

# Error bounds for rank constrained optimization problems and applications\*

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## Abstract

This paper is concerned with the rank constrained optimization problem whose feasible set is the intersection of the rank constraint set  $\mathcal{R} = \{X \in \mathbb{X} \mid \text{rank}(X) \leq \kappa\}$  and a closed convex set  $\Omega$ . We establish the local (global) Lipschitzian type error bounds for estimating the distance from any  $X \in \Omega$  ( $X \in \mathbb{X}$ ) to the feasible set and the solution set, respectively, under the calmness of a multifunction associated to the feasible set at the origin, which is specially satisfied by three classes of common rank constrained optimization problems. As an application of the local Lipschitzian type error bounds, we show that the penalty problem yielded by moving the rank constraint into the objective is exact in the sense that its global optimal solution set coincides with that of the original problem when the penalty parameter is over a certain threshold. This particularly offers an affirmative answer to the open question whether the penalty problem (32) in [7] is exact or not. As another application, we derive the error bounds of the iterates generated by a multi-stage convex relaxation approach to those three classes of rank constrained problems and show that the bounds are nonincreasing as the number of stages increases.

**Key words:** rank constrained optimization; error bounds; calmness; exact penalty

**AMS Subject Classifications (2010):** 90C26, 90C22

## 1 Introduction

Let  $\mathbb{X}$  denote the vector space  $\mathbb{R}^{n_1 \times n_2}$  of all  $n_1 \times n_2$  real matrices or the vector space  $\mathbb{H}^n$  of all  $n \times n$  Hermitian matrices, both endowed with the trace inner product  $\langle \cdot, \cdot \rangle$  and its

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induced norm  $\|\cdot\|_F$ . Given a positive integer  $\kappa$  and a suitable continuous loss function  $f: \mathbb{X} \rightarrow \mathbb{R}$ , we are concerned with the following rank constrained optimization problem

$$\min_{X \in \mathbb{X}} \left\{ f(X) \mid \text{rank}(X) \leq \kappa, X \in \Omega \right\}, \quad (1)$$

where  $\Omega$  is a convex compact subset of  $\mathbb{X}$ . Such a problem has many applications in a host of fields including statistics, signal and image processing, system identification and control, collaborative filtering, quantum tomography, finance, and so on (see, e.g., [2, 5, 6, 7, 17, 22, 25, 30]). In the sequel, we denote by  $\mathcal{F}$  the feasible set of (1) and assume that  $\mathcal{F} \neq \emptyset$ , which implies that the solution set of (1), denoted by  $\mathcal{F}^*$ , is nonempty.

Due to the combinatorial nature of rank function, problem (1) is generally NP-hard, and it is almost impossible to seek a global optimal solution. A common way to resolve this class of problems is to adopt convex relaxation technique, which yields a desirable local optimal even feasible solution by solving a single or a sequence of tractable convex optimization problems. The popular nuclear norm convex relaxation method proposed in [5] belongs to the single-stage convex relaxation class, which received active research in the past several years from many fields such as optimization, statistics, information, computer science, and so on (see, e.g., [2, 3, 8, 11, 22, 23, 27]). Although the nuclear norm promotes low rank solutions, it has a big difference from the rank function in a general setting since the former is convex whereas the latter is nonconvex even concave (in the positive semidefinite cone). Hence, when the set  $\Omega$  is characterizing some structure conflicted with low rank, such as the correlation or density matrix structure, the nuclear norm relaxation method will fail in yielding a low rank solution. In view of this, many researchers recently develop effective solution methods based on the sequential convex relaxation models arising from the penalty problems [7, 12], the nonconvex surrogate problems [6, 10, 18, 19], and the rank constrained optimization problem itself [16, 26, 28]. We notice that to measure the distance from any given point to the feasible set or the solution set plays a key role in the analysis of these methods. Motivated by this, we in this work take the first step towards the study on Lipschitzian type error bounds for (1).

In this paper, we show that the calmness of a multifunction associated to the feasible set  $\mathcal{F}$  at the origin is a sufficient and necessary condition for the local Lipschitzian error bounds to estimate the distance from any  $X \in \Omega$  to  $\mathcal{F}$ , which is specially satisfied by three classes of common rank constrained optimization problems (1) where  $\Omega$  is a ball set, a density matrix set or a correlation matrix set, and under this condition derive the global error bound for estimating the distance from any  $X \in \mathbb{X}$  to  $\mathcal{F}$ . In addition, under an additional mild assumption for the objective function  $f$ , we also establish the local (global) Lipschitzian error bounds for estimating the distance from any  $X \in \Omega$  ( $X \in \mathbb{X}$ ) to the solution set  $\mathcal{F}^*$ . To the best of our knowledge, this paper is the first one to study the Lipschitzian type error bounds for low-rank optimization problems, though there are many works on error bounds for the system of linear inequalities and (nondifferentiable) convex inequalities (see, e.g., [13, 14, 15, 21, 31] and references therein). To illustrate the

potential applications of the derived error bounds, we show that the penalty problem

$$\min_{X \in \Omega} \left\{ f(X) + \rho \sum_{i=\kappa+1}^n \sigma_i(X) \right\} \quad (2)$$

is exact in the sense that its global optimal solution set coincides with that of (1) when the penalty parameter  $\rho$  is over a certain threshold. This does not only affirmatively answer the open question proposed in [7] about whether the penalty problem (32) there is exact or not for the rank constrained correlation matrix problem, but also provides a platform for designing convex relaxation algorithms for (1). In addition, we also establish the error bounds of the iterates generated by a multi-stage convex relaxation approach to the rank constrained optimization problems with  $\Omega$  being a ball set, a density matrix set or a correlation matrix set, and show that the error bound sequence is nonincreasing.

To close this section, we provide a brief summary of notations used in this paper:

- $\mathbb{H}_+^n$  denotes the cone of Hermitian positive semidefinite matrices. For  $X \in \mathbb{H}^n$ , we assume that its eigenvalue decomposition takes the form of  $X = \sum_{i=1}^n \lambda_i(X) u_i u_i^\top$  where  $\lambda_1(X) \geq \dots \geq \lambda_n(X)$  and all  $u_i$  are complex orthonormal column vectors.
- For  $X \in \mathbb{R}^{n_1 \times n_2}$ , we denote by  $\|X\|_*$  the nuclear norm of  $X$ , and assume that its singular value decomposition (SVD) takes the form of  $X = \sum_{i=1}^n \sigma_i(X) u_i v_i^\top$  where  $\sigma_1(X) \geq \dots \geq \sigma_n(X)$  with  $n = \min(n_1, n_2)$ , and all  $u_i \in \mathbb{R}^{n_1}$  and  $v_i \in \mathbb{R}^{n_2}$  are orthonormal column vectors.
- For a given vector  $x \in \mathbb{R}^n$ ,  $\text{Diag}(x)$  denotes the diagonal matrix with  $i$ th diagonal entry being  $x_i$ . For a vector  $u \in \mathbb{R}_{++}^n$ ,  $\frac{1}{\sqrt{u}}$  means the vector  $(\frac{1}{\sqrt{u_1}}, \dots, \frac{1}{\sqrt{u_n}})^\top \in \mathbb{R}^n$ .
- The notation  $\mathbb{B}_{\mathbb{X}}$  denotes a closed unit ball of the space  $\mathbb{X}$  centered at the origin. For a closed subset  $\mathcal{S} \subseteq \mathbb{X}$ ,  $\Pi_{\mathcal{S}}(X)$  means a projection point of  $X$  onto the set  $\mathcal{S}$ .

## 2 Lipschitzian type error bounds

Let  $\mathbb{Y}$  and  $\mathbb{Z}$  be the finite dimensional vector spaces which are both endowed with the norm  $\|\cdot\|$ . Recall that a multifunction  $\Upsilon: \mathbb{Y} \rightrightarrows \mathbb{Z}$  is calm at  $\bar{y}$  for  $\bar{z} \in \Upsilon(\bar{y})$  if there exist a constant  $\alpha \geq 0$  along with neighborhoods  $\mathcal{U}$  of  $\bar{y}$  and  $\mathcal{V}$  of  $\bar{z}$  such that

$$\Upsilon(y) \cap \mathcal{V} \subseteq \Upsilon(\bar{y}) + \alpha \|y - \bar{y}\| \mathbb{B}_{\mathbb{Z}} \quad \text{for all } y \in \mathcal{U}.$$

By [4, Exercise 3H.4], we know that there is no need at all to mention a neighborhood  $\mathcal{U}$  of  $\bar{y}$  in the description of calmness, i.e., the following equivalent description holds.

**Lemma 2.1** *For a multifunction  $\Upsilon: \mathbb{Y} \rightrightarrows \mathbb{Z}$ , the calmness of  $\Upsilon$  at  $\bar{y}$  for  $\bar{z} \in \Upsilon(\bar{y})$  is equivalent to the existence of a constant  $\alpha \geq 0$  and a neighborhood  $\mathcal{V}$  of  $\bar{z}$  such that*

$$\Upsilon(y) \cap \mathcal{V} \subseteq \Upsilon(\bar{y}) + \alpha \|y - \bar{y}\| \mathbb{B}_{\mathbb{Z}} \quad \text{for all } y \in \mathbb{Y},$$

or the existence of a constant  $\alpha \geq 0$  and a neighborhood  $\mathcal{V}$  of  $\bar{z}$  such that

$$\text{dist}(z, \Upsilon(\bar{y})) \leq \alpha \text{dist}(\bar{y}, \Upsilon^{-1}(z)) \quad \text{for all } z \in \mathcal{V}.$$

Define a multifunction  $\Gamma: \mathbb{R} \rightrightarrows \mathbb{X}$  associated to the feasible set of problem (1) as follows

$$\Gamma(\omega) := \left\{ X \in \Omega \mid \sum_{i=\kappa+1}^n \sigma_i(X) = \omega \right\} \quad \text{for } \omega \in \mathbb{R}. \quad (3)$$

In this section, we show that the calmness of the multifunction  $\Gamma$  at 0 for each  $X \in \Gamma(0)$  is a sufficient and necessary condition for the local Lipschitzian error bounds to estimate the distance from any  $X \in \Omega$  to the feasible set  $\mathcal{F}$ , which is specially satisfied for three classes of popular rank constrained optimization problems, and then establish the global error bounds for estimating the distance from any  $X \in \mathbb{X}$  to  $\mathcal{F}$  under the calmness of  $\Gamma$ , and derive the local and global Lipschitzian error bounds for estimating the distance to  $\mathcal{F}^*$  under an additional restricted strong convexity for the objective function  $f$ .

## 2.1 Local error bounds

The following theorem states that the distance from any  $Z \in \Omega$  to  $\Gamma(0) = \Omega \cap \mathcal{R}$  can be bounded above by  $\sum_{i=\kappa+1}^n \sigma_i(Z)$  iff the multifunction  $\Gamma$  is calm at 0 for each  $X \in \Gamma(0)$ .

**Theorem 2.1** *Let  $\Gamma$  be the multifunction defined by (3). The multifunction  $\Gamma$  is calm at 0 for each  $X \in \Gamma(0)$  if and only if there exists a constant  $c > 0$  such that*

$$\text{dist}(Z, \Gamma(0)) \leq c \text{dist}(0, \Gamma^{-1}(Z)) = c \sum_{i=\kappa+1}^n \sigma_i(Z) \quad \text{for all } Z \in \Omega. \quad (4)$$

**Proof:** “ $\implies$ ”. By the calmness of  $\Gamma$  at 0 for each  $X \in \Gamma(0) = \Omega \cap \mathcal{R}$  and Lemma 2.1, it follows that for each  $X \in \Omega \cap \mathcal{R}$ , there exist constants  $\alpha(X) \geq 0$  and  $\epsilon(X) > 0$  such that

$$\text{dist}(Y, \Gamma(0)) \leq \alpha(X) \text{dist}(0, \Gamma^{-1}(Y)) \quad \forall Y \in \mathbb{B}(X, \epsilon(X)), \quad (5)$$

where  $\mathbb{B}(X, \epsilon(X))$  is a closed ball of radius  $\epsilon(X)$  centered at  $X$ . Notice that the compact set  $\Omega \cap \mathcal{R}$  is covered by the set  $\bigcup_{X \in \Omega \cap \mathcal{R}} (X + \frac{\epsilon(X)}{2} \mathbb{B}_{\mathbb{X}}^{\circ})$ , where  $\mathbb{B}_{\mathbb{X}}^{\circ}$  denotes the open unit ball around the origin in  $\mathbb{X}$ . By the Heine-Borel theorem, there exist a finite number of points  $X^1, X^2, \dots, X^m \in \Omega \cap \mathcal{R}$  such that  $\Omega \cap \mathcal{R} \subseteq \bigcup_{i=1}^m (X^i + \frac{\epsilon(X^i)}{2} \mathbb{B}_{\mathbb{X}}^{\circ})$ . Write

$$\bar{\epsilon} := \min\{\epsilon(X^1), \dots, \epsilon(X^m)\} \quad \text{and} \quad \bar{\alpha} := \max\{\alpha(X^1), \dots, \alpha(X^m)\}.$$

Let  $Z$  be an arbitrary point from  $\Omega$ . We proceed the arguments by two cases as below.

**Case 1:**  $\text{dist}(Z, \Omega \cap \mathcal{R}) \leq \bar{\epsilon}/2$ . Since the set  $\Omega \cap \mathcal{R}$  is closed, there must exist  $\bar{Z} \in \Omega \cap \mathcal{R}$  such that  $\|Z - \bar{Z}\|_F \leq \bar{\epsilon}/2$ . Since  $\bar{Z} \in \Omega \cap \mathcal{R}$ , there exists a  $k \in \{1, 2, \dots, m\}$  such that  $\|\bar{Z} - X^k\|_F < \epsilon(X^k)/2$ . Consequently,  $\|Z - X^k\|_F \leq \|Z - \bar{Z}\|_F + \|\bar{Z} - X^k\|_F \leq \epsilon(X^k)$ . Together with (5),  $\text{dist}(Z, \Gamma(0)) \leq \bar{\alpha} \text{dist}(0, \Gamma^{-1}(Z))$ . This shows that (4) holds with  $c = \bar{\alpha}$ .

**Case 2:**  $\text{dist}(Z, \Omega \cap \mathcal{R}) > \bar{\epsilon}/2$ . Now there must exist an  $\eta > 0$  such that  $\sum_{i=\kappa+1}^n \sigma_i(Y) \geq \eta$  for all  $Y \in \Omega$  with  $\text{dist}(Y, \Omega \cap \mathcal{R}) > \bar{\epsilon}/2$ . If not, one may select a sequence  $\{Z^k\} \subseteq \Omega$  with  $\text{dist}(Z^k, \Omega \cap \mathcal{R}) > \bar{\epsilon}/2$  such that  $\sum_{i=\kappa+1}^n \sigma_i(Z^k) \leq \eta^k$  for all  $k$ , where  $\{\eta^k\}$  is a sequence of positive numbers with  $\lim_{k \rightarrow +\infty} \eta^k = 0$ . Since  $\Omega$  is compact, we without loss of generality assume that  $\{Z^k\}$  converges to  $Z^* \in \Omega$ . Then, from the locally Lipschitz continuity of  $\sigma_i(\cdot)$ , it follows that  $\sum_{i=\kappa+1}^n \sigma_i(Z^*) \leq 0$ , and then  $Z^* \in \Omega \cap \mathcal{R}$ . On the other hand, from  $\text{dist}(Z^k, \Omega \cap \mathcal{R}) > \bar{\epsilon}/2$  for all  $k$ , we have  $\text{dist}(Z^*, \Omega \cap \mathcal{R}) > \bar{\epsilon}/2$ . Thus, we obtain a contradiction, and the above statement holds. Since  $\Omega$  is bounded, it follows that  $\text{dist}(\cdot, \Omega \cap \mathcal{R})$  is bounded above on  $\Omega$ , say, by some  $M > 0$ . Thus, for all  $Z \in \Omega$  with  $\text{dist}(Z, \Omega \cap \mathcal{R}) > \bar{\epsilon}/2$ , one has that  $\text{dist}(Z, \Omega \cap \mathcal{R}) \leq M \leq (M/\eta) \sum_{i=\kappa+1}^n \sigma_i(Z)$ . By taking  $c = M/\eta$ , the desired inequality (4) then follows.

“ $\Leftarrow$ ”. Let  $X$  be an arbitrary point from  $\Gamma(0)$  and  $\epsilon \in (0, 1)$  be an arbitrary constant. By Lemma 2.1, we only need to argue that there must exist a constant  $c' > 0$  such that

$$\text{dist}(Z, \Gamma(0)) \leq c' \text{dist}(0, \Gamma^{-1}(Z)) \quad \forall Z \in \mathbb{B}(X, \epsilon). \quad (6)$$

Indeed, for any  $Z \in \mathbb{B}(X, \epsilon)$ , if  $Z \notin \Omega$ , then  $\Gamma^{-1}(Z) = \emptyset$  by noting that  $\text{dom } \Gamma^{-1} \subseteq \Omega$ , and inequality (6) holds for any  $c' > 0$ ; if  $Z \in \Omega$ , then by taking  $c' = c$ , inequality (6) follows directly from (4). Until now, the proof is completed.  $\square$

Next, we show that the sufficient and necessary condition in Theorem 2.1 is especially satisfied by three classes of common rank constrained optimization problems in which  $\Omega$  is a ball, a density matrix set or a correlation matrix set, and provide the Lipschitzian type error bounds to estimate the distance from any  $X \in \Omega$  to  $\Omega \cap \mathcal{R}$  for them.

**(1) Rank constrained optimization problems over a ball.** The feasible set of this class of rank constrained optimization problems takes the following form

$$\mathcal{F} := \left\{ X \in \mathbb{R}^{n_1 \times n_2} \mid \text{rank}(X) \leq \kappa, \|X\| \leq \gamma \right\}, \quad (7)$$

where  $\|\cdot\|$  is a matrix norm and  $\gamma > 0$  is a given constant. The following proposition provides a Lipschitzian type bound for the error  $\text{dist}(X, \mathcal{F})$  with  $\|X\| \leq \gamma$ .

**Proposition 2.1** *Let  $\Omega = \{X \in \mathbb{R}^{n_1 \times n_2} \mid \|X\| \leq \gamma\}$ . Then, for any  $X \in \Omega$ , by assuming that  $X$  has the SVD of the form  $\sum_{i=1}^n \sigma_i(X) u_i v_i^\top$ , it holds that*

$$\text{dist}(X, \mathcal{F}) \leq \sqrt{1 + c_u^2/c_l^2} \|X - \Pi_{\mathcal{R}}(X)\|_F \quad \text{with} \quad \Pi_{\mathcal{R}}(X) := \sum_{i=1}^{\kappa} \sigma_i(X) u_i v_i^\top, \quad (8)$$

where  $c_l$  and  $c_u$  are positive constants such that  $c_l \|\cdot\|_F \leq \|\cdot\| \leq c_u \|\cdot\|_F$ . In particular,

$$\Gamma(t) \subseteq \Gamma(0) + \sqrt{1 + c_u^2/c_l^2} |t| \mathbb{B}_{\mathbb{R}^{n_1 \times n_2}} \quad \forall t \in \mathbb{R}. \quad (9)$$

When  $\|\cdot\|$  is unitarily invariant, the above constant  $\sqrt{1 + c_u^2/c_l^2}$  can be replaced by 1.

**Proof:** Let  $X$  be an arbitrary point in  $\Omega$  with the SVD given by  $\sum_{i=1}^n \sigma_i(X) u_i v_i^\top$ . Define

$$\widehat{X}_{\mathcal{F}} := \frac{\gamma}{\max(\gamma, \|\Pi_{\mathcal{R}}(X)\|)} \Pi_{\mathcal{R}}(X). \quad (10)$$

It is easy to verify that  $\widehat{X}_{\mathcal{F}} \in \mathcal{F} = \Gamma(0)$ . Thus, to establish (8), it suffices to argue that

$$\|X - \widehat{X}_{\mathcal{F}}\|_F \leq \sqrt{1 + c_u^2/c_l^2} \|X - \Pi_{\mathcal{R}}(X)\|_F. \quad (11)$$

When  $\|\Pi_{\mathcal{R}}(X)\| \leq \gamma$ , inequality (11) holds since  $\|X - \widehat{X}_{\mathcal{F}}\|_F = \|X - \Pi_{\mathcal{R}}(X)\|_F$ . We next consider the case where  $\|\Pi_{\mathcal{R}}(X)\| > \gamma$ . Now we have that

$$\begin{aligned} \|X - \widehat{X}_{\mathcal{F}}\|_F^2 &= \left\| X - \Pi_{\mathcal{R}}(X) + \Pi_{\mathcal{R}}(X) \left(1 - \frac{\gamma}{\|\Pi_{\mathcal{R}}(X)\|}\right) \right\|_F^2 \\ &= \|X - \Pi_{\mathcal{R}}(X)\|_F^2 + \|\Pi_{\mathcal{R}}(X)\|_F^2 \left(\frac{\|\Pi_{\mathcal{R}}(X)\| - \gamma}{\|\Pi_{\mathcal{R}}(X)\|}\right)^2 \\ &\leq \|X - \Pi_{\mathcal{R}}(X)\|_F^2 + \left(\frac{c_u \|X - \Pi_{\mathcal{R}}(X)\|_F \|\Pi_{\mathcal{R}}(X)\|_F}{\|\Pi_{\mathcal{R}}(X)\|}\right)^2 \\ &\leq \|X - \Pi_{\mathcal{R}}(X)\|_F^2 + \left(\frac{c_u \|X - \Pi_{\mathcal{R}}(X)\|_F}{c_l}\right)^2 \end{aligned}$$

where the second equality is by  $\langle X - \Pi_{\mathcal{R}}(X), \Pi_{\mathcal{R}}(X) \rangle = 0$ , the first inequality is since

$$\gamma < \|\Pi_{\mathcal{R}}(X)\| \leq \|X\| + \|X - \Pi_{\mathcal{R}}(X)\| \leq \gamma + c_u \|X - \Pi_{\mathcal{R}}(X)\|_F,$$

and the second one is due to  $c_l \|\Pi_{\mathcal{R}}(X)\|_F \leq \|\Pi_{\mathcal{R}}(X)\|$ . The last inequality implies (11). We next prove that (9) holds. Indeed, if  $t \in (-\infty, 0)$ , then the inclusion (9) immediately holds since  $\Gamma(t) = \emptyset$ . Now let  $t$  be an arbitrary point from  $[0, +\infty)$ . Let  $Z$  be an arbitrary point from  $\Gamma(t)$  and define  $\widehat{Z}_{\mathcal{F}}$  as in (10). Then, by noting that  $\Gamma(t) \subseteq \Omega$ , we have

$$\begin{aligned} \text{dist}(Z, \Gamma(0)) &\leq \|Z - \widehat{Z}_{\mathcal{F}}\|_F \leq \sqrt{1 + c_u^2/c_l^2} \|Z - \Pi_{\mathcal{R}}(Z)\|_F \\ &\leq \sqrt{1 + c_u^2/c_l^2} \|Z - \Pi_{\mathcal{R}}(Z)\|_* = \sqrt{1 + c_u^2/c_l^2} t, \end{aligned} \quad (12)$$

where the equality is due to  $\|Z - \Pi_{\mathcal{R}}(Z)\|_* = \sum_{i=\kappa+1}^n \sigma_i(Z)$  and  $Z \in \Gamma(t)$ . Equation (12) means that  $Z \in \Gamma(0) + \sqrt{1 + c_u^2/c_l^2} t \mathbb{B}_{\mathbb{R}^{n_1 \times n_2}}$ . Thus, (9) follows by the arbitrariness of  $t$  in  $\mathbb{R}$ .

When  $\|\cdot\|$  is unitarily invariant, it is easy to check that  $\Pi_{\mathcal{R}}(X) \in \Gamma(0)$  for any  $X \in \Omega$  (see [9, Corollary 3.5.9]). Then, by letting  $\widehat{X}_{\mathcal{F}} = \Pi_{\mathcal{R}}(X)$  and using the same arguments as above, we have the second part of conclusions. The proof is completed.  $\square$

The result of Proposition 2.1 is not trivial when  $\|\cdot\|$  is not unitarily invariant. Taking  $\|\cdot\| = \|\cdot\|_{\infty}$ , the infinity norm of matrices, since  $\frac{1}{\sqrt{n_1 n_2}} \|\cdot\|_F \leq \|\cdot\|_{\infty} \leq \|\cdot\|_F$ , we have

$$\text{dist}(X, \mathcal{F}) \leq \sqrt{1 + n_1 n_2} \sum_{i=\kappa+1}^n \sigma_i(X) \quad \text{for } \|X\|_{\infty} \leq \gamma.$$

**(2) Rank constrained density matrix optimization problems.** The feasible set of this class of rank constrained optimization problems takes the following form

$$\mathcal{F} := \left\{ X \in \mathbb{H}_+^n \mid \text{rank}(X) \leq \kappa, \text{tr}(X) = 1 \right\}, \quad (13)$$

we provide a Lipschitzian bound for the error  $\text{dist}(X, \mathcal{F})$  with  $X \in \mathbb{H}_+^n$  and  $\text{tr}(X) = 1$ .

**Proposition 2.2** *Let  $\Omega = \{X \in \mathbb{H}_+^n \mid \text{tr}(X) = 1\}$ . Then, for any  $X \in \Omega$ , by assuming that  $X$  has the eigenvalue decomposition of the form  $\sum_{i=1}^n \lambda_i(X) u_i u_i^\top$ , it holds that*

$$\text{dist}(X, \mathcal{F}) \leq \sqrt{\|X - \Pi_{\mathcal{R}}(X)\|_F^2 + \|X - \Pi_{\mathcal{R}}(X)\|_*^2} \quad \text{with } \Pi_{\mathcal{R}}(X) := \sum_{i=1}^{\kappa} \lambda_i(X) u_i u_i^\top.$$

In particular, the inclusion  $\Gamma(t) \subseteq \Gamma(0) + \sqrt{2}|t| \mathbb{B}_{\mathbb{H}^n}$  holds for any  $t \in \mathbb{R}$ .

**Proof:** Let  $X$  be an arbitrary point from  $\Omega$  with  $X = \sum_{i=1}^n \lambda_i(X) u_i u_i^\top$ . Define

$$\widehat{X}_{\mathcal{F}} := \frac{\Pi_{\mathcal{R}}(X)}{\text{tr}(\Pi_{\mathcal{R}}(X))}. \quad (14)$$

Notice that  $\text{tr}(\Pi_{\mathcal{R}}(X)) > 0$  since  $\lambda_1(X) > 0$ . Hence,  $\widehat{X}_{\mathcal{F}}$  is well defined and  $\widehat{X}_{\mathcal{F}} \in \mathcal{F}$ . Thus, by the definition of  $\widehat{X}_{\mathcal{F}}$ , we have

$$\begin{aligned} (\text{dist}(X, \mathcal{F}))^2 &\leq \|X - \widehat{X}_{\mathcal{F}}\|_F^2 = \left\| X - \Pi_{\mathcal{R}}(X) + \Pi_{\mathcal{R}}(X) \left(1 - \frac{1}{\text{tr}(\Pi_{\mathcal{R}}(X))}\right) \right\|_F^2 \\ &= \|X - \Pi_{\mathcal{R}}(X)\|_F^2 + \|\Pi_{\mathcal{R}}(X)\|_F^2 \left(1 - \frac{1}{\text{tr}(\Pi_{\mathcal{R}}(X))}\right)^2 \\ &= \|X - \Pi_{\mathcal{R}}(X)\|_F^2 + \left( \frac{\|\Pi_{\mathcal{R}}(X)\|_F (1 - \text{tr}(\Pi_{\mathcal{R}}(X)))}{\text{tr}(\Pi_{\mathcal{R}}(X))} \right)^2 \\ &\leq \|X - \Pi_{\mathcal{R}}(X)\|_F^2 + \|X - \Pi_{\mathcal{R}}(X)\|_*^2 \end{aligned}$$

where the inequality is due to  $\text{tr}(\Pi_{\mathcal{R}}(X)) = \|\Pi_{\mathcal{R}}(X)\|_* \geq \|\Pi_{\mathcal{R}}(X)\|_F$  and  $1 - \text{tr}(\Pi_{\mathcal{R}}(X)) = \|X\|_* - \|\Pi_{\mathcal{R}}(X)\|_* = \|X - \Pi_{\mathcal{R}}(X)\|_*$ . Thus, we complete the proof of the inequality.

Now let  $Z$  be an arbitrary point from  $\Gamma(t)$ . Noting that  $Z \in \Gamma(t) \subseteq \Omega$ , we have

$$\text{dist}(Z, \Gamma(0)) \leq \|Z - \widehat{Z}_{\mathcal{F}}\|_F \leq \sqrt{2} \|Z - \Pi_{\mathcal{R}}(Z)\|_* = \sqrt{2} \sum_{i=\kappa+1}^n \sigma_i(Z) = \sqrt{2}t$$

where  $\widehat{Z}_{\mathcal{F}}$  is defined as in (14), and the second equality is due to  $Z \in \Gamma(t)$ . This shows that  $Z \in \Gamma(0) + \sqrt{2}t \mathbb{B}_{\mathbb{H}^n}$ . From the arbitrariness of  $t$ , the desired inclusion follows.  $\square$

**(3) Rank constrained correlation matrix optimization problems.** The feasible set of this class of rank constrained optimization problems takes the following form

$$\mathcal{F} := \left\{ X \in \mathbb{H}_+^n \mid \text{rank}(X) \leq \kappa, \text{diag}(X) = e \right\} \quad (15)$$

where  $e \in \mathbb{R}^n$  is the vector of all ones. The following proposition provides a Lipschitzian bound for the error  $\text{dist}(X, \mathcal{F})$  with  $X \in \mathbb{H}_+^n$  and  $\text{diag}(X) = e$ .

**Proposition 2.3** Let  $\Omega = \{X \in \mathbb{H}_+^n \mid \text{diag}(X) = e\}$ . Then, for any  $X \in \Omega$ , by assuming that  $X$  has the eigenvalue decomposition of the form  $\sum_{i=1}^n \lambda_i(X) u_i u_i^\top$ , it holds that

$$\text{dist}(X, \mathcal{F}) \leq (1 + 2n) \|X - \Pi_{\mathcal{R}}(X)\|_* \quad \text{with} \quad \Pi_{\mathcal{R}}(X) := \sum_{i=1}^{\kappa} \lambda_i(X) u_i u_i^\top. \quad (16)$$

In particular, the inclusion  $\Gamma(t) \subseteq \Gamma(0) + (1 + 2n) |t| \mathbb{B}_{\mathbb{H}^n}$  holds for any  $t \in \mathbb{R}$ .

**Proof:** Let  $X$  be an arbitrary point from the set  $\Omega$  with  $X = \sum_{i=1}^n \lambda_i(X) u_i u_i^\top$ . Define

$$\widehat{X}_{\mathcal{F}} := \begin{cases} D(X) \Pi_{\mathcal{R}}(X) D(X) & \text{if } \text{diag}(\Pi_{\mathcal{R}}(X)) > 0 \\ ee^\top & \text{otherwise} \end{cases} \quad (17)$$

with  $D(X) = \text{Diag}\left(\frac{1}{\sqrt{\text{diag}(\Pi_{\mathcal{R}}(X))}}\right)$ . From the expression of  $\Pi_{\mathcal{R}}(X)$ , it follows that

$$[\text{diag}(\Pi_{\mathcal{R}}(X))]_j = 1 - \sum_{i=\kappa+1}^n \lambda_i(X) |u_{ij}|^2 \geq 0, \quad j = 1, 2, \dots, n \quad (18)$$

where  $u_{ij}$  means the  $j$ th entry of  $u_i$ . It is easy to check that  $\widehat{X}_{\mathcal{F}} \in \mathcal{F}$ , which implies that  $\text{dist}(X, \mathcal{F}) \leq \|X - \widehat{X}_{\mathcal{F}}\|_F$ . Thus, to prove the desired result, we only need to argue that

$$\|X - \widehat{X}_{\mathcal{F}}\|_F \leq (1 + 2n) \|X - \Pi_{\mathcal{R}}(X)\|_*. \quad (19)$$

If there is an index  $j$  such that  $[\text{diag}(\Pi_{\mathcal{R}}(X))]_j = 0$ , by the definition of  $\widehat{X}_{\mathcal{F}}$  we have

$$\|X - \widehat{X}_{\mathcal{F}}\|_F = \|X - ee^\top\|_F \leq \|X\|_F + \|ee^\top\|_F \leq n + n \leq 2n \sum_{i=\kappa+1}^n \lambda_i(X),$$

where the last inequality is since  $\sum_{i=\kappa+1}^n \lambda_i(X) \geq \sum_{i=\kappa+1}^n \lambda_i(X) |u_{ij}|^2 = 1$ . Thus, (19) follows. We next consider the case where  $\text{diag}(\Pi_{\mathcal{R}}(X)) > 0$ . For convenience, we denote  $D(X)$  by  $D$  and write its  $i$ th diagonal entry as  $D_{ii}$ . From (18), it is clear that  $D_{jj} \geq 1$  for all  $j$ . Notice that  $(\widehat{X}_{\mathcal{F}})_{kl} = (D \Pi_{\mathcal{R}}(X) D)_{kl} = D_{kk} D_{ll} (\Pi_{\mathcal{R}}(X))_{kl}$ . Then,

$$\begin{aligned} \|\widehat{X}_{\mathcal{F}} - \Pi_{\mathcal{R}}(X)\|_F^2 &= \sum_{k=1}^n \sum_{l=1}^n ((\widehat{X}_{\mathcal{F}})_{kl} - (\Pi_{\mathcal{R}}(X))_{kl})^2 = \sum_{k=1}^n \sum_{l=1}^n (D_{kk} D_{ll} - 1)^2 [(\Pi_{\mathcal{R}}(X))_{kl}]^2 \\ &\leq \max_{1 \leq j \leq n} (D_{jj}^2 - 1)^2 \|\Pi_{\mathcal{R}}(X)\|_F^2 \leq n^2 \max_{1 \leq j \leq n} (D_{jj}^2 - 1)^2, \end{aligned} \quad (20)$$

where the last inequality is due to  $\|\Pi_{\mathcal{R}}(X)\|_F \leq \|X\|_F \leq n$ . By using (20), we have

$$\begin{aligned} \|X - \widehat{X}_{\mathcal{F}}\|_F &\leq \|X - \Pi_{\mathcal{R}}(X)\|_F + \|\Pi_{\mathcal{R}}(X) - \widehat{X}_{\mathcal{F}}\|_F \\ &\leq \|X - \Pi_{\mathcal{R}}(X)\|_F + n \max_{1 \leq j \leq n} (D_{jj}^2 - 1) \\ &= \|X - \Pi_{\mathcal{R}}(X)\|_F + n \max_{1 \leq j \leq n} \left( \frac{1}{[\text{diag}(\Pi_{\mathcal{R}}(X))]_j} - 1 \right) \\ &\leq \|X - \Pi_{\mathcal{R}}(X)\|_F + \frac{n \sum_{i=\kappa+1}^n \lambda_i(X)}{\min_{1 \leq j \leq n} [\text{diag}(\Pi_{\mathcal{R}}(X))]_j} \\ &\leq \sum_{i=\kappa+1}^n \lambda_i(X) \left( 1 + \frac{n}{\min_{1 \leq j \leq n} [\text{diag}(\Pi_{\mathcal{R}}(X))]_j} \right), \end{aligned} \quad (21)$$



where the third inequality is due to  $|u_{ij}| \leq 1$ . If  $\sum_{i=\kappa+1}^n \lambda_i(X) \leq 0.5$ , equation (18) implies that  $[\text{diag}(\Pi_{\mathcal{R}}(X))]_j \geq 0.5$  for all  $j$ . Then by using (21) we obtain

$$\|X - \widehat{X}_{\mathcal{F}}\|_F \leq (1 + 2n) \sum_{i=\kappa+1}^n \lambda_i(X).$$

Now, we assume that  $\sum_{i=\kappa+1}^n \lambda_i(X) > 0.5$ . Since  $(\widehat{X}_{\mathcal{F}})_{kl} = D_{kk} D_{ll} (\Pi_{\mathcal{R}}(X))_{kl}$  and  $D_{kk} \geq 1$  for all  $k$ , we have  $\|\widehat{X}_{\mathcal{F}} - \Pi_{\mathcal{R}}(X)\|_F \leq \|\widehat{X}_{\mathcal{F}}\|_F \leq n$ . Consequently,

$$\|X - \widehat{X}_{\mathcal{F}}\|_F \leq \|X - \Pi_{\mathcal{R}}(X)\|_F + \|\Pi_{\mathcal{R}}(X) - \widehat{X}_{\mathcal{F}}\|_F \leq \|X - \Pi_{\mathcal{R}}(X)\|_* + n,$$

which along with  $\sum_{i=\kappa+1}^n \lambda_i(X) > 0.5$  implies that  $\|X - \widehat{X}_{\mathcal{F}}\|_F \leq (1 + 2n) \sum_{i=\kappa+1}^n \lambda_i(X)$ . Thus, we show that inequality (19) holds. The first part of the conclusions follows. By inequality (19), using the same arguments as those for the second part of Proposition 2.2, we obtain the desired inclusion. The proof is completed.  $\square$

From Propositions 2.1-2.3, we see that for the above three class of rank constrained problems, there exists a constant  $\alpha$  such that the associated multifunction  $\Gamma$  satisfies

$$\Gamma(t) \subseteq \Gamma(0) + \alpha |t| \mathbb{B}_{\mathbb{X}} \quad \forall t \in \mathbb{R}.$$

By [24, Definition 1], this is actually the upper Lipschitzian at 0 of  $\Gamma$  with respect to the set  $\mathbb{R}$ , which clearly implies the calmness of  $\Gamma$  at 0 for each  $X \in \Gamma(0)$ .

Next, we establish a local error bound for estimating the distance  $\text{dist}(X, \mathcal{F}^*)$  with  $X \in \Omega$ , under the calmness of  $\Gamma$  at 0 for each  $X \in \Gamma(0)$  and a suitable assumption on  $f$ .

**Theorem 2.2** *Suppose that the multifunction  $\Gamma$  in (3) is calm at 0 for each  $X \in \Gamma(0)$ , and  $f$  is a smooth convex function such that*

$$\vartheta := \inf_{X \in \mathcal{F}, Y \in \mathcal{F}^*, X \neq Y} \frac{f(X) - f(Y) - \langle \nabla f(Y), X - Y \rangle}{\|X - Y\|_F^2} > 0. \quad (22)$$

Then, there exists a constant  $c > 0$  such that for any  $X \in \Omega$ ,

$$\text{dist}(X, \mathcal{F}^*) \leq c \sum_{i=\kappa+1}^n \sigma_i(X) + \frac{1}{\vartheta} \|\nabla f(\Pi_{\mathcal{F}}(X)) - \nabla f(X^*)\|_F, \quad (23)$$

where  $X^*$  is an arbitrary point from  $\mathcal{F}^*$ . When  $\mathcal{F}$  takes the form of (7), (13) and (15), respectively, the constant  $c$  can be specified as  $c = \sqrt{1 + c_u^2/c_l^2}$ ,  $\sqrt{2}$  and  $1 + 2n$ , respectively.

**Proof:** From the definition of  $\vartheta$  and the convexity of the function  $f$ , it follows that

$$\begin{aligned} \vartheta \|X^* - \Pi_{\mathcal{F}}(X)\|_F^2 &\leq f(\Pi_{\mathcal{F}}(X)) - f(X^*) - \langle \nabla f(X^*), \Pi_{\mathcal{F}}(X) - X^* \rangle \\ &\leq \langle \nabla f(\Pi_{\mathcal{F}}(X)) - \nabla f(X^*), \Pi_{\mathcal{F}}(X) - X^* \rangle \\ &\leq \|\nabla f(\Pi_{\mathcal{F}}(X)) - \nabla f(X^*)\|_F \|\Pi_{\mathcal{F}}(X) - X^*\|_F, \end{aligned}$$

which implies that  $\|X^* - \Pi_{\mathcal{F}}(X)\|_F \leq \frac{1}{\vartheta} \|\nabla f(\Pi_{\mathcal{F}}(X)) - \nabla f(X^*)\|_F$ . Thus, we have that

$$\begin{aligned} \text{dist}(X, \mathcal{F}^*) &\leq \|X - X^*\|_F \leq \|X - \Pi_{\mathcal{F}}(X)\|_F + \|X^* - \Pi_{\mathcal{F}}(X)\|_F \\ &\leq c \sum_{i=\kappa+1}^n \sigma_i(X) + \frac{1}{\vartheta} \|\nabla f(\Pi_{\mathcal{F}}(X)) - \nabla f(X^*)\|_F, \end{aligned} \quad (24)$$

where the last inequality is using Theorem 2.1. By Propositions 2.1-2.3, when  $\mathcal{F}$  is given by (7), (13) and (15),  $c$  can be specified as  $\sqrt{1+c_u^2/c_l^2}$ ,  $\sqrt{2}$  and  $1+2n$ , respectively.  $\square$

**Remark 2.1 (a)** *The assumption  $\vartheta > 0$  in Theorem 2.2 is a kind of restricted strong convexity condition, which extends the sparse reconstruction condition used for the analysis of sparse constrained optimization problems (see, e.g., [1, 29]). One can check that*

$$0 < \inf_{1 \leq \text{rank}([X \ Y]) \leq 2\kappa, X \neq Y} \frac{f(X) - f(Y) - \langle \nabla f(Y), X - Y \rangle}{\|X - Y\|_F^2} \quad (25)$$

implies  $\vartheta > 0$ , where (25) is an extension of sparse reconstruction condition in [1, 29].

(b) When  $f$  is twice continuously differentiable, it is not difficult to check that

$$\frac{1}{\vartheta} \|\nabla f(\Pi_{\mathcal{F}}(X)) - \nabla f(X^*)\|_F \leq \frac{\vartheta_{\max}}{\vartheta_{\min}} \|\Pi_{\mathcal{F}}(X) - X^*\|_F$$

where  $\vartheta_{\min} = \min_{X \in \mathcal{F}} \lambda_{\min}(\nabla^2 f(X))$  and  $\vartheta_{\max} = \max_{X \in \mathcal{F}} \lambda_{\max}(\nabla^2 f(X))$ . This shows that the bound in (23) is related to the condition number of the Hessian matrix  $\nabla^2 f(X)$  restricted over the set  $\mathcal{F}$ , which is clearly smaller than the condition number of  $\nabla^2 f(X)$  restricted over the set  $\mathcal{R}$ . Together with Theorem 2.1, we actually have that

$$\frac{1}{\vartheta} \|\nabla f(\Pi_{\mathcal{F}}(X)) - \nabla f(X^*)\|_F \leq \frac{\vartheta_{\max}}{\vartheta_{\min}} \left( c \sum_{i=\kappa+1}^n \sigma_i(X) + \|X - X^*\|_F \right) \quad \forall X \in \Omega.$$

## 2.2 Global error bounds

Generally, for a nonconvex feasible set, it is almost impossible to get a global error bound for estimating the distance of any point to the feasible set (see [21]). However, under the calmness of  $\Gamma$  at 0 for each  $X \in \Gamma(0)$ , we can establish a global error bound for estimating the distance from any  $X \in \mathbb{X}$  to the feasible set  $\mathcal{F}$  and the solution set  $\mathcal{F}^*$ .

**Theorem 2.3** *Suppose that the multifunction  $\Gamma$  in (3) is calm at 0 for each  $X \in \Gamma(0)$ . Then, there exists a constant  $c > 0$  such that for any  $X \in \mathbb{X}$ ,*

$$\text{dist}(X, \mathcal{F}) \leq (1 + c\sqrt{n}) \text{dist}(X, \Omega) + c \sum_{i=\kappa+1}^n \sigma_i(X). \quad (26)$$

When  $\mathcal{F}$  takes the form of (7), (13) and (15), respectively, the constant  $c$  can be specified as  $\sqrt{1+c_u^2/c_l^2}$ ,  $\sqrt{2}$  and  $1+2n$ , respectively. In particular, for the set  $\mathcal{F}$  in (7), when  $\|\cdot\|$  is unitarily invariant, one may take  $c = 1$ .

**Proof:** Let  $X$  be an arbitrary point from  $\mathbb{X}$ . Then, the following inequalities hold

$$\begin{aligned}
\text{dist}(X, \mathcal{F}) &\leq \|X - \Pi_{\mathcal{F}}(\Pi_{\Omega}(X))\|_F \\
&\leq \|X - \Pi_{\Omega}(X)\|_F + \|\Pi_{\Omega}(X) - \Pi_{\mathcal{F}}(\Pi_{\Omega}(X))\|_F \\
&\leq \|X - \Pi_{\Omega}(X)\|_F + c \|\Pi_{\Omega}(X) - \Pi_{\mathcal{R}}(\Pi_{\Omega}(X))\|_* \\
&= \|X - \Pi_{\Omega}(X)\|_F + c \min_{\text{rank}(Y) \leq \kappa} \|\Pi_{\Omega}(X) - Y\|_* \\
&\leq \|X - \Pi_{\Omega}(X)\|_F + c(\|\Pi_{\Omega}(X) - X\|_* + \|X - \Pi_{\mathcal{R}}(X)\|_*) \\
&\leq \|X - \Pi_{\Omega}(X)\|_F + c(\sqrt{n}\|\Pi_{\Omega}(X) - X\|_F + \|X - \Pi_{\mathcal{R}}(X)\|_*) \\
&= (1 + c\sqrt{n})\text{dist}(X, \Omega) + c\|X - \Pi_{\mathcal{R}}(X)\|_*,
\end{aligned}$$

where the third inequality is using Theorem 2.1 and the fourth one is due to the inequality  $\min_{\text{rank}(Y) \leq \kappa} \|\Pi_{\Omega}(X) - Y\|_* \leq \|\Pi_{\Omega}(X) - \Pi_{\mathcal{R}}(X)\|_*$ . The proof is completed.  $\square$

**Remark 2.2** For the term  $\text{dist}(X, \Omega)$  in (26), when the set  $\mathcal{F}$  is given by (7), we have

$$\text{dist}(X, \Omega) \leq \frac{1}{c_l} \max(\|X\| - \gamma, 0) \quad \text{for any } X \in \mathbb{R}^{n_1 \times n_2};$$

when  $\mathcal{F}$  takes the form of (13), using [31, Theorem 2.1] with  $\mathcal{L} = \{Y \in \mathbb{H}^n \mid \text{tr}(Y) = 0\}$ ,  $B = \frac{1}{n}I$  and  $\mathcal{K} = \mathbb{H}_+^n$  shows that there exists a constant  $\nu \geq 1$  such that

$$\begin{aligned}
\text{dist}(X, \Omega) &\leq \nu(\text{dist}(X, B + \mathcal{L}) + \|X - \Pi_{\mathbb{H}_+^n}(X)\|_F) \\
&= \frac{\nu}{\sqrt{n}} |1 - \text{tr}(X)| + \nu \|X - \Pi_{\mathbb{H}_+^n}(X)\|_F \quad \text{for any } X \in \mathbb{H}^n;
\end{aligned}$$

and when  $\mathcal{F}$  takes the form of (15), using [31, Theorem 2.1] with  $\mathcal{L} = \{Y \in \mathbb{H}^n \mid \text{diag}(Y) = 0\}$ ,  $B = I$  and  $\mathcal{K} = \mathbb{H}_+^n$  yields that there exists a constant  $\nu \geq 1$  such that

$$\begin{aligned}
\text{dist}(X, \Omega) &\leq \nu(\text{dist}(X, B + \mathcal{L}) + \|X - \Pi_{\mathbb{H}_+^n}(X)\|_F) \\
&= \nu(\|e - \text{diag}(X)\| + \|X - \Pi_{\mathbb{H}_+^n}(X)\|_F) \quad \text{for any } X \in \mathbb{H}^n.
\end{aligned}$$

Now following the same arguments as for inequality (24) and using Theorem 2.3 to bound  $\|X - \Pi_{\mathcal{F}}(X)\|_F$ , we can establish a global error bound for estimating the distance from any point  $X \in \mathbb{X}$  to the solution set  $\mathcal{F}^*$  under the assumptions of Theorem 2.2.

**Theorem 2.4** Suppose that the multifunction  $\Gamma$  in (3) is calm at 0 for each  $X \in \Gamma(0)$ , and  $f$  is a smooth convex function such that  $\vartheta > 0$ , where  $\vartheta$  is defined by (22). Then, there exists a constant  $c > 0$  such that for any  $X \in \mathbb{X}$ ,

$$\text{dist}(X, \mathcal{F}^*) \leq (1 + c\sqrt{n})\text{dist}(X, \Omega) + c \sum_{i=\kappa+1}^n \sigma_i(X) + \frac{1}{\vartheta} \|\nabla f(\Pi_{\mathcal{F}}(X)) - \nabla f(X^*)\|_F,$$

where  $X^*$  is an arbitrary point from  $\mathcal{F}^*$ . When  $\mathcal{F}$  takes the form of (7), (13) and (15), respectively, the constant  $c$  can be specified as  $\sqrt{1 + c_u^2/c_l^2}$ ,  $\sqrt{2}$  and  $1 + 2n$ , respectively.

### 3 Applications of local error bounds

This section illustrates the applications of the local Lipschitzian type error bounds in establishing the exact penalty for the rank constrained optimization problem (1), and deriving an error bound for a multi-stage convex relaxation approach to problem (1) in which  $\Omega$  is a ball set, a density matrix set and a correlation matrix set, respectively.

#### 3.1 Exact penalty for problem (1)

With the help of the local error bounds in Theorem 2.1, we show that (2) is an exact penalty problem of (1) in the sense that the global optimal solution set of (2) coincides with that of problem (1) when the penalty parameter is over a threshold. This result is stated in the following theorem. Its proof technique is similar to that of [13, Theorem 2.1.2]. Notice that the latter focuses on the subanalytic compact set and employs an error bound derived by the Lojasiewicz' inequality for subanalytic function, but our result is stated for a general compact set  $\Omega \cap \mathcal{R}$ . For completeness, we here include the proof.

**Theorem 3.1** *Suppose that the multifunction  $\Gamma$  in (3) is calm at 0 for each  $X \in \Gamma(0)$ . Let  $L > 0$  be the Lipschitz constant of the function  $f$  over  $\Omega$ . Then, when  $\rho > cL$  with  $c$  same as the one in Theorem 2.1, the following statements hold.*

- (a) *The global optimal solution set of (1) coincides with that of (2);*
- (b)  *$\bar{X} \in \mathcal{F}$  is a local optimal solution of (1) if and only if  $\bar{X}$  is locally optimal to (2).*

**Proof:** (a) Since  $\Omega$  is compact and  $f$  is continuous, the optimal solution set of (2) is nonempty. Let  $X^*$  be an arbitrary optimal solution of (1). Then, for any  $X \in \Omega$ ,

$$\begin{aligned} f(X) + \rho \sum_{i=\kappa+1}^n \sigma_i(X) &\geq f(X) + cL \sum_{i=\kappa+1}^n \sigma_i(X) \geq f(X) + L\|X - \Pi_{\mathcal{F}}(X)\|_F \\ &\geq f(\Pi_{\mathcal{F}}(X)) \geq f(X^*) = f(X^*) + \rho \sum_{i=\kappa+1}^n \sigma_i(X^*) \end{aligned} \quad (27)$$

where the second inequality is due to Theorem 2.1. The last inequality shows that  $X^*$  is an optimal solution of (2). Now, let  $\bar{X}$  be an arbitrary optimal solution of (2). From the feasibility of  $X^*$  to (2) and Theorem 2.1, it follows that

$$\begin{aligned} f(X^*) &\geq f(\bar{X}) + \rho \sum_{i=\kappa+1}^n \sigma_i(\bar{X}) \geq f(\bar{X}) + cL \sum_{i=\kappa+1}^n \sigma_i(\bar{X}) \\ &\geq f(\bar{X}) + L\|\bar{X} - \Pi_{\mathcal{F}}(\bar{X})\|_F \geq f(\Pi_{\mathcal{F}}(\bar{X})) \geq f(X^*). \end{aligned} \quad (28)$$

This means that  $f(X^*) = f(\bar{X}) + \rho \sum_{i=\kappa+1}^n \sigma_i(\bar{X})$  and  $\rho \sum_{i=\kappa+1}^n \sigma_i(\bar{X}) = cL \sum_{i=\kappa+1}^n \sigma_i(\bar{X})$ . Since  $\rho > cL$ , we must have  $\sum_{i=\kappa+1}^n \sigma_i(\bar{X}) = 0$ , and then  $\bar{X} \in \mathcal{F}$ . Along with  $f(X^*) = f(\bar{X})$ , it follows that  $\bar{X}$  is an optimal solution of problem (1). Thus, part (a) follows.

(b) Since  $\bar{X} \in \mathcal{F}$ , it is easy to show that if  $\bar{X}$  is a local optimal solution of (2), then  $\bar{X}$  is locally optimal of (1). Now, we assume that  $\bar{X}$  is a local optimal solution of (1). Then there exists a neighborhood of  $\bar{X}$ , denoted by  $\mathcal{N}(\bar{X}, \bar{\varepsilon})$  with  $\bar{\varepsilon} > 0$ , such that

$$f(\bar{X}) \leq f(X) \text{ for all } X \in \mathcal{N}(\bar{X}, \bar{\varepsilon}) \cap \mathcal{F}. \quad (29)$$

Notice that  $\sum_{i=\kappa+1}^n \sigma_i(X)$  is continuous and  $\sum_{i=\kappa+1}^n \sigma_i(\bar{X}) = 0$ . Hence, there must exist  $\tilde{\varepsilon} > 0$  such that  $\sum_{i=\kappa+1}^n \sigma_i(X) < \frac{\bar{\varepsilon}}{2c}$  for all  $X \in \mathcal{N}(\bar{X}, \tilde{\varepsilon})$ . In the following, we show that for any  $X \in \mathcal{N}(\bar{X}, \hat{\varepsilon}) \cap \Omega$  where  $0 < \hat{\varepsilon} \leq \min(\frac{\bar{\varepsilon}}{2}, \tilde{\varepsilon})$ , it holds that

$$f(\bar{X}) + \rho \sum_{i=\kappa+1}^n \sigma_i(\bar{X}) = f(\bar{X}) \leq f(X) + \rho \sum_{i=\kappa+1}^n \sigma_i(X).$$

Notice that for any  $X \in \mathcal{N}(\bar{X}, \hat{\varepsilon}) \cap \Omega$ , by Theorem 2.1 and  $\|X - \bar{X}\|_F \leq \hat{\varepsilon}$  we have

$$\|\Pi_{\mathcal{F}}(X) - \bar{X}\|_F \leq \|\Pi_{\mathcal{F}}(X) - X\|_F + \|X - \bar{X}\|_F \leq c \sum_{i=\kappa+1}^n \sigma_i(X) + \hat{\varepsilon} \leq \bar{\varepsilon}.$$

Thus, it holds  $\Pi_{\mathcal{F}}(X) \in \mathcal{N}(\bar{X}, \bar{\varepsilon}) \cap \mathcal{F}$ . This, together with inequality (29), yields that

$$f(\bar{X}) = f(\bar{X}) + \rho \sum_{i=\kappa+1}^n \sigma_i(\bar{X}) \leq f(\Pi_{\mathcal{F}}(X)) \leq f(X) + \rho \sum_{i=\kappa+1}^n \sigma_i(X) \quad \forall X \in \mathcal{N}(\bar{X}, \hat{\varepsilon}) \cap \Omega,$$

where the last inequality is due to (27). Then  $\bar{X}$  is a local optimal solution of (2).  $\square$

### 3.2 Error bound for a multi-stage convex relaxation method

Let  $\Omega$  be one of the sets in Propositions 2.1-2.3 and  $f$  be a convex function. By Theorem 3.1, when  $\rho > cL$  with  $L > 0$  being the Lipschitz constant of  $f$  over  $\Omega$ , problem (2), i.e.

$$\min_{X \in \Omega} \left\{ f(X) + \rho(\|X\|_* - \|X\|_{\kappa}) \right\} \quad (30)$$

has the same global optimal solution set as that of (1), where  $\|X\|_{\kappa}$  means the Ky Fan  $\kappa$ -norm, i.e., the sum of the  $\kappa$ -largest singular values of  $X$ . Motivated by this, at the current iterate  $X^{k-1}$  we replace the concave function  $-\|X\|_{\kappa}$  in (30) by a linear majorization function  $-\langle W^{k-1}, X \rangle$ , and transform the solution of (1) or its exact penalty problem (30) into the solution of a sequence of convex minimization problems. This leads to a multi-stage convex relaxation approach to problem (1). The idea of replacing  $-\|X\|_{\kappa}$  by the linear majorization function  $-\langle W^{k-1}, X \rangle$  first appears in the majorized penalty approach proposed in [7], where  $W^{k-1} \in \partial \|\cdot\|_{\kappa}(X^{k-1})$  is used. Here we consider the multi-stage convex relaxation approach where the linear majorization function  $-\langle W^{k-1}, X \rangle$  is given with  $W^{k-1} \in \partial \|\cdot\|_{\kappa}(\hat{X}_{\mathcal{F}}^{k-1})$ , where  $\hat{X}_{\mathcal{F}}^{k-1}$  is defined by (10), (14) and (17) respectively with  $X = X^{k-1}$ . Also, different from the majorized penalty approach [7], our multi-stage convex relaxation approach is using  $f$  itself instead of its majorization function.

**Multi-stage convex relaxation approach for problem (1)**

- (S.0) Let  $\rho_0 > 0$  be given. Choose a starting point  $X^0 \in \mathbb{X}$ . Set  $k := 0$ .
- (S.1) Compute  $\widehat{X}_{\mathcal{F}}^k$  by (10), (14) and (17) with  $X = X^k$  and the associated  $\Omega$ .
- (S.2) Seek  $W^k \in \partial \|\cdot\|_{\kappa}(\widehat{X}_{\mathcal{F}}^k)$  where  $\partial \|\cdot\|_{\kappa}$  is the subdifferential map of  $\|\cdot\|_{\kappa}$ .
- (S.3) Find  $X^{k+1} \in \arg \min_{X \in \Omega} \left\{ f(X) + \rho_k (\|X\|_* - \langle W^k, X \rangle) \right\}$ .
- (S.4) Set  $\rho_{k+1} := \tau_k \rho_k$  with  $\tau_k \geq 1$ . Let  $k \leftarrow k + 1$ , and go to Step (S.1).

Assume that  $\widehat{X}_{\mathcal{F}}^k$  for  $k \geq 0$  has the SVD of the form  $\widehat{X}_{\mathcal{F}}^k = \sum_{i=1}^n \sigma_i(\widehat{X}_{\mathcal{F}}^k) u_i^k (v_i^k)^\top$  and let  $U_1^k = [u_1^k \ u_2^k \ \dots \ u_{\kappa}^k]$  and  $V_1^k = [v_1^k \ v_2^k \ \dots \ v_{\kappa}^k]$ . Then,  $W^k = U_1^k (V_1^k)^\top \in \partial \|\cdot\|_{\kappa}(\widehat{X}_{\mathcal{F}}^k)$ . The above multi-stage convex relaxation approach generates the sequences  $\{\widehat{X}_{\mathcal{F}}^k\}_{k \geq 1} \subseteq \mathcal{F}$  and  $\{X^k\}_{k \geq 1} \subseteq \Omega$ . In the following, we bound the distance from  $\widehat{X}_{\mathcal{F}}^k$  (respectively,  $X^k$ ) to the solution set  $\mathcal{F}^*$  with a bound sequence nonincreasing as the number of stages.

**Theorem 3.2** *Suppose that  $f$  is a nonnegative smooth convex function with Lipschitz continuous gradient over  $\Omega$  such that  $\vartheta$  defined by (22) is positive. Let  $\{\widehat{X}_{\mathcal{F}}^k\}$  and  $\{X^k\}$  be the sequences generated by the multi-stage convex relaxation approach with  $\rho_0 > \max(cL, f(\widehat{X}_{\mathcal{F}}^0), \frac{c^2(\vartheta + \bar{L})^2}{4\bar{L}})$ , where  $L > 0$  and  $\bar{L} > 0$  are the Lipschitz constants of  $f$  and  $\nabla f$  over  $\Omega$ , respectively, and  $c$  is same as the one in Theorem 2.2. Then,*

$$\text{dist}(\widehat{X}_{\mathcal{F}}^k, \mathcal{F}^*) \leq \frac{M + \sqrt{M^2 + 4\vartheta(f(\widehat{X}_{\mathcal{F}}^k) - f(X^*))}}{2\vartheta} \leq \dots \leq \frac{M + \sqrt{M^2 + 4\vartheta(f(\widehat{X}_{\mathcal{F}}^0) - f(X^*))}}{2\vartheta} \quad (31)$$

where  $M := \max_{Z \in \mathcal{F}^*} \|\nabla f(Z)\|$  and  $X^* \in \mathcal{F}^*$ , and with  $\rho_{-1} := cL$ , it holds that

$$\text{dist}(X^k, \mathcal{F}^*) \leq \Xi^k \leq \Xi^{k-1} \leq \dots \leq \Xi^1 \leq \Xi^0 \quad \text{for } k \geq 1, \quad (32)$$

where  $\Xi^l := \frac{2\sqrt{2\bar{L}}}{\vartheta} \sqrt{f(X^l) + \rho_{l-1} \sum_{i=\kappa+1}^n \sigma_i(X^l) + f(X^*)}$ .

**Proof:** We first argue that the sequence  $\{f(\widehat{X}_{\mathcal{F}}^k)\}_{k \geq 0}$  is nonincreasing. Indeed, from the proof of Propositions 2.1-2.3 we have  $\|X^k - \widehat{X}_{\mathcal{F}}^k\|_F \leq c \sum_{i=\kappa+1}^n \sigma_i(X^k)$ . This implies that

$$f(X^k) + \rho_{k-1} \sum_{i=\kappa+1}^n \sigma_i(X^k) \geq f(X^k) + cL \sum_{i=\kappa+1}^n \sigma_i(X^k) \geq f(\widehat{X}_{\mathcal{F}}^k), \quad k \geq 0. \quad (33)$$

Since  $\|X\|_{\kappa} = \sum_{i=1}^{\kappa} \sigma_i(X) = \sup_{\|W\| \leq 1, \text{rank}(W) \leq \kappa} \langle W, X \rangle$  for any  $X \in \mathbb{X}$ , it follows that

$$\sum_{i=\kappa+1}^n \sigma_i(X^k) = \|X^k\|_* - \sup_{\|W\| \leq 1, \text{rank}(W) \leq \kappa} \langle W, X^k \rangle \leq \|X^k\|_* - \langle W^{k-1}, X^k \rangle \quad \text{for } k \geq 1. \quad (34)$$

From equation (34) and the definitions of  $X^k$  and  $X_{\mathcal{F}}^{k-1} \in \Omega$  for  $k \geq 1$ , we have that

$$f(X^k) + \rho_{k-1} \sum_{i=\kappa+1}^n \sigma_i(X^k) \leq f(X^k) + \rho_{k-1} (\|X^k\|_* - \langle W^{k-1}, X^k \rangle) \leq f(\widehat{X}_{\mathcal{F}}^{k-1}), \quad (35)$$

where the last inequality is due to  $\langle W^{k-1}, \widehat{X}_{\mathcal{F}}^{k-1} \rangle - \|\widehat{X}_{\mathcal{F}}^{k-1}\|_{\kappa} = 0$ , implied by  $W^{k-1} \in \partial\|\cdot\|_{\kappa}(\widehat{X}_{\mathcal{F}}^{k-1})$ . From (33) and (35),  $\{f(\widehat{X}_{\mathcal{F}}^k)\}_{k \geq 0}$  is nonincreasing. By the definition of  $\vartheta$ ,

$$\vartheta \|\widehat{X}_{\mathcal{F}}^k - X^*\|_F^2 \leq f(\widehat{X}_{\mathcal{F}}^k) - f(X^*) - \langle \nabla f(X^*), \widehat{X}_{\mathcal{F}}^k - X^* \rangle \leq f(\widehat{X}_{\mathcal{F}}^k) - f(X^*) + M \|\widehat{X}_{\mathcal{F}}^k - X^*\|,$$

which implies that  $\|\widehat{X}_{\mathcal{F}}^k - X^*\|_F \leq \frac{1}{2\vartheta} \left( M + \sqrt{M^2 + 4\vartheta(f(\widehat{X}_{\mathcal{F}}^k) - f(X^*))} \right)$ . This, along with the nonincreasing of the sequence  $\{f(\widehat{X}_{\mathcal{F}}^k)\}_{k \geq 0}$ , yields the desired result in (31).

Notice that  $X^k \in \Omega$  for all  $k \geq 1$ . Therefore, from Theorem 2.2 it follows that

$$\begin{aligned} \text{dist}(X^k, \mathcal{F}^*) &\leq c \sum_{i=\kappa+1}^n \sigma_i(X^k) + \frac{1}{\vartheta} (\|\nabla f(\Pi_{\mathcal{F}}(X^k)) - \nabla f(X^k)\|_F + \|\nabla f(X^k) - \nabla f(X^*)\|_F) \\ &\leq c \sum_{i=\kappa+1}^n \sigma_i(X^k) + \frac{\bar{L}}{\vartheta} \|\Pi_{\mathcal{F}}(X^k) - X^k\|_F + \frac{1}{\vartheta} \|\nabla f(X^k) - \nabla f(X^*)\|_F \\ &\leq c \sum_{i=\kappa+1}^n \sigma_i(X^k) + \frac{\bar{L}}{\vartheta} \|\Pi_{\mathcal{F}}(X^k) - X^k\|_F + \frac{2\sqrt{\bar{L}}}{\vartheta} \sqrt{f(X^k) + f(X^*)} \\ &\leq c \left(1 + \frac{\bar{L}}{\vartheta}\right) \sum_{i=\kappa+1}^n \sigma_i(X^k) + \frac{2\sqrt{\bar{L}}}{\vartheta} \sqrt{f(X^k) + f(X^*)} \\ &\leq \frac{2\sqrt{\bar{L}}}{\vartheta} \left[ \frac{c(\vartheta + \bar{L})}{2\sqrt{\bar{L}}} \sqrt{\sum_{i=\kappa+1}^n \sigma_i(X^k)} + \sqrt{f(X^k) + f(X^*)} \right] \\ &\leq \frac{2\sqrt{2\bar{L}}}{\vartheta} \sqrt{f(X^k) + \rho_{k-1} \sum_{i=\kappa+1}^n \sigma_i(X^k) + f(X^*)} \end{aligned}$$

where the second inequality is using the Lipschitz continuity of  $\nabla f$  over  $\Omega$ , the third one is using [20, Equation (2.1.10)] implied by the assumption that  $f$  is a nonnegative smooth convex function with Lipschitz continuous gradient, the fifth one is using  $\sum_{i=\kappa+1}^n \sigma_i(X^k) < 1$  implied by  $\rho_{k-1} > f(\widehat{X}_{\mathcal{F}}^0) \geq f(\widehat{X}_{\mathcal{F}}^k)$  and (35), and the last one is due to  $\rho_{k-1} > \frac{c^2(\vartheta + \bar{L})^2}{4\bar{L}}$ . Thus, to establish inequality (32), it suffices to show that  $\Xi^k \leq \Xi^{k-1} \leq \dots \leq \Xi^0$  for  $k \geq 1$ . Indeed, by using equations (33) and (35), we have that

$$f(X^k) + \rho_{k-1} \sum_{i=\kappa+1}^n \sigma_i(X^k) \leq f(\widehat{X}_{\mathcal{F}}^{k-1}) \leq f(X^{k-1}) + \rho_{k-2} \sum_{i=\kappa+1}^n \sigma_i(X^{k-1}) \quad \text{for } k \geq 1.$$

This implies that the sequence  $\{f(X^k) + \rho_{k-1} \sum_{i=\kappa+1}^n \sigma_i(X^k)\}_{k \geq 0}$  is nonincreasing. By the definition of  $\Xi^k$ , it follows that  $\Xi^k \leq \Xi^{k-1} \leq \dots \leq \Xi^1 \leq \Xi^0$ . The proof is completed.  $\square$

From the proof of Theorem 3.2, we have  $\sum_{i=\kappa+1}^n \sigma_i(X^k) \leq f(\widehat{X}_{\mathcal{F}}^{k-1})/\rho_{k-1}$ . By noting that  $\sum_{i=\kappa+1}^n \sigma_i(X^k) = 0$  means  $X^k \in \mathcal{F}$ , this shows that  $X^k$  is an approximate feasible solution to (1), and the infeasibility violation becomes smaller as the number of stages increases, since the sequence  $\{f(\widehat{X}_{\mathcal{F}}^{k-1})\}$  is nonincreasing and  $\rho_k$  is nondecreasing.

## 4 Conclusions

In this paper we have provided a sufficient and necessary condition to establish the Lipschitzian type error bounds for estimating the distance from any  $X \in \Omega$  to the feasible set  $\mathcal{F}$  of the rank constrained optimization problem (1), and showed that this condition is specially satisfied by three classes of common  $\Omega$ . With the help of this result and the error bound for the convex feasibility system, we also derived the global error bound for estimating the distance from any  $X \in \mathbb{X}$  to  $\mathcal{F}$ . Under an additional suitable restricted strong convexity for the objective function  $f$ , the error bounds for estimating the distance from  $X \in \Omega$  (or  $X \in \mathbb{X}$ ) to the solution set  $\mathcal{F}^*$  are also derived. To illustrate the applications of these error bounds, we have showed that the penalty problem yielded by moving the rank constraint  $\text{rank}(X) \leq \kappa$  into the objective is exact, which affirmatively answers the open question proposed in [7] about whether the penalty problem (32) there is exact or not, and provided the error bound of the iterates generated by a multi-stage convex relaxation approach to (1) with three classes of special  $\Omega$ .

To the best of our knowledge, this paper is the first one to touch the error bound and exact penalty for the NP-hard low-rank optimization problems. Clearly, the error bound and exact penalty results also hold for the corresponding classes of zero norm constrained optimization problems, such as the sparse principal components analysis and the sparse portfolio selection problems. Our future research work will focus on the applications of these Lipschitzian error bounds to the convergence and iteration complexity of algorithms for low-rank constrained optimization problems, especially the error bound of the iterates yielded by the majorized penalty approach [7].

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