The uncapacitated p-hub center problem under the existence of zero flows

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Abstract

In this study, we consider the special case of the uncapacitated p-hub center problem, where the weight/flow matrix includes 0 entities. In some real life networks, such as airline networks, cargo networks etc., the zero flow might exist between certain demand points. Typically in airline networks, nobody travels from city i to city i. In the literature, the general assumption is that all entries of the weight/flow matrix are positive. Therefore, the optimal solution of the hub center problem might correspond to a path from origin to destination with a zero flow. In this study, we discuss the effects of zero flows on the optimal solution and modified the current formulation of the uncapacitated p-hub center problem in order to handle this case. We also analyze several scenarios under which the longest path should have enough volumes of flow in order to be involved in investment decisions. We present our computational results.

Keywords: Location, hub center, hub networks, airline networks.

1. Introduction

The p-hub center problem aims to minimize the maximum travel time/distance between any origin-destination pair in a network in order to increase the customer service level. This problem is important for on-time delivery, rapid transit or emergency services systems. As stated by Aykın and Brown [2] airlines and express delivery companies try to maintain the fastest service because of the competition and the sensitivity of demand. Tan and Kara [17] also indicate that the speed and reliability are more important factors than cost in cargo delivery. Because of that, p-hub centers are located in the network and non-hub nodes are allocated to hubs. There are two common allocation strategies in the literature: one is to allow only single allocation and the other is to allow multiple allocations. We assume that the number of hubs is fixed to p and there is no capacity restriction.

The hub center problem was addressed by O'Kelly and Miller [15] and Campbell [3]. Campbell [3] formulated the uncapacitated single allocation p-hub center problem (USApHCP) as a quadratic integer problem. Campbell [3] also provides the linearization of this model, which requires huge number of binary variables as a disadvantage. Kara and Tansel [13] proved that the USApHCP is NP-hard and proposed an improved model. Campbell et al. [5] considered the p-hub center allocation problem and showed that many versions of the problem are NP-hard. They also presented polynomial time algorithms for some special cases. Recently, Ernst et al. [8] provided the novel formulation for the USApHCP based on the hub radius.

Hamacher and Meyer (2006) presented a binary search algorithm to solve the USApHCP. Meyer et al. (2009) proposed a two-phase algorithm, which first makes the location decision and then the allocation decision, for the USApHCP. Jütte et al. (2007) analyses the polyhedral properties of the radius formulation.

Ernst et al. [10] also presented four-index and three-index formulations for the uncapacitated multiple allocation p-hub center problem (UMApHCP). However, they addressed that both formulations are inefficient but the four-index formulation might be preferable. They also proposed a branch-and-bound algorithm, which outperforms the standard solver of formulations.

The detailed literature review of the hub location problems is given by Campbell et al. [4], Alumur and Kara [1], Campbell and O'Kelly [6], Farahani et al. [9] and Contreras [7].

In the literature, the general assumption is that a positive flow is associated with every (i, j) origin-destination pair, i.e. $w_{ij} > 0 \;\; \forall (i, j)$. In some real life network structures such as airline networks, logistics networks etc., the zero flow might exist between certain demand points. The typical example is that in an airline network nobody travels from city i to city i, i.e. $w_{ii} = 0 \quad \forall i$. The CAB data set [16] also includes zero flow on the diagonal of the flow matrix. In some logistics networks, the inner-city flow is directly delivered without needing to send it to hub. Campbell [3] also mentioned that the zero-flow might exist in the network and this can be handled by restricting (i, j) pairs such that $w_{ij} > 0$ in the proposed four-index formulation. In this study, we explain the role of zero flow on the optimal solution of the uncapacitated p-hub center problem, modify current formulations and discuss several scenarios considering flow over a threshold.

In Section 2, we explain the problem and modify current formulations. In Section 3, we provide our computational results on the modified formulations. The paper ends with a conclusion in Section 4.

2. Problem Formulation

Let $G = (N, E)$ be a connected network with the node set $N = \{1,2, \ldots, n\}$, which corresponds to origins and destinations, and the arc set E, which corresponds to pathways. Any pair of nodes $i, j \in N$ is connected with an arc (i, j) and t_{ij} is the length/travel time of arc (i, j). We assume that the length $\boldsymbol{t}$ satisfies the triangle inequality and $t_{ii} = 0 \;\; \forall i \in N$ and $t_{ij} > 0 \;\; \forall i, j \in N,\; i \neq j$. There exists a flow w_{ij} over arc(i, j). Assume that there is no innercity flow and the flow exists only if to other cities (nodes), i.e. $w_{ii} = 0 \quad \forall i \in N$ and $w_{ij} > 0 \;\; \forall i, j \in N,\; i \neq j$. Although this assumption only avoids innercity flow, in a more generalized version some node pairs may have zero flow in the network. The latter case is also discussed in the next section.

The flow is routed through hub(s) associated with nodes i and j. Let (i,k,m,j) be a path from origin i to destination j via hubs k and m. The travel time over this path, denoted T_{ij}, is $T_{ij} = t_{ik} + \alpha t_{km} + t_{mj}$ where α is a discount parameter for the inter-hub transfer.

In the USApHCP, the aim is to minimize the longest path, i.e. $min. max_{\forall i,j \in N} T_{ij}$. Ernst et al. [8] present the radius formulation of the USApHCP as follows.

Decision variables

$$x_{ik} = \begin{cases} 1 & if\ node\ i\ is\ allocated\ to\ hub\ k \\ 0 & otherwise \end{cases} \qquad \forall i,k \in N$$

If $x_{kk} = 1$, then node k is a hub.

r_k is the maximum distance (radius) to hub k among nodes allocated to it, $\forall k \in N$

$$minimize\ \ T_{max} \tag{1}$$

Subject to

$$\sum_{k=1}^{n} x_{ik} = 1 \qquad \forall i = 1,2,\dots,n \tag{2}$$

$$x_{ik} \le x_{kk} \qquad \forall i,k = 1,2,\dots,n \tag{3}$$

$$\sum_{k=1}^{n} x_{kk} = p \tag{4}$$

$$r_k \ge t_{ik} x_{ik} \qquad \forall i,k = 1,2,\dots,n \tag{5}$$

$$T_{max} \ge r_k + r_m + \alpha t_{km} \qquad \forall k \le m = 1,2,\dots,n \tag{6}$$

$$x_{ik} \in \{0,1\} \qquad \forall i,k = 1,2,\dots,n \tag{7}$$

$$r_k \ge 0 \qquad \forall k = 1,2,\dots,n \tag{8}$$

Where the objective function is to minimize the longest travel time T_{max}. The constraint set (2) ensures that each node is allocated to exactly one-hub (single allocation restriction). The constraint set (3) requires that a node i can only be allocated to node k if node k is a hub. The constraint (4) states that exactly p hubs to be established. The constraint set (5) determines the radius of a hub k. The constraint set (6) ensures that the objective value should be greater than all travel times via hub(s).

Ernst et al. [8] also presented a strengthened formulation by changing the constraint (6) with the following one.

$$T_{max} \ge r_k + r_l + \alpha t_{kl} \ + (1-\alpha)(1-x_{kk}) \min_{i:i \ne k} t_{ik} + (1-\alpha)(1-x_{ll}) \min_{i:i \ne l} t_{il} \qquad \forall k,l = 1,2,\dots,n \tag{9}$$

The above formulations are suitable if inner-city transportation is applicable in reality, i.e. $w_{ij} > 0 \ \ \forall i, j \in N$. Otherwise, the optimum solution might correspond to the path i-hub(i)-i, which is not applicable in some real life networks. Let consider the optimum network structure given in Figure 1(a) for CAB ($n = 10, p = 3, \alpha = 0.2$) data set. According to the optimum solution of the radius formulation, nodes 5, 7 and 9 are assigned as hubs. The node 6 is allocated to hub 9, while other nodes are allocated to hub 5. The longest path is due to flow from node 0 to node 0 where $T_{00} = t_{05} + t_{50} = 1119.54$. In this case, the constraint (6) is active for $k = m = 5$ as $T_{max} \geq 2r_5$. However, in CAB data set there is no flow from node 0 to node 0, i.e. $w_{00} = 0$. In order to handle this case, we modify the Ernst et al. [8] formulation as follows.

$$T_{max} \geq r_k + r_m + \alpha t_{km}(x_{kk} + x_{mm} - 1) \qquad \forall k \neq m = 1,2,\dots,n \qquad (6')$$

$$T_{max} \geq \sum_k (t_{ik}x_{ik} + t_{kj}x_{jk}) \qquad \forall i \neq j = 1,2,\dots,n \qquad (10)$$

(1)-(5), (7), (8)

The constraint set (6) is modified as (6'), which evaluates the longest travel times via two hubs. On the other hand, the new constraint set (10) evaluates the travel time from node i to node j if both assigned to hub k.

After solving the CAB 10x3x0.2 data set with the modified formulation we observed that hub set, allocations and the longest path have changed as given in Figure 1(b).

Three cases might appear here.

Case 1. Both k and m are hubs ($x_{kk} = x_{mm} = 1$)

Then $T_{max} \geq r_k + r_m + \alpha t_{km}$, which is the travel time between the farthest nodes allocated to hubs k and m.

Assume that the node i is assigned to hub k and the node j is assigned to hub m ($x_{ik} = 1, x_{jm} = 1$). Then due to constraint (10) $T_{max} \geq t_{ik}x_{ik} + t_{mj}x_{jm}$. In this case, it is clear that $r_k + r_m + \alpha t_{km} \geq t_{ik}x_{ik} + t_{mj}x_{jm}$. Hence, $T_{max} \geq t_{ik}x_{ik} + t_{mj}x_{jm}$ would be inactive. However, if both i and j are assigned to hub k ($x_{ik} = x_{jk} = 1$), then $T_{max} \geq t_{ik}x_{ik} + t_{kj}x_{jk}$ might be active. The optimum solution of the modified model given in Figure 1 (b) is an example of the latter case. Here both node 0 and node 2 are assigned to hub 5 ($x_{05} = x_{25} = 1$). Hence the constraint $T_{max} \geq 559.77x_{05} + 556.07x_{25}$ is active and the optimal objective function value is $T^*_{max} = 1115.84$.

Case 2. Node k is a hub but node m is not a hub ($x_{kk} = 1, x_{mm} = 0$)

Then, $T_{max} \geq r_k$.

Two subcases arise here.

Case 2.1. The node m is allocated to hub k ($x_{mk} = 1$).

In this case, $r_k \geq t_{km}x_{mk}$ and $T_{max} \geq r_k$. Since $t_{ik}x_{ik} + t_{km}x_{mk} \geq t_{km}x_{mk}$ and $T_{max} \geq t_{ik}x_{ik} + t_{km}x_{mk}$, the constraint $T_{max} \geq r_k$ would be inactive.

Case 2.2. The node m is allocated to another hub s ($x_{ms} = 1$).

In this case, there is a constraint $T_{max} \geq r_k + r_s + \alpha c_{ks}$. Hence the constraint $T_{max} \geq r_k$ would be inactive.

Case 3. Both k and m are not hubs ($x_{kk} = x_{mm} = 0$)

Then, the constraint $T_{max} \geq -\alpha t_{km}$ is inactive.

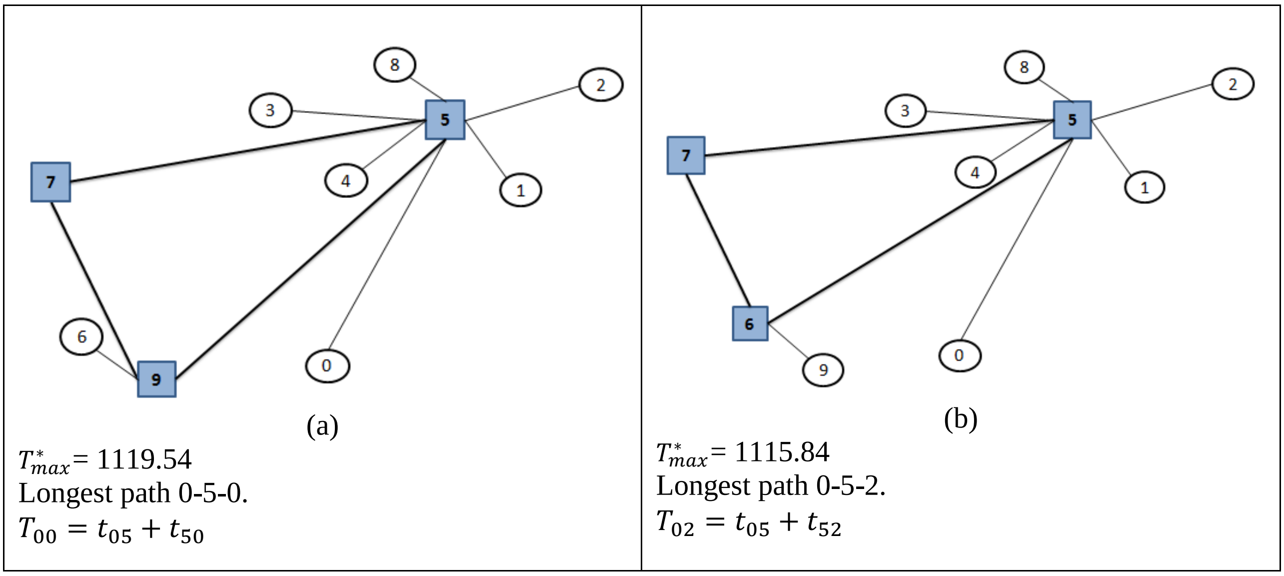

Figure 1. Optimum solution of CAB 10x3x0.2 data set with assumptions (a) $w_{ij} > 0 \quad \forall i,j \in N$ (b) $w_{ii} = 0 \quad \forall i \in N$ and $w_{ij} > 0 \ \forall i,j \in N, \ i \neq j$.

Instead of constraint (6'), the modified version of the strengthened constraint (9) can be written by only replacing the αt_{kl} parameter as follows.

$$T^*_{max} \geq r_k + r_l + \alpha t_{kl}(x_{kk} + x_{ll} - 1) \ + (1-\alpha)(1-x_{kk}) \min_{i:i\neq k} t_{ik} + (1-\alpha)(1-x_{ll}) \min_{i:i\neq l} t_{il}$$

$$\forall k \neq l = 1,2,\dots,n \qquad (9')$$

Note that in the modified version of the USApHCP model, the number of constraints increases by n^2 due to constraint (10).

2.1. The generalized version: $w_{ij} = 0$ for some (i, j)

At this point, we consider the generalized version of the problem where there is no flow between some pair of nodes (i, j). The $w_{ii} = 0$ is the special case of this problem. In this case, the optimum solution might correspond to the path from node i to node j but there is no flow over this path. Then the constraint (10) would be insufficient and it should be replaced with the following one, which is exactly the p-HC2LIN constraint proposed by Kara and Tansel [13] except the $w_{ij} > 0$ condition.

$$T^*_{max} \geq \sum_k [(t_{ik} + \alpha t_{km})x_{ik}] + t_{mj}x_{jm} \qquad \forall i, j, m = 1,2, \dots, n \ \ and \ w_{ij} > 0 \qquad (11)$$

(1)-(4), (7)

However, in this case the p-HC2LIN model would be more efficient since it does not require constraint sets (5), (6') and variables $r_k, \ \forall k \in N$.

In the four-index formulations of the hub center problems, it would be enough only considering (i, j) pairs such that $w_{ij} > 0$.

Let consider the example given in Figure 1(b). Assume that the flow on the longest path (i.e. the path 0-5-2) is as low as to be neglected. Although this is the case, this path affects the hub investment decision. Now, assume that we set $w_{02} = 0$. Then, the longest path would be 0-5-6-9 with T^*_{max}= 983.126.

At this point, an interesting strategy could be determining a threshold (lb) on the flow volumes and not considering any pair of nodes of which flow is less than lb for the hub center decision. Then the restriction in the constraint set (11) will be updated as $w_{ij} > lb$. In computational results section, we performed some experiments over this strategy.

3. Computational Results

We performed experiments on the CAB data set (O'Kelly [16]) since innercity flow is not applicable. Each problem instance is coded as $n \times p \times \alpha$, where n is the number of nodes in the network, p is the number of hubs will be installed and α is the discount factor for the inter-hub transportation cost.

All numerical experiments were carried out on an Intel Core Duo 2.26 GHz machine equipped with 4GB ram. We used the Cplex 12.6.1 (IBM ILOG [11]) solver in order to solve problem instances.

We tested two versions of the modified radius formulation for the USApHCP: one with the constraint set (6') and the other with the strengthened constraint set (9').

We also presented our experimental results of the four-index formulation of the Ernst et al. [8] for the UMApHCP. Here we simply modified the formulation by only considering (i, j) pairs such that $w_{ij} > 0$.

The following measures are reported.

Obj. pre.: the optimum objective value of the formulation presented by Ernst et al. [8].

Obj.0: the optimum objective value of the modified formulation.

CPU_MR(6'): the CPU time (seconds) required for solving the modified radius formulation with the constraint set (6').

CPU_MR(9'): the CPU time (seconds) required for solving the modified strengthened radius formulation with the constraint set (9').

CPU_4-index: the CPU time (seconds) required for solving the modified four-index formulation.

%Gap: the gap between the optimum objective function values of the mixed-integer problem and the relaxed LP problem, i.e. $\left(\frac{obj.0-relaxed\ obj.0}{obj.0} * 100\right)$.

Computational results on the USApHCP are presented in Table 1. Accordingly, the optimum solutions of 16 instances have changed as a result of using modified formulations. These instances are marked with a "*" sign in Table 1. We also observe that all CPU times are very low with a maximum 3.38 sec. There is no significant difference between solution times of MR(6') and MR(9') formulations.

Table 1. Computational results on the modified formulations of the USApHCP for CAB instances

Instance	Obj. pre.	Obj.0	CPU_MR(6')	%gap	CPU_MR(9')	%gap
10x2x.2	1425.58	1425.58	0.11	35.6	0.19	35.6
10x2x.4	1627.52	1627.52	0.16	43.6	0.26	43.6
10x2x.6	1759.13	1671.49*	0.11	45.0	0.08	45.0
10x2x.8	1759.13	1743.86*	0.08	47.3	0.06	47.3
10x2x1	1839.65	1839.65	0.08	50.1	0.06	50.1
10x3x.2	1119.54	1115.83*	0.09	46.7	0.11	46.7
10x3x.4	1185.07	1185.07	0.09	49.9	0.11	49.9
10x3x.6	1387	1387	0.11	57.2	0.08	57.1
10x3x.8	1588.94	1588.94	0.08	62.6	0.08	62.6
10x3x1	1790.55	1790.55	0.09	66.8	0.08	66.8
10x4x.2	830.25	809.37*	0.14	51.8	0.13	48.6
10x4x.4	968.2	968.2	0.09	59.7	0.09	57.7
10x4x.6	1146.19	1146.19	0.09	65.9	0.06	64.9

10x4x.8	1454.44	1454.44	0.06	73.1	0.06	72.7
10x4x1	1764.79	1764.79	0.06	77.6	0.06	77.6
15x2x.2	2005.02	2005.02	0.31	33.0	0.28	33.0
15x2x.4	2160.75	2017.41*	0.31	33.4	0.20	33.4
15x2x.6	2214.09	2102.18*	0.25	36.1	0.16	36.1
15x2x.8	2423.8	2423.8	0.25	44.5	0.14	44.5
15x2x1	2609.18	2609.18	0.17	48.5	0.13	48.5
15x3x.2	1749.04	1641.7*	0.33	43.8	0.47	43.6
15x3x.4	1760.15	1738.32*	0.42	46.9	0.34	46.9
15x3x.6	1844.92	1844.92	0.38	50.0	0.16	50.0
15x3x.8	2166.54	2166.54	0.17	57.4	0.20	57.4
15x3x1	2600.08	2600.08	0.16	64.5	0.16	64.5
15x4x.2	1340.96	1322.92*	0.36	48.2	0.3	46.8
15x4x.4	1434.38	1434.38	0.19	52.3	0.33	51.8
15x4x.6	1754.51	1754.51	0.16	61.0	0.17	61.0
15x4x.8	2080.06	2080.06	0.16	67.1	0.13	67.1
15x4x1	2600.08	2600.08	0.17	73.7	0.14	73.7
20x2x.2	1892.99	1854.89*	0.59	27.5	0.67	27.5
20x2x.4	2160.75	2067.04*	0.67	34.9	0.69	34.9
20x2x.6	2274.67	2255.06*	0.97	40.4	0.47	40.4
20x2x.8	2501.93	2487.15*	0.39	45.9	0.38	45.9
20x2x1	2609.18	2609.18	0.47	48.5	0.39	48.5
20x3x.2	1551.25	1551.25	0.84	40.1	0.92	40.1
20x3x.4	1760.15	1744.68*	1.14	46.8	0.73	46.8
20x3x.6	1997.79	1997.79	1.02	53.5	0.69	53.5
20x3x.8	2263.54	2263.54	0.58	59.0	0.50	59.0
20x3x1	2600.08	2600.08	0.28	64.3	0.31	64.3
20x4x.2	1355.41	1355.19*	1.08	48.6	1.05	47.8
20x4x.4	1472.71	1472.71	1.34	52.7	1.01	52.7
20x4x.6	1834.83	1834.83	0.74	62.1	0.95	62.1
20x4x.8	2153	2153	0.34	67.7	0.30	67.7
20x4x1	2600.08	2600.08	0.27	73.2	0.27	73.2
25x2x.2	2131.2	2109.08*	1.27	30.1	1.22	30.1
25x2x.4	2402.55	2402.55	1.44	38.7	1.61	38.7
25x2x.6	2558.74	2558.74	1.42	42.4	1.22	42.4
25x2x.8	2714.93	2714.93	0.94	45.7	1.34	45.7
25x2x1	2827.16	2827.16	0.78	47.9	0.81	47.9
25x3x.2	1923.12	1923.12	2.84	43.8	1.84	43.8
25x3x.4	2100.47	2091.13*	2.44	48.3	3.17	48.3
25x3x.6	2340.25	2340.25	2.03	53.8	2.08	53.8
25x3x.8	2554.13	2554.13	0.94	57.7	1.27	57.7
25x3x1	2758.39	2758.39	0.80	60.8	0.80	60.8
25x4x.2	1619.48	1619.48	2.39	49.3	3.38	49.2
25x4x.4	1884.84	1884.84	3.25	56.5	2.63	56.5
25x4x.6	2182.49	2182.49	2.61	62.4	2.75	62.4
25x4x.8	2454.35	2454.35	0.94	66.6	1.27	66.6
25x4x1	2726.28	2726.28	0.75	69.9	0.69	69.9

* instances of which optimum solutions change in the modified formulation

Table 2 presents the computational results on the four-index formulation of the UMApHCP. Similar to the above case the optimum solutions of some instances have changed. Additionally, as the problem size increases, the CPU time increases as expected.

Table 2. Computational results on the modified formulations of the UMApHCP for CAB instances

Instance	Obj. pre.	Obj.0	CPU_4-index	%gap
10x2x.2	1421.89	1421.89	0.73	26.6
10x2x.4	1548.37	1548.37	0.71	22.2
10x2x.6	1749.04	1629.17*	0.56	15.4
10x2x.8	1749.04	1629.17*	0.47	5.9
10x2x1	1764.79	1764.79	0.21	0.0
10x3x.2	1119.54	1115.84*	0.63	32.0
10x3x.4	1181.37	1181.37	0.91	20.2
10x3x.6	1308.85	1308.85	0.56	11.3
10x3x.8	1502.14	1502.14	0.45	8.6
10x3x1	1764.79	1764.79	0.21	0.0
10x4x.2	809.36	809.36	0.94	30.6
10x4x.4	968.20	968.20	0.55	19.5
10x4x.6	1146.19	1146.19	0.67	7.6
10x4x.8	1411.83	1411.83	0.38	0.0
10x4x1	1764.79	1764.79	0.22	0.0
15x2x.2	2005.02	2005.02	9.81	24.2
15x2x.4	2027.69	2017.41*	4.95	14.8
15x2x.6	2081.04	2081.04	3.28	5.7
15x2x.8	2335.82	2335.82	2.91	3.7
15x2x1	2600.08	2600.08	1.03	0.0
15x3x.2	1716.14	1641.70*	17.63	30.8
15x3x.4	1738.32	1738.32	8.02	21.9
15x3x.6	1823.1	1823.1	3.78	8.6
15x3x.8	2141.83	2141.83	2.81	2.6
15x3x1	2600.08	2600.08	1.27	0.0
15x4x.2	1287.78	1287.78	12.58	29.2
15x4x.4	1395.88	1395.88	6.94	17.3
15x4x.6	1751.45	1751.45	5.88	10.9
15x4x.8	2080.06	2080.06	2.31	0.0
15x4x1	2600.08	2600.08	1.06	0.0
20x2x.2	1892.99	1854.89*	66.45	17.8
20x2x.4	2027.69	2027.69	51.17	13.7
20x2x.6	2248.13	2248.13	30.98	11.1
20x2x.8	2335.99	2335.99	25.88	3.3
20x2x1	2600.08	2600.08	6.64	0.0
20x3x.2	1551.25	1551.25	66.41	25.7
20x3x.4	1738.32	1738.32	50.66	19.5
20x3x.6	1916.16	1916.16	41.86	10.0
20x3x.8	2195.22	2175.71*	25.63	0.0
20x3x1	2600.08	2600.08	5.52	0.0
20x4x.2	1287.78	1287.78	68.66	26.4
20x4x.4	1472.71	1472.71	63.42	16.5

20x4x.6	1808.7	1808.70	34.31	11.1
20x4x.8	2128.11	2128.11	34.16	2.3
20x4x1	2600.08	2600.08	5.63	0.0
25x2x.2	2049.48	2049.48	764.78	14.9
25x2x.4	2402.55	2402.55	669.81	13.3
25x2x.6	2558.74	2558.74	208.11	7.9
25x2x.8	2714.93	2714.93	92.23	6.1
25x2x1	2739.22	2739.22	104.03	0.5
25x3x.2	1911.6	1765.12*	462.67	23.1
25x3x.4	2064.67	2064.67	248.66	18.6
25x3x.6	2243.77	2243.77	158.58	8.2
25x3x.8	2515.58	2515.58	91.67	4.7
25x3x1	2725.79	2725.79	52.72	0.0
25x4x.2	1619.48	1619.48	640.33	30.3
25x4x.4	1774.75	1774.75	360.88	16.8
25x4x.6	2127.13	2127.13	193.88	12.2
25x4x.8	2437.71	2437.71	102.47	6.0
25x4x1	2725.79	2725.79	32.21	0.0

* instances of which optimum solutions change in the modified formulation

We performed further experiments using several thresholds such as lb=1000, 2000, 3000 and 4000. Here we assume that paths, on which flow are less than lb, are ignored in the decision process. The results are given in Table 3 for the USApHCP. Instances, whose optimum solutions change depending on the threshold, are marked with a "*" sign. For example, let consider the first instance of which objective value reduces to 1421.89 when lb is set to 3000. Originally hubs 5 and 6 are installed in the optimal solution and the longest path (1425.58) is due to the path 0-5-6-7. However, since the flow $w_{07} = 2243$ is less than the threshold, the optimal solution has changed to another path 2-5-6-7 with the length 1421.89. Then, the latter is the solution should be focused on.

Table 3. Computational results on the p-HC2LIN model [13] with $w_{ij} > lb$ restriction for the USApHCP and CAB instances.

			lb=1000		**lb=2000**		**lb=3000**		**lb=4000**	
N	p	α	Obj.	CPU	Obj.	CPU	Obj.	CPU	Obj.	CPU
10	2	0.2	1425.58	0.27	1425.58	0.99	1421.89*	0.41	1421.89*	0.61
		0.4	1627.52	0.27	1627.52	0.64	1623.82*	0.33	1623.82*	0.52
		0.6	1671.49	0.23	1671.49	0.57	1671.49	0.27	1671.49	0.53
		0.8	1743.86	0.28	1743.86	0.5	1743.86	0.28	1743.86	0.43
		1	1839.65	0.27	1839.65	0.52	1839.65	0.27	1839.65	0.48
10	3	0.2	1115.84	0.3	1115.84	0.68	1115.84	0.25	1077.74*	0.51
		0.4	1185.07	0.3	1185.07	0.52	1185.07	0.29	1185.07	0.44
		0.6	1387	0.27	1387	0.57	1387	0.28	1387	0.47
		0.8	1588.94	0.25	1588.94	0.53	1588.94	0.22	1588.94	0.36
		1	1790.55	0.23	1790.55	0.51	1790.55	0.25	1790.55	0.45
10	4	0.2	809.37	0.46	809.37	0.5	809.36	0.3	802.92*	0.73
		0.4	968.2	0.27	968.2	0.33	968.2	0.22	947.06*	0.33
		0.6	1146.19	0.19	1146.19	0.49	1146.19	0.16	1146.19	0.39
		0.8	1454.44	0.23	1454.44	0.61	1454.44	0.25	1454.44	0.46

		1	1764.79	0.14	1764.79	0.49	1764.79	0.14	1764.79	0.5
15	2	0.2	2005.02	2.51	2005.02	4.02	2005.02	2	1959.94*	2.9
		0.4	2017.41	2.7	2017.41	3.81	2017.41	2.17	1959.94*	3.03
		0.6	2102.18	1.69	2102.18	3.08	2102.18	1.36	2102.18	2.49
		0.8	2423.8	1.56	2423.8	2.52	2423.8	1.28	2423.8	2.04
		1	2609.18	0.45	2609.18	1.51	2609.18	0.4	2609.18	1.22
15	3	0.2	1641.7	1.96	1641.7	3.67	1641.7	1.62	1580.89*	2.45
		0.4	1738.32	1.27	1738.32	2.32	1738.32	0.94	1580.89*	2.03
		0.6	1844.92	1.4	1844.92	1.72	1844.92	0.95	1844.92	1.91
		0.8	2166.54	1.22	2166.54	2.67	2166.54	0.83	2166.54	1.95
		1	2600.08	0.44	2600.08	1.49	2600.08	0.33	2600.08	1.26
15	4	0.2	1322.92	1.14	1322.92	4.59	1322.92	1.77	1322.92	3.54
		0.4	1434.38	1.58	1434.38	3.05	1434.38	1.45	1434.38	2.18
		0.6	1754.51	0.92	1754.51	2.27	1754.51	0.83	1754.51	1.56
		0.8	2080.06	0.67	2080.06	1.43	2080.06	0.6	2080.06	1.19
		1	2600.08	0.45	2600.08	1.54	2600.08	0.55	2600.08	1.22
20	2	0.2	1854.89	10.87	1854.89	12.98	1854.89	8.5	1826.88*	9.24
		0.4	2067.04	8.2	2067.04	10.32	2067.04	5.73	2067.04	6.82
		0.6	2255.06	5.66	2255.06	6.59	2230.42*	5.37	2230.42*	5.97
		0.8	2487.15	5.23	2487.15	7.75	2487.15	5.2	2487.15	7.02
		1	2609.18	1.49	2609.18	2.78	2609.18	1.36	2609.18	2.34
20	3	0.2	1551.25	9.7	1551.25	13.03	1551.25	7.12	1551.25	11.07
		0.4	1744.68	7.71	1744.68	9.66	1744.68	5.03	1731.15*	7.09
		0.6	1997.79	6.28	1997.79	11.17	1997.79	8.68	1997.79	5.31
		0.8	2263.54	3.63	2263.54	5.44	2263.54	4.07	2263.54	4.33
		1	2600.08	1.25	2600.08	2.15	2600.08	1.08	2600.08	1.78
20	4	0.2	1355.19	7.26	1355.19	11.01	1355.19	10.36	1294.93*	7.88
		0.4	1472.71	10.24	1472.71	15.65	1472.71	7.35	1472.71	10.29
		0.6	1834.83	5.01	1823.1*	7.15	1823.1*	3.75	1823.1*	4.91
		0.8	2153	5.05	2153	6.86	2153	4.12	2153	4.3
		1	2600.08	1.96	2600.08	2.79	2600.08	1.43	2600.08	2.83
25	2	0.2	2109.08	31.02	2109.08	41.28	2109.08	17.82	2109.08	36.95
		0.4	2402.55	30.94	2335.27*	40.65	2335.27*	27.94	2335.27*	25.21
		0.6	2558.74	17.44	2536.62*	17.18	2536.62*	12.08	2536.62*	22.74
		0.8	2714.93	14.23	2692.81*	14.2	2692.81*	6.59	2692.81*	8.49
		1	2827.16	6.56	2827.16	8.07	2827.16	5.71	2827.16	5.89
25	3	0.2	1923.12	40.29	1784.05*	34.19	1784.05*	46.45	1784.05*	44.69
		0.4	2091.13	26.7	2031.7*	36.92	2031.7*	26.99	2031.7*	46.7
		0.6	2340.25	45.15	2299.69*	33.44	2299.69*	19.82	2299.69*	31.42
		0.8	2554.13	30.52	2518.11*	25.98	2518.11*	14.89	2518.11*	17.54
		1	2758.39	13.61	2758.39	12.77	2758.39	7.47	2758.39	10.26
25	4	0.2	1619.48	51.39	1586.78*	54.81	1586.78*	55.53	1580.89*	41.51
		0.4	1884.85	28.57	1884.85	65.49	1884.85	40.55	1884.85	35.06
		0.6	2182.49	31.42	2182.49	28	2182.49	20.02	2182.49	19.81
		0.8	2454.35	23.19	2454.35	25.76	2454.35	16	2454.35	17.12
		1	2726.28	7.87	2726.28	8.99	2726.28	5.52	2726.28	5.95

The results for the UMApHCP under the $w_{ij} > lb$ restriction are given in Table 4. Similar to the above table several instances were affected from the lb setting. As the threshold and the problem size increase, the number of instances affected increases.

Table 4. Computational results on the four-index formulation [8] with $w_{ij} > lb$ restriction for the UMApHCP and CAB instances.

			lb=1000		**lb=2000**		**lb=3000**		**lb=4000**	
N	p	α	Obj.	CPU	Obj.	CPU	Obj.	CPU	Obj.	CPU
10	2	0.2	1421.89	0.72	1421.89	1.17	1421.89	0.6	1421.89	0.83
		0.4	1548.37	0.58	1548.37	1.06	1548.37	0.52	1548.37	0.93
		0.6	1629.17	0.48	1629.17	0.78	1629.17	0.45	1629.17	0.72
		0.8	1629.17	0.37	1629.17	0.75	1629.17	0.37	1629.17	0.52
		1	1764.79	0.17	1764.79	0.25	1764.79	0.17	1764.79	0.3
10	3	0.2	1115.84	0.62	1115.84	0.87	1115.84	0.84	1077.74	1.25
		0.4	1181.37	0.73	1181.37	1.19	1181.37	0.6	1181.37	0.6
		0.6	1308.85	0.57	1308.85	0.76	1308.85	0.53	1308.85	0.59
		0.8	1502.14	0.42	1502.14	0.54	1502.14	0.36	1466.22	0.55
		1	1764.79	0.17	1764.79	0.22	1764.79	0.25	1764.79	0.22
10	4	0.2	809.36	1.03	809.36	1.42	809.36	1.14	802.92*	0.94
		0.4	968.2	0.47	968.2	0.81	938.18*	0.66	916.63*	0.69
		0.6	1146.19	0.69	1146.19	0.95	1146.19	0.56	1146.19	0.58
		0.8	1411.83	0.34	1411.83	0.36	1411.83	0.3	1411.83	0.41
		1	1764.79	0.17	1764.79	0.2	1764.79	0.16	1764.79	0.17
15	2	0.2	2005.02	14.64	2005.02	10.7	2005.02	10.55	1959.94*	5.15
		0.4	2017.41	5.34	2017.41	5.51	2017.41	4.58	1959.94*	4.97
		0.6	2081.04	4.01	2081.04	4.29	2081.04	3	2081.04	2.98
		0.8	2335.82	3.69	2335.82	3.34	2335.82	2.8	2335.82	2.69
		1	2600.08	1.09	2600.08	1.31	2600.08	0.83	2600.08	0.8
15	3	0.2	1641.7	24.42	1641.7	24.78	1641.7	17.94	1580.89*	15.64
		0.4	1738.32	13.36	1738.32	10.21	1738.32	11.84	1580.89*	9.96
		0.6	1823.1	4.81	1823.1	4.92	1823.1	3.16	1808.7*	3.63
		0.8	2141.83	3.77	2141.83	4.01	2141.83	2.75	2141.83	3.33
		1	2600.08	1.01	2600.08	1.75	2600.08	0.86	2600.08	1.01
15	4	0.2	1287.78	19.04	1287.78	27.41	1287.78	7.94	1287.78	18.09
		0.4	1395.88	12.97	1395.88	12.3	1395.88	9.19	1367.2*	11.46
		0.6	1751.44	11.33	1751.44	8.4	1751.44	6.04	1751.44	8.68
		0.8	2080.06	2.04	2080.06	2.92	2080.06	1.43	2080.06	2.21
		1	2600.08	1	2600.08	1.91	2600.08	0.78	2600.08	0.86
20	2	0.2	1854.89	147.1	1854.89	84.22	1854.89	58	1826.88*	33.8
		0.4	2027.69	45.58	2027.69	43.74	2027.69	25.03	1933.98*	21.14
		0.6	2248.13	33.71	2241.69	35.21	2081.04	21.36	2081.04	24.59
		0.8	2335.99	30.49	2335.82	24.67	2335.82	14.88	2335.82	19.04
		1	2600.08	6.64	2600.08	6.27	2600.08	4.5	2600.08	4.96
20	3	0.2	1551.25	134.79	1551.25	117.63	1551.25	102.17	1551.25	82.03
		0.4	1738.32	67.44	1738.32	69.96	1738.32	42.11	1702.43*	50.34
		0.6	1916.16	53.16	1916.16	51.63	1853.4	27.36	1853.4	25.01
		0.8	2175.71	27.37	2175.71	27	2153*	19.9	2153*	15.66
		1	2600.08	5.83	2600.08	5.68	2600.08	3.82	2600.08	3.91

20	4	0.2	1287.78	86.94	1287.78	94.05	1287.78	77.84	1287.78	71.94
		0.4	1472.71	76.59	1472.71	79.52	1472.71	54.26	1472.71	48.75
		0.6	1808.7	39.11	1808.7	45.09	1706.57*	47.02	1706.57*	26.82
		0.8	2128.11	39.99	2128.11	56.68	2097.42*	30.62	2097.42*	34.06
		1	2600.08	6.14	2600.08	5.62	2600.08	4.52	2600.08	4.02
25	2	0.2	2049.48	311.44	2027.69*	621.83	2027.69*	366.89	1959.94*	184.08
		0.4	2402.55	578.98	2335.27*	611.49	2258.99*	205.62	2258.99*	200.18
		0.6	2558.74	184.01	2536.62*	198.49	2536.62*	135.29	2536.62*	150.14
		0.8	2714.93	97.43	2639.27*	98.52	2639.27*	78.06	2639.27*	59.82
		1	2739.22	91.4	2703.4*	92	2703.4*	56.38	2703.4*	50.97
25	3	0.2	1765.12	420.54	1738.32*	347.31	1738.32*	321.55	1733.64*	251.09
		0.4	2064.67	201.7	2031.7*	293.71	2022.37*	230.48	2022.37*	170.09
		0.6	2243.77	148.66	2243.77	217.83	2243.77	130.06	2243.77	113.74
		0.8	2515.58	94.64	2515.58	114.96	2515.58	89.77	2515.58	69.64
		1	2725.79	32.49	2703.4*	33.71	2703.4*	21.98	2703.4*	17.17
25	4	0.2	1619.48	629.46	1586.78*	350.6	1586.78*	426.85	1580.89*	246.99
		0.4	1774.45	380.43	1774.45	371.49	1774.45	245.02	1774.45	186.25
		0.6	2127.13	183.88	2127.13	175.72	2127.13	149.51	2127.13	122.19
		0.8	2437.71	109.17	2433.37*	116.26	2433.37*	88.6	2433.37*	70.48
		1	2725.79	35.75	2703.4*	34.77	2703.4*	25.91	2703.4*	20.59

4. Conclusion

In this study we take into consideration the zero flow for uncapacitated p-hub center problems. We addressed that the optimum solution of the problem may correspond to an o-d path not in use in reality. This may affect possible investments to reduce the longest path.

We performed experiments on the CAB data set, which includes zero entities in the weight/flow matrix. We observed that the optimum solutions of some instances of the CAB data set have changed after solving the modified formulations. We suggest to use the appropriate formulation when the flow matrix includes zero entities.

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