: Network for flow algorithm.

4.3. *Optimality of the slice selection*

Each slice is searched independently for nondominated points by our branch-and-cut algorithm and our experiments in Section 6 clearly indicate that the smaller the area of the slice rectangle, the faster the branch-and-cut algorithm terminates. The path approach described in the previous subsection finds the slice division, which leads to the minimal maximal rectangle. The division is however dependent on the quality of the points used, and in our experiments we use a set of points found using a heuristic.

5. An application to the biobjective traveling salesman problem

In this section we will show how to use the parallel branch-and-bound algorithm described in Sections 3 and 4 to solve the biobjective traveling salesman problem.

The traveling salesman problem consists in finding a shortest Hamiltonian cycle through n cities. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote a graph with nodeset $\mathcal{V} = \{1, 2, \cdots, n\}$ and edgeset $\mathcal{E} = \{(i,j) \mid i,j \in \mathcal{V}\}$. In the biobjective case (bTSP) we have two distance matrices $\{(c^1_{ij}, c^2_{ij}) \mid i,j \in \mathcal{V}\}$, and the problem is to find a cyclic permutation π of $\{1, 2, \cdots, n\}$ that minimizes $(\sum_{i=1}^{n} c^1_{i,\pi(i)}, \sum_{i=1}^{n} c^2_{i,\pi(i)})$. Here we are only interested in the symmetric case, that is $c^k_{ij} = c^k_{ji},\ i,j \in \mathcal{V},\ k = 1, 2$. If we let the binary variables x_{ij} be equal to 1 if city j is visited immediately after city i the problem can be formulated as shown below.

$$\begin{aligned}
\min\quad & z_1(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} c^1_{ij}x_{ij} \\
\min\quad & z_2(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} c^2_{ij}x_{ij} \\
\text{s.t.}\quad & \\
& \sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, \cdots, n \\
& \sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, \cdots, n \\
& \sum_{i\in S, j\in \mathcal{V}\setminus S} x_{ij} \geq 2, \quad S \subset \mathcal{V},\ 2 \leq |S| \leq |\mathcal{V}| - 2 \\
& x_{ij} \in \{0, 1\}, \quad i,j \in \mathcal{V}
\end{aligned} \tag{10}$$

The biobjective TSP is $\mathcal{NP}$-complete and intractable, see [26]. In particular, even though all entries in both distance matrices are positive, the number of nondominated points may be equal to $(n-1)!$ (all Hamiltonian cycles are efficient).

5.1. *Previous Approaches to Bi-Objective TSP*

As the single-objective TSP has been a testbed for a number of different single-objective optimization techniques, multiobjective TSP could be used in a similar way to develop new multiobjective optimization techniques. However, this seems not to be the case. We have only been able to find **one** article [1] which solves the bi-objective TSP problem. This article is very interesting, both because it is the first to solve the bi-objective TSP and because there are similarities with our approach. In [1] the previously developed method AUGMECON2 [6] is tested on bi-objective TSP problems and bi-objective Set-Covering problems. To speed up solution, the AUGMECON2 algorithm is parallelized to three threads, simply by starting the same algorithm with different parameter settings. In our result Section 6.1 we will compare with the results in [1] and show that our approach leads to a very significant speedup.

We will briefly mention two somewhat similar biobjective problems which has been solved to optimality, namely the Multilabel Traveling Salesman Problem mTSP [8] and the Traveling Salesman Problem with profits TSPp [7].

5.2. The overall procedure

Given that program (10) contains an exponential number of sub-tour elimination constraints, typically branch-and-cut is applied. Indeed, as stated earlier, branch-and-cut was *invented* for TSP solution [22].

Our procedure for solving BITSP is divided into three steps:

- Use a heuristic procedure to find an initial set of nondominated points. The two points z^{UL} and z^{LR} are included in the initial set. To reduce the length of this paper we have used the previously found best solutions to BITSP problems instead.
- Determine the slices as described in Subsection 4.2.
- Use the parallel branch-and-bound method described in Sections 3 and 4.
  - Include cuts.

5.3. Cuts for BITSP

Notice that set of feasible solutions to BITSP (10) is not affected by the fact that there are two objective functions. The importance of this becomes clear when one realizes that then all previously developed cuts for the traveling salesman problem can be applied in a specialized BITSP branch-and-cut algorithm.

In the past several years a number of cuts have been developed for the TSP. It is beyond the scope of this paper to describe the cut-research for TSP, and instead we refer to the book by Applegate, Bixby, Chvatal and Cook [28]. For the single-objective TSP their Concorde [23, 29] branch-and-cut solver holds the current record for solving large-scale single-objective TSP, by a wide margin.

Since we can apply all the previously developed cuts and the Concorde software is open-source, we simply apply the same cuts in the parallel biobjective branch-and-bound algorithm. In Algorithm 3 we describe the cutting plane procedure (procedure Cutting-Plane-Procedure()). This procedure takes as input an LP-program. Then a series of separation algorithms for finding cuts are attempted on the optimal LP solution. If one or more cuts are found it is added to the current LP problem, and the problem is reoptimized. Then we continue the search for new cuts. The procedure stops when no more cuts can be found.

We have divided the separation algorithms into two parts: Sub-tour cuts and Comb cuts. For the sub-tour cuts the separation algorithms are attempted in the following order: Sub-Tour-Connect-Cuts(), Sub-Tour-Segment-Cuts(), Sub-Tour-Shrink-Cuts() and Sub-Tour-Exact-Cuts().The Comb cuts are attempted in the order: Ghfast-Blossom(), Fast-Blossom(), Exact-Blossom(), Block-Combs(), Edge-Comb-Grower() and Cliquetree(). All the cuts and the separation algorithms to find the cuts are from the Concorde software.

As all generated cuts are valid for all the nodes in the branching tree, a separate issue arises, namely if the cuts should only be used locally (on a particular node in the branching tree) or used globally (on all the nodes in the branching tree). During our experiments we observed that when the cuts were used globally the solution time for solving the LP problems were much higher than when just using the cuts locally, even though in the latter case we might generate the same cuts several times. Furthermore, only up to about 10 % of the execution time is used for finding cuts.

5.4. Biobjective branch-and-cut algorithm for BITSP

The full biobjective branch-and-cut algorithm for BITSP is shown in Algorithm 5. It takes as input the biobjective traveling salesman problem (10) and a parameter γ. First an initial set of points are formed in line 2. Using these points k slices are constructed in line 3. Next, a weighted LP (5) for each of the k slices are formulated in line 4. The set of nondominated points within each of the k slices are found in lines 6-8, and finally the full set of nondominated points are returned in line 11. Algorithm 4 describes how to find the set of nondominated points in a particular slice. It is simply the branch-and-bound algorithm described in Algorithm 1, with the addition of the cutting plane procedure described in Algorithm 3.

6. Tests

The classic site for TSP test problems, the TSPLIB [30] contains a large number of TSP instances of various types, all of which are single-objective and all of which have a known optimal value. Unfortunately, such a site does not exist for BITSP problems. Instead, the only BITSP test set which has been used in several articles [31, 32, 33, 34, 1] is obtained by combining 5 symmetric TSP problems from TSPLIB with 100 cities: kroA100.tsp, kroB100.tsp, kroC100.tsp, kroD100.tsp, and kroE100.tsp. These 5 instances are then combined to obtain 10 BITSP instances, e.g. kro100ab.motsp, where the first cost matrix $\{c^1_{ij}\}$ comes from kroA100.tsp and the second cost matrix $\{c^2_{ij}\}$ comes from kroB100.tsp. All the cost matrixes in symmetric TSP instances have positive integer values. The big benefit of having integer costs is that rounding errors are avoided and comparisons of results become easier.

The algorithms are implemented in C++ using the g++ (GCC) compiler, 4.4.7, the ILOG CPLEX 12.1.1 callable library. The biobjective branch-and-cut algorithm (Algorithm 5) is implemented using the ILOG CPLEX 12.1.1 call back routines. The cuts added, as shown in Algorithm 3, are found using separation routines from Concorde [23, 29]. The Concorde

Algorithm 3: Cutting-plane-procedure()

```
    Input: An LP for BITSP
    Output: A strengthened LP
 1  Optimize LP
 2  while cuts added or first do
 3      if Sub-Tour-Connect-Cuts() then
 4          add cuts and break
 5      end
 6      else if Sub-Tour-Segment-Cuts() then
 7          add cuts and break
 8      end
 9      else if Sub-Tour-Shrink-Cuts() then
10          add cuts and break
11      end
12      else if Sub-Tour-Exact-Cuts() then
13          add cuts and break
14      end
15      else if Ghfast-Blossom() then
16          add cuts and break
17      end
18      else if Fast-Blossom() then
19          add cuts and break
20      end
21      else if Exact-Blossom() then
22          add cuts and break
23      end
24      else if Block-Combs() then
25          add cuts and break
26      end
27      else if Edge-Comb-Grower() then
28          add cuts and break
29      end
30      else if Cliquetree() then
31          add cuts and break
32      end
33      Re-optimize LP
34  end
```

Algorithm 4: Branch-and-Cut-for-TSP()

```
   Input: A weighted LP (5) and initial archive
   Output: Set of nondominated points for a slice
 1 Cutting-Plane-Procedure()
 2 Add the solution of the LP as a node to tree
 3 while tree not empty do
 4     get LP from tree
       /* fathom section                                   */
 5     if node infeasible then
 6         fathom
 7         continue
 8     end
 9     else if node bounded then
10         fathom
11         continue
12     end
       /* branching section                                */
13     if node integer then
14         if not dominated: save solution in set of nondominated points
15         perform integer branching
16         Cutting-Plane-Procedure()
17         add children to tree
18         continue
19     end
20     else if node dominated then
21         perform Pareto branching
22         Cutting-Plane-Procedure()
23         add children to tree
24         continue
25     end
26     else
27         standard variable branching
28         Cutting-Plane-Procedure()
29         add children to tree
30     end
31 end
32 Return set of nondominated points for a slice
```

Algorithm 5: Biobjective branch-and-cut algorithm for BITSP

```
   Input: Problem (10) and $\gamma$
   Output: Full set of nondominated points for problem (10)
1  $\bar{\mathcal{Z}}_{\mathcal{N}} = \{\}$
2  Initial-Set-of-Points()
3  Generate $k$ Slices as described in Subsection 4.2
4  Formulate a weighted LP (5) for each of the $k$ slices. Denote it
   $slice^i,\ i = 1, 2, \cdots, k$
5  $\bar{\mathcal{Z}}^i_N = \bar{\mathcal{Z}}_N,\ i = 1, 2, \cdots, k$
6  for $i = 1$ to $k$ do
7      Branch-and-Cut-for-TSP($Slice^i, \bar{\mathcal{Z}}^i_N$)
8  end
9  $\mathcal{Z}_N = \cup_{i=1}^k \bar{\mathcal{Z}}^i_N$
10 Remove dominated points from $\mathcal{Z}_N$
11 Return full set of nondominated points for problem (10)
```

program is Open Source and can be freely downloaded. We have, however, only used the separation routines, not the entire framework.

The tests are performed on a cluster of Linux machines using up to 60 machines in parallel, each machine consisting of 2 Xeon X5550 2.67 Ghz quadcore CPU's with 8 Mb. of cache and 24 Gb. 1333 Mhz DDR3 ECC ram. By using a batch system it is ensured that all tests are performed with exclusive processor rights, meaning that the programs cannot be interrupted or preempted by other processes.

6.1. Results

All results for the 10 test problems are shown in Table 1. The first column is the name of the datasets, ab to de. Column 2 is the single threaded solution time (in seconds) to find the initial set of points, and column 3 is the number of points in the initial set of points (line 3 in Algorithm 5). Based on the initial set of points, the Flow algorithm described in Subsection 4.2 is used to find good division angles for the RCNP algorithm. The computation time of the Flow algorithm is, for all test instances, less than 5 seconds, and these data are not included into Table 1. The RCNP algorithm then calculates the slice end points for each slice. This computation takes less than 20 seconds per point and can be performed in parallel and is hence not shown in Table 1. The biobjective branch-and-cut algorithm is then applied in parallel to the slices (line 7 in Algorithm 5). The total computation time (in seconds) and the maximal computation time for a slice is shown for 10 slices in column 4 and column 5, for 30 slices in column 6 and column 7 and for 60 slices in column 8 and column 9. Column 10 shows the number of nondominated points in $\mathcal{Z}_N$. Finally, column 11 and 12 shows the total computation time and the maximal computation time (for 3 threads) achieved in [1]. In [1] computation time is given as hours, but we have converted it to seconds. The bottom row of Table 1 is the average of each column.

	Heuristic		10 slices		30 slices		60 slices		Opt.	Florios&Mavrotas [1]	
Prob	Ti	P-size	Tot-Ti	Max-Time	Tot-Ti	Max-Ti	Tot-Ti	Max-Ti	P-Size	Tot-Ti	Max-Ti
ab	6380	1184	20888	4565	17183	1119	16291	587	3332	482400	208800
ac	6147	1168	22257	3089	13042	928	13003	471	2458	266400	108000
ad	4328	1253	12614	2145	12232	1482	11673	366	2351	176400	75600
ae	4914	1197	11849	1596	9961	943	9601	574	2112	-	-
bc	6114	1246	22962	6286	15112	807	15462	574	2752	277200	100800
bd	6234	1454	21754	2697	15522	970	14817	1023	2657	237600	82800
be	5855	1255	24675	6272	14574	999	15069	597	2842	-	-
cd	4100	1150	13853	2198	10063	701	10458	396	2044	140400	75600
ce	3524	1155	18978	4071	11140	659	11697	1059	1991	-	-
de	4107	1211	22737	2829	16896	1037	16302	528	2751	-	-
avg	5170	1227	19257	3575	13572	964	13437	617	2529	263400	108600

Table 1: Results from the tests on the datasets kroab100, ..., krode100.tsp

The results in Table 1 first and foremost show that the 10 BITSP problems can be solved to optimality in a reasonable time. Compared to [1] our approach is usually faster by a factor of more than 10, for the BITSP problems tested. Unfortunately, the BITSP problems ae, be, ce and de were not tested by in [1]. We acknowledge that we are using more powerfull computer recourse's than in [1], but still our 10 slice approach compared to the 3 threads used in [1] is 16 to 45 **times** faster. There are a number of differences in the approach which we think can explain the massive difference:

1. We use state-of-art TSP cuts from Concorde [23] whereas [1] utilize the branch-and-cut facility built in GAMS [35]. This probably leads to a significant speed improvement.
2. In [1] the program is parallelized into 3 threads, by running the same algorithm with different parameter settings. This in fact leads to what we term slices, but the slices used in [1] are rather un-balanced regarding objective area, leading to un-balanced solution times. This can be seen i Figure 6 page 13 in [1].
3. In [1] the AUGMECON2 [6] algorithm is used in each thread/slice. We speculate that our slice-solving algorithm, described in [10] is faster, particularly because of a faster cutting plane generation and utilization.

7. Conclusion

In this article we have extended the biobjective branch-and-bound algorithm described in [10] such that it can take advantage of cuts and such that the set of nondominated points within each slice can be found in parallel. Given an initial set of points in criterion space (not necessarily nondominated points) we have presented an optimal method to construct the set of slices in a way that may speed up the solution time for the biobjective branch-and-cut procedure. We have applied the extended biobjective branch-and-cut algorithm to the biobjective traveling salesman problem. The approach presented in this article can easily be applied to other BOMIPs where at most one of the objective functions contains continuous variables.

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