Combinatorial Optimization Problems in Engineering Applications

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Abstract. This paper deals with several combinatorial optimization problems. The most challenging such problem is the quadratic assignment problem. It is considered in both two dimensions (QAP) and in three dimensions (Q3AP) and in the context of communication engineering. Semidefinite relaxations are used to derive lower bounds for the optimum while heuristics are applied to either find upper bounds or good feasible solutions. Semidefinite relaxations also yield bounds for questions related to binary and spherical codes including for the kissing number problem. Finally, two combinatorial problems are solved exactly, a Q3AP from communications and a directional sensor location problem.

Keywords: combinatorial optimization, quadratic assignment problem, index assignment problem, modulation diversity, binary codes, kissing number, spherical codes, sensor location

1 Introduction

As is well-known many engineering applications call for continuous mathematical models such as ordinary or partial differential equations. The problems can be challenging due to their nonlinearity or other features and sophisticated approaches are required. This work, however, is exclusively addressing problems of importance and timeliness but with a need to solve a discrete optimization problem. The characterizing feature of these problems is that a combinatorial set of feasible solutions exists, such as $n!$ or n^k. While they have finitely many solutions it is deceptive to regard them as more easily solvable than complex but continuous problems. One obvious reason is that the number of solutions increases very rapidly and enumeration is out of the question already for moderate size. It is thus the goal to find the optimal solutions without evaluating the bulk of the possible solutions.

We will cover in the following two challenging and previously unsolved problems for which we were able to determine the exact solution and do it with reasonable effort. All other problems will, however, still be out of reach and for them at present only upper and lower bounds on the optimum can be given. If the problems are constrained minimization resp maximization problems, finding lower resp upper bounds to the optimal value is frequently the more challenging task requiring special methods while feasible solutions that are frequently easy to find provide upper resp lower bounds. For this latter task we will utilize available methods while for the former we have developed algorithms which were successfully applied to engineering applications resulting in the best currently known bounds. Apart from presenting some new and unpublished results, another purpose of the paper is to compile for the first time results for closely related problems.

The by far best-known combinatorial optimization problem is the traveling salesperson problem (TSP) of finding the shortest closed loop that passes through all n given locations exactly once. It has $n!$ possible tours of which in general just one is the optimal one. While $n!$ grows extremely rapidly, it is still possible with advanced methods that have been developed over decades to determine the exact optimum for thousands of locations. This is in sharp contrast to a very closely related problem, the quadratic assignment problem (QAP). In this case in addition to locations there are, in the most basic application, also n facilities given and a prescribed flow between them. The task is to assign the facilities to locations, one by one, such that the overall flow is minimized. Again, there are $n!$ possible assignments but the exact solution is already very challenging for n larger than about 30 and there are unsolved problems of this size [1].

While both the TSP and the QAP are very easy to understand and while they can be seen completely in an abstract framework as a challenging mathematical riddle, they both occur in important applications and need to be solved partly frequently. Every delivery schedule of a shipping company can benefit from TSP solutions, for example. It is less obvious where the QAP has similar applications except in the decision where to put, factories, storage facilities etc in order to minimize operational cost. This would likely not have to be solved frequently per day but a few times during the life of a company. These examples do not give a real picture of the prevalence of these problems.

The following sections will cover the determination of lower bounds for generic QAPs with semidefinite optimization methods (SDP), section 2, and the application of these methods to the so-called index assignment problem in communication engineering and to modulation design, section 3. In section 4 a fundamental cominatorial question of relevance in the index assignment problem will be addressed, the question how many binary numbers exist of length n that have a minimal Hamming distance of d. This is of relevance for binary codes. Then, a series of new bounds for nonbinary codes are presented. Next, a problem from spherical codes is addressed, namely the kissing number problem. Finally in section 5 two problems will be solved exactly, one is a higher dimensional QAP while the other deals with sensor location.

2 Semidefinite Relaxations by Matrix Splitting (SDRMS)

As with most important discrete optimization problems also for the QAP there is an established library of such problems for researchers to apply their algorithms to. The QAPLIB [1] has been established in 1997 and in the mean time only a relatively small set of instances remains unsolved. This is in contrast to other libraries, such as MIPLIB, available at `http://miplib.zib.de` , for which over time a number of updates have been posted. Still, these unsolved problems present a challenge and they are not of one type but occur in four of the fifteen groups (by author) of QAPLIB. To quote the outcome of the research summarized in the following: Of the 33 presently unsolved problems in QAPLIB, the SDRMS method has produced the best known lower bounds in 18 cases.

The standard quadratic assignment problem takes the following form

$$\min_{X \in \Pi} \mathrm{Tr}\big(\mathrm{XAX^TB}\big) \tag{1}$$

where $A, B \in \Re^{n\times n}$, and Π is the set of permutation matrices. This problem was first introduced by Koopmans and Beckmann [2] for facility location. The model covers many scenarios arising from various applications such as in chip design [3], image processing [4], and keyboard design [5]. For more applications of QAPs, we refer to the survey paper [6] where many interesting QAPs from numerous fields are listed. In practical applications one of the matrices is the distance matrix recording mutual distances between locations and the other the flow matrix. We assume that both matrices are symmetric. To compute lower bounds for this and other similar combinatorial optimization problems advantageously semidefinite relaxations (SDP) have been used. Using the Kronecker product we have

$$\mathrm{Tr}\big(\mathrm{XAX^TB}\big) = \mathrm{x^T(B \otimes A)x} = \mathrm{Tr}\big(\mathrm{(B \otimes A)xx^T}\big), \mathrm{x} = \mathrm{vec(X)} \tag{2}$$

where vec(X) is obtained from X by stacking its columns into a vector of order n^2. Many existing SDP relaxations of QAPs are derived by relaxing the rank-1 matrix xx^T to be positive semidefinite with additional constraints on the matrix elements. For convenience, we call such a relaxation the classical SDP relaxation of QAPs. As pointed out in [6], the SDP bounds are tighter compared with bounds based on other relaxations, but usually much more expensive to compute due to the large number $O(n^4)$ of variables and constraints in this classical SDP relaxation.

Our goal is the derivation of SDP based lower bounding methods requiring only the minimal $O(n^2)$ variables and constraints. In [7] and [8] this was accomplished. While in [7] special relaxations for the case of Hamming and Manhattan distance matrices are considered and successfully applied to large problems from QAPLIB in [8] general splittings were given. We outline just the basic approach. The first observation is that with a nonsingular matrix X the product $Y = XBX^T$ is positive semidefinite when B has this property. Let B be one of the two matrices in (1). We have the freedom to choose the matrix for which a larger lower bound is obtained. Now let B be split into the difference of two positive semidefinite matrices B^+ and B^-. Such a splitting is always possible but it is not unique. The QAP has then become

$$\min_{X \in \Pi} \mathrm{Tr}\big(\mathrm{A}(\mathrm{Y}^+ - \mathrm{Y}^-)\big) \tag{3}$$

It is necessary to link the auxiliary matrices Y^+ and Y^- to the matrix B and to make sure the matrix X is a relaxed permutation matrix. All this is done in the following basic SDRMS method.

Let e be the all-1 vector and $\min(B)$ the minimum element of B.

$$\min \mathrm{Tr}\big(\mathrm{A}(\mathrm{Y}^+ - \mathrm{Y}^-)\big) \tag{4}$$

subject to

$$\begin{aligned}
Y^+e = XB^+e, Y^-e = XB^-e, && \text{(5a)}\\
\mathrm{diag}\,(Y^+) = X\mathrm{diag}\,(B^+), Y^+ \geq \min(B^+), && \text{(5b)}\\
\mathrm{diag}\,(Y^-) = X\mathrm{diag}\,(B^-), Y^- \geq \min(B^-), && \text{(5c)}\\
Y^+ - XB^+X^T \succeq 0, Y^- - XB^-X^T \succeq 0, && \text{(5d)}\\
Xe = X^Te = e, X \geq 0 && \text{(5e)}
\end{aligned}$$

There are five lines of constraints. The symbol $\succeq$ in the fourth line denotes positive semidefiniteness while the last line defines X as required. This is not a standard SDP problem because the fourth line defines a quadratic and not a linear condition. This can be overcome by taking the matrix square roots R^+ and R^- of the positive semidefinite matrices B^+ and B^-. The fourth line is then replaced by the constraints

$$\begin{aligned}
[I, Z^+T; Z^+, Y^+] \succeq 0, && \text{(6a)}\\
[I, Z^-T; Z^-, Y^-] \succeq 0 && \text{(6b)}
\end{aligned}$$

where $Z^+ = XR^+$ and $Z^- = XR^-$.

The resulting linear SDP is advantageously phrased and solved in a modeling language. There are several available packages that have both such a language and have implemented available SDP solvers. Initially we have used [9] and the built-in solvers SeDuMi and SDPT3. For larger problems, the sizes in QAPLIB go up to 256, this requires both too much memory and too much time. So, we have linked with the package [10] and reduced to acceptable levels. Another issue is the question whether with unavoidable numerical errors the obtained lower bounds are indeed such. For this we have postprocessed with an interval arithmetic package. These measures were applied in particular for an engineering application to be described next. Here the largest size that had to be considered is 512. For details and results on the QAPLIB problems we refer to [8].

3 Index Assignment and Modulation Design in Communications Engineering

3.1 Bounds for the Index Assignment Problem

In digital communications information is coded in binary words of a certain length, transmitted. and decoded on the receiver side. However, this sounds deceptively simple and the transmission is also prone to errors. Without so-called channel errors the decoding could proceed in the reverse way to the decoding. But this cannot be assumed and one has to provide the possibility that single and multiple bit errors can be corrected. A binary word representing a piece of information, say, from a digital picture, is not transmitted itself but it is associated with a word in a codebook that must contain sufficiently many such words. This codeword is also not transmitted but its index within the codebook. This index is itself also a binary word.
Depending on how the codewords are arranged physically in the channel, an erroneous transmission will likely have confused a word with one nearby. In all practical so-called modulations these are words which differ in very few bits from the codeword. One thus has to make sure that codewords that are close to each other are indexed by binary numbers of large Hamming distance, i.e. which differ in many bit positions. It is not surprising the chance for error correction is maximized when a QAP is solved in which the distance matrix is the Hammimg distance matrix of the codewords and the flow matrix the matrix of codeword transition probabilities. For one-bit correction the flow matrix simplifies to the adjacency matrix of the n-dimensional hypercube. For the sake of completeness we derive the QAP which arises in this context. Details can be found in [11].

A basic element of a signal compression and communication system is the quantizer Q, either scalar or vector. We focus on index assignment of vector quantizers (VQ) for the superior source coding performance of VQ. All our results apply to index assignment of scalar quantizers (SQ) as well, because SQ is just a special case of VQ. A vector quantizer $Q : \Re^d \to \mathbb{C}$ maps a continuous source vector $\mathbf{x} \in \Re^d$ to a codeword $\mathbf{c}_i \in \Re^d$ in the VQ codebook $\mathbb{C} = \{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_N\}$ by the nearest neighbor rule. The index i rather than the codeword $\mathbf{c}_i$ itself is transmitted via the channel. Upon receiving i correctly, the VQ decoder can reconstruct $\mathbf{x}$ to $\mathbf{c}_i$ by inverse quantizer mapping Q^{-1} (a simple table lookup operation). Typically, the size N of the codebook $\mathbb{C}$ is made an integer power of two, $N = 2^n$ so that the codeword index i is a binary number of n bits. An index assignment of $\mathbb{C}$ is a bijection map $\pi : \mathbb{C} \leftrightarrow \{0, 1\}^n$.

If the transmission is error free, then Q^{-1} is invariant with respect to the index assignment of the N codewords in $\mathcal{C}$, hence the overall system performance is independent of codeword index assignment. However, in the event of a transmission error that an index $\pi(\mathbf{c}_i)$ is received as $\pi(\mathbf{c}_j)$, an input vector $\mathbf{x}$ such that $\mathbf{w}_i = Q(\mathbf{x})$ will be reconstructed as $\mathbf{w}_j$, incurring an extra channel distortion $d(\mathbf{c}_i, \mathbf{c}_j)$ that does depend on the index assignment π. Let $P(\jmath|\imath)$ be the probability of transmitting index $\imath$ but receiving index $\jmath$, and $P(\mathbf{c}_i)$ be the prior probability of codeword $\mathbf{c}_i \in \mathbb{C}$. Given an index assignment π, the expected channel distortion is

$$\bar{d}_\pi = \sum_{i=1}^{N} P(\mathbf{c}_i) \sum_{j=1}^{N} P(\pi(\mathbf{w}_j)|\pi(\mathbf{c}_i)) d(\mathbf{c}_i, \mathbf{c}_j). \tag{7}$$

Adopting the common probability model of binary symmetric channel (BSC), we have

$$P(\pi(\mathbf{w}_j)|\pi(\mathbf{c}_i)) = (1-p)^{n-h(\pi(\mathbf{w}_j),\pi(\mathbf{c}_i))} p^{h(\pi(\mathbf{w}_j),\pi(\mathbf{c}_i))} \tag{8}$$

where p is the BSC crossover probability, and $h(\cdot,\cdot)$ is the Hamming distance. To minimize the expected BSC channel distortion $\bar{d}_\pi$ one would like to find an optimal index assignment defined by the following objective function

$$\pi_* = \arg\min_\pi \sum_{i=1}^{N} P(\mathbf{c}_i) \sum_{j=1}^{N} (1-p)^{n-h(\pi(\mathbf{w}_j),\pi(\mathbf{c}_i))} p^{h(\pi(\mathbf{w}_j),\pi(\mathbf{c}_i)) d(\mathbf{c}_i,\mathbf{c}_j)}. \tag{9}$$

For convenience, we rewrite (7) in a matrix form. Let

$$\mathbf{W} = (P(\mathbf{c}_1), P(\mathbf{c}_2), \cdots, P(\mathbf{c}_N))^T \mathbf{I} \tag{10}$$

be the diagonal matrix consisting of prior probabilities of the VQ codewords, and let

$$\mathbf{B} = \{(1-p)^{n-h(i,j)} p^{h(i,j)}\}_{1 \le i \le N, 1 \le j \le N} \tag{11}$$

be the symmetric matrix whose elements $B(i,j)$ are the codeword transition probabilities $P(\pi(\mathbf{w}_j)|\pi(\mathbf{c}_i))$ due to BSC bit errors of probability p. Also, denote by $\mathbf{D} = \{d(\mathbf{c}_i, \mathbf{c}_j)\}_{1 \le i \le N, 1 \le j \le N}$ the symmetric distance matrix between pairs of codewords, and use the $N \times N$ permutation matrix $\mathbf{X}$ to specify π. Now, the expected channel distortion of (7) has the following matrix form

$$\begin{aligned}
\bar{d}_\pi &= \sum_{i=1}^{N} P(\mathbf{c}_i) \sum_{j=1}^{N} \{\mathbf{X}\mathbf{B}\mathbf{X}^T\}_{i,j} d(\mathbf{c}_i, \mathbf{c}_j) \\
&= \mathrm{trace}(\mathbf{W}\mathbf{X}\mathbf{B}\mathbf{X}^T\mathbf{D}) \\
&= \mathrm{trace}(\mathbf{D}\mathbf{W}\mathbf{X}\mathbf{B}\mathbf{X}^T) \\
&= \mathrm{trace}(\tilde{\mathbf{D}}\mathbf{X}\mathbf{B}\mathbf{X}^T), \tilde{\mathbf{D}} = \frac{1}{2}(\mathbf{D}\mathbf{W} + \mathbf{D}^T\mathbf{W}^T)
\end{aligned} \tag{12}$$

This is the QAP which is solved using our SDRMS method of the previous section. To test this approach a training set of 18 natural images is used to design 16-dimensional vector quantizers of various fixed integer rates n, i.e. generating codebooks of size 2^n. n was varied from 5 to 9. In order to assess the quality of the computed lower bounds a heuristic was applied to compute upper bounds. The gaps between the bounds was very satisfactory for engineering standards, in the range of $.71dB$ to $1.78dB$ for one bit errors and $.96dB$ to $1.82dB$ for multiple errors and small transition probability.

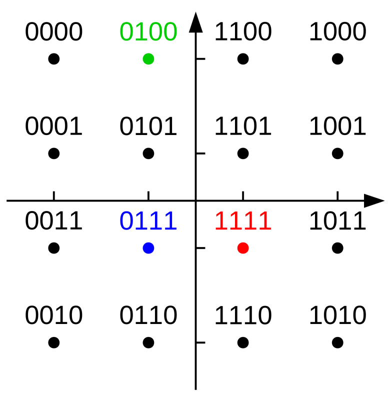
3.2 Automatic Repeat Requests and Modulation Diversity

In case even a multiple bit error correction is not possible, repeated transmissions have to be made. This is typically triggered by an automatic repeat request (ARQ). But, in order that the likelihood of a successful decoding is increased, a different index assignment has to be chosen in each of possible more than one repetition. In our recent work [12] and [13] two such scenarios were investigated in detail. The so-called modulation diversity issue was addressed first for a two-way amplify-and-forward relay. To find the optimal modulation design for a given number of repetitions in this case a sequence of standard QAPs as considered in the previous section has to be solved. In fact, with each retransmission a fully general procedure would involve QAPs of dimensions $2, 3, 4$, etc with tensors instead of matrices. This would become too costly and is not necessary as our results have shown.

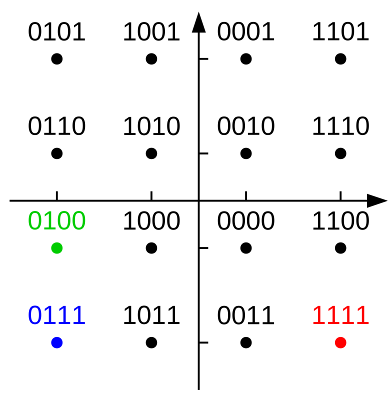
Here the task is not to encode high-resolution pictures but to wirelessly communicate. The length of the codewords does not go to n^9 or n^{10} but a typical size is n^6. The modulation technique is also different. In the previous application it was phase-shift keying (PSK), here it is quadrature amplitude modulation (QAM). In the former case the symmetries are those induced by the sphere, in the latter those of a square. To explain the need for retransmissions one considers the case of 16QAM where in the standard numbering according to the Gray mapping the constellation points 0111 and 1111 are neighbors but they can easily be confused with each other. 0111 is easily distinguishable from 0100, so a rearrangement as in the picture to the right is beneficial, see Figure 1.

Fig. 1: Constellation Rearrangement

(a) Original transmission



(b) First Retransmission



The emphasis in the study was less on assessing the quality of the lower and upper bounds obtained but more on finding feasible solutions, in other words, on near-optimal index assignments and then also to make sure they are not too far from optimal. So, again both methods were applied, an SDP based lower bounding technique and the best available heuristics for finding upper bounds. For the latter a robust tabu search method was suitably adapted while our SDP lower bounds were computed as before. As mentioned in [12] the gap between the bounds was typically between 10% and 20%. The industry standard approach to improve modulation diversity is CoRe or constellation rearrangement. Extensive comparisons of this to the method suggested here in [12] show that CoRe is outperformed in all ranges of practical interest. In the more complicated setting of a coordinated MIMO multi-point scenario a suitable generalized approach showed significantly reduced packet loss and a clear performance gain over simple retransmissions and over a heuristic CoRe from the literature, [13]. Due to the MIMO framework, however, two-dimensional QAPs are not sufficient. We need a six-dimensional matrix as in the Lawler form of the QAP.

Let $\mathbf{x}^{(m)}$ be the 3d permutation matrix in retransmission m, so $\mathbf{x}^{(m)} \in \mathcal{S}$ where

$$\mathcal{S} = \left\{ \mathbf{x} : \sum_{p=1}^{n} x_{pij} = \sum_{i=1}^{n} x_{pij} = \sum_{j=1}^{n} x_{pij} = 1 \right\}. \tag{13}$$

Then the minimization of the so-called bit error rate (BER) can be formulated into a Q3AP as follows:

$$\min_{\mathbf{x}^{(m)} \in \mathcal{S}} \sum_{p=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{q=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} c_{pijqkl}^{(m)} x_{pij}^{(m)} x_{qkl}^{(m)}, \tag{14}$$

Fortunately, it turns out that the matrix c is composed of a two-dimensional (flow) matrix f dependent on the index of the retransmission and a four-dimensional constant (distance) matrix d.

$$\min_{\mathbf{x}^{(m)} \in \mathcal{S}} \sum_{p=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{q=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} f_{pq}^{(m)} d_{ikjl} x_{pij}^{(m)} x_{qkl}^{(m)}, \tag{15}$$

Thus, the effort becomes very reasonable and a sequence of up to $m = 4$ retransmissions is considered with low effort. The upper bounds this time are computed with iterated local search [14].

4 Semidefinite Bounds for Binary and Spherical Codes

4.1 A Fundamental Question from Binary Codes

In the previous section an underlying principle was to maximize the Hamming distance of certain binary words. The Hamming distance is the number of bit positions in which such words differ and an important fundamental problem is to find the number $A(n, d)$ of binary words of length n having at least Hamming distance d. This is also a very hard problem comparable to the QAP and in general only upper and lower bounds can be obtained. Various methods have been proposed for both purposes. Here, the determination of upper bounds is the more challenging problem and SDP based methods had already yielded improved such bounds. However, the SDP problems obtained are huge and SDP solution techniques do not scale as well as in other related optimization tasks such as second-order cone programming (SOCP). Due to their high symmetry the problems can be reduced to manageable size but it turns out these SDPs are ill-conditioned. With standard double precision computations insufficient accuracy is obtained which does not allow to make rigorous statements about the value of the bounds or even no statement at all.

It is necessary to do higher precision SDP solves and in particular one solver, SDPA, has been supplied with several such features and this has even implemented by us for interactive use [15]. Precision higher than quadruple still takes considerable time. It was also a major difficulty to find parameters with which the method did not diverge. It was a major but worthwile effort to obtain the new upper bounds [16]. They did improve many of the bounds in the regularly updated list [17] and proved even one conjecture, namely that the quadruply shortened binary Golay code is optimal, by pushing the upper bound to the value of the best-known lower bound hereby establishing an exact value, namely 256 for $A(20, 8)$, see Table 1.

For the definition of the SDP whose optimum provides an upper bound on $A(n, d)$ we refer to section 2 of [16]. The quadruple precision solution of the SDPs took up to several months of compute time.

4.2 The Kissing Number Problem

A problem related not to binary codes but to so-called spherical ones [22] is the kissing number problem. Given n-dimensional spheres of equal radius, the question is how many can be placed on the surface of a central one. For $n = 2$ the answer is obviously 6 as an experiment with coins confirms. However, the answer in three dimensions was the subject of a life-long dispute between Sir Isaac Newton and the scientist David Gregory [24]. It took until 1953 [25] to prove the bound of 12 conjectured by Newton and until 2008 [26] to prove the value of 24 for four dimensions.

We sketch the SDPs that have to be solved to obtain strong upper bounds for the kissing number.

The kissing number is the *stability* number of an *infinite* graph. It is is bounded by the Lovasz *theta* number. The Lovasz theta number is the solution of an SDP. One considers the graph $\Gamma(S^{n-1}, (0, \pi/3))$ on the vertex set $S^{n-1} = \{x \in \mathbb{R}^n : x \cdot x = 1\}$, edges whose angular distance is less then $\pi/3$, and with inner product greater then $1/2$. The bounds from this basic SDP are then

Table 1: Upper and lower bounds for binary codes

n	d	known lower bound	known upper bound	new upper bound	$A_4(n,d)$
17	4	2720	3276		3276.800
18	4	5312	6552		6553.600
19	4	10496	13104		13107.200
20	4	20480	26168		26214.400
21	4	36864	43688		43690.667
17	6	256	340		351.506
18	6	512	680	673	673.005
19	6	1024	1280	1237	1237.939
20	6	2048	2372	2279	2279.758
21	6	2560	4096		4096.000
22	6	4096	6941		6943.696
23	6	8192	13766	13674	13674.962
17	8	36	37		38.192
18	8	64	72		72.998
19	8	128	142	135	135.710
20	8	256	274	256	256.000
25	8	4096	5477	5421	5421.499
26	8	4096	9672	9275	9275.544
21	10	42	48	47	47.007
22	10	64	87	84	84.421
23	10	80	150		151.324
24	10	128	280	268	268.812
25	10	192	503	466	466.809
26	10	384	886	836	836.669
27	10	512	1764	1585	1585.071
28	10	1024	3170	2817	2817.313
25	12	52	56	55	55.595
26	12	64	98	96	96.892
27	12	128	169		170.667
28	12	178	288		288.001

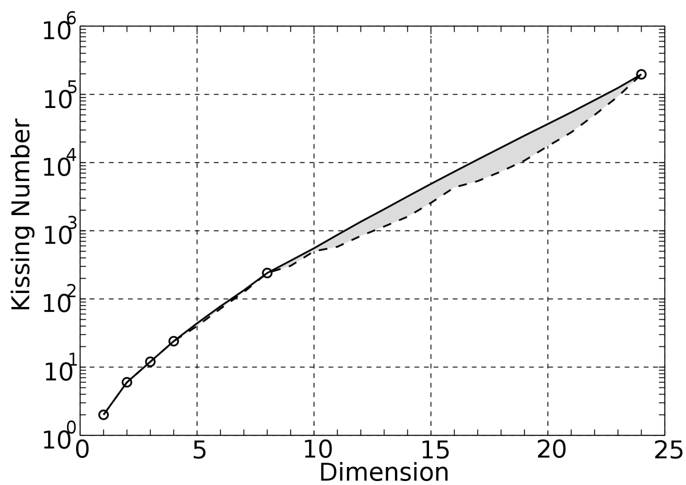
strengthened using *symmetries* and a *Lasserre hierarchy.*

$$\begin{aligned}\vartheta'(\Gamma(S^{n-1},(0,\pi/3))) = \inf \Big\{ \lambda : & K \in \mathcal{C}(S^{n-1}\times S^{n-1})_{\succeq 0}, \\ & K(x,x) = \lambda - 1, \text{ for all } x \in S^{n-1}, \\ & K(x,y) \le -1, \text{ for all } x,y \in S^{n-1} \\ & \quad \text{with } x\cdot y \le 1/2 \Big\},\end{aligned} \tag{16}$$

$\mathcal{C}(S^{n-1}\times S^{n-1})_{\succeq 0}$ is the cone of positive definite *Hilbert-Schmidt kernels*

Over many years slowly upper and lower bounds had been established, mostly by methods that differed considerably in different dimensions. It is thus remarkable that our approach in [23] did improve on *all* known bounds in dimensions 5 to 23 with the exception of 8 for which an exact value is known, see Figure 2. Just as in our work on binary codes, also for the kissing number problem we proved a conjecture. The new upper bound of 7355 in dimension 16 shows that a conjecture made in ch. 7, p. 190 of [27], is true, namely that there is no periodic point with a given average theta series.

Fig. 2: Upper and lower bounds for the kissing number



5 Two Exactly Solved Combinatorial Optimization Problems

5.1 A Three Dimensional Quadratic Assignment Problem

In this final section we describe how two problems of combinatorial nature, one of polynomial complexity and one from optimal index assignment were solved exactly.

In his work Peter Hahn had addressed many variants of the QAP with a number of different approximation and solution methods. One problem provided to him by the engineering collaborators of [28] was left unsolved. For some time he made its datafile available on his personal webpage. It is a Q3AP from optimal index assignment and the datafile contains the six-dimensional matrix of the form (14) for $n = 16$. Independent of the numerical method used the shear facts that of the 16^{16} elements over 12 million are nonzero and that their range is $3.6*10^{12}$ makes this a very challenging problem. To treat it directly as the binary quadratic problem that it is does not permit a solution with reasonable effort. What was exploited is the power available in mixed-integer linear programming (MILP) codes due to both very substantial progress on the software and on the hardware side during recent decades. As a first step a big-M method is applied to formally rewrite the problem as a MILP.

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} w_{ijk} \tag{17}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ijk} = 1, \sum_{i=1}^{n} \sum_{k=1}^{n} x_{ijk} = 1, \sum_{j=1}^{n} \sum_{k=1}^{n} x_{ijk} = 1 \tag{18}$$

$$w_{ijk} \geq \sum_{p=1}^{n} \sum_{q=1}^{n} \sum_{r=1}^{n} c_{ijkpqr} x_{pqr} - M(1 - x_{ijk}) \tag{19}$$

$$x_{ijk} \in \{0, 1\}, w_{ijk} \geq 0 \tag{20}$$

What is remarkable about this MILP is that it has of order n^3 variables and constraints. However, its dual bound is much two weak. As the next step two sets of cutting planes are added. The first has the form

$$w_{ijk} \geq L_{ijk} x_{ijk} \tag{21}$$

Here L_{ijk} is the optimum of the linear assignment problem (3AP)

$$\min \sum_{p=1}^{n} \sum_{q=1}^{n} \sum_{r=1}^{n} c_{ijkpqr} x_{pqr} \tag{22}$$

$$\sum_{p=1}^{n} \sum_{q=1}^{n} x_{pqr} = 1, \forall r \in \{1, ..., n\} \tag{23}$$

$$x_{pqr} \in \{0, 1\}, x_{ijk} = 1 \tag{24}$$

It turns out that this 3AP can be solved quite efficiently, but it improves the dual bound only very little. A somewhat stronger improvement is obtained by adding cutting planes of the form

$$w_{ijk} + w_{pqr} \geq T_{ijkpqr}(x_{ijk} + x_{pqr} - 1) \tag{25}$$

where T_{ijkpqr} is the optimal value of the MILP

$$\min(w_{ijk} + w_{pqr}) \tag{26}$$

$$\sum_{s=1}^{n}\sum_{t=1}^{n} x_{pqr} = 1, \forall u \in \{1, ..., n\} \tag{27}$$

$$\sum_{s=1}^{n}\sum_{u=1}^{n} x_{pqr} = 1, \forall t \in \{1, ..., n\} \tag{28}$$

$$\sum_{t=1}^{n}\sum_{u=1}^{n} x_{pqr} = 1, \forall s \in \{1, ..., n\} \tag{29}$$

$$w_{ijk} \geq \sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{u=1}^{n} c_{ijkstu} x_{stu} \tag{30}$$

$$w_{pqr} \geq \sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{u=1}^{n} c_{pqrstu} x_{stu} \tag{31}$$

$$x_{stu} \in \{0, 1\} \tag{32}$$

$$x_{ijk} = 1, x_{pqr} = 1 \tag{33}$$

Again, the solution of these MILPs is not hard but their number is potentially of the order n^6, so as few as possible should be solved. With the basic MILP formulation and two sets of cutting planes only the minimal groundwork has been done to solve the problem. A key property is its high symmetry [29]. As mentioned before the modulation PSK introduces the symmetry of the sphere and this is heavily exploited in [30].

A key notion in symmetry is that of an *orbit* and the binary variables of the problem can be partitioned into 6 orbits. One introduces for an orbit an aggregated variable and with the help of the symbolic package Gecode all possible aggregated solutions can be enumerated, there are 85 of them. This is called orbital shrinking. Still, several other techniques need to be applied such as isomorphism pruning, primal heuristics, parameter tuning, and especially scaling due to the wide range of the matrix elements. For details we refer to [30]. Most of the 85 problems can be solved in minutes, while the 15 hardest ones take about up to half a day. But, all the problems are independent and can be run in parallel, so that the entire problem was solved in less than a day.

5.2 The Directional Sensor Problem

The second problem that was solved exactly and efficiently is one of polynomial complexity but still challenging. It was considered in [31] in a nonconvex formulation which made it difficult to obtain the globally optimal solution. The solution by the best heuristic considered in [31] also was relatively expensive. However, the treatment in [32] confirmed that it could in some cases determine the global optimum. We consider sensors such as cameras, lasers etc which are in fixed locations and have a limited field-of-view (FOV). For simplicity the setting is two-dimensional and the FOV is a certain angle range. The sensors observe n targets in the plane whose location χ_j (of target j) is not known with precision. It is described instead by a prior distribution $\mathcal{N}(a_j, A_j)$. Sensor i, there are m of them total, has location s_i and can be pointed in one of K possible directions. The problem consists in choosing a direction for each sensor to maximize the expected information gain over a number of possible scenarios resp. easignments. For a justification of this approach, see [32]. If there was only one scenario and no measurement error, we would just direct the sensors to cover as many targets as possible. This corresponds to a set covering problem. For several scenarios and perfect measurement we would look for the best coverage on average which is still similar to a set covering problem. If target j is in the FOV of sensor i, when it is pointed in direction u_i we get the measure z_{ij}:

$$z_{ij} = H\chi_j + \eta_{ij} \tag{34}$$

where H is the observation model and η_{ij} is the measurement noise, assumed to be normally distributed according to the distribution $\mathcal{N}(0, R(s_i, u_i, \chi_j)$, with R being the measurement error covariance matrix. No measurement is obtained if the target is not in the FOV o fthe sensor. For a given scenario, the measurements of all sensors are fused as a posterior distribution in order to obtain a global estimate for each target. This distribution is not Gaussian in general and it cannot easily be computed, instead it is approximated as Gaussian distribution $\mathcal{N}(y_j, P_j)$. Here, the arguments y_j and P_j are computed as

$$P_j = \left(A_j^{-1} + \sum_i H^T(R(s_i, u_i, a_j))^{-1}H\right)^{-1} \tag{35}$$

and

$$y_j = P_j\left(A_j a_j + \sum_i H^T(R(s_i, u_i, a_j))^{-1} z_{ij}\right) \tag{36}$$

with the summations done over the sensors that generated a measurement for target j.

Given a control vector $u = (u_1, ...u_n)$, the corresponding objective to be maximized is then

$$E\left[\sum_{j=1}^{n} -\log\left(\frac{\det(P_j(u))}{\det(A_j)}\right]\right. \tag{37}$$

This expectation is approximated by Monte Carlo methods. We generate a number of samples from the joint prior distribution of the target state and compute the average (over the samples) objetcive values for a given control action.

As given above the directional sensor control problem is nonconvex. Heuristics and rigorous algorithms for this nonlinear integer optimization problem (MINLP) will in general only produce local optima. Global optimization requires either the use of algorithms that through spatial branch and bound or other techniques can guarantee that or convexification. The latter is possible in this case and was done in [32]. That then allows the use of software which is more efficient since it is not aiming at global optima.

We proceed by reformulating the problem and precomputing some quantities. Given the set S of samples s, we can write the problem as

$$\max \sum_{s\in S}\sum_{j=1}^{n}\Big[\log(\det(\overline{P}_{js})) + \log(\det(A_j))\Big]/|S| \tag{38}$$

$$\sum_{k=1}^{m} u_{ik} = 1, \forall i \tag{39}$$

$$\overline{P}_{js} = A_j^{-1} + \sum_i \sum_k R_{ijks} u_{ik}, \forall j, s \tag{40}$$

$$u_{ik} \in \{0,1\}, \forall i, k \tag{41}$$

Here $\overline{P}_{js}$ is the inverse of the posterior covariance matrix of target j in scenario s.
R_{ijks} is the inverse of the measurement covariance matrix between sensor i pointing in direction k and target j in scenario s if the target is within the FOV of the sensor, or the null matrix otherwise. This matrix is precomputed as is A_j^{-1}. Finally, u_{ik} is a binary variable whose value is one if and only if sensor i is pointing in direction k.

Since the only nonlinearities left are the log det terms in the objective but log det is a concave function of its argument and the variable u_{ik} enters linearly, so the above problem is a nonlinear convex binary optimization problem. A solver such as KNITRO [33] can be used for its global optimization. However, it turns out that the solution with KNITRO is relatively costly for an increasing number of sensors. Therefore, a Bender's decomposition was developed in [32] which reduced the effort considerably, especially if called in a fashion where the best metaheuristic tried in [32] for a comparison of such methods is used together with the best solution found to generate outer approximation cuts for the initial master formulation.

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