Two New Weak Constraint Qualifications for Mathematical Programs with Equilibrium Constraints and Applications

Alberto Ramos*

May, 2017 / Revisted: December, 2017

Abstract

We introduce two new weaker Constraint Qualifications (CQs) for Mathematical Programs with Equilibrium (or Complementarity) Constraints, MPEC for short. One of them is a tailored version of the Constant Rank of Subspace Component (CRSC) and the other is a relaxed version of the MPEC-No Nonzero Abnormal Multiplier Constraint Qualification (MPEC-NNAMCQ). Both incorporate the exact set of gradients of inequality constraints whose properties have to be preserved locally. MPEC-RNNAMCQ and MPEC-CRSC have nice properties: they have the local preservation property and imply the error bound property under mild assumptions. Thus, they can be used to extend some known results on perturbation analysis and sensitivity. Both conditions can also be used in the convergence analysis of several methods for solving MPECs. Relations to other MPEC-CQs will also be discussed.

1 Introduction

Mathematical program with equilibrium (complementary) constraints (MPEC) is a difficult class of optimization problems that appears naturally in several applications and play an important role in many fields. See [1, 2, 3, 4]. The main difficulty, from a theoretical and a practical point of view, comes from the complementarity constraints. Indeed, the feasible set has a special structure and most of the standard CQs fail at every feasible point. Classical CQs like the Linear Independence CQ (LICQ) and the Mangasarian Fromovitz CQ (MFCQ) are violated at any feasible point. Even weaker CQs cannot be expected to hold, as Abadie's CQ. Without CQs the Karush-Kuhn-Tucker (KKT) condition could not be a necessary optimality condition (even in the case where all constraints are linear functions) and convergence assumptions of several standard methods for solving optimization problems are not satisfied. For this reason, several stationary concepts for MPECs have been defined over the years. For instance, we have the strong stationarity (S-stationarity), the Clarke-stationarity (C-stationarity), the Mordukhovich stationarity (M-stationarity), see, e. g. [5, 6, 7, 8, 9]. To ensure that a local minimizer of MPEC is stationary in one of the above senses, we need CQs. Since many standard CQs do not work for MPECs, tailored MPEC-CQs have been introduced over the years, most of them analogues to the standard CQs for non-linear mathematical programs (NLPs). Since M-stationary has been shown to be an important and useful stationary concept when we deal with MPECs, c. f., [9, 10, 11, 12, 13, 14], we will pay a special attention to CQs for M-stationarity.

In this paper we introduce two new weaker CQs for M-stationarity that can be used to extend some results in the perturbation analysis and sensitivity for MPECs. One of them is an MPEC version of the *Constant Rank of the Subspace Component* (CRSC) which was introduced in [15, 16]. We called it MPEC-CRSC which is weaker than the recently introduced MPEC-Relaxed Constant Positive Linear Dependence (MPEC-RCPLD), see [17]. MPEC-CRSC implies the existence of the error bound property (see Definition 5.1) under the strict complementary assumption at the basis point and it also has the local preservation property (see Definition 3.2). We also introduce a relaxed version of MPEC-NNAMCQ called MPEC-*Relaxed No Nonzero Abnormal Multiplier* CQ (MPEC-RNNAMCQ for short). MPEC-RNNAMCQ has the local preservation property and

*Department of Mathematics, Federal University of Paraná, Curitiba, PR, Brazil. e-mail: albertoramos@ufpr.br.

implies the error bound property without any additional assumptions. Since MPEC-RNNAMCQ and MPEC-CRSC are stronger than the recently introduced MPEC-Cone Continuity Property (MPEC-CCP), [18], both conditions can be used in the convergence analysis of several methods for solving MPECs such as some relaxation methods [19, 20, 21] or complementary-penalty methods [22, 23] by substituting more stringent MPEC-CQs. Furthermore, since they imply the error bound property, they could be useful in the perturbation analysis of MPECs.

The paper is organized as follows: Section 2 contains some basic definitions and results. Section 3 gives new weaker CQs for M-stationarity and it is devoted to the study of the local preservation property. In Section 4, we discuss the relationships among some CQs for M-stationarity and in Section 5, we focus on the existence of the error bound property for the new weaker MPEC-CQs. Finally, we give some remarks and conclusions in Section 6.

2 Preliminaries and basic assumptions

We denote by $\mathbb{B}$ the closed unit ball in $\mathbb{R}^n$ and $\mathbb{B}(x,\eta) := x + \eta\mathbb{B}$ the closed ball with center x and radius $\eta > 0$. $\mathbb{R}_+$ is the set of all non-negative numbers, $\mathbb{R}_- := -\mathbb{R}_+$. We denote the euclidean inner product by $\langle \cdot, \cdot \rangle$ and by $\| \cdot \|$ the associated norm. We denote by $\| \cdot \|_1$ the l_1-norm and by dist_1 the distance with respect to this norm. Given a finite set $\mathcal{A}$, we denote by $|\mathcal{A}|$ the cardinality of $\mathcal{A}$. For every $a \in \mathbb{R}^s$, the support of a is given by $\mathrm{supp}(a) := \{i : a_i \neq 0\}$. Given a set-valued mapping $\Gamma : \mathbb{R}^s \rightrightarrows \mathbb{R}^d$, the *sequential outer limit* of $\Gamma(z)$ as $z \to z^*$ is denoted by $\limsup_{z \to z^*} \Gamma(z) := \{w^* \in \mathbb{R}^d : \exists \ (z^k, w^k) \to (z^*, w^*) \text{ with } w^k \in \Gamma(z^k)\}$. We say that Γ is *outer semi-continuous* (osc) at z^* if $\limsup_{z \to z^*} \Gamma(z) \subset \Gamma(z^*)$. Given $X \subset \mathbb{R}^n$ and $z^* \in X$, define the *tangent cone* to X at z^* by

$$T_X(z^*) := \limsup_{t \downarrow 0} \frac{X - z^*}{t} = \{d \in \mathbb{R}^n : \exists \ \ t_k \downarrow 0, d^k \to d \text{ with } z^* + t_k d^k \in X\}.$$

The *regular normal cone* to X at $z^* \in X$ is defined by

$$\widehat{N}_X(z^*) := \{w \in \mathbb{R}^n : \lim_{z \to z^*, z \in X} \frac{\langle w, z - z^* \rangle}{\|z - z^*\|} \leq 0\}.$$

The *limiting normal cone* to X at $x^* \in X$ is defined by

$$N_X(z^*) := \limsup_{z \to z^*, z \in X} \widehat{N}_X(z) = \{w \in \mathbb{R}^n : \exists \ \ (w^k, z^k) \to (w, z^*), w^k \in \widehat{N}_X(z^k)\}.$$

We consider MPECs of the form

$$\begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & g_j(x) \leq 0 \ \ \forall j \in \mathcal{P} := \{1, \dots, p\} \\ & h_i(x) = 0 \ \ \forall i \in \mathcal{E} := \{1, \dots, q\} \\ & 0 \leq H_i(x) \perp G_i(x) \geq 0 \ \ \forall i \in \mathcal{M} := \{1, \dots, m\}, \end{array} \tag{1}$$

where f, h_i, g_j, H_ℓ, $G_\ell : \mathbb{R}^n \to \mathbb{R}$ $\forall i \in \mathcal{E}, \forall j \in \mathcal{P}, \forall \ell \in \mathcal{M}$ are continuously differentiable functions. The notation $0 \leq u \perp v \geq 0$ for vectors u and v in $\mathbb{R}^n$ is a short-cut for $u \geq 0, v \geq 0$ and $\langle u, v \rangle = 0$.

In order to exploit the very special structure of the complementary constraints, we write (1) as an optimization problem with geometric constraints:

$$\text{Minimize } f(x) \text{ subject to } F(x) \in \Lambda, \tag{2}$$

where

$$\begin{array}{c} F(x) := (g(x), h(x), \Psi(x)), \quad \Psi(x) := ((-H_1(x), -G_1(x)), \dots, (-H_m(x), -G_m(x))) \\ \Lambda := \mathbb{R}^p_- \times \{0\}^q \times \mathcal{C}^m \quad \text{and} \quad \mathcal{C} = \{(c_1, c_2) \in \mathbb{R}^2 : 0 \leq -c_1 \perp -c_2 \geq 0\}. \end{array} \tag{3}$$

The feasible set of (2) is $\Omega := \{x \in \mathbb{R}^n : F(x) \in \Lambda\}$. From [24, 9], we have

Proposition 2.1. *For every $c = (c_1, c_2) \in \mathcal{C}$, we have that the limiting normal cone at (c_1, c_2) is*

$$N_{\mathcal{C}}((c_1,c_2)) := \left\{ d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \in \mathbb{R}^2 : \begin{array}{ll} d_1 \in \mathbb{R}, d_2 = 0 & \textit{if } c_1 = 0, c_2 < 0 \\ d_1 = 0, d_2 \in \mathbb{R} & \textit{if } c_1 < 0, c_2 = 0 \\ \textit{either } d_1 d_2 = 0 & \textit{if } c_1 = 0, c_2 = 0 \\ \ \textit{or } d_1 > 0, d_2 > 0 & \end{array} \right\}. \tag{4}$$

Following the proof of Proposition 6.41 of [25], we have the next lemma

Lemma 2.2. *Let $\Lambda := \mathbb{R}^p_- \times \mathbb{R}^q \times \mathcal{C}^m$ and $z := (a, b, (c_1^1, c_2^1), \dots, (c_1^m, c_2^m)) \in \Lambda$. The limiting normal cone to Λ at z is*

$$N_\Lambda(z) = \prod_{j=1}^{p} N_{\mathbb{R}_-}(a_j) \times \prod_{j=1}^{q} N_{\{0\}}(b_j) \times \prod_{i=1}^{m} N_{\mathcal{C}}((c_1^i, c_2^i)). \tag{5}$$

Now, we will define some crucial index sets that will occur frequently in the subsequent analysis. Since, we will deal with several MPECs, these index sets must explicitly depend on the feasible constraint sets. Consider the set $\Lambda = \mathbb{R}^p_- \times \{0\}^q \times \mathcal{C}^m$ for some $p, q, m \in \mathbb{N}$. Now, for $z := (a, b, -(c_1^1, c_2^1), \dots, -(c_1^m, c_2^m))$ in $\Lambda = \mathbb{R}^p_- \times \{0\}^q \times \mathcal{C}^m$, we use the notation

$$\begin{array}{l} \mathcal{I}(z,\Lambda) := \{i \in \{1,\dots,m\} : c_1^i = 0, c_2^i > 0\}, \\ \mathcal{J}(z,\Lambda) := \{i \in \{1,\dots,m\} : c_1^i = 0, c_2^i = 0\}, \\ \mathcal{K}(z,\Lambda) := \{i \in \{1,\dots,m\} : c_1^i > 0, c_2^i = 0\}. \end{array} \tag{6}$$

When Λ is clear from the context, we write $\mathcal{I}(z)$, $\mathcal{J}(z)$ and $\mathcal{K}(z)$ instead of $\mathcal{I}(z,\Lambda)$, $\mathcal{J}(z,\Lambda)$ and $\mathcal{K}(z,\Lambda)$ respectively. Given a feasible point $x^* \in \Omega$, set

$$\begin{array}{l} A(x^*,\Omega) := \{j \in \{1,\dots,p\} : g_j(x^*) = 0\}, \\ \mathcal{I}(x^*,\Omega) := \mathcal{I}(F(x^*),\Lambda) = \{i \in \{1,\dots,m\} : H_i(x^*) = 0, G_i(x^*) > 0\}, \\ \mathcal{J}(x^*,\Omega) := \mathcal{J}(F(x^*),\Lambda) = \{i \in \{1,\dots,m\} : H_i(x^*) = 0, G_i(x^*) = 0\}, \\ \mathcal{K}(x^*,\Omega) := \mathcal{K}(F(x^*),\Lambda) = \{i \in \{1,\dots,m\} : H_i(x^*) > 0, G_i(x^*) = 0\}. \end{array} \tag{7}$$

Similarly, when Ω is clear from the context, we use $A(x^*)$, $\mathcal{I}(x^*)$, $\mathcal{J}(x^*)$ and $\mathcal{K}(x^*)$ instead of $A(x^*,\Omega)$, $\mathcal{I}(x^*,\Omega)$, $\mathcal{J}(x^*,\Omega)$ and $\mathcal{K}(x^*,\Omega)$ respectively. There is no risk of confusion, between $\mathcal{I}(x)$ and $\mathcal{I}(z)$ since we reserve the letter z for elements of Λ. The same considerations hold for the other index sets.

The index set $A(x^*)$ is the set of indexes of all active inequalities and the index sets $\mathcal{I}(x^*)$, $\mathcal{J}(x^*)$ and $\mathcal{K}(x^*)$ form a partition of the set of of indexes of all complementary constraints. The set $\mathcal{J}(x^*)$ is called the bi-active set. When $\mathcal{J}(x^*) = \emptyset$, we say that the strict complementarity condition holds at x^*.

We proceed with the definition of M-stationarity.

Definition 2.1. We say that a feasible point x^* of (1) is a M-stationary point of problem (1) iff there exist multipliers $(\mu, \lambda, u, v) \in \mathbb{R}^p_+ \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m$ such that

$$\nabla f(x^*) + \sum_{j=1}^{p} \mu_j \nabla g_j(x^*) + \sum_{i=1}^{q} \lambda_i \nabla h_i(x^*) - \sum_{\imath=1}^{m} u_\imath \nabla H_\imath(x^*) - \sum_{\jmath=1}^{m} v_\jmath \nabla G_\jmath(x^*) = 0$$

with $\mathrm{supp}(\mu) \subset A(x^*)$, $u_\imath = 0$ ($\imath \in \mathcal{K}(x^*)$), $v_\jmath = 0$ ($\jmath \in \mathcal{I}(x^*)$) and either $u_\ell > 0, v_\ell > 0$ or $u_\ell v_\ell = 0$ for all $\ell \in \mathcal{J}(x^*)$

Before introducing the new MPEC-CQs, let us recall the definition of MPEC-RPLCD. Given a point $x \in \mathbb{R}^n$ and four index sets $I_1 \subset \{1,\dots,p\}$, $I_2 \subset \{1,\dots,q\}$, $I_3 \subset \{1,\dots,m\}$ and $I_4 \subset \{1,\dots,m\}$, we define the set

$$\mathcal{G}(x; I_1, I_2, I_3, I_4) := \{\nabla g_j(x), \nabla h_i(x), \nabla H_\imath(x), \nabla G_\jmath(x) : j \in I_1, i \in I_2, \imath \in I_3, \jmath \in I_4\}.$$

The linear subspace generated by the set $\mathcal{G}(x; I_1, I_2, I_3, I_4)$ is denoted by span $\mathcal{G}(x; I_1, I_2, I_3, I_4)$, and we denote by $\mathrm{span}_+ \mathcal{G}(x; I_1, I_2, I_3, I_4)$ the set of all linear combinations of $\mathcal{G}(x; I_1, I_2, I_3, I_4)$ with positive multipliers for all indexes in I_1.

Definition 2.2. Let x^* be a feasible point and $\mathcal{E}^{'} \subset \mathcal{E}$, $\mathcal{I}^{'} \subset \mathcal{I}(x^*)$, $\mathcal{K}^{'} \subset \mathcal{K}(x^*)$ be index sets such that the set $\mathcal{G}(x^*; \emptyset, \mathcal{E}^{'}, \mathcal{I}^{'}, \mathcal{K}^{'})$ is a linear basis for span $\mathcal{G}(x^*; \emptyset, \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$. We say that MPEC-Relaxed Constant Positive Linear Dependence constraint qualification (MPEC-RCPLD) holds at x^* iff there is a $\delta > 0$ such that

1. $\mathcal{G}(y; \emptyset, \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$ has the same rank for each $y \in \mathbb{B}(x^*, \delta)$;
2. For each index sets $A^{'} \subset A(x^*)$ and $\mathcal{J}_H^{'}, \mathcal{J}_G^{'} \subset \mathcal{J}(x^*)$, if there are non-zero multipliers $\{(\mu, \lambda, u, v)\} \in \mathbb{R}_+^p \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m$ with $\mathrm{supp}(\mu) \subset A^{'}$, $\mathrm{supp}(\lambda) \subset \mathcal{E}^{'}$, $\mathrm{supp}(u) \subset \mathcal{I}^{'} \cup \mathcal{J}_H^{'}$, $\mathrm{supp}(v) \subset \mathcal{K}^{'} \cup \mathcal{J}_G^{'}$, and either $u_\ell v_\ell = 0$ or $u_\ell > 0, v_\ell > 0$ for each $\ell \in \mathcal{J}(x^*)$ such that

$$\sum_{i \in A^{'}} \mu_i \nabla g_i(x^*) + \sum_{j \in \mathcal{E}^{'}} \lambda_j \nabla h_j(x^*) + \sum_{\imath \in \mathcal{I}^{'} \cup \mathcal{J}_H^{'}} u_\imath \nabla H_\imath(x^*) + \sum_{\jmath \in \mathcal{K}^{'} \cup \mathcal{J}_G^{'}} v_\jmath \nabla G_\jmath(x^*) = 0.$$

   Then, for any $y \in \mathbb{B}(x^*, \delta)$, the set $\mathcal{G}(y; A^{'}, \mathcal{E}^{'}, \mathcal{I}^{'} \cup \mathcal{J}_H^{'}, \mathcal{K}^{'} \cup \mathcal{J}_G^{'})$ is a set of vectors linearly dependent.

The next lemma is a modified version of Caratheódory's lemma [26].

Lemma 2.3. *[27] If $v = \sum_{i \in \mathcal{I}} \alpha_i p_i + \sum_{j \in \mathcal{J}} \beta_j q_j$ with $p_i, q_j \in \mathbb{R}^n$, $i \in \mathcal{I}$, $j \in \mathcal{J}$; $\{p_i\}_{i \in \mathcal{I}}$ to be linearly independent and α_i, β_j are non-zero for $i \in \mathcal{I}, j \in \mathcal{J}$. Then, there exist $\mathcal{J}^{'} \subset \mathcal{J}$ and scalars $\hat{\alpha}_i, \hat{\beta}_j$ for all $i \in \mathcal{I}$, $j \in \mathcal{J}^{'}$ such that $v = \sum_{i \in \mathcal{I}} \hat{\alpha}_i p_i + \sum_{j \in \mathcal{J}^{'}} \hat{\beta}_j q_j$ and $\{p_i, q_j\}_{i \in \mathcal{I}, j \in \mathcal{J}^{'}}$ is a linearly independent set.*

3 New Constraint Qualifications for M-stationarity

Here we introduce two new weaker CQs for M-stationarity and show that both MPEC-CQs have the local preservation property. Consider the closed convex cone $\mathrm{span}_+ \mathcal{G}(x; A(x), \mathcal{E}, \mathcal{I}(x), \mathcal{K}(x))$ which is the set of all linear combinations of $\mathcal{G}(x; A(x), \mathcal{E}, \mathcal{I}(x), \mathcal{K}(x))$ with positive multipliers for all indexes in $A(x)$. Now, for every feasible point x, define the next sets of indexes

$$A_-(x) := \{j \in A(x) : -\nabla g_i(x) \in \mathrm{span}_+ \mathcal{G}(x; A(x), \mathcal{E}, \mathcal{I}(x), \mathcal{K}(x))\} \tag{8}$$

and $A_+(x) := A(x) \setminus A_-(x)$. Take $x^* \in \Omega$. Let $A_-^{'} \subset A_-(x^*)$, $\mathcal{E}^{'} \subset \mathcal{E}$, $\mathcal{I}^{'} \subset \mathcal{I}(x^*)$, $\mathcal{K}^{'} \subset \mathcal{K}(x^*)$ be index sets such that $\mathcal{G}(x^*; A_-^{'}, \mathcal{E}^{'}, \mathcal{I}^{'}, \mathcal{K}^{'})$ is a linear basis for span $\mathcal{G}(x^*; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$. Consider the next assumptions:

(H1) $\mathcal{G}(y; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$ has the same rank for each $y \in \mathbb{B}(x^*, \delta)$;

(H2) For each $A_+^{'} \subset A_+(x^*)$ and $\mathcal{J}_H^{'}, \mathcal{J}_G^{'} \subset \mathcal{J}(x^*)$, if there are non-zero multipliers $\{(\mu, \lambda, u, v)\} \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m$ with $\mathrm{supp}(\mu) \subset A_+^{'} \cup A_-^{'}$, $\mu_j \geq 0$ $(j \in A_+(x^*))$, $\mathrm{supp}(\lambda) \subset \mathcal{E}^{'}$, $\mathrm{supp}(u) \subset \mathcal{I}^{'} \cup \mathcal{J}_H^{'}$, $\mathrm{supp}(v) \subset \mathcal{K}^{'} \cup \mathcal{J}_G^{'}$ and either $u_\ell v_\ell = 0$ or $u_\ell > 0, v_\ell > 0$ for each $\ell \in \mathcal{J}(x^*)$, such that

$$\sum_{j \in A_+^{'} \cup A_-^{'}} \mu_j \nabla g_j(x^*) + \sum_{i \in \mathcal{E}^{'}} \lambda_i \nabla h_i(x^*) - \sum_{\imath \in \mathcal{I}^{'} \cup \mathcal{J}_H^{'}} u_\imath \nabla H_\imath(x^*) - \sum_{\jmath \in \mathcal{K}^{'} \cup \mathcal{J}_G^{'}} v_\jmath \nabla G_\jmath(x^*) = 0.$$

Then, we have that the vectors $\{\nabla g_j(y) : j \in A_+^{'} \cup A_-^{'}\}$, $\{\nabla h_i(y) : i \in \mathcal{E}^{'}\}$, $\{\nabla H_\imath(y) : \imath \in \mathcal{I}^{'} \cup \mathcal{J}_H^{'}\}$ and $\{\nabla G_\jmath(y) : \jmath \in \mathcal{K}^{'} \cup \mathcal{J}_G^{'}\}$ are linearly dependent for every $y \in \mathbb{B}(x^*, \delta)$.

(H3) There are no non-zero multipliers $\{(\mu, \lambda, u, v)\} \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m$ with $\mathrm{supp}(\mu) \subset A_+(x^*) \cup A_-^{'}$, $\mu_j \geq 0$ $(j \in A_+(x^*))$, $\mathrm{supp}(\lambda) \subset \mathcal{E}^{'}$, $\mathrm{supp}(u) \subset \mathcal{I}^{'} \cup \mathcal{J}(x^*)$, $\mathrm{supp}(v) \subset \mathcal{K}^{'} \cup \mathcal{J}(x^*)$ and either $u_\ell v_\ell = 0$ or $u_\ell > 0, v_\ell > 0$ for each $\ell \in \mathcal{J}(x^*)$, such that

$$\sum_{j \in A_+(x^*) \cup A_-^{'}} \mu_j \nabla g_j(x^*) + \sum_{i \in \mathcal{E}^{'}} \lambda_i \nabla h_i(x^*) - \sum_{\imath \in \mathcal{I}^{'} \cup \mathcal{J}(x^*)} u_\imath \nabla H_\imath(x^*) - \sum_{\jmath \in \mathcal{K}^{'} \cup \mathcal{J}(x^*)} v_\jmath \nabla G_\jmath(x^*) = 0.$$

Now, we proceed with the definitions of the new MPEC-CQs.

Definition 3.1. Let $x^* \in \Omega$ be a feasible point. Let $A'_- \subset A_-(x^*)$, $\mathcal{E}' \subset \mathcal{E}$, $\mathcal{I}' \subset \mathcal{I}(x^*)$, $\mathcal{K}' \subset \mathcal{K}(x^*)$ be index sets such that $\mathcal{G}(x^*; A'_-, \mathcal{E}', \mathcal{I}', \mathcal{K}')$ is a linear basis for span $\mathcal{G}(x^*; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$.

a) We say that MPEC-Constant Rank of the Subspace Component Condition (MPEC-CRSC) holds at x^* if there is a $\delta > 0$ such that (H1) and (H2) hold.

b) We say that the MPEC-Relaxed No Nonzero Abnormal Multiplier Constraint Qualification (MPEC-RNNAMCQ) holds at x^* if there is a $\delta > 0$ such that (H1) and (H3) hold.

From the definition, MPEC-RNNAMCQ implies MPEC-CRSC. Actually, MPEC-RNNAMCQ is strictly stronger than MPEC-CRSC as the example 4.2 will show. When $m = 0$ (i. e. there are no complementary constraints), both MPEC-RNNAMCQ and MPEC-CRSC coincide with CRSC.

Remark 1. Clearly, we have for every feasible point x in Ω, than the linear subspace span $\mathcal{G}(x; A_-(x), \mathcal{E}, \mathcal{I}(x), \mathcal{K}(x))$ is included into $\nabla F(x)^\top N_\Lambda(F(x))$. This subspace has been used in the definition of several MPEC-CQs. See [17, 28]. MPEC-CRSC and MPEC-RNNAMCQ incorporate into this subspace the gradients of the inequality constraints and still works as a CQ for M-stationarity.

Remark 2. The definition of MPEC-CRSC (similarly for MPEC-RNNAMCQ) does not depend on the specific choice of the index set that select the basis for span $\mathcal{G}(x^*; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$. That is, both definitions are independent of choice of the basis.

Remark 3. From the definition of $A_-(x^*)$, we can infer that the convex cone $\text{span}_+ \mathcal{G}(x^*; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$ is actually a linear subspace. In fact, $\text{span}_+ \mathcal{G}(x^*; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*)) = \text{span}\, \mathcal{G}(x^*; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$.

Now, we formally state the local preservation property.

Definition 3.2. We say that a condition (C) for X has the local preservation property at $x^* \in X$ if there is a $\delta > 0$ such that (C) is satisfied at every $x \in \mathbb{B}(x^*, \delta) \cap X$, i.e. the set of feasible points where (C) holds is open in X.

Several MPEC-CQs have the local preservation property. For instance, we can mention MPEC-LICQ, MPEC-MFCQ and MPEC-CRCQ, [29]. Recently, Guo et al. [17] proved the preservation property for MPEC-CPLD and Chieu and Lee [28] proved the preservation property for MPEC-RCLPD. Here, we will prove that MPEC-CRSC and MPEC-RNNAMCQ also have the local preservation property. The result is based on the following lemmas

Lemma 3.1. *Let $x^* \in \Omega$ such that (H3) holds. Then, there exists $\delta > 0$, such that $A_-(x^*) \subset A_-(y)$ for every feasible point $y \in \Omega \cap \mathbb{B}(x^*, \delta)$.*

Proof. Define $\widetilde{\Omega} := \{x \in \mathbb{R}^n : g(x) \leq 0, h(x) = 0, H_i(x) = 0, G_j(x) = 0, i \in \mathcal{I}(x^*), j \in \mathcal{K}(x^*)\}$. From (H3), $\widetilde{\Omega}$ conforms CRSC at x^*. By Lemma 5.4 of [15], there is a $\delta_1 > 0$ such that $A_-(x^*, \widetilde{\Omega}) \subset A_-(y, \widetilde{\Omega})$ for every $y \in \widetilde{\Omega} \cap \mathbb{B}(x^*, \delta_1)$. Now, take $\delta_2 > 0$, such that $\mathcal{I}(x^*) \subset \mathcal{I}(y)$ and $\mathcal{K}(x^*) \subset \mathcal{K}(y)$ for every $y \in \Omega \cap \mathbb{B}(x^*, \delta_2)$. Then, $\Omega \cap \mathbb{B}(x^*, \delta_2) \subset \widetilde{\Omega}$. From the definition of $A_-(x^*, \widetilde{\Omega})$, we get that $A_-(x^*, \widetilde{\Omega}) = A_-(x^*)$ and $A_-(y, \widetilde{\Omega}) \subset A_-(y)$ for every $y \in \Omega \cap \mathbb{B}(x^*, \delta_2)$. Define $\delta_3 := \min\{\delta_1, \delta_2\}$. Then, for every $y \in \Omega \cap \mathbb{B}(x^*, \delta_3)$ we get $A_-(x^*) = A_-(x^*, \widetilde{\Omega}) \subset A_-(y, \widetilde{\Omega}) \subset A_-(y)$ as we wanted to prove. □

When there are no complementary constraints, Lemma 5.4 of [15] shows that $A_-(x^*) = A_-(y)$ for every feasible point y near x^*. In the presence of complementary constraints this is no longer true as the next example shows.

Example 3.1. In $\mathbb{R}^2$, consider the point $x^* = (0, 0)$ and the constraint system $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 : g_1(x_1, x_2) := x_1 \leq 0, \;\; 0 \leq x_1 \perp x_2 \geq 0\}$.

In this example, (H3) holds since the constraints are given by linear functions. Clearly, $A_-(x^*) = \emptyset$, since $\mathcal{I}(x^*)$ and $\mathcal{K}(x^*)$ are empty sets. Now, take $x^k := (0, 1/k) \in \Omega$, $\forall k \in \mathbb{N}$. By calculations, $\mathcal{I}(x^k) = \{1\}$, $\mathcal{K}(x^k) = \emptyset$, $\mathcal{J}(x^k) = \emptyset$ and $A(x^k) = \{1\}$. Hence, $A_-(x^k) = \{1\}$. Thus, the inclusion in the Lemma 3.1 can be strict.

We continue with the next lemma.

Lemma 3.2. *If MPEC-CRSC holds at the point $x^* \in \Omega$. Then, there is a positive scalar δ such that, for each feasible point $y \in \Omega \cap \mathbb{B}(x^*, \delta)$, the set $\mathcal{G}(x; A_-(y), \mathcal{E}, \mathcal{I}(y), \mathcal{K}(y))$ has the same rank for every x near y.*

The proof is based on induction over the number of complementarity constraints. A formal proof is given in the appendix. Since MPEC-RNNAMCQ implies MPEC-CRSC, we have the next result.

Corollary 3.3. *Let $x^* \in \Omega$ such that MPEC-RNNAMCQ holds at x^*. Then, there is a $\delta > 0$ such that, for each feasible point $y \in \Omega \cap \mathbb{B}(x^*, \delta)$, the set $\mathcal{G}(x; A_-(y), \mathcal{E}, \mathcal{I}(y), \mathcal{K}(y))$ has the same rank for every x near y.*

The next theorems will show that MPEC-CRSC and MPEC-RNNAMCQ have the local preservation property. We start with MPEC-CRSC.

Theorem 3.4. *MPEC-CRSC has the local preservation property.*

Proof. Suppose that MPEC-CRSC holds at x^*. We will show that MPEC-CRSC holds for every feasible point close to x^*. By Lemma 3.2, (H1) holds for every feasible point close to x^*. Thus, we only need to show that (H2) holds for every $y \in \Omega$ sufficiently close to x^*.

Suppose by contradiction, that is not true. Thus, there is a feasible sequence $\{y^k\} \subset \Omega$ with $y^k \to x^*$ such that (H2) does not hold at y^k. For each $k \in \mathbb{N}$, take index subsets $A_-^{'}(y^k) \subset A_-(y^k)$, $\mathcal{E}^{'}(y^k) \subset \mathcal{E}$, $\mathcal{I}^{'}(y^k) \subset \mathcal{I}(y^k)$ and $\mathcal{K}^{'}(y^k) \subset \mathcal{K}(y^k)$ such that $\mathcal{G}(y^k; A_-^{'}(y^k)), \mathcal{E}^{'}, \mathcal{I}^{'}(y^k), \mathcal{K}^{'}(y^k))$ is a basis for span $\mathcal{G}(y^k; A_-(y^k)), \mathcal{E}, \mathcal{I}(y^k), \mathcal{K}(y^k))$ and $\mathcal{G}(y^k; \emptyset, \mathcal{E}^{'}, \mathcal{I}^{'}(y^k), \mathcal{K}^{'}(y^k))$ is a basis for span $\mathcal{G}(y^k; \emptyset, \mathcal{E}, \mathcal{I}(y^k), \mathcal{K}(y^k))$. Since MPEC-CRSC does not hold at y^k. There are index sets $A_+^{'}(y^k) \subset A_+(y^k)$, $\mathcal{J}_H^{'}(y^k), \mathcal{J}_G^{'}(y^k) \subset \mathcal{J}(y^k)$, and multipliers $\{(\lambda^k, \mu^k, u^k, v^k)\}$ which are not all zero, $\mu_j^k \in \mathbb{R}_+$, $j \in A_+^{'}(y^k)$ and either $u_\ell^k v_\ell^k = 0$ or $u_\ell^k > 0, v_\ell^k > 0$ for each $\ell \in \mathcal{J}(y^k)$, such that

$$\sum_{A_+^{'}(y^k) \cup A_-^{'}(y^k)} \mu_j^k \nabla g_j(y^k) + \sum_{\mathcal{E}^{'}(y^k)} \lambda_i^k \nabla h_i(y^k) - \sum_{\mathcal{I}^{'}(y^k) \cup \mathcal{J}_H^{'}(y^k)} u_\imath^k \nabla H_\imath(y^k) - \sum_{\mathcal{K}^{'}(y^k) \cup \mathcal{J}_G^{'}(y^k)} v_\jmath^k \nabla G_\jmath(y^k) = 0 \tag{9}$$

and for every positive sequence $\delta^k \to 0$, $\mathcal{G}(x^k; A_+^{'}(y^k) \cup A_-^{'}(y^k), \mathcal{E}, \mathcal{I}^{'}(y^k) \cup \mathcal{J}_H^{'}(y^k), \mathcal{K}^{'}(y^k) \cup \mathcal{J}_G^{'}(y^k))$ is linearly independent at some point x^k in $\mathbb{B}(y^k, \delta^k)$.

Now, define the next vector

$$\omega^k := \sum_{A_-^{'}(y^k)} \mu_j^k \nabla g_j(y^k) + \sum_{\mathcal{E}^{'}(y^k)} \lambda_i^k \nabla h_i(y^k) - \sum_{\mathcal{I}^{'}(y^k)} u_\imath^k \nabla H_\imath(y^k) - \sum_{\mathcal{K}^{'}(y^k)} v_\jmath^k \nabla G_\jmath(y^k) \tag{10}$$

From definition of $A_-(y^k)$, ω^k is in $\text{span}_+ \mathcal{G}(y^k, A_-(y^k), \mathcal{E}, \mathcal{I}(y^k), \mathcal{K}(y^k))$, (see Remark 3). Then, ω^k can be written as

$$\omega^k = \sum_{j \in A_+^{''}(y^k)} \hat{\mu}_j^k \nabla g_j(y^k) + \sum_{i \in \mathcal{E}^{'}(y^k)} \hat{\lambda}_i^k \nabla h_i(y^k) - \sum_{\imath \in \mathcal{I}^{'}(y^k)} \hat{u}_\imath^k \nabla H_\imath(y^k) - \sum_{\jmath \in \mathcal{K}^{'}(y^k)} \hat{v}_\jmath^k \nabla G_\jmath(y^k) \tag{11}$$

where $A_+^{''}(y^k) \subset A_-(y^k)$ and $\hat{\mu}_j^k \geq 0$, for $j \in A_+^{''}(y^k)$. By Lemma 2.3, we can take $A_+^{''}(y^k)$ such that $\mathcal{G}(y^k; A_+^{''}(y^k), \mathcal{E}^{'}(y^k), \mathcal{I}^{'}(y^k), \mathcal{K}^{'}(y^k))$ is a linearly independent set. Define the following multipliers:

$$\begin{array}{lll} \tilde{\lambda}_i^k := \hat{\lambda}_j^k (i \in \mathcal{E}^{'}(y^k)), & \tilde{\mu}_j^k := \mu_j^k (j \in A_+^{'}(y^k)), & \tilde{\mu}_j^k := \hat{\mu}_j^k (j \in A_+^{''}(y^k)), \\ & \tilde{u}_j^k := u_j^k (j \in \mathcal{J}_H^{'}(y^k)), & \tilde{u}_j^k := \hat{\mu}_j^k (j \in \mathcal{I}^{'}(y^k)), \\ & \tilde{v}_j^k := v_j^k (j \in \mathcal{J}_G^{'}(y^k)), & \tilde{v}_j^k := \hat{v}_j^k (j \in \mathcal{K}^{'}(y^k)) \end{array} \tag{12}$$

with $\text{supp}(\tilde{\mu}) \subset A_-^{''}(y^k) \cup A_+^{'}(y^k)$, $\text{supp}(\tilde{\lambda}) \subset \mathcal{E}^{'}(y^k)$, $\text{supp}(\tilde{u}) \subset \mathcal{I}^{'}(y^k) \cup \mathcal{J}_H^{'}(y^k)$ and $\text{supp}(\tilde{v}) \subset \mathcal{K}^{'}(y^k) \cup \mathcal{J}_G^{'}(y^k)$. Substituting (11) into (9), we get

$$\sum_{A_+^{'}(y^k) \cup A_+^{''}(y^k)} \tilde{\mu}_j^k \nabla g_j(y^k) + \sum_{\mathcal{E}^{'}(y^k)} \tilde{\lambda}_i^k \nabla h_i(y^k) - \sum_{\mathcal{I}^{'}(y^k) \cup \mathcal{J}_H^{'}(y^k)} \tilde{u}_\imath^k \nabla H_\imath(y^k) - \sum_{\mathcal{K}^{'}(y^k) \cup \mathcal{J}_G^{'}(y^k)} \tilde{v}_\jmath^k \nabla G_\jmath(y^k) = 0 \tag{13}$$

After dividing (13) by some scalar, we assume that $\|(\tilde{\lambda}^k, \tilde{\mu}^k, \tilde{u}^k, \tilde{v}^k)\| = 1$.

Now, since there is only a finite number of index sets, we assume without loss of generality (possibly after taking an adequate subsequence), that $\{A_+^{'}(y^k), A_+^{''}(y^k), A_-^{'}(y^k), \mathcal{E}^{'}(y^k), \mathcal{I}^{'}(y^k), \mathcal{J}_H^{'}(y^k), \mathcal{K}^{'}(y^k), \mathcal{J}_G^{'}(y^k)\}$ does depend on k, namely, $\{A_+^{'}, A_+^{''}, A_-^{'}, \mathcal{E}^{'}, \mathcal{I}^{'}, \mathcal{J}_H^{'}, \mathcal{K}^{'}, \mathcal{J}_G^{'}\}$. Certainly, $\tilde{\gamma}^k := (\tilde{\mu}^k, \tilde{\lambda}^k, (\tilde{u}_1^k, \tilde{v}_1^k), \ldots, (\tilde{u}_m^k, \tilde{v}_m^k)) \in N_\Lambda(F(y^k))$, where F is given by (3). Since $\|\tilde{\gamma}^k\| = 1$, we assume that $\tilde{\gamma}^k \to \hat{\gamma}$ with $\|\tilde{\gamma}\| = 1$. By the outer semi-continuity of normal cone, we get that $\tilde{\gamma} \in N_\Lambda(F(x^*))$. Denote $\tilde{\gamma}$ by $(\tilde{\mu}, \tilde{\lambda}, (\tilde{u}_1, \tilde{v}_1), \ldots, (\tilde{u}_m, \tilde{v}_m))$. Clearly, $\tilde{\mu} \geq 0$ and either $\tilde{u}_\ell \tilde{v}_\ell = 0$ or $\tilde{u}_\ell > 0, \tilde{v}_\ell > 0$ for each $\ell \in \mathcal{J}(x^*)$. Furthermore, taking limit in (13) we get

$$\sum_{j \in A_+^{'} \cup A_+^{''}} \tilde{\mu}_j \nabla g_j(x^*) + \sum_{i \in \mathcal{E}^{'}} \tilde{\lambda}_i \nabla h_i(x^*) - \sum_{\imath \in \mathcal{I}^{'} \cup \mathcal{J}_H^{'}} \tilde{u}_\imath \nabla H_\imath(x^*) - \sum_{\jmath \in \mathcal{K}^{'} \cup \mathcal{J}_G^{'}} \tilde{v}_\jmath \nabla G_\jmath(x^*) = 0 \tag{14}$$

By MPEC-CRSC, the set $\mathcal{G}(x; A_+^{'} \cup A_+^{''}, \mathcal{E}^{'}, \mathcal{I}^{'} \cup \mathcal{J}_H^{'}, \mathcal{K}^{'} \cup \mathcal{J}_G^{'})$ is linearly dependent for every x near x^*. But, since (i) $\mathcal{G}(x; A_+^{''}, \mathcal{E}^{'}, \mathcal{I}^{'}, \mathcal{K}^{'})$ is a linearly independent set for every x such that $\|x - y^k\| \leq \delta_1^k$ (for some $\delta_1^k > 0$ and k large enough) and (ii) span $\mathcal{G}(x; A_+^{''}, \mathcal{E}^{'}, \mathcal{I}^{'}, \mathcal{K}^{'})$ is a subset of span $\mathcal{G}(x; A_-^{'}, \mathcal{E}^{'}, \mathcal{I}^{'}, \mathcal{K}^{'})$ where $\mathcal{G}(x; A_-^{'}, \mathcal{E}^{'}, \mathcal{I}^{'}, \mathcal{K}^{'})$ is a basis of $\mathcal{G}(x; A_-(y^k), \mathcal{E}, \mathcal{I}(y^k), \mathcal{K}(y^k))$ for every x such that $\|x - y^k\| \leq \delta_2^k$ for some $\delta_2^k > 0$ (this follows from the inclusion $A_+^{''} \subset A_-(y^k)$ and Lemma 3.2). We get that $\mathcal{G}(x; A_+^{'} \cup A_-^{'}, \mathcal{E}^{'}, \mathcal{I}^{'} \cup \mathcal{J}_H^{'}, \mathcal{K}^{'} \cup \mathcal{J}_G^{'})$ is linearly dependent for every x with $\|x - y^k\| \leq \min\{\delta_1^k, \delta_2^k\}$ for k sufficiently large. Now, choose $\delta^k := \min\{\delta_1^k, \delta_2^k, 1/k\}$. Clearly, $\delta^k \to 0$ and for every x with $\|x - y^k\| \leq \delta^k \leq \min\{\delta_1^k, \delta_2^k\}$, the set $\mathcal{G}(x; A_+^{'} \cup A_+^{''}, \mathcal{E}^{'}, \mathcal{I}^{'} \cup \mathcal{J}_H^{'}, \mathcal{K}^{'} \cup \mathcal{J}_G^{'})$ is linearly dependent, which is a contradiction. □

Theorem 3.5. *MPEC-RNNAMCQ has the local preservation property.*

Proof. By Lemma 3.2, we only need to prove that (H3) holds for every $y \in \Omega$ near to x^*. Suppose by contradiction that MPEC-RNNAMCQ fails for some sequence $y^k \in \Omega$ with $y^k \to x^*$. Using the same reasoning as in the proof of the above theorem, we get multipliers $\tilde{\gamma} := (\tilde{\mu}, \tilde{\lambda}, (\tilde{u}_1, \tilde{v}_1), \ldots, (\tilde{u}_m, \tilde{v}_m))$ with $\tilde{\gamma} \in N_\Lambda(F(x^*))$ and $\|\tilde{\gamma}\| = 1$ such that

$$\sum_{j \in A_+^{'} \cup A_+^{''}} \tilde{\mu}_j \nabla g_j(x^*) + \sum_{i \in \mathcal{E}^{'}} \tilde{\lambda}_i \nabla h_i(x^*) - \sum_{\imath \in \mathcal{I}^{'} \cup \mathcal{J}_H^{'}} \tilde{u}_\imath \nabla H_\imath(x^*) - \sum_{\jmath \in \mathcal{K}^{'} \cup \mathcal{J}_G^{'}} \tilde{v}_\jmath \nabla G_\jmath(x^*) = 0,$$

which is a contradiction with MPEC-RNNAMCQ. □

4 Relationship between old and new MPEC-CQs

4.1 MPEC-RCPLD, MPEC-RNNAMCQ and MPEC-CRSC

Here we will show that MPEC-CRSC is strictly weaker than MPEC-RCPLD and there is no relation between MPEC-RNNAMCQ and MPEC-RCPLD.

Theorem 4.1. *MPEC-RCPLD implies MPEC-CRSC.*

The proof of this result is presented in the appendix. MPEC-CRSC is strictly weaker than MPEC-RCPLD. Indeed, in the next example, MPEC-RNNAMCQ holds (and hence MPEC-CRSC) but MPEC-RCPLD fails.

Example 4.1. (MPEC-RNNAMCQ and MPEC-CRSC do not imply MPEC-RCPLD) In $\mathbb{R}^2$, consider the point $x^* = (0,0)$ and the constraint system defined by the functions $g_1(x_1,x_2) = x_1 + x_2$; $g_2(x_1,x_2) = -x_1 - x_2$; $g_3(x_1,x_2) = x_1^2 + x_2^2$; $0 \leq H_1(x_1,x_2) := x_1 - x_2 \perp G_1(x_1,x_2) := x_2 + 1 \geq 0$.

In this example MPEC-RCPLD does not hold. Indeed, since $\mathcal{I}(x^*) = \{1\}$, the next set of gradient $\{\nabla g_3(x_1,x_2), \nabla H_1(x_1,x_2)\}$ is positive linearly dependent at $x^* = (0,0)$, but not in any neighborhood of x^*. On the other hand, it is not difficult that MPEC-RNNAMCQ holds at x^*. In fact, since $A_-(x^*) = \{1,2,3\}$ and $\mathcal{I}(x^*) = \{1\}$, we have that $\mathcal{G}(y, A_-(x^*), \emptyset, \mathcal{I}(x^*), \emptyset)$ has the same rank for y near to x^* and (H3) holds. Thus, MPEC-RNNAMCQ holds at x^*.

Example 4.2. (MPEC-RCPLD holds but MPEC-RNNAMCQ may fails) In $\mathbb{R}^2$, set $x^* = (0,0)$ and consider the constraint system defined by the functions: $g_1(x_1,x_2) = x_1 + x_2$, $0 \leq H_1(x_1,x_2) := x_1 + x_2 \perp G_1(x_1,x_2) := x_1 + x_2 \geq 0$.

Since all the constraints are given by linear functions, MPEC-RCPLD holds (Corollary 4.3 of [30]). To see that MPEC-RNNAMCQ fails at x^*, note that $A_-(x^*) = \emptyset$, $\mathcal{J}(x^*) = \{1\}$ and the equation $\mu\nabla g(x_1, x_2) - u\nabla H_1(x_1, x_2) - v\nabla G_1(x_1, x_2) = (0, 0)$ has a nontrivial solution $(\mu, u, v) = (2, 1, 1)$. Since, MPEC-RCPLD implies MPEC-CRSC, this example also shows that MPEC-CRSC is strictly weaker than MPEC-RNNAMCQ.

4.2 MPEC-RNNAMCQ, MPEC-CRSC and MPEC-CCP

Here, we will prove that MPEC-RNNAMCQ and MPEC-CRSC are CQs for M-stationarity. In fact, we will show that MPEC-RNNAMCQ and MPEC-CRSC imply the recently introduced MPEC-Cone Continuity Property (MPEC-CCP). which is a tailored MPEC version of the Cone Continuity Property defined in [31]. Such MPEC-CQ is weaker than MPEC-RCPLD and it can be used in the global convergence analysis for several methods for solving MPECs, [18]. Let us recall, the definition of MPEC-CCP .

Definition 4.1. Let x^* be a feasible point of (1). We say that MPEC-CCP holds at x^* if the next inclusion $\limsup_{(x,z)\to(x^*,F(x^*))} \nabla F(x)^\top N_\Lambda(z) \subset \nabla F(x^*)^\top N_\Lambda(F(x^*))$ holds.

Now, we proceed with the next theorem.

Theorem 4.2. *MPEC-CRSC implies MPEC-CCP.*

The proof of this result is also presented in the appendix. As corollary we have

Corollary 4.3. *MPEC-CRSC and MPEC-RNNAMCQ are CQs for M-stationarity.*

The Figure 1 shows the relations among several CQs for M-stationarity. Note that MPEC-RNNAMCQ is independent of MPEC-pseudonormality and MPEC-quasinormality. In fact, we can use the example 4 and example 5 of [18]. For more information about other MPEC-CQs found in the literature, see [17, 32] and reference therein.

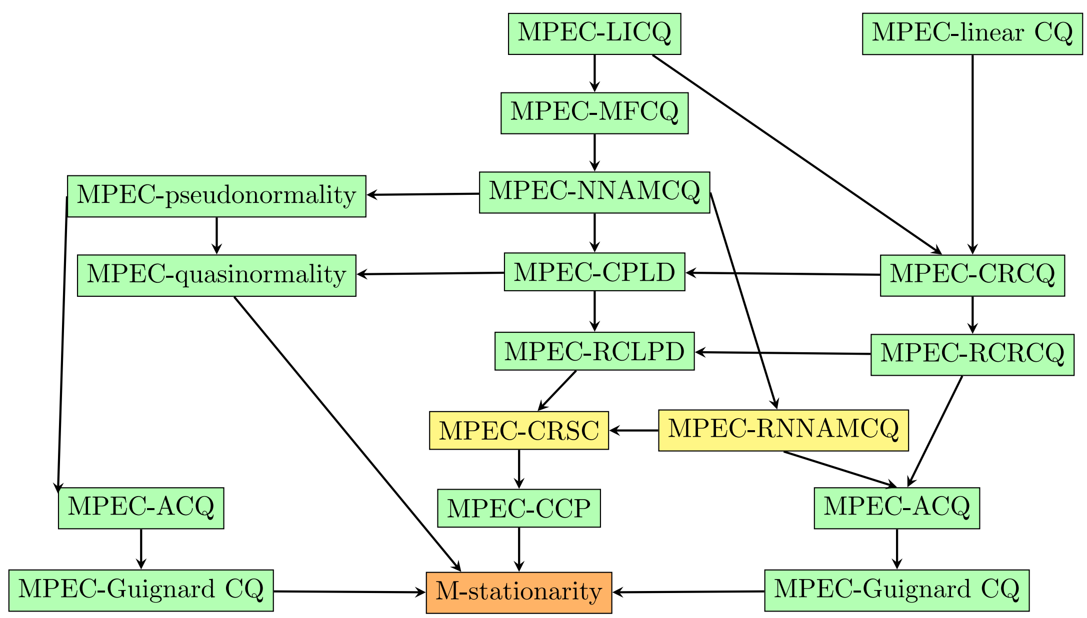

Figure 1: Relations among several CQs for M-stationarity. Arrows indicate implications.

5 Applications of MPEC-RNNAMCQ and MPEC-CRSC

One interesting question about a CQ is whether it implies the existence of the error bound property, a property which has a large range of applications in variational analysis and optimization. See [33, 34, 4, 14] and the references therein. Here, we will show that MPEC-CRSC is sufficient to guarantee the existence of the error bound property under the strict complementary condition and MPEC-RNNAMCQ always implies the error bound property without any additional assumptions. Let us recall the definition of the error bound property.

Definition 5.1. We say that the *error bound property* holds at the feasible point $x^* \in \Omega = \{x \in \mathbb{R}^n : h(x) = 0, g(x) \leq 0, 0 \leq H(x) \perp G(x) \geq 0\}$, if there are scalars δ and $L > 0$ such that for all $x \in \mathbb{B}(x^*, \delta)$,

$$\operatorname{dist}(x, \Omega) \leq L \max\{|h_i(x)|_{i\in\mathcal{E}}, \max\{g_j(x), 0\}_{j\in\mathcal{P}}, \sum_{i=1}^{m} \operatorname{dist}_1((H_i(x), G_i(x)), \mathcal{C})\}. \tag{15}$$

Clearly, using the definition of F (see (3)), the expression (15) is equivalent to saying that there are scalars $L, \delta > 0$ such that

$$\operatorname{dist}(x, \Omega) \leq L \operatorname{dist}(F(x), \Lambda) \text{ for all } x \in \mathbb{B}(x^*, \delta). \tag{16}$$

Before continuing, consider the abstract constraint system $\mathcal{X} := \{x \in \mathbb{R}^n : \mathcal{F}(x) \in \mathcal{Y}\}$, where $\mathcal{Y} \subset \mathbb{R}^s$ is a closed set and $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^s$ is continuously differentiable mapping. Now, let us introduce the following property.

Definition 5.2. We say that the *Assumption A* holds at $y^* \in \mathcal{Y}$, if for every sequence $\{\omega^k, v^k, \lambda^k\} \in \mathbb{R}^s \times \mathbb{R}^s \times \mathbb{R}^s$ with $\omega^k \notin \mathcal{Y}$, $v^k \in \mathcal{Y}$, $\lambda^k \in N_{\mathcal{Y}}(v^k)$, $\omega^k \to y^*$, $v^k \to y^*$ and $\{\lambda^k\}$ bounded, we have

$$\limsup_{k\to\infty} \frac{\langle \lambda^k, z^k - v^k \rangle}{\|z^k - \omega^k\|} < \infty, \tag{17}$$

where $z^k \in \mathcal{Y}$ satisfies $\|z^k - \omega^k\| = \operatorname{dist}(\omega^k, \mathcal{Y})$.

Clearly, (17) is equivalent to require $\limsup_k \langle \lambda^k, \omega^k - v^k \rangle / \operatorname{dist}(\omega^k, \mathcal{Y}) < \infty$. Motivated by [35, 36], we consider the next proposition.

Proposition 5.1. *Consider the following assertions:*

1. *The error bound property holds at $x^* \in \mathcal{X}$, that is, there exist positive scalars δ and L such that dist$(x, \mathcal{X}) \leq L$ dist$(\mathcal{F}(x), \mathcal{Y})$ for all $x \in \mathbb{B}(x^*, \delta)$.*

2. *For every sequence $\{y^k\}$ with $y^k \to x^*$ and $y^k \notin \mathcal{X}$, we can take a subsequence $\{y^k\}_{k\in\mathbb{N}'}$ and a number $M > 0$ such that, for every $k \in \mathbb{N}'$*

$$\left\{\gamma \in N_{\mathcal{Y}}(\mathcal{F}(x^k)) : \left\|\frac{x^k - y^k}{\|x^k - y^k\|} + \nabla\mathcal{F}(x^k)^\top \gamma\right\| \leq \varepsilon_k \text{ and } \|\gamma\| \leq M\right\} \neq \emptyset \tag{18}$$

   for some $x^k \in$ argmin dist$(y^k, \mathcal{X})$ and $\varepsilon_k \in \mathbb{R}_+$ with $\varepsilon_k \to 0$.

Then, (1) *implies* (2)*. If $\mathcal{Y}$ satisfies the Assumption A at $\mathcal{F}(x^*)$, then* (2) *implies* (1)*.*

Proof. First, we will prove that (1) implies (2). Take a sequence $\{y^k\}$ with $y^k \notin \mathcal{X}$ and $y^k \to x^*$. Hence, $\mathcal{X}$ is closed, there is a $x^k \in \mathcal{X}$ such that $\operatorname{dist}(y^k, \mathcal{X}) = \|y^k - x^k\|$. By Fermat's rule, $0 \in \|x^k - y^k\|^{-1}(x^k - y^k) + N_\Omega(x^k)$. Since the error bound property holds at x^*, by Theorem 3 of [34], there is a $K > 0$ such that $N_\Omega(x^k) \subset \{\nabla\mathcal{F}(x^k)^\top\gamma^k : \|\gamma^k\| \leq K\|\nabla\mathcal{F}(x^k)^\top\gamma^k\| \text{ and } \gamma^k \in N_\Lambda(\mathcal{F}(x^k))\}$, $\forall k \in \mathbb{N}$. Thus, the vectors x^k and γ^k satisfy (18) with $\varepsilon_k := 0$.

Now, we will show that (2) implies (1). Suppose by contradiction that the error bound property does not hold at x^*. Then, there are sequences $\{y^k\}$ with $y^k \to x^*$ and $y^k \notin \mathcal{X}$ such that $\operatorname{dist}(y^k, \mathcal{X}) > k \operatorname{dist}(\mathcal{F}(y^k), \mathcal{Y})$

for every $k \in \mathbb{N}$. By (18), there are sequences $\{x^k\}$, $\{\varepsilon_k\}$, $\{r^k\}$ and $\{\gamma^k\}$ with $x^k \to x^*$, $x^k \in \Omega$, $\operatorname{dist}(y^k, \Omega) = \|x^k - y^k\|$, $\varepsilon_k \to 0$ such that $\|\gamma^k\| \le M$ and

$$r^k := \frac{x^k - y^k}{\|x^k - y^k\|} + \nabla \mathcal{F}(x^k)^\top \gamma^k \quad \text{with} \quad \gamma^k \in N_{\mathcal{Y}}(\mathcal{F}(x^k)) \quad \text{and} \quad \|r^k\| \le \varepsilon_k. \tag{19}$$

We obtain from (19) that, for each k sufficiently large,

$$\begin{aligned}\|y^k - x^k\| &= \langle \nabla \mathcal{F}(x^k)^\top \gamma^k, y^k - x^k \rangle + \langle -r^k, y^k - x^k \rangle \\ &= \langle \gamma^k, \mathcal{F}(y^k) - \mathcal{F}(x^k) \rangle + \langle \gamma^k, o(\|y^k - x^k\|) \rangle + \langle -r^k, y^k - x^k \rangle \\ &\le \langle \gamma^k, \mathcal{F}(y^k) - \mathcal{F}(x^k) \rangle + \frac{1}{2}\|y^k - x^k\|,\end{aligned}$$

where the last inequality follows from $o(\|y^k - x^k\|)/\|y^k - x^k\| \to_k 0$, $\|r^k\| \to_k 0$ and the boundedness of the sequence $\{\gamma^k\}$. Thus, we get that $k \operatorname{dist}(\mathcal{F}(y^k), \mathcal{Y}) < \operatorname{dist}(y^k, \mathcal{X}) = \|y^k - x^k\| \le 2\langle \gamma^k, \mathcal{F}(y^k) - \mathcal{F}(x^k) \rangle$ and $k < 2\langle \gamma^k, \mathcal{F}(y^k) - \mathcal{F}(x^k) \rangle / \operatorname{dist}(\mathcal{F}(y^k), \mathcal{Y})$.

Taking limit in the last expression we obtain a contradiction, since $\mathcal{Y}$ satisfies the Assumption A at $\mathcal{F}(x^*)$. ☐

Now, we discuss some specific cases where Assumption A holds. When $\mathcal{Y}$ is a closed convex set, we have that Assumption A holds at every point of $\mathcal{Y}$. In fact, by convexity, we always have $\langle \lambda^k, z^k - v^k \rangle \le 0$ for every $\lambda^k \in N_{\mathcal{Y}}(v^k)$ and every $z^k \in \mathcal{Y}$. Thus, (17) holds. The Assumption A may holds even for non-convex set. Indeed, consider the closed set $\mathcal{Y} = \{(y_1, y_2) \in \mathbb{R}^2 : (y_1^2 + y_2^2 - 1)y_2 \ge 0\}$ and $y^* = (0,0)$. Clearly, $\mathcal{Y}$ is not a convex set and Assumption A holds for $\mathcal{Y}$ at $y^* = (0,0)$. For our purpose, it is important to note that the Assumption A always holds when $\mathcal{Y}$ is the form $\mathbb{R}^p_- \times \{0\}^q \times \mathcal{C}^m$, whenever the basis point satisfies the strict complementary condition.

Theorem 5.2. *Let $\mathcal{Y} := \mathbb{R}^p_- \times \{0\}^q \times \mathcal{C}^m$. Then, the Assumption A holds for every point in $\mathcal{Y}$ that satisfies the strictly complementarity condition.*

Proof. Let $\bar{z} := (\bar{a}, \bar{b}, -(\bar{c}_1^1, \bar{c}_2^1), \ldots, -(\bar{c}_1^m, \bar{c}_2^m)) \in \mathcal{Y}$ such that the strictly complementarity assumption holds, i. e. , $\mathcal{J}(\bar{z}, \Lambda) = \{i \in \{1, \ldots, m\} : \bar{c}_1^i = 0, \bar{c}_2^i = 0\} = \emptyset$. Take every sequence $\{(\omega^k, v^k, \gamma^k)\}$ with $\omega^k \notin \mathcal{Y}$, $\nu^k \in \mathcal{Y}$, $\gamma^k \in N_{\mathcal{Y}}(v^k)$, $\omega^k \to \bar{z}$, $\nu^k \to \bar{z}$ and $\{\gamma^k\}$ bounded. Set $\gamma^k := (\mu^k, \lambda^k, (u_1^k, v_1^k), \ldots, (u_m^k, v_m^k))$ and for every $k \in \mathbb{N}$ define $\nu^k := (a^k, b^k, -(c_1^{1k}, c_2^{1k}), \ldots, -(c_1^{mk}, c_2^{mk}))$ and $\omega^k := (\alpha^k, \beta^k, (\eta_1^k, \tau_1^k), \ldots, (\eta_m^k, \tau_m^k))$. Now, we will proceed to to show that $\limsup_k \langle \gamma^k, \omega^k - \nu^k \rangle / \operatorname{dist}(\omega^k, \mathcal{Y}) < \infty$. Computing the inner product, we have that $\langle \gamma^k, \omega^k - \nu^k \rangle$ is equals to

$$\sum_{j \in \operatorname{supp}(a^k)} \mu_j^k \alpha_j^k + \sum_{i=1}^{q} \lambda_i^k \beta_i^k + \sum_{i \in \mathcal{I}(\nu^k)} u_i^k \eta_i^k + \sum_{i \in \mathcal{K}(\nu^k)} v_i^k \tau_i^k + \sum_{i \in \mathcal{J}(\nu^k)} \left(u_i^k \eta_i^k + v_i^k \tau_i^k \right). \tag{20}$$

For k sufficiently large, we have $\mathcal{I}(\bar{z}) \subset \mathcal{I}(\nu^k)$ and $\mathcal{K}(\bar{z}) \subset \mathcal{K}(\nu^k)$. Thus, by the strict complementary condition of $\bar{z}$, we get $\mathcal{J}(\nu^k) = \emptyset$, $\mathcal{K}(\bar{z}) = \mathcal{K}(\nu^k)$ and $\mathcal{I}(\bar{z}) = \mathcal{I}(\nu^k)$. Then, (20) is simplified to:

$$\langle \gamma^k, \omega^k - \nu^k \rangle = \sum_{j \in \operatorname{supp}(a^k)} \mu_j^k \alpha_j^k + \sum_{i=1}^{q} \lambda_i^k \beta_i^k + \sum_{i \in \mathcal{I}(\bar{z})} u_i^k \eta_i^k + \sum_{i \in \mathcal{K}(\bar{z})} v_i^k \tau_i^k. \tag{21}$$

Now, we will find an upper bound for (21). Define $M := \sup\{\|\gamma^k\| : k \in \mathbb{N}\}$. The first two sums are bounded by $\sum_{j=1}^{p} M \max\{\alpha_j^k, 0\} + \sum_{i=1}^{q} M|\beta_i^k|$. To bound the last two sums, let us recall that using the norm l_1, (Lemma 4.1 of [11]), we have that $\operatorname{dist}_1((\alpha, \beta), \mathcal{C}) = \max\{-\alpha, -\beta, -\alpha - \beta, \min\{\alpha, \beta\}\}$ for every $(\alpha, \beta) \in \mathbb{R}^2$. Now, take $i \in \mathcal{I}(\bar{z})$. Since $\omega^k \to \bar{z}$, we have $(\eta_i^k, \tau_i^k) \to_k -(\bar{c}_1^i, \bar{c}_2^i)$. Thus, there is $\varepsilon_i > 0$ such that $\tau_i^k < -\varepsilon_i$ and $|\eta_i^k| \le \varepsilon_i$ for k large enough. Now, if $\eta_i^k \le 0$ then $u_i^k \eta_i^k \le M(-\eta_i^k) \le M\operatorname{dist}_1((\eta_i^k, \tau_i^k), \mathcal{C})$ and if $\eta_i^k > 0$, from $|\eta_i^k| < -\tau_i^k$, we get $u_i^k \eta_i^k \le M(-\tau_i^k) \le M\operatorname{dist}_1((\eta_i^k, \tau_i^k), \mathcal{C})$. For both cases, we conclude that $u_i^k \eta_i^k \le M\operatorname{dist}_1((\eta_i^k, \tau_i^k), \mathcal{C})$ for k large enough. Similarly, for every $i \in \mathcal{K}(\bar{z})$, we deduce that $v_i^k \tau_i^k \le M\operatorname{dist}_1((\eta_i^k, \tau_i^k), \mathcal{C})$. From the above results and (21), we get

$$\begin{aligned}\langle \gamma^k, \omega^k - \nu^k \rangle &\le \sum_{j=1}^{p} M \max\{\alpha_j^k, 0\} + \sum_{i=1}^{q} M|\beta_i^k| + \sum_{i=1}^{m} M \mathrm{dist}_1((\eta_i^k, \tau_i^k), \mathcal{C}) \\ &\le M \mathrm{dist}_1(\omega^k, \mathcal{Y}),\end{aligned} \tag{22}$$

where the last inequality comes from Lemma 4.1 of [11]. Since all norms are equivalent, the Assumption A holds at $\bar{z}$. □

The Assumption A may not hold without the strictly complementarity condition, as the next example will show

Example 5.1. Consider $\mathcal{Y} = \mathcal{C} = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \le 0, y_2 \le 0, y_1 y_2 = 0\}$, $y^* = (0,0)$, $v^k := (0,0)$, $\gamma^k := -(1,0)$ and $\omega^k := -(1/k, 1/k^2)$. Clearly, $\omega^k \to y^*$ and $v^k \to y^*$. By Proposition 2.1, $\gamma^k \in N_{\mathcal{C}}(v^k)$. Hence $\mathrm{dist}(\omega^k, \mathcal{C}) = 1/k^2$, we get $\langle \gamma^k, \omega^k - v^k \rangle / \mathrm{dist}(\omega^k, \mathcal{C}) = k$. Thus, the Assumption A fails at y^*.

Let us come back to analysis of MPECs, i.e., $\mathcal{Y} = \Lambda$, $\mathcal{F} = F$ and $\mathcal{X} = \Omega$. We start by proving that MPEC-CRSC is sufficient to guarantee the error bound property if the basis point satisfies the strict complementary condition.

Theorem 5.3. *MPEC-CRSC implies the error bound property at any basis point where the strict complementary condition holds.*

Proof. Take $x^* \in \Omega$ such that $\mathcal{J}(x^*)$ is an empty set and MPEC-CRSC holds. By Proposition 5.1, we need to show that for every sequence $\{y^k\}$ with $y^k \to x^*$ and $y^k \notin \Omega$, there is a subsequence $\{y^k\}_{k \in K_1}$ of $\{y^k\}$ and a scalar $M > 0$, such that for every $k \in K_1$, $\{\gamma \in N_\Lambda(F(x^k)) : \| \|x^k - y^k\|^{-1}(x^k - y^k) + \nabla F(x^k)^\top \gamma\| \le \varepsilon_k, \text{ and } \|\gamma\| \le M\}$ is a non empty set, for some sequence $x^k \to x^*$ with $\|x^k - y^k\| = \mathrm{dist}(y^k, \Omega)$ and $\varepsilon_k \to 0$. Consider the set:

$$\widehat{\Omega} := \left\{ x \in \mathbb{R}^n : \begin{array}{l} h_i(x) = 0, i \in \mathcal{E}, \quad g_j(x) = 0, j \in A_-(x^*) \\ g_j(x) \le 0, \quad j \in \{1, \dots, p\} \setminus A_-(x^*), \\ 0 \le H_i(x) \perp G_i(x) \ge 0 \text{ for } i \in \{1, \dots, m\} \end{array} \right\}. \tag{23}$$

Since $\widehat{\Omega}$ is a subset of Ω and MPEC-CRSC holds for Ω at x^*, the constraint system $\widehat{\Omega}$ conforms MPEC-RCLPD at x^*. Write $\widehat{\Omega}$ as $\{x \in \mathbb{R}^n : \widehat{F}(x) \in \widehat{\Lambda}\}$, where $\widehat{F}(x) := (\widehat{g}(x), \widehat{h}(x), \Psi(x))$ and $\widehat{\Lambda} := \mathbb{R}_-^{p-|A_-(x^*)|} \times \{0\}^{|\mathcal{E}|+|A_-(x^*)|} \times \mathcal{C}^m$. Here, $\hat{g}$ and $\hat{h}$ represent the inequality and equality constraints of $\widehat{\Omega}$ respectively. Now, take a sequence $\{y^k\}$ with $y^k \to x^*$ and $y^k \notin \Omega$. For each $k \in \mathbb{N}$, consider the minimization problem

$$\text{Minimize } \|y^k - x\| \text{ subject to } x \in \widehat{\Omega}. \tag{24}$$

Take $x^k \in \Omega$ with $\mathrm{dist}(y^k, \Omega) = \|x^k - y^k\|$. By Lemma 3.1, both sets Ω and $\widehat{\Omega}$ coincide in a neighbourhood of x^*. Thus, for k large enough, x^k is a local minimizer of (24). Now, since MPEC-RCPLD has the local preservation property, MPEC-RCPLD holds at x^k for k large enough. Since x^k is a local minimizer and MPEC-RCPLD is a CQ for M-stationarity, we find $\widehat{\gamma}^k \in N_{\widehat{\Lambda}}(\widehat{F}(x^k))$ such that $\|x^k - y^k\|^{-1}(x^k - y^k) + \nabla \widehat{F}(x^k)^\top \widehat{\gamma}^k = 0$. By the proof of Theorem 4.1 of [17], we can assume that $\{\widehat{\gamma}^k\}$ is a bounded sequence. Only rest to show, that there is a subsequence $\{x^k\}_{k \in K_1}$ and a bounded sequence $\{\gamma^k\}_{k \in K_1} \subset N_\Lambda(F(x^k))$ such that $\nabla F(x^k)^\top \gamma^k = \nabla \widehat{F}(x^k)^\top \widehat{\gamma}^k + r^k$ for all $k \in K_1$ and $\lim_{k \in K_1} \|r^k\| = 0$. Suppose $\widehat{\gamma}^k := (\widehat{\mu}^k, \widehat{\lambda}^k, (\widehat{u}_1^k, \widehat{v}_1^k), \dots, (\widehat{u}_m^k, \widehat{v}_m^k))$. Thus, $\omega^k := \nabla \widehat{F}(x^k)^\top \widehat{\gamma}^k$ can be written as

$$\omega^k = \sum_{j \in A(x^k, \widehat{\Omega}) \cup A_-(x^*)} \hat{\mu}_j^k \nabla g_j(x^k) + \sum_{i \in \mathcal{E}} \hat{\lambda}_i^k \nabla h_i(x^k) - \sum_{\imath \in \mathcal{I}(x^k) \cup \mathcal{J}(x^k)} \hat{u}_\imath^k \nabla H_\imath(x^k) - \sum_{\jmath \in \mathcal{K}(x^k) \cup \mathcal{J}(x^k)} \hat{v}_\jmath^k \nabla G_\jmath(x^k) \tag{25}$$

where $\hat{\mu}_j^k \in \mathbb{R}_+$ for $j \in A(x^k, \widehat{\Omega})$, $\hat{\mu}_j^k \in \mathbb{R}$ for $j \in A_-(x^*)$, $\mathrm{supp}(\hat{\mu}^k) \subset A(x^k, \widehat{\Omega}) \cup A_-(x^*)$ and either $\hat{u}_\ell^k \hat{v}_\ell^k = 0$ or $\hat{u}_\ell^k > 0, \hat{v}_\ell^k > 0$ for each $\ell \in \mathcal{J}(x^k)$. Furthermore, $\{(\hat{\mu}^k, \hat{u}^k, \hat{v}^k)\}$ is a bounded sequence, since $\widehat{\gamma}^k$ is bounded.

Note that $A(x^k, \widehat{\Omega}) = A(x^k) \cap (\{1,\ldots,p\} \setminus A_-(x^*)) \subset A(x^k) \cap A_+(x^*)$ for every $x^k \in \widehat{\Omega}$ with k large enough. Now, define:

$$\omega_-^k := \omega^k - \sum_{j \in A(x^k,\widehat{\Omega})} \hat{\mu}_j^k \nabla g_j(x^k) + \sum_{\imath \in \mathcal{J}(x^k)\cup[\mathcal{I}(x^k)\setminus\mathcal{I}(x^*)]} \hat{u}_\imath^k \nabla H_\imath(x^k) + \sum_{\jmath \in \mathcal{J}(x^k)\cup[\mathcal{K}(x^k)\setminus\mathcal{K}(x^*)]} \hat{v}_\jmath^k \nabla G_\jmath(x^k). \quad (26)$$

Clearly, we have

$$\omega_-^k = \sum_{j \in A_-(x^*)} \hat{\mu}_j^k \nabla g_j(x^k) + \sum_{i\in\mathcal{E}} \hat{\lambda}_i^k \nabla h_i(x^k) - \sum_{\imath \in \mathcal{I}(x^*)} \hat{u}_\imath^k \nabla H_\imath(x^k) - \sum_{\jmath\in\mathcal{K}(x^*)} \hat{v}_\jmath^k \nabla G_\jmath(x^k) \quad (27)$$

and $\omega_-^k \in \text{span}\, \mathcal{G}(x^k; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*)) = \text{span}_+ \mathcal{G}(x^k; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$. Since $\{\omega_-^k\}$ is a bounded sequence, we can assume possibility after taking a subsequence, that $\omega_-^k \to_k \omega_-$ for some point ω_-. By the MPEC-CRSC, the set $\mathcal{G}(x; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$ has the same rank for every x near x^*. Thus, $\omega_-^k \in \text{span}\, \mathcal{G}(x^k; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$ and ω_- belongs to the subspace $\text{span}\, \mathcal{G}(x^*; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*)) \subset \text{span}_+ \mathcal{G}(x^*; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$.

Now, take index sets $\mathcal{E}' \subset \mathcal{E}$, $\mathcal{I}' \subset \mathcal{I}(x^*)$ and $\mathcal{K}' \subset \mathcal{K}(x^*)$ such that $\mathcal{G}(x^*; \emptyset, \mathcal{E}', \mathcal{I}', \mathcal{K}')$ is a basis of $\text{span}\, \mathcal{G}(x^*; \emptyset, \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$. Using Lemma 2.3, we find an index set $A_-' \subset A_-(x^*)$ and multipliers $\{(\bar{\mu}, \bar{\lambda}, \bar{u}, \bar{v})\}$ such that

$$\omega_- = \sum_{j\in A_-'} \bar{\mu}_j \nabla g_j(x^*) + \sum_{i\in\mathcal{E}'} \bar{\lambda}_i \nabla h_i(x^*) - \sum_{\imath\in\mathcal{I}'} \bar{u}_\imath^k \nabla H_\imath(x^*) - \sum_{\jmath\in\mathcal{K}'} \bar{v}_\jmath^k \nabla G_\jmath(x^*) \quad (28)$$

where $\bar{\mu}_j > 0$, for $j \in A_-'$ and $\mathcal{G}(x^*; A_-', \mathcal{E}', \mathcal{I}', \mathcal{K}')$ is a linearly independent set. Now, take an index set $A_-'' \subset A_-(x^*) \setminus A_-'$ such that $\mathcal{G}(x^*; A_-'' \cup A_-', \mathcal{E}', \mathcal{I}', \mathcal{K}')$ is a basis of $\text{span}\, \mathcal{G}(x^*; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$.

Again, using MPEC-CRSC, we have (for k large enough) that

$$\omega_-^k = \sum_{j\in A_-''} \check{\mu}_j^k \nabla g_j(x^k) + \sum_{j\in A_-'} \bar{\mu}_j^k \nabla g_j(x^k) + \sum_{i\in\mathcal{E}'} \bar{\lambda}_i^k \nabla h_i(x^k) - \sum_{\imath\in\mathcal{I}'} \bar{u}_\imath^k \nabla H_\imath(x^k) - \sum_{\jmath\in\mathcal{K}'} \bar{v}_\jmath^k \nabla G_\jmath(x^k) \quad (29)$$

for some multipliers $(\check{\mu}^k, \bar{\mu}^k, \bar{\lambda}^k, \bar{u}^k, \bar{v}^k)$. Furthermore, $\{(\check{\mu}^k, \bar{\mu}^k, \bar{\lambda}^k, \bar{u}^k, \bar{v}^k)\}$ is a bounded sequence. From (29), (28), $\omega_-^k \to \omega_-$ and since $\mathcal{G}(x^*; A_-'' \cup A_-', \mathcal{E}', \mathcal{I}', \mathcal{K}')$ is a basis, we get that $\check{\mu}_j^k \to_k 0$, $\bar{\mu}_j^k \to_k \bar{\mu}_j$, $\bar{\lambda}_i^k \to_k \bar{\lambda}$, $\bar{u}_\imath^k \to_k \bar{u}_\imath$ and $\bar{v}_\jmath^k \to_k \bar{v}_\jmath$.

Define $r^k := -\sum_{j\in A_-''} \check{\mu}_j^k \nabla g_j(x^k)$. It is clear that $r^k \to 0$. Substituing (29) into (26), we get

$$\omega^k = \sum_{j\in A_-' \cup A(x^k,\hat{\Omega})} \mu_j^k \nabla g_j(x^k) + \sum_{i\in\mathcal{E}} \lambda_i^k \nabla h_i(x^k) - \sum_{\imath\in\mathcal{I}(x^k)} u_\imath^k \nabla H_\imath(x^k) - \sum_{\jmath\in\mathcal{K}(x^k)} v_\jmath^k \nabla G_\jmath(x^k) - r^k \quad (30)$$

where the multipliers $(\mu^k, \lambda^k, u^k, v^k)$ are given by

$$\begin{array}{lll} \mu_j^k := \bar{\mu}_j^k (j \in A_-'), & \mu_j^k := \hat{\mu}_j^k (j \in A(x^k, \hat{\Omega})), & \lambda_j^k := \bar{\lambda}_j^k (j \in \mathcal{E}'), \\ u_j^k := \bar{u}_j^k (j \in \mathcal{I}'), & u_j^k := \hat{u}_j^k (j \in \mathcal{J}(x^k) \cup [\mathcal{I}(x^k) \setminus \mathcal{I}(x^*)]), & \\ v_j^k := \bar{v}_j^k (j \in \mathcal{K}'), & v_j^k := \hat{v}_j^k (j \in \mathcal{J}(x^k) \cup [\mathcal{K}(x^k) \setminus \mathcal{K}(x^*)]) & \end{array} \quad (31)$$

with $\mu^k \in \mathbb{R}_+^p$, $\text{supp}(\mu^k) \subset A_-' \cup A(x^k, \hat{\Omega}) \subset A(x^k)$, $\text{supp}(\lambda^k) \subset \mathcal{E}'$, $\text{supp}(u^k) \subset \mathcal{I}(x^k) \cup \mathcal{J}(x^k)$ and $\text{supp}(v^k) \subset \mathcal{K}(x^k) \cup \mathcal{J}(x^k)$.

Define $\gamma^k := (\mu^k, \lambda^k, (u_1^k, v_1^k), \ldots, (u_m^k, v_m^k))$ and $z^k := F(x^k)$. Clearly, $\gamma^k \in N_\Lambda(F(x^k))$, $\{\gamma^k\}$ is a bounded sequence and $\omega^k = \nabla F(x^k)^\top \gamma^k - r^k$. Put $\varepsilon_k := \|r^k\|$. Thus, substituting $\omega^k = \nabla \widehat{F}(x^k)^\top \widehat{\gamma}^k = \nabla F(x^k)^\top \gamma^k - r^k$ into $\|x^k - y^k\|^{-1}(x^k - y^k) + \nabla \widehat{F}(x^k)^\top \widehat{\gamma}^k = 0$, we obtain

$$\left\| \frac{x^k - y^k}{\|x^k - y^k\|} + \nabla F(x^k)^\top \gamma^k \right\| = \|r^k\| = \varepsilon_k \to 0, \text{ and } \gamma^k \in N_\Lambda(F(x^k)), \quad (32)$$

where $\{\gamma^k\}$ is a bounded sequence and $\|x^k - y^k\| = \text{dist}(y^k, \Omega)$. Thus, from Proposition 5.1, we conclude that MPEC-CRSC implies the existence of the error bound property whenever x^* satisfies the strict complementary condition. □

Now, we continue with the next theorem.

Theorem 5.4. *MPEC-RNNAMCQ implies the error bound property.*

Proof. We will use induction. Our hypothesis is that every constraint system with m complementary constraints, MPEC-RNNAMCQ implies the error bound property. When $m = 0$, MPEC-RNNAMCQ coincides with CRSC and the theorem is valid in the absence of complementary constraints. Now, assume that the theorem holds whenever the number of complementary constraints is less than $m_0 \in \mathbb{N}$. We will show that the theorem holds when the number of complementary constraints is m_0. Consider the following cases:

1. Case: $\mathcal{J}(x^*) = \emptyset$. In this case, x^* satisfies the strict complementarity condition. Hence, MPEC-CRSC is implied by MPEC-RNNAMCQ, we use the Theorem 5.3 to ensure that the error bound property holds.

2. Case: $\mathcal{J}(x^*) \neq \emptyset$ and the number of elements of $J(x^*)$ is less that m_0. Given a vector $(p, q, r, s) \in \mathbb{B}_\epsilon(0) \subset \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m$, define the set $\Omega(p, q, r, s) := \{x \in \mathbb{R}^n : g(x) \leq p, h(x) = q, 0 \leq H(x)+r \perp G(x)+s \geq 0\}$. Obviously, $\Omega = \Omega(0, 0, 0, 0)$. Now, choose $i_0 \in \{1, \ldots, m\} \setminus \mathcal{J}(x^*)$ and consider the following sets:

$$\Omega_1(p,q,r,s) := \left\{ x \in \mathbb{R}^n : \begin{array}{l} g(x) \leq p, h(x) = q, \\ H_{i_0}(x) + r_{i_0} = 0, G_{i_0}(x) + s_{i_0} \geq 0 \\ 0 \leq H_i(x) + r_i = 0 \perp G_i(x) + s_i \geq 0, \forall i \neq i_0 \end{array} \right\}$$

$$\Omega_2(p,q,r,s) := \left\{ x \in \mathbb{R}^n : \begin{array}{l} g(x) \leq p, h(x) = q, \\ H_{i_0}(x) + r_{i_0} \geq 0, G_{i_0}(x) + s_{i_0} = 0 \\ 0 \leq H_i(x) + r_i = 0 \perp G_i(x) + s_i \geq 0, \forall i \neq i_0 \end{array} \right\}.$$

Set $\Omega_1 := \Omega(0, 0, 0, 0)$ and $\Omega_2 := \Omega_2(0, 0, 0, 0)$. Clearly, $\Omega(p, q, r, s) = \Omega_1(p, q, r, s) \cup \Omega_2(p, q, r, s)$ and $\Omega = \Omega_1 \cup \Omega_2$. Note that the error bound property is equivalent to the existence of $L > 0$ and $\epsilon > 0$ such that $\operatorname{dist}(x, \Omega) \leq L\|(p, q, r, s)\|$, for all $(p, q, r, s) \in \mathbb{B}_\epsilon(0) \subset \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m$ and for all $x \in \Omega(p, q, r, s) \cap \mathbb{B}_\epsilon(x^*)$.

Now, since $i_0 \notin \mathcal{J}(x^*)$, i_0 is in $\mathcal{K}(x^*)$ or in $\mathcal{I}(x^*)$. If $i_0 \in \mathcal{K}(x^*)$ then $x^* \notin \Omega_1$. In this case, we will show that MPEC-RNNAMCQ holds for Ω_2 at x^*. Indeed, since $\mathcal{K}(x^*, \Omega_2) = \mathcal{K}(x^*) \setminus \{i_0\}$, $\mathcal{I}(x^*, \Omega_2) = \mathcal{I}(x^*)$, $\mathcal{J}(x^*, \Omega_2) = \mathcal{J}(x^*)$, $\mathcal{E}(x^*, \Omega_2) = \mathcal{E}(x^*) \cup \{i_0\}$ and $A_-(x^*, \Omega_2) = A_-(x^*)$. A simple inspection show that the set $\mathcal{G}(y, A_-(x^*, \Omega_2), \mathcal{E}(x^*, \Omega_2), \mathcal{I}(x^*, \Omega_2), \mathcal{K}(x^*, \Omega_2))$ is the same as $\mathcal{G}(y, A_-(x^*), \mathcal{E}(x^*), \mathcal{I}(x^*), \mathcal{K}(x^*))$ and hence it has the same rank for y sufficiently near to x^*. Thus, (H1) holds. Furthermore, since $A_+(x^*, \Omega_2) = A_+(x^*)$ and $\mathcal{J}(x^*, \Omega_2) = \mathcal{J}(x^*)$, (H3) holds. Thus, Ω_2 conforms MPEC-RNNAMCQ at x^*. Seeing that Ω_2 have less complementarity constraints that Ω, by induction, the set Ω_2 has the error bound property and hence there are $L > 0$ and $\epsilon > 0$ such that $\operatorname{dist}(x, \Omega_2) \leq L\|(p, q, r, s)\|$, for all $(p, q, r, s) \in \mathbb{B}_\epsilon(0) \subset \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m$ and for all $x \in \Omega_2(p, q, r, s) \cap \mathbb{B}_\epsilon(x^*)$. Now, since $i_0 \in \mathcal{K}(x^*)$, we can choose ϵ sufficiently small such that $\Omega \cap \mathbb{B}_\epsilon = \Omega_2 \cap \mathbb{B}_\epsilon$ and for every $(p, q, r, s) \in \mathbb{B}_\epsilon(0)$, $\Omega(p, q, r, s) \cap \mathbb{B}_\epsilon(x^*) = \Omega_2(p, q, r, s) \cap \mathbb{B}_\epsilon(x^*)$. With this choice of ϵ, we get that $\operatorname{dist}(x, \Omega) \leq L\|(p, q, r, s)\|$, for all $(p, q, r, s) \in \mathbb{B}_\epsilon(0) \subset \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m$ and for all $x \in \Omega(p, q, r, s) \cap \mathbb{B}_\epsilon(x^*)$. Thus, the error bound property holds for Ω at x^*. When $i_0 \in \mathcal{I}(x^*)$, using a similar technique, we can conclude that the error bound property holds for Ω at x^*.

3. Case: $\mathcal{J}(x^*) \neq \emptyset$ and the number of elements of $J(x^*)$ is equals to m_0. As MPEC-RNNAMCQ holds, take index sets such that (H3) holds. Now, consider the set

$$\Omega_0 := \left\{ x \in \mathbb{R}^n : \begin{array}{l} h_i(x) = 0, \quad (i \in \mathcal{E}'), \quad g_j(x) = 0, \quad (j \in A'_-) \\ g_j(x) \leq 0, \quad (j \in A_+(x^*)) \\ 0 \leq H_i(x) \perp G_i(x) \geq 0, \quad i \in \mathcal{J}(x^*) \end{array} \right\} \tag{33}$$

Now, since $\mathcal{J}(x^*) = \{1, \ldots, m\}$, we can use Lemma 5.3 of [15], to guaranteed that $g_j(x) = 0$, for $j \in A_-(x^*)$ for x near to x^*. Then, $\Omega \subset \Omega_0$ near to the basis point x^* and hence $\operatorname{dist}(x, \Omega) \leq \operatorname{dist}(x, \Omega_0)$ for x near to x^*. We continue to bound $\operatorname{dist}(x, \Omega_0)$. Since MPEC-NNAMCQ holds for Ω_0, the error bound property is valid and thus, there are scalars $L > 0$ and $\epsilon > 0$ such that for all $x \in \mathbb{B}_\epsilon(x^*)$ we get

$$\operatorname{dist}(x, \Omega_0) \leq L \max\{|h_i(x)|_{i \in \mathcal{E}'}, |g_j(x)|_{j \in A'_-}, \max\{g_j(x), 0\}_{j \in A_+}, \sum_{i=1}^{m} \operatorname{dist}((H_i(x), G_i(x)), \mathcal{C})\}. \tag{34}$$

Clearly, $A_+ \subset \{1,\ldots,m\} \setminus A'_-$ and from the expression (34) we obtain that $\mathrm{dist}(x,\Omega_0)$ is bounded by

$$L\max\{|h_i(x)|_{i\in\mathcal{E}}, |g_j(x)|_{j\in A'_-}, \max\{g_j(x),0\}_{j\in\{1,\ldots,m\}\setminus A'_-}, \sum_{i=1}^{m}\mathrm{dist}((H_i(x),G_i(x)),\mathcal{C}). \tag{35}$$

Thus, we only need to estimate $|g_j(x)|$, for all $j \in A'_-$. Following the proof of Theorem 5.5 of [15], we can find a number $\hat{L} > 0$ such that $|g_j(x)| \le \hat{L}\max\{|h_i(x)|_{i\in\mathcal{E}}, \max\{g_j(x),0\}_{j\in A(x^*)}\}$, for all $j \in A'_-$ and for x sufficiently near to x^*. From this and by (35) we obtain the desired result.

□

6 Conclusions

We presented two new CQs that are weaker than the previous MPEC-CQs based on constant positive linear dependence and the no non-zero abnormal multiplier. The new CQs for M-stationarity have the local preservation property and the error bound property, which may be useful for the stability and perturbation analysis of MPECs. We also establish some new relations among the new weaker MPEC-CQs and the others in the literature.

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Appendix

Proof of Lemma 3.2 The proof is based on induction over the number of complementary constraints. Our assumption is that for every constraint system Ω with m complementary constraints such that MPEC-CRSC holds at some feasible point x^*, there is a $\delta > 0$ (which may depends on Ω and x^*) such that for each feasible point $y \in \Omega \cap \mathbb{B}(x^*, \delta)$, the set $\mathcal{G}(x; A_-(y), \mathcal{E}, \mathcal{I}(y), \mathcal{K}(y))$ has the same rank for every x near y.

When $m = 0$, MPEC-CRSC coincides with CRSC and since CRSC has a local preservation property, the conclusion of the lemma is valid for $m = 0$. Now, assume that the lemma holds whenever the number of complementary constraints is less than $m_0 \in \mathbb{N}$. Assume $m = m_0$. Now, we will analyse the following cases:

1. Case: $\mathcal{J}(x^*)$ is an empty set. Consider the following constraint system

$$\widehat{\Omega} := \{x \in \mathbb{R}^n : g(x) \leq 0, h(x) = 0, H_i(x) = 0, G_j(x) = 0, i \in \mathcal{I}(x^*), j \in \mathcal{K}(x^*)\}.$$

   Since MPEC-CRSC holds for Ω at x^*, CRSC holds for $\widehat{\Omega}$ at x^*. Then, by Theorem 5.4 of [15], there is a $\delta_1 > 0$ such that for each feasible point $y \in \widehat{\Omega} \cap \mathbb{B}(x^*, \delta)$, $\mathcal{G}(x; A_-(y, \widehat{\Omega}), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$ has the same rank for every x near y. Now, by Theorem 5.4 of [15], $A_-(y, \widehat{\Omega}) = A_-(x^*, \widehat{\Omega})$, for all y in $\widehat{\Omega} \cap \mathbb{B}(x^*, \delta)$. Now, take $\delta_2 > 0$ such that $\mathcal{I}(x^*) \subset \mathcal{I}(y)$, $\mathcal{K}(x^*) \subset \mathcal{K}(y)$ and $\mathcal{J}(y) \subset \mathcal{J}(x^*)$ for every $y \in \Omega \cap \mathbb{B}(x^*, \delta_2)$. Observe that $\mathcal{I}(y) \setminus \mathcal{I}(x^*) \subset \mathcal{J}(x^*)$ and $\mathcal{K}(y) \setminus \mathcal{K}(x^*) \subset \mathcal{J}(x^*)$. Since $J(x^*)$ is empty, $\mathcal{I}(x^*) = \mathcal{I}(y)$ and $\mathcal{K}(x^*) = \mathcal{K}(y)$. Since, $A(y, \widehat{\Omega}) = A(y)$, we get that $A_-(y, \widehat{\Omega}) = A_-(y)$ for every $y \in \Omega \cap \mathbb{B}(x^*, \delta_2)$. Thus, for any $y \in \Omega \cap \mathbb{B}(x^*, \min\{\delta_1, \delta_2\})$, $\mathcal{G}(x; A_-(y, \widehat{\Omega}), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*)) = \mathcal{G}(x; A_-(y), \mathcal{E}, \mathcal{I}(y), \mathcal{K}(y))$ has the same rank for every x near y.

2. Case: $\mathcal{J}(x^*)$ is a non empty set. Suppose by contradiction that there is sequence $\{y^k\} \subset \Omega$ with $y^k \to x^*$ such that $\mathcal{G}(x; A_-(y^k), \mathcal{E}, \mathcal{I}(y^k), \mathcal{K}(y^k))$ does not have the same rank for some x near y^k. In addition, we can assume that $\mathcal{I}(x^*) \subset \mathcal{I}(y^k)$, $\mathcal{K}(x^*) \subset \mathcal{K}(y^k)$ $\mathcal{J}(y^k) \subset \mathcal{J}(x^*)$ and $A_-(x^*) \subset A_-(y^k)$ (Lemma 3.1), $\forall k \in \mathbb{N}$. Without loss of generality (possibility after taken an adequate subsequence) we can assume that $\mathcal{I}(y^k)$, $\mathcal{K}(y^k)$, $\mathcal{J}(y^k)$ and $A_-(y^k)$ are constant index sets, namely, $\mathcal{I}$, $\mathcal{K}$, $\mathcal{J}$ and A_- respectively. Depending if $\mathcal{J}$ is an empty set or not, we have the following sub-cases:

   (a) Sub-case: $\mathcal{J}$ is not an empty set. Here, consider the constraint system

$$\widetilde{\Omega_1} := \left\{ x \in \mathbb{R}^n : \quad \begin{array}{l} g(x) \leq 0, h(x) = 0, \\ 0 \leq H_i(x) \perp G_i(x) \geq 0, i \in \{1, \ldots, m_0\} \setminus \mathcal{J} \end{array} \right\}. \tag{36}$$

   Here MPEC-CRSC holds at x^* for $\widetilde{\Omega}_1$. In fact, it is enough to see that $\mathcal{I}(x^*, \widetilde{\Omega}_1) = \mathcal{I}(x^*)$, $\mathcal{K}(x^*, \widetilde{\Omega}_1) = \mathcal{K}(x^*)$, $A(x^*, \widetilde{\Omega_1}) = A(x^*)$ and $A_-(x^*, \widetilde{\Omega}_1) = A_-(x^*)$. Then, since $\widetilde{\Omega}_1$ has less complementary constraints that Ω we conclude, by induction, that there is a $\delta_1 > 0$ such that for every $y \in \widetilde{\Omega}_1 \cap$

$\mathbb{B}(x^*,\delta_1)$, $\mathcal{G}(x; A_-(y,\widetilde{\Omega}_1),\mathcal{E},\mathcal{I}(y,\widetilde{\Omega}_1),\mathcal{K}(y,\widetilde{\Omega_1}))$ has the same rank for every x near y. In particular, for $y = y^k \in \Omega$ for k large enough. But, since $\mathcal{J} = \mathcal{J}(y^k)$, we get $\mathcal{I}(y^k,\widetilde{\Omega}_1) = \mathcal{I}(y^k)\cap(\{1,\dots,m_0\}\setminus\mathcal{J}) = \mathcal{I}(y^k)$, $\mathcal{K}(y^k,\widetilde{\Omega}_1) = \mathcal{K}(y^k)\cap(\{1,\dots,m_0\}\setminus\mathcal{J}) = \mathcal{K}(y^k)$ and hence $A_-(y^k,\widetilde{\Omega_1}) = A_-(y^k)$. Thus, we get that $\mathcal{G}(x; A_-(y^k,\widetilde{\Omega}_1),\mathcal{E},\mathcal{I}(y^k,\widetilde{\Omega_1}),\mathcal{K}(y^k,\widetilde{\Omega_1})) = \mathcal{G}(x; A_-(y^k),\mathcal{E},\mathcal{I}(y^k),\mathcal{K}(y^k))$ has the same rank for every x near y^k, which is a contradiction.

(b) Sub-case: $\mathcal{J}$ is an empty set. Consider the constraint system

$$\widetilde{\Omega}_2 := \left\{ x\in\mathbb{R}^n : \begin{array}{l} g_j(x)\le 0, j\in A_+(x^*),\\ g_j(x)=0, j\in A_-(x^*), h(x)=0\\ 0\le H_i(x)\perp G_i(x)\ge 0 \text{ for } i\in\{1,\dots,m_0\}\end{array}\right\}, \tag{37}$$

Note that MPEC-RCPLD holds on $\widetilde{\Omega_2}$ at x^*. By Theorem 4.3 of [28], MPEC-RCPLD holds on $\widetilde{\Omega_2}$ for every $y\in\widetilde{\Omega_2}$ close to x^*. In particular, for $y = y^k\in\Omega$ for k large enough due to $A_-(x^*)\subset A_-(y^k)$. Now, since $\mathcal{J}(y^k) = \mathcal{J} = \emptyset$, MPEC-RCPLD holds on $\widetilde{\Omega_2}$ at y^k, RCPLD holds for $\widetilde{\Omega}_{2a}$ at y^k, where

$$\widetilde{\Omega}_{2a} := \left\{ x\in\mathbb{R}^n : \begin{array}{l} g_j(x)\le 0, j\in A_+(x^*), g_j(x)=0, j\in A_-(x^*)\\ h(x)=0, H_i(x)=0 \text{ for } i\in\mathcal{I}, G_j(x)=0 \text{ for } j\in\mathcal{K}\end{array}\right\}, \tag{38}$$

By Theorem 4.3 of [15], RCPLD implies CRSC. Thus, from CRSC, we conclude that $\mathcal{G}(x; A_-(y^k,\widetilde{\Omega}_{2a})\cup A_-(x^*),\mathcal{E},\mathcal{I},\mathcal{K})$ has the same rank for every x near y^k.

Now, we will show that $A_-(y^k,\widetilde{\Omega}_{2a}) = A_-(y^k)\cap A_+(x^*)$. Take $j\in A_-(y^k)\cap A_+(x^*)$. Then, $-\nabla g_j(y^k)\in \operatorname{span}_+\mathcal{G}(y^k; A_-(y^k),\mathcal{E},\mathcal{I}(y^k),\mathcal{K}(y^k))$ but since the cone $\operatorname{span}_+\mathcal{G}(y^k; A_-(y^k),\mathcal{E},\mathcal{I}(y^k),\mathcal{K}(y^k))$ is included in $\operatorname{span}_+\mathcal{G}(y^k; A_-(y^k)\setminus A_-(x^*),\mathcal{E},\mathcal{I}(y^k),\mathcal{K}(y^k)) + \operatorname{span}\{\nabla g_i(y^k) : i\in A_-(x^*)\}$, which is in $\operatorname{span}_+\mathcal{G}(y^k; A(y^k,\widetilde{\Omega}_{2a}), A_-(x^*)\cup\mathcal{E},\mathcal{I}(y^k),\mathcal{K}(y^k))$. We have that $j\in A_-(y^k,\widetilde{\Omega}_{2a})$. Now, let us prove the other inclusion. Take $j\in A_-(y^k,\widetilde{\Omega}_{2a})$. By (38), $j\in A_+(x^*)$ and $-\nabla g_j(y^k)\in \operatorname{span}_+\mathcal{G}(y^k; A_-(y^k)\cap A_+(x^*),\mathcal{E},\mathcal{I}(y^k),\mathcal{K}(y^k)) + \operatorname{span}\{\nabla g_i(y^k) : i\in A_-(x^*)\}$. But, this set is included into $\operatorname{span}_+\mathcal{G}(y^k; A_-(y^k),\mathcal{E},\mathcal{I}(y^k),\mathcal{K}(y^k))$, because $A_-(x^*)\subset A_-(y^k)$. Thus, we conclude that $j\in A_-(y^k)\cap A_+(x^*)$.

Now, from $A_-(y^k,\widetilde{\Omega}_{2a}) = A_-(y^k)\cap A_+(x^*)$, $\mathcal{I}(y^k) = \mathcal{I}$ and $\mathcal{K}(y^k) = \mathcal{K}$ we have that $\mathcal{G}(x; A_-(y^k,\widetilde{\Omega}_{2a})\cup A_-(x^*),\mathcal{E},\mathcal{I},\mathcal{K}) = \mathcal{G}(x; [A_-(y^k)\cap A_+(x^*)]\cup A_-(x^*),\mathcal{E},\mathcal{I},\mathcal{K})$ has the same rank for every x near y^k. But, the last set is, actually, $\mathcal{G}(x; A_-(y^k),\mathcal{E},\mathcal{I}(y^k),\mathcal{K}(y^k))$. Thus, we obtain a contradiction with the choice of the sequence $\{y^k\}$.

From all the cases, there is a $\delta>0$ such that, for each feasible point $y\in\Omega\cap\mathbb{B}(x^*,\delta)$, $\mathcal{G}(x; A_-(y),\mathcal{E},\mathcal{I}(y),\mathcal{K}(y))$ has the same rank for every x near y.

Proof of Theorem 4.1. We will use induction. Our hypothesis is that every constraint system with m complementary constraints, MPEC-RCPLD implies MPEC-CRSC. Clearly, the theorem is true when $m=0$, since RCPLD implies CRSC. Now, assume that the theorem holds whenever the number of complementary constraints is less than m_0. To show that the theorem holds when the number of complementary constraints is m_0. we consider the cases:

1. Case: $\mathcal{J}(x^*)$ is an empty set. Consider the following constraint system

$$\Omega_1 := \left\{ x\in\mathbb{R}^n : \begin{array}{l} g(x)\le 0, h(x)=0\\ H_i(x)=0, G_j(x)=0 \text{ for } i\in\mathcal{I}(x^*), j\in\mathcal{K}(x^*)\end{array}\right\}, \tag{39}$$

Since MPEC-RCPLD holds for Ω at x^*, RCPLD and CRSC hold for Ω_1 at x^*. But, if Ω_1 conforms CRSC at x^* then Ω holds MPEC-CRSC at x^*.

2. Case: $\mathcal{J}(x^*)$ is a non empty set. Define the next constraint system

$$\Omega_2 := \left\{ x\in\mathbb{R}^n : \begin{array}{l} g(x)\le 0, h(x)=0\\ 0\le H_i(x)\perp G_i(x)\ge 0 \text{ for } i\in\{1,\dots,m_0\}\setminus\mathcal{J}(x^*)\end{array}\right\}, \tag{40}$$

Hence MPEC-RCPLD holds at x^* for Ω, MPEC-RCPLD holds for Ω_2 at x^*. But, since Ω_2 has less complementary constraints that Ω, by induction we have that MPEC-CRSC holds for Ω_2 at x^*. Thus, the set $\mathcal{G}(y; A_-(x^*,\Omega_2), \mathcal{E}, \mathcal{I}(x^*,\Omega_2), \mathcal{K}(x^*,\Omega_2))$ has the same rank for y near x^*. But, since $\mathcal{I}(x^*,\Omega_2) = \mathcal{I}(x^*)$, $\mathcal{K}(x^*,\Omega_2) = \mathcal{K}(x^*)$ and $A_-(x^*,\Omega_2) = A_-(x^*)$, the set $\mathcal{G}(y; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$ has the same rank for y sufficiently close x^*. Thus, (H1) holds.

Now, take four index subsets $A_-^{'} \subset A_-(x^*)$, $\mathcal{E}^{'} \subset \mathcal{E}$, $\mathcal{I}^{'} \subset \mathcal{I}(x^*)$ and $\mathcal{I}^{'} \subset \mathcal{K}(x^*)$ such that $\mathcal{G}(x^*; A_-^{'}, \mathcal{E}^{'}, \mathcal{I}^{'}, \mathcal{K}^{'})$ is a basis of the subspace span $\mathcal{G}(x^*; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$. Furthermore, we can assume that the set $\mathcal{G}(x^*; \emptyset, \mathcal{E}^{'}, \mathcal{I}^{'}, \mathcal{K}^{'})$ is a linear basis of the span $\mathcal{G}(x^*; \emptyset, \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$. In order to complete the proof of the theorem, we will show that for each $A_+^{'} \subset A_+(x^*)$ and $\mathcal{J}_H^{'}, \mathcal{J}_G^{'} \subset \mathcal{J}(x^*)$, if there are multipliers $\{\lambda, \mu, u, v\}$ which are not all zero, $\mu_j \in \mathbb{R}_+$, $\forall j \in A_+^{'}$, and either $u_\ell v_\ell = 0$ or $u_\ell > 0, v_\ell > 0$ for each $\ell \in \mathcal{J}(x^*)$, such that

$$\sum_{j \in A_+^{'} \cup A_-^{'}} \mu_j \nabla g_j(x^*) + \sum_{i \in \mathcal{E}^{'}} \lambda_i \nabla h_i(x^*) + \sum_{\imath \in \mathcal{I}^{'} \cup \mathcal{J}_H^{'}} u_\imath \nabla H_\imath(x^*) + \sum_{\jmath \in \mathcal{K}^{'} \cup \mathcal{J}_G^{'}} v_\jmath \nabla G_\jmath(x^*) = 0. \tag{41}$$

Then, $\mathcal{G}(y; A_+^{'} \cup A_-^{'}, \mathcal{E}^{'}, \mathcal{I}^{'} \cup \mathcal{J}_H^{'}, \mathcal{K}^{'} \cup \mathcal{J}_G^{'})$ is linearly dependent for every $y \in \mathbb{B}(x^*, \delta)$. Assume that (41) holds for some multipliers $\{\lambda, \mu, u, v\}$, not all zero, such that $\mu_j \in \mathbb{R}_+$, $j \in A_+^{'}$ and either $u_\ell v_\ell = 0$ or $u_\ell > 0, v_\ell > 0$ for each $\ell \in \mathcal{J}(x^*)$. Denote

$$\omega := \sum_{j \in A_-^{'}} \mu_j \nabla g_j(x^*) + \sum_{i \in \mathcal{E}^{'}} \lambda_i \nabla h_i(x^*) + \sum_{\imath \in \mathcal{I}^{'}} u_\imath \nabla H_\imath(x^*) + \sum_{\jmath \in \mathcal{K}^{'}} v_\jmath \nabla G_\jmath(x^*). \tag{42}$$

Since, $\text{span}_+(x^*; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*)) = \text{span}(x^*; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$ by Remark 3, we have that ω is in $\text{span}_+(x^*; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$. Then, there is a subset $A_-^{''} \subset A_-(x^*)$ such that ω can be written as

$$\omega = \sum_{j \in A_-^{''}} \mu_j \nabla g_j(x^*) + \sum_{i \in \mathcal{E}^{'}} \lambda_i \nabla h_i(x^*) + \sum_{\imath \in \mathcal{I}^{'}} u_\imath \nabla H_\imath(x^*) + \sum_{\jmath \in \mathcal{K}^{'}} v_\jmath \nabla G_\jmath(x^*). \tag{43}$$

with $\mu_j \geq 0$, $j \in A_-^{''}$. By Lemma 2.3, we assume that $\mathcal{G}(x^*; A_-^{''}, \mathcal{E}^{'}, \mathcal{I}^{'}, \mathcal{K}^{'})$ is linearly independent and so $\mathcal{G}(y; A_-^{''}, \mathcal{E}^{'}, \mathcal{I}^{'}, \mathcal{K}^{'})$ for every $y \in \mathbb{B}(x^*, \delta_2)$ for some $\delta_2 > 0$. Substituting (43) into (41) and by MPEC-RCPLD, the set $\mathcal{G}(y; A_+^{'} \cup A_-^{''}, \mathcal{E}^{'}, \mathcal{I}^{'} \cup \mathcal{J}_H^{'}, \mathcal{K}^{'} \cup \mathcal{J}_G^{'})$ is linearly dependent for every y near x^*. Sum up, we have (i) $\mathcal{G}(y; A_+^{'} \cup A_-^{''}, \mathcal{E}^{'}, \mathcal{I}^{'} \cup \mathcal{J}_H^{'}, \mathcal{K}^{'} \cup \mathcal{J}_G^{'})$ is linearly dependent for every y close to x^*; (ii) $\mathcal{G}(y; A_-^{''}, \mathcal{E}^{'}, \mathcal{I}^{'}, \mathcal{K}^{'})$ is linearly independent for y close to x^* and (iii) span $\mathcal{G}(y; A_-^{''}, \mathcal{E}^{'}, \mathcal{I}^{'}, \mathcal{K}^{'})$ is a subset of span $\mathcal{G}(y; A_-^{'}, \mathcal{E}^{'}, \mathcal{I}^{'}, \mathcal{K}^{'})$ for every y close to x^*. From these results, we have that $\mathcal{G}(y; A_+^{'} \cup A_-^{''}, \mathcal{E}^{'}, \mathcal{I}^{'} \cup \mathcal{J}_H^{'}, \mathcal{K}^{'} \cup \mathcal{J}_G^{'})$ is linearly dependent for every y near x^*.

From, all the cases mentioned we conclude that MPEC-CRSC is valid at x^*.

Proof of Theorem 4.2. Let ω^* be an element of $\limsup_{(x,z)\to(x^*,F(x^*))} \nabla F(x)^\top N_\Lambda(z)$. Thus, there are sequences $\{x^k\}$, $\{z^k\}$, $\{\omega^k\}$ and $\{\gamma^k\}$ such that $x^k \to x^*$, $z^k \to F(x^*)$ and $\omega^k \to \omega^*$ with $\omega^k := \nabla F(x^k)^\top \gamma^k$ and $\gamma^k \in N_\Lambda(z^k)$. Furthermore, by Remark 2 of [18] we can assume $\gamma^k \in \widehat{N}_\Lambda(z^k)$. Denote γ^k by $(\mu^k, \lambda^k, (u_1^k, v_1^k), \ldots, (u_m^k, v_m^k))$. Then, for k large enough, we have that

$$\omega^k = \sum_{j \in A(x^*)} \mu_j^k \nabla g_j(x^k) + \sum_{i \in \mathcal{E}} \lambda_i^k \nabla h_i(x^k) - \sum_{\imath \in \mathcal{I}(z^k) \cup \mathcal{J}(z^k)} u_\imath^k \nabla H_\imath(x^k) - \sum_{\jmath \in \mathcal{K}(z^k) \cup \mathcal{J}(z^k)} v_\jmath^k \nabla G_\jmath(x^k) \tag{44}$$

where $\mu^k \in \mathbb{R}_+^p$, $\text{supp}(\mu^k) \subset A(x^*)$ and $u_\ell^k \geq 0, v_\ell^k \geq 0$, $\forall \ell \in \mathcal{J}(z^k)$. Besides, we can assume that $\mathcal{I}(x^*) \subset \mathcal{I}(z^k)$, $\mathcal{K}(x^*) \subset \mathcal{K}(z^k)$ and $\mathcal{J}(z^k) \subset \mathcal{J}(x^*)$, for all $k \in \mathbb{N}$. Write ω^k as $\omega_+^k + \omega_-^k$, where

$$\omega_+^k := \sum_{j \in A_+(x^*)} \mu_j^k \nabla g_j(x^k) - \sum_{\imath \in [\mathcal{I}(z^k) \setminus \mathcal{I}(x^*)] \cup \mathcal{J}(z^k)} u_\imath^k \nabla H_\imath(x^k) - \sum_{\jmath \in [\mathcal{K}(z^k) \setminus \mathcal{K}(x^*)] \cup \mathcal{J}(z^k)} v_\jmath^k \nabla G_\jmath(x^k) \tag{45}$$

and

$$\omega_-^k := \sum_{j\in A_-(x^*)} \mu_j^k \nabla g_j(x^k) + \sum_{i\in\mathcal{E}} \lambda_i^k \nabla h_i(x^k) - \sum_{\imath\in\mathcal{I}(x^*)} u_\imath^k \nabla H_\imath(x^k) - \sum_{\jmath\in\mathcal{K}(x^*)} v_\jmath^k \nabla G_\jmath(x^k). \tag{46}$$

Now, take index sets $A_-^{'} \subset A_-(x^*)$, $\mathcal{E}^{'} \subset \mathcal{E}$, $\mathcal{I}^{'} \subset \mathcal{I}(x^*)$ and $\mathcal{I}^{'} \subset \mathcal{K}(x^*)$ such that $\mathcal{G}(x^*; A_-^{'}, \mathcal{E}^{'}, \mathcal{I}^{'}, \mathcal{K}^{'})$ is a basis of span $\mathcal{G}(x^*; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$. By MPEC-CRSC, the set $\mathcal{G}(x; A_-^{'}, \mathcal{E}^{'}, \mathcal{I}^{'}, \mathcal{K}^{'})$ is a basis of the linear subspace span $\mathcal{G}(x; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*))$ for every x near x^*, in particular for $x = x^k$. Thus, we can write ω_-^k as

$$\omega_-^k = \sum_{j\in A_-^{'}} \hat{\mu}_j^k \nabla g_j(x^k) + \sum_{i\in\mathcal{E}^{'}} \hat{\lambda}_i^k \nabla h_i(x^k) - \sum_{\imath\in\mathcal{I}^{'}} \hat{u}_\imath^k \nabla H_\imath(x^k) - \sum_{\jmath\in\mathcal{K}^{'}} \hat{v}_\jmath^k \nabla G_\jmath(x^k) \tag{47}$$

for some $(\hat{\mu}^k, \hat{\lambda}^k, \hat{u}^k, \hat{v}^k) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m$ with $\operatorname{supp}(\hat{\mu}^k) \subset A_-^{'}$, $\operatorname{supp}(\hat{\lambda}^k) \subset \mathcal{E}^{'}$, $\operatorname{supp}(\hat{u}^k) \subset \mathcal{I}^{'}$ and $\operatorname{supp}(\hat{v}^k) \subset \mathcal{K}^{'}$. For each $k \in \mathbb{N}$, by using Lemma 2.3, we find index subsets, $A_+^{'}(k) \subset A_+(x^*)$, $\mathcal{I}_+^{'}(k) \subset \mathcal{I}(x^k) \setminus \mathcal{I}(x^*)$, $\mathcal{K}_+^{'}(k) \subset \mathcal{K}(x^k) \setminus \mathcal{K}(x^*)$, $\mathcal{J}_H^{'}(k), \mathcal{J}_G^{'}(k) \subset \mathcal{J}(x^k)$ such that

$$\omega_+^k = \sum_{j\in A_+^{'}(k)} \tilde{\mu}_j^k \nabla g_j(x^k) - \sum_{\imath\in\mathcal{I}_+^{'}(k)\cup\mathcal{J}_H^{'}(k)} \tilde{u}_\imath^k \nabla H_\imath(x^k) - \sum_{\jmath\in\mathcal{K}_+^{'}(k)\cup\mathcal{J}_G^{'}(k)} \tilde{v}_\jmath^k \nabla G_\jmath(x^k) \tag{48}$$

for some multipliers $(\tilde{\mu}^k, \tilde{u}^k, \tilde{v}^k) \in \mathbb{R}_+^p \times \mathbb{R}^m \times \mathbb{R}^m$ with $\operatorname{supp}(\tilde{\mu}^k) \subset A_+^{'}(k)$, $\operatorname{supp}(\tilde{u}^k) \subset \mathcal{I}_+^{'}(k) \cup \mathcal{J}_H^{'}(k)$, $\operatorname{supp}(\tilde{v}^k) \subset \mathcal{K}_+^{'}(k) \cup \mathcal{J}_G^{'}(k)$, and $\tilde{u}_\ell^k, \tilde{v}_\ell^k \geq 0$, $\ell \in \mathcal{J}(z^k)$ such that for each $k \in \mathbb{N}$, the set of vectors $\mathcal{G}(x^k; A_+^{'}(k) \cup A_-^{'}, \mathcal{E}^{'}, \mathcal{I}^{'} \cup \mathcal{I}_+^{'}(k) \cup \mathcal{J}_H^{'}(k), \mathcal{K}^{'} \cup \mathcal{K}_+^{'}(k) \cup \mathcal{J}_H^{'}(k))$ is linearly independent. Since there is only a finite number of index subset, we can assume, after taken an adequate subsequence, that $A_+^{'}(k)$, $\mathcal{I}_+^{'}(k)$, $\mathcal{K}_+^{'}(k)$, $\mathcal{J}_H^{'}(k)$ and $\mathcal{J}_G^{'}(k)$ are all independent of $k \in \mathbb{N}$. Denote them by $A_+^{'}$, $\mathcal{I}_+^{'}$, $\mathcal{K}_+^{'}$, $\mathcal{J}_H^{'}$ and $\mathcal{J}_G^{'}$ respectively. Thus, we get

$$\omega^k = \sum_{j\in A_+^{'}\cup A_-^{'}} \bar{\mu}_j^k \nabla g_j(x^k) + \sum_{i\in\mathcal{E}^{'}} \bar{\lambda}_i^k \nabla h_i(x^k) - \sum_{\imath\in\mathcal{I}^{'}\cup\mathcal{I}_+^{'}\cup\mathcal{J}_H^{'}} \bar{u}_\imath^k \nabla H_\imath(x^k) - \sum_{\jmath\in\mathcal{K}^{'}\cup\mathcal{K}_+^{'}\cup\mathcal{J}_G^{'}} \bar{v}_\jmath^k \nabla G_\jmath(x^k) \tag{49}$$

where the multipliers $(\bar{\mu}^k, \bar{\lambda}^k, \bar{u}^k, \bar{v}^k) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m$ are given by

$$\begin{array}{llll} \bar{\lambda}_i^k := \hat{\lambda}_i^k (i \in \mathcal{E}^{'}), & \bar{\mu}_j^k := \tilde{\mu}_j^k (j \in A_+^{'}), & \bar{\mu}_j^k := \hat{\mu}_j^k (j \in A_-^{'}) & \\ \bar{u}_\imath^k := \hat{u}_\imath^k (\imath \in \mathcal{I}^{'}), & \bar{u}_\imath^k := \tilde{u}_\imath^k (\imath \in \mathcal{I}_+^{'} \cup \mathcal{J}_H^{'}) & \bar{v}_\jmath^k := \hat{v}_\jmath^k (\jmath \in \mathcal{K}^{'}), & \bar{v}_\jmath^k := \tilde{v}_\jmath^k (\jmath \in \mathcal{K}_+^{'} \cup \mathcal{J}_G^{'})) \end{array} \tag{50}$$

with $\operatorname{supp}(\bar{\mu}) \subset A_-^{'} \cup A_+^{'}$, $\operatorname{supp}(\bar{\lambda}) \subset \mathcal{E}^{'}$, $\operatorname{supp}(\bar{u}) \subset \mathcal{I}^{''} \cup \mathcal{J}_H^{'}$ and $\operatorname{supp}(\bar{v}) \subset \mathcal{K}^{''} \cup \mathcal{J}_G^{'}$. It is not difficult to see, rearranging, that $\bar{\gamma}^k := (\bar{\mu}^k, \bar{\lambda}^k, (\bar{u}_1^k, \bar{v}_1^k), \dots, (\bar{u}_m^k, \bar{v}_m^k)) \in N_{\bar{\Lambda}}(\bar{z}^k)$ where $\bar{\Lambda} := \mathbb{R}_-^{p-|A_(x^*)|} \times \{0\}^{|\mathcal{E}|+|A_-(x^*)|} \times \mathcal{C}^m$, $\bar{z}_i^k := z_i^k$, $\forall i \notin A_-(x^*)$ and $\bar{z}_i^k = 0$ otherwise. Now, the sequence $\{(\bar{\mu}^k, \bar{\lambda}^k, \bar{u}^k, \bar{v}^k)\}$ has a subsequence bounded, otherwise, dividing the expression (49) by $M_k := \|(\bar{\mu}^k, \bar{\lambda}^k, \bar{u}^k, \bar{v}^k)\|$ and taking an adequate convergent subsequence of $M_k^{-1}(\bar{\mu}^k, \bar{\lambda}^k, \bar{u}^k, \bar{v}^k)$, says $\{(\bar{\mu}, \bar{\lambda}, \bar{u}, \bar{v})\}$, we obtain that

$$\sum_{j\in A_+^{'}\cup A_-^{'}} \bar{\mu}_j \nabla g_j(x^*) + \sum_{i\in\mathcal{E}^{'}} \bar{\lambda}_i \nabla h_i(x^*) - \sum_{\imath\in\mathcal{I}^{'}\cup\mathcal{I}_+^{'}\cup\mathcal{J}_H^{'}} \bar{u}_\imath \nabla H_\imath(x^*) - \sum_{\jmath\in\mathcal{K}^{'}\cup\mathcal{K}_+^{'}\cup\mathcal{J}_G^{'}} \bar{v}_\jmath \nabla G_\jmath(x^*) = 0, \tag{51}$$

where $\{(\bar{\mu}, \bar{\lambda}, \bar{u}, \bar{v}))\}$ is different to zero. From the outer semi-continuity of the normal cone $N_{\bar{\Lambda}}$ and $\bar{z}^k \to F(x^*)$, we get that $(\bar{\mu}, \bar{\lambda}, (\bar{u}_1, \bar{v}_1), \dots, (\bar{u}_m, \bar{v}_m)) \in N_{\bar{\Lambda}}(F(x^*))$. Thus, $\bar{\mu}_j \geq 0$ $(j \in A_+^{'})$ and either $\bar{u}_\ell \bar{v}_\ell = 0$ or $\bar{u}_\ell > 0, \bar{v}_\ell > 0$ for each $\ell \in \mathcal{J}(x^*)$. Since (51) holds, by MPEC-CRSC, the set $\mathcal{G}(y; A_+^{'} \cup A_-^{'}, \mathcal{E}^{'}, \mathcal{I}^{'} \cup \mathcal{I}_+^{'} \cup \mathcal{J}_H^{'}, \mathcal{K}^{'} \cup \mathcal{K}_+^{'} \cup \mathcal{J}_H^{'})$ is linearly dependent for every y near x^*, in particular for $y = x^k$, which is a contradiction.

Thus, $\{(\bar{\mu}^k, \bar{\lambda}^k, \bar{u}^k, \bar{v}^k)\}$ has a convergent subsequence and without loss of generality, we assume that $(\bar{\mu}^k, \bar{\lambda}^k, \bar{u}^k, \bar{v}^k)$ itself converges to $(\bar{\mu}, \bar{\lambda}, \bar{u}, \bar{v})$. Furthermore, by (5) and the outer semi-continuity of the normal cone, we get that $(\bar{\mu}, \bar{\lambda}, (\bar{u}_1, \bar{v}_1), \dots, (\bar{u}_m, \bar{v}_m))$ is in $N_{\bar{\Lambda}}(F(x^*))$ and

$$\omega^* = \sum_{j\in A_+^{'}\cup A_-^{'}} \bar{\mu}_j \nabla g_j(x^*) + \sum_{i\in\mathcal{E}^{'}} \bar{\lambda}_i \nabla h_i(x^*) - \sum_{\imath\in\mathcal{I}^{'}\cup\mathcal{I}_+^{'}\cup\mathcal{J}_H^{'}} \bar{u}_\imath \nabla H_\imath(x^*) - \sum_{\jmath\in\mathcal{K}^{'}\cup\mathcal{K}_+^{'}\cup\mathcal{J}_G^{'}} \bar{v}_\jmath \nabla G_\jmath(x^*). \tag{52}$$

where $\bar{\mu}_j \in \mathbb{R}_+$, $(j \in A'_+)$, $\bar{\mu}_j \in \mathbb{R}$, $(j \in A'_-)$ and either $\bar{u}_\ell \bar{v}_\ell = 0$ or $\bar{u}_\ell > 0, \bar{v}_\ell > 0$ for each $\ell \in \mathcal{J}(x^*)$. Moreover, $\text{supp}(\bar{\mu}) \subset A'_+ \cup A'_-$, $\text{supp}(\bar{\lambda}) \subset \mathcal{E}'$ $\text{supp}(\bar{u}) \subset \mathcal{I}' \cup \mathcal{I}'_+ \cup \mathcal{J}'_H$ and $\text{supp}(\bar{v}) \subset \mathcal{K}' \cup \mathcal{K}'_+ \cup \mathcal{J}'_G$. But, since $\mathcal{I}(x^*)$, $\mathcal{K}(x^*)$ and $\mathcal{J}(x^*)$ is a partition,

$$\sum_{j \in A'_-} \bar{\mu}_j \nabla g_j(x^*) + \sum_{i \in \mathcal{E}'} \bar{\lambda}_i \nabla h_i(x^*) + \sum_{\imath \in \mathcal{I}'} \bar{u}_\imath \nabla H_\imath(x^*) + \sum_{\jmath \in \mathcal{K}'} \bar{v}_\jmath \nabla G_\jmath(x^*) \in \text{span}_+ \mathcal{G}(x^*; A_-(x^*), \mathcal{E}, \mathcal{I}(x^*), \mathcal{K}(x^*)) \quad (53)$$

and $A'_+ \subset A_+(x^*)$, $\mathcal{I}'_+ \cup \mathcal{J}'_H \subset \mathcal{J}(x^*)$, $\mathcal{K}'_+ \cup \mathcal{J}'_G \subset \mathcal{J}(x^*)$ with $\bar{\mu}_j \geq 0$ $(j \in A'_+)$, we get from (53) and from (52), that ω^* is an element of $\nabla F(x^*)^\top N_\Lambda(F(x^*))$ as we wanted to prove.