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Abstract

Interior point methods (IPM) are the most popular approaches to solve Second Order Cone Optimization (SOCO) problems, due to their theoretical polynomial complexity and practical performance. In this paper, we present a warm-start method for primal-dual IPMs to reduce the number of IPM steps needed to solve SOCO problems that appear in a Branch and Conic Cut (BCC) tree when solving Mixed Integer Second Order Cone Optimization (MISOCO) problems. Our method exploits the optimal Jordan frame of a related subproblem and provides a conic feasible primal-dual initial point for the self-dual embedding model by solving two auxiliary linear optimization problems. Numerical results on test problems in the CBLIB library show on average around 61% reduction of the IPM iterations for a variety of MISOCO problems.

Keywords— mixed integer optimization, conic optimization, interior point method, branch and bound, warm-start

1 Introduction

Second order cone optimization (SOCO) is the optimization of a linear objective function over a set of linear and second order cone constraints. Second order cone constraints are special cases of nonlinear constraints and generalizations of linear constraints. Interest to solve SOCO problems and their mixed integer counterpart MISOCO problems grew immensely in the last two decades, since a variety of problems can be formulated as SOCO and MISOCO problems. Convex quadratic optimization, problems involving p-norms, hyperbolic constraints, robust least-squares are shown to be SOCO-representable, see e.g. Ben-Tal and Nemirovski [9], Lobo et al. [28], and, Alizadeh and Goldfarb [2]. Recent studies on portfolio optimization [13], options pricing [36], robust delay-constrained routing [25], machine-job assignment [1], facility location and inventory management [5] are some examples from the literature where problems are modeled with MISOCO formulations. On top of these, recent advances in cut generation [6, 7, 8] and available off-the-shelf solvers [3, 18, 24, 26] has drawn the attention of both researchers and practitioners.

Without loss of generality, we can write a SOCO problem in the standard form as

$$\begin{array}{lrcl} \min & c^\top x & & \\ \text{s.t.} & Ax & = & b, \\ & x & \in & \mathcal{K}, \end{array} \tag{P-SOCO}$$

and its dual as

$$\begin{array}{llll} \max & b^\top y & & \\ \text{s.t.} & A^\top y + z & = & c, \\ & z & \in & \mathcal{K}^*, \end{array} \tag{D-SOCO}$$

where $A \in \mathbb{R}^{m\times n}$ is a full rank row matrix, $c \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$. Also, $\mathcal{K}$ is the Cartesian product of second order cones of various dimensions, i.e., $\mathcal{K} = \mathbb{L}^{n_1} \times \mathbb{L}^{n_2} \times \cdots \times \mathbb{L}^{n_k}$, where $\mathbb{L}^{n_i} = \{x^i \in \mathbb{R}^{n_i} \mid x^i_1 \geq \|x^i_{2:n_i}\|\}$, for $i = 1,\ldots,k$, with $\sum_{i=1}^k n_i = n$ for $x = \left((x^1)^\top, (x^2)^\top, \ldots, (x^k)^\top\right)^\top$, and $x^i \in \mathbb{R}^{n_i}$. Due to self-duality of second order cones (SOC), we have $\mathcal{K} = \mathcal{K}^*$.

Closely related SOCO problems appear at every node of Branch and Bound based methods when solving MISOCO problems. There are mainly three methodologies to solve SOCO problems. The first one is using IPMs developed specifically for SOCOs [2, 4, 33]. The second approach is to employ IPM methods for general nonlinear optimization problems [11]. The third method is the use polyhedral relaxations of SOCO problems and solve them approximately as linear optimization problems [10]. A recent survey on solution methods, developments and applications of MISOCO are presented by Benson and Sağlam [11].

Despite the efficiency of solving SOCO problems with IPMs, they are not clearly dominating other methods as the default methods for solving subproblems when solving MISOCO problems inside off-the-shelf solvers. Indeed, polyhedral relaxations are still a prominent opponent of IPMs when solving such subproblems inside BB and BCC algorithms. There are a variety of factors in play here. First of all, subproblems obtained from the polyhedral approximation are linear optimization (LO) problems. Methodologies for solving mixed integer linear optimization (MILO) problems are well-developed, hence these software benefit from the well-established literature of MILO, such as cuts and heuristics. Second, preprocessing for LO problems is far better developed than for SOCO. We have yet to see a significant work on preprocessing techniques for SOCO problems. Most importantly, warm-starting LO problems usually requires only a few dual simplex iterations in practice if the problem has changed slightly. This efficiency brings the total solution time down significantly for MILO problems. Despite recent interest in MISOCO and the existence of solvers deploying IPM-based MISOCO strategies [3, 18, 24, 26], there are still need and considerable opportunities to improve the efficiency of these methods. Warm-starting is one of the main concerns for MISOCO, due to lack of efficient warm-starting methods. The issue of warm-starting is being carried over unsolved for three decades on IPMs for solving LO problems.

Any method of exploiting information obtained from a problem instance to solve a closely related one is called warm-starting [14]. Warm-starting of IPMs gained attention of researchers for a long time. Due to the success of IPMs to solve large scale problems efforts have focused on improving the efficiency of IPM based solution methods for LO problems. We have two lines of studies in the literature where warm-starting is needed [22]. The first one is warm-starting for perturbation based changes, where all or some of the problem parameters A, b, c are perturbed by a small magnitude. These types of problems appear when one needs to solve problems of the same structure with slightly different parameters many times, such as finding the efficient frontier of a portfolio optimization problem [23, 30, 39]. Warm-starting of LO for perturbation based changes is a well-studied problem in the literature [12, 15, 22, 27, 43]. The second one is warm-starting for modification based changes, where new constraints and/or variables are added to the problem. Such transformations occur in cutting plane, column generation, and BCC algorithms. These methods focus on using the optimal value or an IPM iteration of the original problem to warm-start a similar instance. Warm-starting for modification based changes also gathered attention [16, 21, 30, 31, 32]. Readers may find the extensive literature review of Engau et al. [16] on warm-starting of IPMs for LO problems in the context of combinatorial optimization useful.

Warm-starting of IPMs for SOCO is required to solve MISOCO problems more efficiently and it stays as an open research problem. Majority of the previous work focused on perturbation based changes [23, 38, 39, 42]. Only methods presented by Sivaramakrishnan et al. [37], Oskoorouchi and Mitchell [34] and Oskoorouchi et al. [35] can be considered as warm-starting of IPMs for modification based changes, although their motivations are different from ours. Engau et al. [16] give straightforward extensions of some warm-start approaches originally developed for LO problems. Skajaa et al. [39] presented a simple warm-starting approach for homogeneous and self-dual IPMs when problems are perturbed by a limited magnitude. They use the optimal solution of the original problem and take a convex combination of it with the default initial point of the homogeneous IPM. The new point is then fed into the self-dual embedding IPM to warm-start the algorithm for a perturbed instance.

The main purpose of this study is to take a step towards filling the gap in the MISOCO literature to warm-start self-dual embedding IPMs. We provide an efficient warm-start method that exploits the Jordan frame of a related instance, and efficiently warm-starts IPMs after adding linear cuts or after branching. Our proposed approach is based on solving so-called primal and dual rounding problems for IPMs. This way, we generate a primal-dual feasible initial point. Then we use a new point, which is a convex combination of this primal-dual feasible point and an earlier IPM iteration of the original instance, to warm-start the new instance. Our way of using convex combinations is similar to Skajaa et al. [39], however we have an extra step to find points that are primal and dual feasible, instead of using the previous optimal solution. This approach is proposed to be used in solving SOCO subproblems inside BB and BCC trees when solving MISOCO problems. We measure the efficiency of our method by comparing the number of IPM iterations to solve new instances using our warm-start approach versus using the default initial point of IPMs, also called as cold-start.

The rest of this paper is structured as follows. In Section 2, we give the fundamentals about Jordan Algebra and introduce the auxiliary rounding LO problems. In Section 3, details of the warm-starting approach is presented. In Section 4, numerical experiments using randomly generated conic instances and the conic benchmark library (CBLIB) are provided. We give an overview of the paper and present our conclusions in Section 5.

2 Preliminaries

We start this section with a quick review of Jordan algebras [17] and the definition of rounding problems before giving details of our warm-start approach in the following section. For a given nonzero vector $x^i \in \mathbb{L}^{n_i}$, we can write it in the form of

$$x^i = f_{P\,i}^+ \lambda_i^+ + f_{P\,i}^- \lambda_i^-$$

where

$$\lambda_i^+ = x_1^i + \|x_{2:n_i}^i\|, \qquad \lambda_i^- = x_1^i - \|x_{2:n_i}^i\|,$$
$$f_{P\,i}^+ = \frac{1}{2}\begin{pmatrix} 1 \\ \frac{x_{2:n_i}^i}{\|x_{2:n_i}^i\|} \end{pmatrix}, \qquad f_{P\,i}^- = \frac{1}{2}\begin{pmatrix} 1 \\ -\frac{x_{2:n_i}^i}{\|x_{2:n_i}^i\|} \end{pmatrix}.$$

In this system, λ^+ and λ^- are called the Jordan values, and f_P^+ and f_P^- are called the Jordan frames of the primal variable x. They corresponds to eigenvalues and eigenvectors in the primal problem. Notice that the norm of both of the Jordan frame vectors is 1/2. Moreover, these Jordan frames are orthogonal, and they are on the boundary of the standard SOC. See Figure 1 for a representation of Jordan frames for the primal variable x.

Similarly, for the dual slack variable z we have

$$z^i = f_{D\,i}^+ \kappa_i^+ + f_{D\,i}^- \kappa_i^-$$

where

$$\kappa_i^+ = z_1^i + \|z_{2:n_i}^i\|, \qquad \kappa_i^- = z_1^i - \|z_{2:n_i}^i\|,$$
$$f_{D\,i}^+ = \frac{1}{2}\begin{pmatrix} 1 \\ \frac{z_{2:n_i}^i}{\|z_{2:n_i}^i\|} \end{pmatrix}, \qquad f_{D\,i}^- = \frac{1}{2}\begin{pmatrix} 1 \\ -\frac{z_{2:n_i}^i}{\|z_{2:n_i}^i\|} \end{pmatrix}.$$

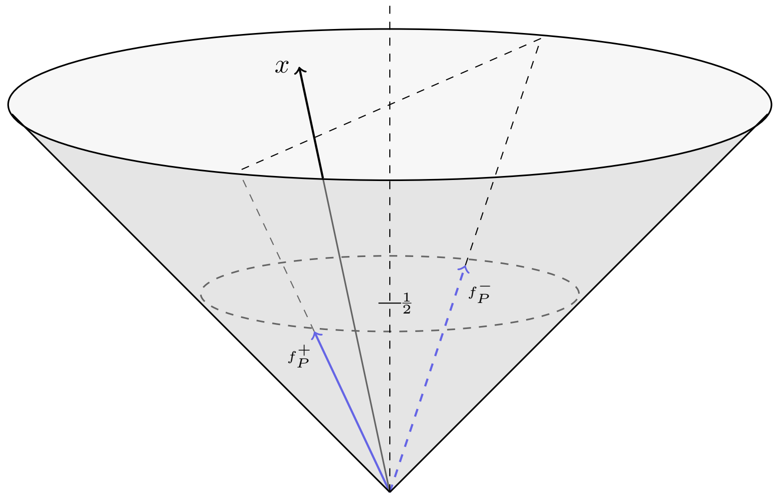

Figure 1: Jordan frames in a cone

2.1 Primal and Dual Rounding Problems

Góez and Pólik [20] presented rounding solutions for SOCO problems. Their method takes a near-optimal solution of an IPM iteration, fixes its Jordan frames and solves the resulting LO problems. These LO problems are called 'rounding problems'.

Suppose we have a conic feasible primal and dual solution $x \in \mathcal{K}, z \in \mathcal{K}$. We will derive primal and dual rounding problems for the given solutions. For notational convenience, we will merge Jordan frames of all cones into matrices $F_P \in \mathbb{R}^{n\times 2k}$ and $F_D \in \mathbb{R}^{n\times 2k}$ for the primal and dual side, respectively, which consist of two block-diagonal parts. Denote

$$F_P = \begin{bmatrix} f_{P\,1}^+ & 0 & \dots & 0 & f_{P\,1}^- & 0 & \dots & 0 \\ 0 & f_{P\,2}^+ & \dots & 0 & 0 & f_{P\,2}^- & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f_{P\,k}^+ & 0 & 0 & \dots & f_{P\,k}^- \end{bmatrix},$$

$$F_D = \begin{bmatrix} f_{D\,1}^+ & 0 & \dots & 0 & f_{D\,1}^- & 0 & \dots & 0 \\ 0 & f_{D\,2}^+ & \dots & 0 & 0 & f_{D\,2}^- & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f_{D\,k}^+ & 0 & 0 & \dots & f_{D\,k}^- \end{bmatrix}.$$

Similarly, denote $\lambda \in \mathbb{R}^{2k}$ and $\kappa \in \mathbb{R}^{2k}$ the vectors of the primal and dual Jordan values, respectively, such as

$$\lambda = \begin{bmatrix} \lambda_1^+ \\ \lambda_2^+ \\ \vdots \\ \lambda_k^+ \\ \lambda_1^- \\ \lambda_2^- \\ \vdots \\ \lambda_k^- \end{bmatrix} \qquad \kappa = \begin{bmatrix} \kappa_1^+ \\ \kappa_2^+ \\ \vdots \\ \kappa_k^+ \\ \kappa_1^- \\ \kappa_2^- \\ \vdots \\ \kappa_k^- \end{bmatrix}.$$

Now we can write $x = F_P\lambda$ and $z = F_D\kappa$, where F_P and F_D represent the primal and dual frames. To find a lower rank solution, we can fix the Jordan frames F_P and F_D at a solution (x, y, z) and rewrite the primal and dual SOCOs as LO problems. The dual variable y, and the Jordan values λ and κ are our decision variables in these rounding problems.

Using the introduced notation, the primal rounding problem (PR) can be written as follows:

$$\begin{array}{rrcl} \min & (c^\top F_P)\lambda & & \\ \text{s.t.} & (AF_P)\lambda & = & b, \\ & \lambda & \geq & 0. \end{array} \tag{PR}$$

The dual rounding problem is written as follows:

$$\begin{array}{rrcl} \max & b^\top y & & \\ \text{s.t.} & A^\top y + F_D\kappa & = & c, \\ & \kappa & \geq & 0. \end{array} \tag{DR}$$

Notice that (PR) and (DR) are not duals of each other. We can write the duals of both the primal and dual rounding problems. The dual of (PR) is written as follows:

$$\begin{array}{rrcl} \max & b^\top y & & \\ \text{s.t.} & F_P^\top A^\top y + u & = & F_P^\top c, \\ & u & \geq & 0. \end{array} \tag{D-PR}$$

Finally, the dual of the (DR) is written as follows:

$$\begin{array}{rrcl} \min & c^\top x & & \\ \text{s.t.} & Ax & = & b, \\ & F_D^\top x & \geq & 0. \end{array} \tag{D-DR}$$

2.2 Relationship between the primal and dual rounding problems

We can derive the relationships between rounding problems and their duals. These relationships are shown in Figure 2 and are explained in this subsection.

We start with showing that any solution to (PR) is always a feasible solution to (P-SOCO).

Lemma 1. *Let F_P be the Jordan frame of a conic feasible solution $x \in \mathcal{K}$. Then, $\tilde{x} = F_P\lambda$ is a feasible solution to (P-SOCO) for any λ that is feasible to (PR).*

Proof. Any feasible solution λ to (PR) satisfies $AF_P\lambda = b$ and $\lambda \geq 0$. In this case, $\tilde{x} = F_P\lambda$ satisfies $A\tilde{x} = A(F_P\lambda) = b$.

For conic feasibility, let us consider a single second order cone $\mathbb{L}^{n_i}$. We have

$$\begin{aligned} \tilde{x}_i = f_{P_i}\lambda_i &= \left[f^+_{P\,i} f^-_{P\,i}\right] \begin{bmatrix} \lambda_i^+ \\ \lambda_i^- \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & 1/2 \\ \frac{x^i_{2:n_i}}{2\|x^i_{2:n_i}\|} & -\frac{x^i_{2:n_i}}{2\|x^i_{2:n_i}\|} \end{bmatrix} \begin{bmatrix} \lambda_i^+ \\ \lambda_i^- \end{bmatrix} \\ &= \begin{bmatrix} \frac{\lambda_i^+ + \lambda_i^-}{2} \\ \frac{(\lambda_i^+ - \lambda_i^-)x^i_{2:n_i}}{2\|x^i_{2:n_i}\|} \end{bmatrix}. \end{aligned}$$

Combining it with $\lambda_i^+ \geq 0, \lambda_i^- \geq 0$ since λ is feasible to (PR) gives

$$\|\tilde{x}_{2:n_i}\| = \sqrt{\sum_{j=2}^{n_i} \frac{\left(\lambda_i^+ - \lambda_i^-\right)^2 {x^i_j}^2}{4\left(\|x_{2:n_i}\|\right)^2}} = \sqrt{\frac{\left(\lambda_i^+ - \lambda_i^-\right)^2}{4}} \leq \frac{\lambda_i^+ + \lambda_i^-}{2} = \tilde{x}^i_1.$$

$$\begin{array}{llcll} \min & (c^\top F_P)\lambda & & & \\ \text{s.t.} & (AF_P)\lambda & = & b, & \text{(PR)} \\ & \lambda & \geq & 0 & \end{array} \quad [m]\times[2k] \qquad \overset{\text{dual}}{\Leftrightarrow} \qquad \begin{array}{llcll} \max & b^\top y & & & \\ \text{s.t.} & F_P^\top A^\top y + u & = & F_P^\top c, & \text{(D-PR)} \\ & u & \geq & 0 & \end{array} \quad [2k]\times[m+2k]$$

$$(\text{for } x = F_P\lambda) \ \rotatebox{90}{$\supseteq$} \qquad\qquad\qquad \rotatebox{90}{$\subseteq$}$$

$$\begin{array}{llcll} \min & c^\top x & & & \\ \text{s.t.} & Ax & = & b, & \text{(P-SOCO)} \\ & x & \in & \mathcal{K} & \end{array} \quad [m]\times[n] \qquad \overset{\text{dual}}{\Leftrightarrow} \qquad \begin{array}{llcll} \max & b^\top y & & & \\ \text{s.t.} & A^\top y + z & = & c, & \text{(D-SOCO)} \\ & z & \in & \mathcal{K} & \end{array} \quad [n]\times[m+n]$$

$$\rotatebox{90}{$\supseteq$} \qquad\qquad\qquad (\text{for } z = F_D\kappa) \ \rotatebox{90}{$\subseteq$}$$

$$\begin{array}{llcll} \min & c^\top x & & & \\ \text{s.t.} & Ax & = & b, & \text{(D-DR)} \\ & F_D^\top x & \geq & 0 & \end{array} \quad [m+2k]\times[n] \qquad \overset{\text{dual}}{\Leftrightarrow} \qquad \begin{array}{llcll} \max & b^\top y & & & \\ \text{s.t.} & A^\top y + F_D\kappa & = & c, & \text{(DR)} \\ & \kappa & \geq & 0 & \end{array} \quad [n]\times[m+2k]$$

Figure 2: Feasibility and duality relationship between original SOCO problems and rounding LO problems

Therefore, $\tilde{x}_i \in \mathbb{L}^{n_i}$. We get $\tilde{x}_i \in \mathbb{L}^{n_i}$ $\forall i = 1, \dots, k$, and therefore $\tilde{x} \in \mathcal{K}$.

Since $\tilde{x}$ satisfies both $A\tilde{x} = b$ and $\tilde{x} \in \mathcal{K}$, it is a feasible solution to (P-SOCO). □

Similarly, we can find a feasible solution to (D-SOCO) from any feasible solution to (DR). In the following lemma, we show the relationship between (P-SOCO) and (D-DR).

Lemma 2. *Let F_D be the Jordan frame of a conic feasible solution $z \in \mathcal{K}$ which is used to define (D-DR). Then, x is a feasible solution to (D-DR) for any x that is feasible to (P-SOCO).*

Proof. Any feasible solution x to (P-SOCO) satisfies $Ax = b$. By definition of dual cones, we have $w^\top x \geq 0$, $\forall w \in \mathcal{K}$. Since $z \in \mathcal{K}$ and hence every column in f_D is in $\mathcal{K}$, we get $F_D^\top x \geq 0$. Since x satisfies both constraints, it is a feasible solution to (D-DR). □

We combine results of these two lemmas in the following theorem, where we show the weak duality relationship between primal and dual rounding problems.

Theorem 3 (Weak duality)**.** *Let $F_P, and F_D$ be the Jordan frames of a conic feasible primal and dual solution (x, y, z). Let λ^* and $y^*_{(DR)}$ be optimal solutions for (PR) and (DR), respectively. Then, $c^\top F_P \lambda^* \geq b^\top y^*_{(DR)}$.*

Proof. Lemma 1 shows that any feasible solution to (PR) corresponds to a feasible solution to (P-SOCO). Therefore, at optimality, we always have $c^\top F_P \lambda^* \geq c^\top x^*_{(\text{P-SOCO})}$. Lemma 2 shows that any feasible solution to (P-SOCO) corresponds to a feasible solution to (D-DR). Therefore at optimality we always have $c^\top x^*_{(\text{P-SOCO})} \geq c^\top x^*_{(\text{D-DR})}$. Due to strong duality of LO, at optimality for (DR) and its dual (D-DR) we have

$$c^\top x^*_{(\text{D-DR})} = b^\top y^*_{(\text{DR})}.$$

Combining all these implications gives

$$c^\top F_P \lambda^* \geq c^\top x^*_{(\text{P-SOCO})} \geq c^\top x^*_{(\text{D-DR})} = b^\top y^*_{(\text{DR})}. \tag{1}$$

This completes the proof. □

Note that, if F_P is the Jordan frame of an optimal solution $x^*_{\text{(P-SOCO)}}$ of (P-SOCO), then the equality $c^\top F_P \lambda^* = c^\top x^*_{\text{(P-SOCO)}}$ holds. Similarly if f_D is the Jordan frame of an optimal solution $z^*_{\text{(D-SOCO)}}$ of (D-SOCO), then the equality $c^\top x^*_{\text{(P-SOCO)}} = c^\top x^*_{\text{(D-DR)}}$ holds.

Corollary 4. *If the objective value of problems (PR) and (DR) are equal to each other for a feasible solution, then all inequalities in (1) hold as equalities. In this case, an optimal solution of (P-SOCO) and (D-SOCO) can be obtained as* $(x, y, z) = \left(F_P\lambda^*, y^*_{(DR)}, F_D\kappa^*\right)$ *by solely solving rounding problems.*

Corollary 5. *If the problem (DR) is dual infeasible, then (P-SOCO) must be infeasible. This includes the case when (DR) is unbounded.*

These results can be also derived for the dual side. Note that we were able to derive these results while not using anything beyond weak duality about the duality properties of the original pair of SOCO problems.

2.3 Self-dual embedding IPM

The warm-start approach we present in this paper is based on self-dual embedding IPMs. For this reason, we implemented a primal-dual self-dual embedding IPM in MATLAB that uses Mehrotra's predictor-corrector method [29]. In our implementation, for algorithmic choices, we follow the directions of Andersen et al. [4] and Sturm [40].

Self-dual embedding IPM starts with a primal (P-SOCO) and dual (D-SOCO) pair. We introduce three auxiliary variables x_0, y_0, z_0 and write the following self-dual problem:

$$\begin{array}{lllllllll}
\min\limits_{x,y,z,x_0,y_0,z_0} & \vartheta y_0 & & & & & & & \\
 & & Ax & -bx_0 & +r_p y_0 & & & = & 0 \\
\text{s.t.} & -A^\top y & & +cx_0 & +r_d y_0 & -z & & = & 0 \\
 & b^\top y & -c^\top x & & +r_g y_0 & & -z_0 & = & 0 \\
 & -r_p^\top y & -r_d^\top x & -r_g x_0 & & & & = & -\vartheta \\
 & \multicolumn{8}{l}{x \in \mathcal{K},\ z \in \mathcal{K}^*,\ x_0 \in \mathbb{R}_+,\ y \in \mathbb{R}^m,\ z_0 \in \mathbb{R}_+,}
\end{array} \tag{2}$$

where

$$r_p := \frac{bx_0 - Ax^{(0)}}{y_0^{(0)}}, \quad r_d := \frac{A^\top y^{(0)} + z^{(0)} - cx_0}{y_0^{(0)}}, \quad r_g := \frac{c^\top x^{(0)} - b^\top y^{(0)} + z_0^{(0)}}{y_0^{(0)}},$$
$$\vartheta = \frac{x^{(0)} z^{(0)} + x_0^{(0)} z_0^{(0)}}{y_0^{(0)}}.$$

Default initial values $(x^{(0)}, y^{(0)}, z^{(0)})$ for a feasible IPM are

$$x^{(0)} = \iota, \quad y^{(0)} = 0, \quad z^{(0)} = \iota, \quad x_0^{(0)} = 1, \quad y_0^{(0)} = 1, \quad z_0^{(0)} = 1,$$

where $\iota^i = (1, 0, \dots, 0)^\top \in \mathbb{R}^{n_i}$ for $i = 1, \dots, k$ and $\iota = (\iota^1; \dots; \iota^k) \in \mathbb{R}^n$. This initial solution is feasible for the self-dual embedding problem and it is also on the central path for $\mu = 1$.

Observe that we can initialize the self-dual embedding IPM with any interior conic feasible solution for (2). This observation is key to our warm-start approach presented in the following section.

3 Warm-starting

Warm-start for optimization methods plays a critical role in reducing solution time, especially in BB and BCC methods, where related instances are being solved consecutively. In fact, the popularity of the dual simplex method for solving MILO can be attributed to its efficiency in warm-start. By warm-start, we can think of using any information obtained from the original instance to solve a related instance. Although warm-start methods for simplex-like methods are only limited to using a previous optimal solution, we need to think more than that for warm-starting IPMs. Using the optimal solution of the original instance for warm-starting IPMs directly is not a good idea. Such a point will be most likely be infeasible, which leaves only infeasible IPMs in play. Moreover, such a point will be on the boundary of the feasible region, which means IPMs are not directly applicable. Contrary to simplex-like methods, a well-centered point could work much better in practice, even if it is away from the previous optimal solution. This is the main reason why warm-starting for IPMs is often seen as unsuccessful or application-specific.

There are various methods researchers have tried to design efficient warm-starting IPMs. Almost all approaches on this topic use either an optimal solution or a stored IPM iteration of the original problem and try to generate a good initial point. There are mainly two ways of approaching warm-starting for IPMs. The first one is modifying the problems to prevent slow progress. These methods are also called shifted barrier methods, since they shift the boundaries of variables temporarily for faster progress [14]. The second one is to use an intermediate procedure to obtain a relatively well-centered point and use it as an initial point to the IPM. There are various methods that have been tried by researchers within this category, such as adding slack variables [16], generating a point by using a feasible point and the previous optimal solution [39], and using an exact primal-dual penalty method [12].

We present a warm-start approach where we use an intermediate procedure to formulate the self-dual embedding IPM with an initial point. Our procedure consists of solving two auxiliary LO problems, which will help us to minimize the primal and dual infeasibilities, respectively. Then, similar to Skajaa et al. [39], we find a point as a convex combination of the optimal solution of the rounding LO problems and an IPM iterate of the original problem.

We present the steps of our proposed warm-start approach in the following subsections.

3.1 Solving rounding problems

Suppose we have an original problem in the form of (P-SOCO). Let (x^*, y^*, z^*) be a primal-dual optimal solution for the original problem. Based on the numerical values of x^* and z^*, each conic component of the solution can belong to one of the four classes: $\mathcal{B}, \mathcal{N}, \mathcal{R}, \mathcal{T}$ [41]. A cone i belongs to $\mathcal{B}$ if x^{*i} is inside the cone and $z^{*i} = 0$, and to $\mathcal{N}$ if z^{*i} is inside the cone and $x^{*i} = 0$. For the class $\mathcal{R}$ both x^{*i} and z^{*i} are on the boundary of the cone i and orthogonal to each other. Finally, for the $\mathcal{T}$ case the sum $x^{*i} + z^{*i}$ is on the boundary of the cone i. If $\mathcal{T} \neq \emptyset$, then the optimal solution is not strictly complementary.

After identifying the classes of the cones, we derive the Jordan frames for both the primal and dual problems. If a cone belongs to the set $\mathcal{B}$, then we use the primal Jordan frame for the dual slack variable z^{*i} as well, and similarly if a cone belongs to the set $\mathcal{N}$, we use the dual Jordan frame for the primal variable x^{*i}, too. If a cone is in $\mathcal{R}$, then the primal and dual Jordan frames are the same. We use this frame for F_P and F_D. If a cone is in $\mathcal{T}$, we use an earlier IPM iteration to choose a Jordan frame.

Denote F_P^* and F_D^* as primal and dual Jordan frames at an optimal solution (x^*, y^*, z^*).

Consider that the new instance is obtained after branching on a variable, say x_j such that

$$x_j \leq \lfloor x_j^* \rfloor$$

is added to the problem. We add a slack variable s to the problem to get the new constraint in the standard form such as

$$x_j + s = \lfloor x_j^* \rfloor, \ s \in \mathbb{L}^1.$$

Denote

$$\bar{x} = \begin{bmatrix} x \\ s \end{bmatrix} \quad \bar{y} = \begin{bmatrix} y \\ y_s \end{bmatrix} \quad \bar{z} = \begin{bmatrix} z \\ z_s \end{bmatrix} \quad \bar{A} = \begin{bmatrix} A & 0 \\ e_j^\top & 1 \end{bmatrix} \quad \bar{b} = \begin{bmatrix} b \\ \lfloor x_j^* \rfloor \end{bmatrix} \quad \bar{c} = \begin{bmatrix} c \\ 0 \end{bmatrix}, \quad \bar{\mathcal{K}} = \mathcal{K} \times \mathbb{L}^1$$

where e_j is the j^{th} unit vector. The new primal-dual problems are

$$\begin{array}{llll} \min & \bar{c}^\top \bar{x} & & \\ \text{s.t.} & \bar{A}\bar{x} & = & \bar{b}, \\ & \bar{x} & \in & \bar{\mathcal{K}}, \end{array} \qquad \begin{array}{llll} \max & \bar{b}^\top \bar{y} & & \\ \text{s.t.} & \bar{A}^\top \bar{y} + \bar{z} & = & \bar{c}, \\ & \bar{z} & \in & \bar{\mathcal{K}}. \end{array}$$

After adding the slack variable into the primal and dual Jordan frames, F_P^* and F_D^*, we obtain Jordan frames for rounding problems. Now, we can replace $\bar{x}$ and $\bar{z}$ with their equivalent $\bar{F}_P\lambda$ and $\bar{F}_D\kappa$ in the primal and dual problems. This gives us primal and dual rounding problems for the new system, namely

$$\begin{array}{llll} \min & (\bar{c}^\top \bar{F}_P)\lambda & & \\ \text{s.t.} & (\bar{A}\bar{F}_P)\lambda & = & \bar{b}, \\ & \lambda & \geq & 0. \end{array} \qquad \begin{array}{llll} \max & \bar{b}^\top \bar{y} & & \\ \text{s.t.} & \bar{A}^\top \bar{y} + \bar{f}_D\kappa & = & \bar{c}, \\ & \kappa & \geq & 0. \end{array}$$

We solve these rounding problems for variables $(\lambda, \bar{y}, \kappa)$ to obtain the rounding solution of the system that is $(\bar{x}^*, \bar{y}^*, \bar{z}^*) = (\bar{F}_P\lambda^*, \bar{y}^*, \bar{f}_D\kappa^*)$. The primal rounding problem consists of m constraints and $2k$ variables, and the dual rounding problem consists of n constraints and $m + 2k$ variables. For a moderately sized MISOCO problem, sizes of the primal and dual rounding problems are considered to be small for any commercial off-the-shelf LO solvers. Solving the rounding problems take a fraction of a second for most cases in practice, so they are negligible compared to the time required to solve the SOCO problems.

3.2 Choosing a convex combination of solutions

Notice that our rounding solutions satisfy $\bar{A}\bar{x}^* = \bar{b}$, and $\bar{A}^\top\bar{y}^* + \bar{z}^* = \bar{c}$ at optimality. Hence, $(\bar{x}^*, \bar{y}^*, \bar{z}^*)$ is a feasible primal-dual solution for the new problem, although it may not satisfy the complementarity condition. This solution is not suitable to start IPMs, since it is on the boundary of the feasible region.

Instead of using the rounding solution, we will find an interior point as the convex combination of the rounding solution and an earlier IPM iteration of the original problem. Here, we know that the default initial point $(x, y, z) = (\iota, 0, \iota)$ are always interior feasible for the new problem after initializing the extra variables as $(s, y_s, z_s) = (1, 0, 1)$ due to the construction of the self-dual embedding model. We can find and choose a conic feasible IPM iteration ℓ, $(x^{(\ell)}, y^{(\ell)}, z^{(\ell)})$, and use it to generate a new initial solution for the IPM.

From a feasible IPM iteration ℓ, we generate an initial point $(\bar{x}^{(0)}, \bar{y}^{(0)}, \bar{z}^{(0)})$such that

$$\bar{x}^0 = \alpha\bar{x}^* + (1-\alpha)\begin{bmatrix} x^{(\ell)} \\ 1 \end{bmatrix},$$

$$\bar{y}^0 = \alpha\bar{y}^* + (1-\alpha)\begin{bmatrix} y^{(\ell)} \\ 0 \end{bmatrix},$$

$$\bar{z}^0 = \alpha\bar{z}^* + (1-\alpha)\begin{bmatrix} z^{(\ell)} \\ 1 \end{bmatrix}.$$

Using an earlier IPM iteration will provide a better centered interior point, although primal-dual infeasibilities may be large. On the other hand, using a later IPM iteration is more likely to give us a smaller μ parameter and smaller primal-dual infeasibility, however may lead to slow progress due to risk of being closer to the boundary. From a practical point of view, one need to consider the trade-off between closeness to optimality versus closeness to boundary when choosing which IPM iteration is used. Based on our numerical experiments, choosing an iterate from one-fourth to one-half of the original IPM iterations is a safe choice, which is usually close to the central path for the new problem and provide a smaller primal-dual infeasibility for our purposes. A moderate choice of α usually works well. We observed that $\alpha = 0.6$ works consistently well across all tests.

3.3 Initialization

After completing the previous steps, now we reached the point that the self-dual embedding model can be initiated by $(\bar{x}^{(0)}, \bar{y}^{(0)}, \bar{z}^{(0)})$. Conic infeasibilities can be fixed by increasing the leading variables with a small magnitude. At the beginning of the IPM steps, the generated initial point is evaluated for centrality and a corrector step is taken if needed. Variables inside the self-dual embedding model are initialized as $(x_0^{(0)}, y_0^{(0)}, z_0^{(0)}) = (1, 1, 1)$.

3.4 Solution approach

Our proposed warm-start method has the ability to detect optimality and infeasibility right after solving rounding problems. There are 4 outcomes in total for each instance in our solution approach, which are;

- Immediately infeasible (II) when problem (DR) is dual infeasible,
- Immediately optimal (IO) when the objectives of (PR) and (DR) are the same,
- Using the warm-start (WS) approach when rounding problems are feasible but does not give a proof of optimality or infeasibility,
- Using the cold-start (CS) approach when rounding problems are infeasible.

We use the following steps as our solution approach after solving the original problem and obtaining the optimal Jordan frames

1. Solve the dual rounding problem.
    - If problem (DR) is dual infeasible, then conclude that problem (P-SOCO) is infeasible and return. (II: Immediately infeasible, see Corollary 5)
    - If problem (DR) is feasible, then continue.
2. Solve the primal rounding problem.
    - If problem (PR) is feasible and the objective values of the rounding problems are equal to each other, then we can conclude that the rounding solution is optimal. (IO: Immediately optimal, see Corollary 4)
    - Else, if problem (PR) is feasible, choose an early iterate of the IPM iterations based on the number of IPM iterations of the original problem, and use $\alpha = 0.6$ to generate an initial point. We can start the IPM from the generated point. (WS: Warm-start)
    - If problem (PR) is infeasible, then use the default initial point of the self-dual embedding system. (CS: Cold-start)

4 Numerical experiments

In this section, we provide details of experiments conducted to test our warm-start approach on a variety of problems. We start with a description of the method, and then provide numerical results.

4.1 Methodology

We use a similar approach to measure efficiency of the proposed warm-start strategy as Engau et al. [16] and Skajaa et al. [39]. We measure the ratio of warm-start iterations to cold-start iterations, and then find the geometric mean covering all test problems. We add one to each metric to be able to include cases where an optimal solution is obtained after solving rounding problems. The ratio for each problem is

$$\mathcal{P}_i = \frac{(\text{\# of IPM iters with warm-start for Problem } i) + 1}{(\text{\# of IPM iters with cold-start for Problem } i) + 1}$$

and the metric for measuring the efficiency of the method is the geometric mean,

$$\mathcal{G}_I = \sqrt[k]{\mathcal{P}_1 \mathcal{P}_2 \dots \mathcal{P}_k}.$$

We mainly use the number of IPM iterations to measure the performance of the warm-start, because solving rounding problems take a negligible amount compared to solving SOCO instances. Number of IPM iterations roughly gives the same ratio, which we found sufficient for our purposes here.

The warm-start approach is implemented in MATLAB on top of a self-dual embedding SOCO solver that we implemented. We used the primal simplex method of IBM ILOG CPLEX 12.7.0.0 to solve the rounding problems. The Conic Benchmark Library (CBLIB) [19] problems and their random fixings are solved at the root node and then two subproblems are generated. Then, both subproblems are solved with cold-start and warm-start IPMs.

As a rule of thumb, we choose one-fourth, one-third and one-half of total IPM iterations if the total IPM iterations to solve the original problem is between 0 and 10, between 10 and 20, and more than 20, respectively. We use $\alpha = 0.6$ for all experiments which works well in practice.

4.2 Performance of warm-start for various branching variable types

We can categorize variables inside SOCO problems in three types: Non-negative variables, leading variables and in-cone variables. Non-negative variables, by definition, belong to $\mathcal{L}^1$ and they do not appear in any other cones. Leading variables are the first index in an SOC that has two or more elements. In-cone variables are the remaining ones, which appear inside the cones that have at least two elements. There are only a few problem in the CBLIB library that has integer variables that appears as leading variables of SOCs. There are no problems in CBLIB that have an integer variables inside SOCs. For these reasons, we generated some variations of CBLIB test problems where we branch on leading and in-cone variables, even if they are not specified as integers, originally. We discarded instances that hit the time or the iteration limit from the results.

Feasibility of the rounding problems heavily depends on which type of variable is used for branching. See Table 1 for the distribution of instances into methods we followed for warm-start based on variable type. We were able to use our warm-start approach (either IO, II or WS) for 496 out of 719 problems when we branch on non-negative variables. This number is 448 out of 700 when branching on leading variables and 256 out of 696 when branching on in-cone variables. Out of these only 44 and 9 of instances are solved as IO or WS for leading and in-cone variables, respectively. These results show that our warm-start approach works best for branching on non-negative variables, which is the case for the majority of CBLIB problems.

Type	Status	IO	WS	CS	II	Total
Non-negative						
	Feasible	99	388	218		705
	Infeasible			5	9	14
Leading						
	Feasible	10	34	238		282
	Infeasible			14	404	418
In-cone						
	Feasible		9	413		422
	Infeasible			27	247	274

Table 1: Distribution of instances into methods based on variable type and problem status.
(IO: Immediately optimal, WS: Warm-start, CS: Cold-start, II: Immediately infeasible)

Instances where we branch on leading and in-cone variables often lead to infeasible rounding problems. The reason for this is the nature of our warm-start. By fixing the optimal Jordan frames, we are assuming that the new optimal solution will have the same or a similar Jordan frame, so that we can get closer to the solution by using a convex combination. However, the Jordan frame corresponding to the cone in which the branching variable appears will most likely change after branching, which gives us infeasible rounding problems. Our use of rounding problems corresponds to outer approximation for the dual side. Adding a new Jordan frame to the problem may make the primal rounding problem feasible, especially for in-cone variable branching, but it is hard to identify which frame will be feasible.

Efficiency of the warm-start approach is given in terms of geometric mean in Table 2. The results show that the we solve the instances about 72% of the iteration compared to cold-start when we branch on non-negative variables. For a warm-start method for modification-based changes, this is a significant value. This percentage is around 48% when we include instances that are solved immediately after rounding problems. On average, we solve new instances with 59% of total IPM iterations compared to cold-start. The average efficiency is worse for leading and in-cone variables when using the warm-start approach. Our biggest gain for these variable types come from infeasible cases. We solve infeasible instances with 11% and 12% of total IPM iterations compared to cold-start when branching on leading and in-cone variables, respectively. Note that we have very few cases where we could apply our warm-start method in those categories.

Type	Status	WS	WS, IO, or II	All	Total
Non-negative					58.88%
	Feasible	71.72%	47.76%	60.02%	
	Infeasible		9.77%	22.43%	
Leading					25.10%
	Feasible	87.62%	55.29%	91.17%	
	Infeasible		9.73%	10.52%	
In-cone					43.83%
	Feasible	99.18%	99.18%	99.98%	
	Infeasible		9.79%	12.31%	

Table 2: Geometric mean of the ratio of warm-start iterations to cold-start iterations among all instances.

4.3 Comparison to cold-start and other warm-start methods

As shown in Section 4.2, our warm-start approach provides two benefits. The first one is when we solve the problem right after solving rounding problems, and the second one is choosing a different initial point for self-dual embedding system if rounding problems provide feasible solutions. A comparison of warm-start iterations to cold-start iterations are given in Figure 3. As seen from the scatter plot with jitter, we have a significant advantage when the rounding problems are feasible.

We compared our method to the warm-start approach of Skajaa et al. [39]. Their warm-start approach performs slightly better than the cold-start for our test problems. Note that their method is originally proposed for perturbation based changes. They use a ratio of 0.99 (previous optimal) to 0.01 (default initial point) to find an initial point for perturbation based changes, but this choice may perform poorly for most test problems since the problems change more significantly than small perturbation after branching. Therefore we tried different values and decided to use 0.15 to 0.75 for comparison. Since we used cold-start for cases where the rounding problems are infeasible, we only compare our method to theirs for cases where we used an initial point to warm-start. A comparison of number of IPM iterations compared to Skajaa et al.'s warm-start method is given in Figure 4 as a scatter plot with jitter. It is worth noting that the geometric mean the improvement of our warm-start approach compared to cold-start for non-negative variables is 71.72%, whereas this number is 99.81% by using the warm-start approach of Skajaa et al..

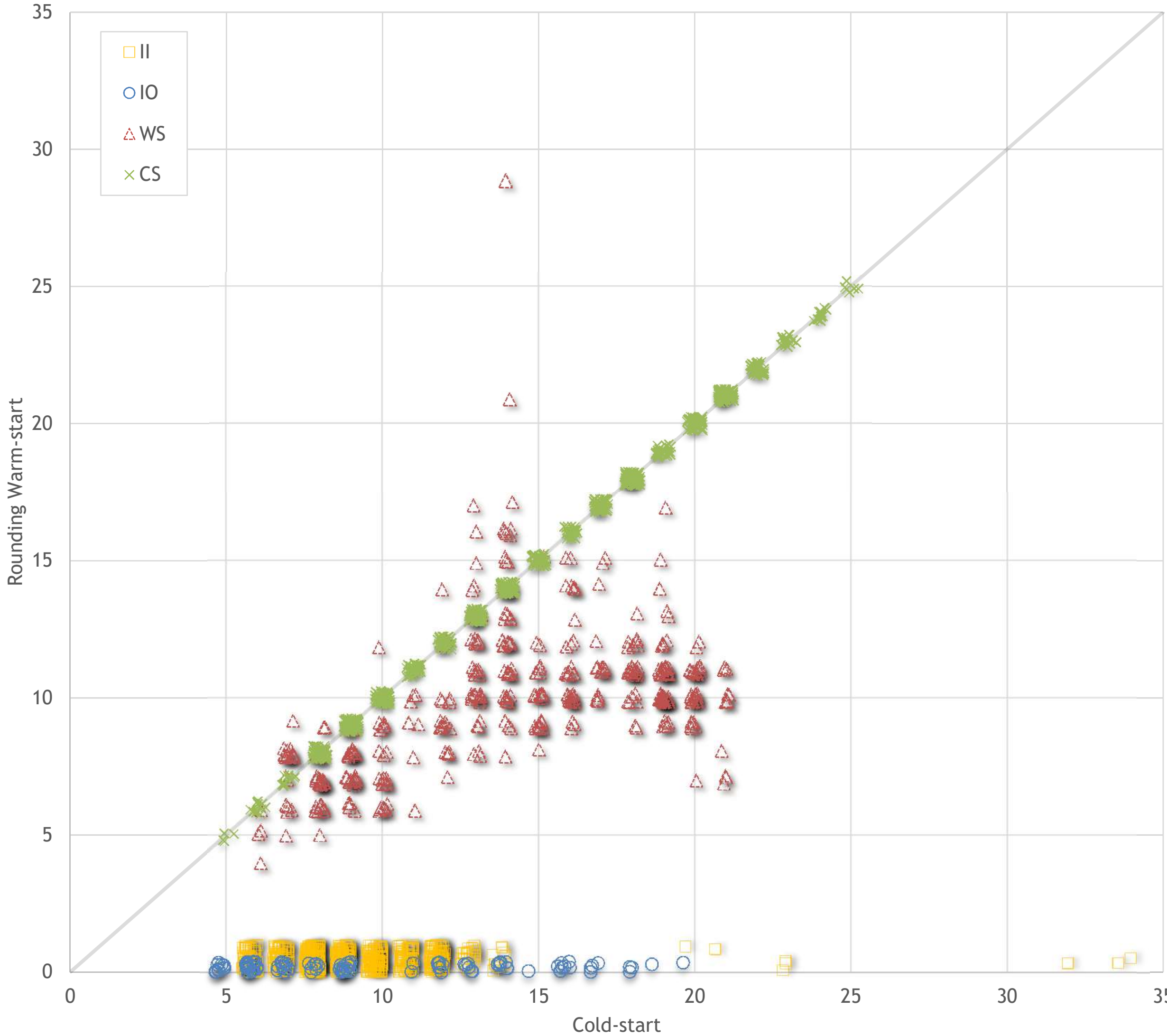

Figure 3: The comparison of warm-start IPM iterations versus cold-start IPM iterations.

4.4 Effect of warm-starting for infeasible cases

We see a huge benefit of solving rounding problems for infeasible cases. To the best of our knowledge, there is no warm-start method for infeasible cases in the literature. Our warm-start procedure is able identify infeasible cases before solving the new instance for a majority of instances we have. For non-negative variable branching 9 out of 14, for leading variable branching 404 out of 418, and for in-cone variable branching 247 out of 274 instances are concluded to be infeasible right after solving the rounding problems. This means a huge saving in terms of solution time and IPM iterations.

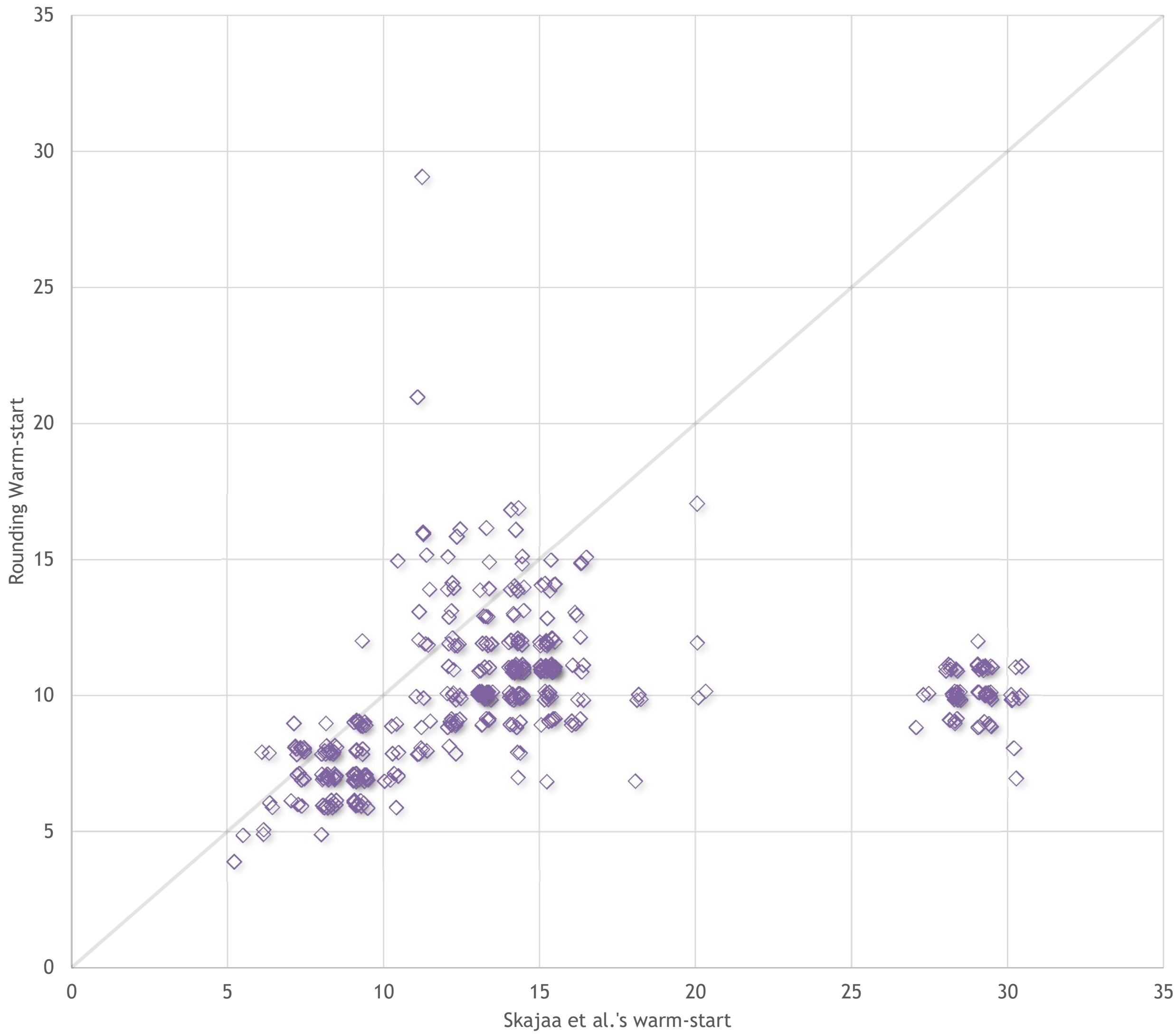

Figure 4: The comparison of IPM iterations of our approach versus warm-start of Skajaa et al. on feasible instances.

5 Conclusion and future work

In this study, we proposed a warm-start approach for self-dual embedding IPMs to solve SOCO problems appearing in a BCC tree when solving MISOCO problems. To our knowledge, this is the first study about warm-starting SOCO instances after a modification based change of the problem instance. Moreover, it is the first study on providing an earlier detection for infeasible cases before starting IPM iterations. Such an approach is not even available for LO problems.

In our experiments, we demonstrate the performance of our approach on the CBLIB test set. We are able to solve test problems taken from CBLIB using around 70% of the total IPM iterations compared to cold-start on feasible instances. The ability of the warm-start approach to identify optimal and infeasible cases is also very significant. We are able to identify the new optimal solution in 99 out of 705 instances when branching on non-negative variables just by solving two LO rounding problems. On top of that we are able to identify infeasible primal SOCO instances in 656 out of 706 infeasible cases. Improvement of such a magnitude is quite uncommon for a warm-start approach to IPMs.

Our approach is limited to the cases where rounding problems are feasible or provide a useful conclusion. Although our approach performs well on infeasible instances to detect infeasibility, the primal rounding problem gets infeasible for a significant number of cases. Such cases might be addressed in the future with an alternative approach of initialization of self-dual embedding IPMs.

There are a few open questions for future studies. First of all, we have yet to try this warm-start approach in a full BCC framework. Only minor change are needed to deploy our proposed warm-start approach after a conic cut is added to the problem.Moreover, we can use the rounding problems for pruning by bound inside a BB tree. If we have a dual feasible solution for (D-SOCO) with an objective value that is worse than the current incumbent objective in the tree, then we can prune the node by bound since, due to weak duality, it cannot yield a better solution. So if the optimal value of problem (DR) is greater than the objective of the incumbent solution, we can draw the same conclusion without solving the node. A final open question is about warm-starting of infeasible instances. The default initialization of the self-dual embedding framework is biased towards feasible instances. When infeasibility is suspected, it may be advantageous to initialize the model differently.

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