A New Voltage Stability-Constrained Optimal Power Flow Model: Sufficient Condition, SOCP Representation, and Relaxation

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Abstract

A simple characterization of the solvability of power flow equations is of great importance in the monitoring, control, and protection of power systems. In this paper, we introduce a sufficient condition for power flow Jacobian nonsingularity. We show that this condition is second-order conic representable when load powers are fixed. Through the incorporation of the sufficient condition, we propose a voltage stability-constrained optimal power flow (VSC-OPF) formulation as a second-order cone program (SOCP). An approximate model is introduced to improve the scalability of the formulation to larger systems. Extensive computation results on Matpower and NESTA instances confirm the effectiveness and efficiency of the formulation.

1 Introduction

Power flow equations are ubiquitous in power system analysis. It is shown in [1] that the singularity of power flow Jacobian is closely related to the loss of steady-state stability. In the paper, we consider the steady-state voltage stability problem as a power flow solvability problem. Reliable numerical tools to calculate the distance to power flow stability boundary are available [2, 3, 4]. However, these tools provide no explicit conditions certifying power flow feasibility, which makes the incorporation of voltage stability conditions in other problems a challenge.

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There has been a resurgence in recent years in the search for explicit conditions quantifying the feasibility of power flow equations along the lines of Wu [5] and Ilić [6]. In [7], a sufficient condition for existence and uniqueness of high-voltage solution in distribution system is obtained using fixed-point argument, which is extended in [8]. Similar techniques have subsequently been applied to yield stronger results in [9] and [10]. Conditions on solution existence and uniqueness in lossless system with PV buses are given in [11, 12]. These methods are to be contrasted with heuristic conditions based on system equivalence [13, 14].

To avoid system instability, security constraints such as voltage magnitude limits and line flow limits are enforced in normal system operation. However, high penetrations of distributed generators (DGs) and extensive deployment of reactive power compensation devices are able to provide support to hold up voltages within operational bounds even when system stability margin is low. Therefore, the effectiveness of the security constraints in safeguarding system stability may be diminished in modern power systems. Another motivation for the inclusion of steady-state stability limit in an optimal power flow (OPF) formulation is the increasing trend to operate power systems ever closer to their operational limits due to increased demand and competitive electricity market. Without stability constraints, the robustness of the OPF solution against voltage instability is not ensured. Voltage stability constraint in an OPF setting can be rigorously represented using the minimum singular value (MSV) of the power flow Jacobian as in [15, 16, 17]. However, the computational cost of this method is high. To achieve a better trade-off between accuracy and efficiency, some conditions quantifying system stress level have been used in voltage stability-related OPF problems that are either heuristic [18, 19], or based on DC [20] or reactive power flow model [21].

We give a new and simplified proof for a sufficient condition for power flow Jacobian nonsingularity that we proposed recently in [22]. The condition can be seen as a generalization to the heuristic condition proposed in [23]. We then formulate a voltage stability-constrained OPF (VSC-OPF) problem in which the voltage stability margin is quantified by the condition. We show that when load powers are fixed, this voltage stability condition describes a second-order conic representable set in a transformed voltage space. Thus second-order cone program (SOCP) reformulation can naturally incorporate the condition. Notice that the formulation does not require the DC or decoupled power flow assumptions. To improve computation time, we sparsify the dense stability constraints while preserving very high accuracy.

The rest of the paper is organized as follows. Section 2 provides background on power system modeling. The sufficient condition for power flow Jacobian nonsingularity is introduced in Section 3. We discuss the VSC-OPF formulation, its convex reformulation, and sparse approximation in Section 4. Section 5 presents results of extensive computational experiments. Concluding remarks are given in Section 6.

2 Background

2.1 Notations

The cardinality of a set or the absolute value of a (possibly) complex number is denoted by $|\cdot|$. $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. $\mathbb{R}$ and $\mathbb{C}$ are the set of real and complex numbers, respectively. For vector $x \in \mathbb{C}^n$, $\|x\|_p$ denotes the p-norm of x where $p \geq 1$ and $\mathrm{diag}(x) \in \mathbb{C}^{n\times n}$ is the associated diagonal matrix. The n-dimensional identity matrix is denoted by $\mathbf{I}_n$. $\mathbf{0}_{n\times m}$ denotes an $n \times m$ matrix of all 0's. For $A \in \mathbb{C}^{n\times n}$, A^{-1} is the inverse of A. For $B \in \mathbb{C}^{m\times n}$, B^T, B^H are respectively the transpose and conjugate transpose of B, and B^* is the matrix with complex conjugate entries. The real and imaginary parts of B are denoted as $\mathrm{Re}\, B$ and $\mathrm{Im}\, B$.

2.2 Power system modeling

We consider a connected single-phase power system with $n + m$ buses operating in steady-state. The underlying topology of the system can be described by an undirected connected graph $G = (\mathcal{N}, \mathcal{E})$, where $\mathcal{N} = \mathcal{N}_G \cup \mathcal{N}_L$ is the set of buses equipped with ($\mathcal{N}_G$) and without ($\mathcal{N}_L$) generators (or generator buses and load buses), and that $|\mathcal{N}_G| = m$ and $|\mathcal{N}_L| = n$. We number the buses such that the set of load buses are $\mathcal{N}_L = \{1, \dots, n\}$ and the set of generator buses are $\mathcal{N}_G = \{n+1, \dots, n+m\}$. Generally, for a complex matrix $A \in \mathbb{C}^{(n+m)\times k}$, define $A_L = (A_{ij})_{i\in\mathcal{N}_L}$. That is, A_L is the first n rows of the matrix A. Similarly, define $A_G = (A_{ij})_{i\in\mathcal{N}_G}$. Every bus i in the system is associated with a voltage phasor $V_i = |V_i|e^{\mathrm{i}\theta_i}$ where $|V_i|$ and θ_i are the magnitude and phase angle of the voltage. We will find it convenient to adopt rectangular coordinates for voltages sometimes, so we also define $V_i = e_i + \mathrm{i} f_i$. The generator buses are modeled as PV buses, while load buses are modeled as PQ buses. For bus i, the injected power is given as $S_i = P_i + \mathrm{i}Q_i$.

The line section between buses i and j in the system is weighted by its complex admittance $y_{ij} = 1/z_{ij} = g_{ij} + \mathrm{i}b_{ij}$. The nodal admittance matrix $Y = G + \mathrm{i}B \in \mathbb{C}^{(n+m)\times(n+m)}$ has components $Y_{ij} = -y_{ij}$ and $Y_{ii} = y_{ii} + \sum_{j=1}^{n+m} y_{ij}$ where y_{ii} is the shunt admittance at bus i.

The nodal admittance matrix relates system voltages and currents as

$$\begin{bmatrix} I_L \\ I_G \end{bmatrix} = \begin{bmatrix} Y_{LL} & Y_{LG} \\ Y_{GL} & Y_{GG} \end{bmatrix} \begin{bmatrix} V_L \\ V_G \end{bmatrix}. \tag{1}$$

We obtain from (1) that

$$V_L = -Y_{LL}^{-1} Y_{LG} V_G + Y_{LL}^{-1} I_L. \tag{2}$$

Define the vector of equivalent voltage to be $E = -Y_{LL}^{-1} Y_{LG} V_G$ and the impedance matrix to be $Z = Y_{LL}^{-1}$ (we assume the invertibility of Y_{LL} and note that this is almost always the case for practical systems). With the definitions, (2) can be rewritten as $V_L = E + Z I_L$.

For practical power systems, the generator buses have regulated voltage magnitudes and small phase angles. It is common in voltage stability analysis to assume that the generator buses have constant voltage phasor V_G [13, 14, 22, 23]. The assumption can be partially justified by the fact that voltage instability are mostly caused by system overloading due to excess demand at load side, irrelevant of generator voltage variations.

Assumption 1. *The vector of generator bus voltages V_G is constant.*

Note that Assumption 1 is always satisfied for uni-directional distribution systems where the only source is modeled as a slack bus with fixed voltage phasor.

The power flow equations in the rectangular form relate voltages and power injections at each bus $i \in \mathcal{N}$ via

$$P_i = \sum_{j=1}^{n+m} [G_{ij}(e_i e_j + f_i f_j) + B_{ij}(e_j f_i - e_i f_j)], \tag{3a}$$

$$Q_i = \sum_{j=1}^{n+m} [G_{ij}(e_j f_i - e_i f_j) - B_{ij}(e_i e_j + f_i f_j)]. \tag{3b}$$

2.3 AC-OPF formulation

Using the power flow equations (3a)-(3b), a standard AC-OPF model can be written as

$$\begin{aligned}
\min \quad & \sum_{i \in \mathcal{N}_G} f_i(P_{G_i}) && && \text{(4a)} \\
\text{s.t.} \quad & P_i(e, f) = P_{G_i} - P_{D_i}, && i \in \mathcal{N} && \text{(4b)} \\
& Q_i(e, f) = Q_{G_i} - Q_{D_i}, && i \in \mathcal{N} && \text{(4c)} \\
& \underline{P}_{G_i} \le P_{G_i} \le \overline{P}_{G_i}, && i \in \mathcal{N}_G && \text{(4d)} \\
& \underline{Q}_{G_i} \le Q_{G_i} \le \overline{Q}_{G_i}, && i \in \mathcal{N}_G && \text{(4e)} \\
& \underline{V}_i^2 \le e_i^2 + f_i^2 \le \overline{V}_i^2, && i \in \mathcal{N} && \text{(4f)} \\
& |P_{ij}(e, f)| \le \overline{P}_{ij}, && (i, j) \in \mathcal{E} && \text{(4g)} \\
& |I_{ij}(e, f)| \le \overline{I}_{ij}, && (i, j) \in \mathcal{E}, && \text{(4h)}
\end{aligned}$$

where $f_i(P_{G_i})$ in (4a) is the variable production cost of generator i, assuming to be a convex quadratic function; P_{G_i} and P_{D_i} in (4b)-(4c) are the real power generation and load at bus i, respectively; Q_{G_i} and Q_{D_i} are the reactive power generation and load at bus i; $P_i(e,f)$ and $Q_i(e,f)$ are given by the power flow equations (3); constraints (4d)-(4e) represent the real and reactive power generation capability of generator i. P_{ij} and I_{ij} in (4g)-(4h) are the real power and current magnitude flowing from bus i to j for line $(i,j) \in \mathcal{E}$, respectively.

3 A Sufficient Condition for Nonsingularity of Power flow Jacobian

A sufficient condition for the nonsingularity of power flow Jacobian is recently proposed in [22] as stated in the following theorem. We will use this result extensively in the paper. Below, we give a new and simplified proof.

Theorem 1. *The power flow Jacobian of (3) is nonsingular if*

$$|V_i| - \|z_i^T \operatorname{diag}(I_L)\|_1 > 0, \quad i \in \mathcal{N}_L. \tag{5}$$

Proof. The Jacobian of power flow equations (3) of load buses is given as

$$\begin{aligned} J = &\begin{bmatrix} \operatorname{diag}(e_L) & -\operatorname{diag}(f_L) \\ \operatorname{diag}(f_L) & \operatorname{diag}(e_L) \end{bmatrix} \begin{bmatrix} G_{LL} & -B_{LL} \\ -B_{LL} & -G_{LL} \end{bmatrix} \\ &+ \operatorname{diag}\left(\begin{bmatrix} G_L & -B_L \\ G_L & -B_L \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} \right) + \operatorname{diag}\left(\begin{bmatrix} B_L & G_L \\ -B_L & -G_L \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} \right) \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_n \\ \mathbf{I}_n & \mathbf{0}_{n\times n} \end{bmatrix}. \end{aligned} \tag{6}$$

Define the matrix $T \in \mathbb{C}^{2n\times 2n}$ as

$$T = \begin{bmatrix} \mathbf{I}_n & \mathbf{I}_n \\ -\mathrm{i}\mathbf{I}_n & \mathrm{i}\mathbf{I}_n \end{bmatrix},$$

then we can construct a matrix M similar to J through T as

$$\begin{aligned} M = T^{-1} J T &= \begin{bmatrix} \operatorname{diag}(I_L^*) & \operatorname{diag}(V_L) Y_L^* \\ \operatorname{diag}(V_L^*)\, Y_L & \operatorname{diag}(I_L) \end{bmatrix} \\ &= I_d^* + \operatorname{diag}(V_{\text{aug}})\, Y_{\text{ant}}^*, \end{aligned}$$

where the matrices I_d and Y_{ant} are defined as

$$I_d = \operatorname{diag}(I_{\text{aug}}), \quad Y_{\text{ant}} = \begin{bmatrix} \mathbf{0}_{n\times n} & Y_{LL} \\ Y_{LL}^* & \mathbf{0}_{n\times n} \end{bmatrix},$$

and the augmented vectors are $V_{\text{aug}} = \begin{bmatrix} V_L \\ V_L^* \end{bmatrix}$ and $I_{\text{aug}} = \begin{bmatrix} I_L \\ I_L^* \end{bmatrix}$. We denote the inverse of Y_{ant}^* by Z_{ant}. That is,

$$(Y_{\text{ant}}^*)^{-1} = Z_{\text{ant}} = \begin{bmatrix} \mathbf{0}_{n\times n} & Z \\ Z^* & \mathbf{0}_{n\times n} \end{bmatrix},$$

and multiply M by the anti-block diagonal matrix Z_{ant} as

$$N = MZ_{\text{ant}} = \operatorname{diag}(V_{\text{aug}}) + I_d^* Z_{\text{ant}}. \tag{8}$$

Since the similarity transformation preserves the eigenstructure of a matrix, and the product of two nonsingular matrices is nonsingular, therefore, J is nonsingular if and only if N is nonsingular.

We then define the matrix J_c as

$$J_c = \begin{bmatrix} \operatorname{diag}(V_L) & Z \operatorname{diag}(I_L) \\ Z^* \operatorname{diag}(I_L^*) & \operatorname{diag}(V_L^*) \end{bmatrix} = \operatorname{diag}(V_{\text{aug}}) + Z_{\text{ant}} I_d^*.$$

Note that when I_L does not have zero elements, J_c can be obtained by a similarity transformation of N as

$$J_c = (I_d^*)^{-1} N I_d^*, \tag{9}$$

which clearly shows the nonsingularity of N since condition (5) ensures the strict diagonal dominance of J_c and the Levy-Desplanques theorem [24, Sect. 5.6] in turn guarantees the nonsingularity of the matrix J_c.

We claim that the nonsingularity of J_c implies that of N even when I_L contains zero elements. To see this, we argue in three cases:

- When I_L does not have zero elements the claim holds since J_c and N are similar.
- When $I_L = \mathbf{0}_{n\times 1}$, $N = \operatorname{diag}(V_{\text{aug}}) = J_c$.
- When $I_i = 0$ for $i \in \mathcal{I} \subset \mathcal{N}_L$ (without loss of generality we may assume $\mathcal{I} = \{1, \ldots, p\}$ and $\mathcal{J} = \{p+1 \ldots, n\}$), we permute the matrix N to separate the buses with and without current injections and apply similarity transformation to the relevant block.

Specifically, define the permutation matrix

$$P = \begin{bmatrix} \mathbf{I}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n-p} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{n-p} \end{bmatrix},$$

where the subscripts of zero matrices are omitted. Then it can be verified that the permuted matrix $P^T NP$ is

$$P^T NP = \begin{bmatrix} \mathrm{diag}(V_I) & \mathbf{0} \\ R & N_{\mathrm{red}} \end{bmatrix},$$

where

$$V_I = \begin{bmatrix} V_{\mathcal{I}} \\ V_{\mathcal{I}}^* \end{bmatrix}, N_{\mathrm{red}} = \begin{bmatrix} \mathrm{diag}(V_{\mathcal{J}}) & \mathrm{diag}(I_{\mathcal{J}}^*) Z_{\mathcal{J}\mathcal{J}} \\ \mathrm{diag}(I_{\mathcal{J}}) Z_{\mathcal{J}\mathcal{J}}^* & \mathrm{diag}(V_{\mathcal{J}}^*) \end{bmatrix}.$$

We can now perform similarity transformation on N_{red} analogous to (9) and condition (5) applied on the resulting matrix ensures its diagonal dominance, which implies the nonsingularity of N_{red}. In addtion, since $\mathrm{diag}(V_I)$ is nonsingular, it is easy to see $P^T NP$, thus N, is also nonsingular. We have then proved the claim.

In summary, we have shown that condition (5) implies the nonsingularity of J_c, which in turn implies the nonsingularity of J. This completes the proof. □

4 A New Model for VSC-OPF

The standard AC-OPF formulation embeds system security constraints as line real power and current limits in (4g) and (4h). However, the parameters in these security-related constraints, such as $\overline{P}_{ij}$ and $\overline{I}_{ij}$, are calculated off-line using possible dispatch scenarios that do not necessarily represent the actual system conditions [25]. This motivates the formulation of VSC-OPF models such as the one using MSV of the power flow Jacobian [15]. However as noted in the literature, it is numerically difficult to handle the singular value constraints [16, 17]. In this section, we propose a new model for VSC-OPF using the voltage condition derived in (5) and show that it has nice convex properties amenable for efficient computation.

4.1 New formulation

We propose the following new VSC-OPF model,

$$\min \quad \sum_{i\in\mathcal{N}_G} f_i(P_{G_i}) \tag{10a}$$

$$\text{s.t.} \quad (4b) - (4f)$$

$$|V_i| - \sum_{j=1}^{n} \frac{|Z_{ij}||S_j|}{|V_j|} \geq \underline{t}_i, \quad i \in \mathcal{N}_L. \tag{10b}$$

The key constraint is (10b), which reformulates the left-hand side of (5) by writing line currents as the ratio of apparent powers that satisfy the power flow equations (3) and voltages, and $\underline{t}_i$ is a preset positive parameter to control the level of voltage stability. To ensure that (10) is a proper formulation with good computational property, we first show that the set of voltages satisfying condition (10b) is voltage stable, and then we show that (10b) is second-order cone (SOC) representable, thus convex, when S_L is constant. The condition of constant S_L is always met in OPF problems.

4.1.1 Connectedness

A necessary condition for voltage instability is the singularity of power flow Jacobian [26, Sect. 7.1.2]. Assume that the zero injection solution of power flow equations (3) is voltage stable with a nonsingular Jacobian (we see from (8) that this implies all load voltage magnitudes are nonzero since otherwise N is singular). We know from (6) that every entry of J is a continuous function of voltages, so the eigenvalues of J are also continuous in voltages. Since a continuous function maps a connected set to another connected set, if a given connected set of power flow solutions contains the zero injection solution (which is voltage stable) and the corresponding power flow Jacobian of every point in the set is nonsingular, then the set characterizes a subset of voltage stable solutions. Define the set $\mathcal{S}_0 := \{V_L| \text{ (5) holds}\}$ and $\mathcal{S}_0 \supseteq \mathcal{S}_{\underline{t}} := \{V_L| \text{ (10b) holds}\}$. We know from Theorem 1 that the power flow Jacobian is nonsingular for $V_L \in \mathcal{S}_0$, we also know the zero injection solution is in $\mathcal{S}_0$. Therefore, in order to show the set $\mathcal{S}_{\underline{t}}$ is voltage stable, we show the more general case that $\mathcal{S}_0$ is voltage stable, which amounts to showing the connectedness of $\mathcal{S}_0$. We give the proof of this property below.

Theorem 2. *The set $\mathcal{S}_0$ is connected.*

Proof. Given any $v_1 \in \mathcal{S}_0$, and let the zero injection solution be $v_0 \in \mathcal{S}_0$. Define V_L parametrized by $t \in [0, 1]$ as $V_L(t) = v_0 + (v_1 - v_0)\, t$. With the definition we find that current injections are linear functions of t, since

$$I_L(t) = Y_{LL}\left(V_L(t) - E\right) \tag{11a}$$
$$= Y_{LL}\left(x_0 + (x_1 - x_0)\,t - x_0\right) \tag{11b}$$
$$= Y_{LL}\left(x_1 - x_0\right) t. \tag{11c}$$

We claim that the derivative of $|V_i|$ has smaller magnitude than the derivative of $\sum_{j=1}^{n} |Z_{ij} I_j|$ for all $i \in \mathcal{N}_L$. Since current injections are linear in t, let $Z_{ij} I_j$ be denoted by $a_{ij} t + \mathrm{i} b_{ij} t$ and E_i by $E_i + \mathrm{i} F_i$. Then $|V_i|$ can be represented by

$$|V_i| = \left|E_i - z_i^T I_i\right| = \sqrt{\left(E_i - \sum_{j=1}^{n} a_{ij} t\right)^2 + \left(F_i - \sum_{j=1}^{n} b_{ij} t\right)^2}, \tag{12}$$

and the derivative of $|V_i|$ with respect to t is (where we shorthand $\sum_{j=1}^{n} a_{ij}$ by a for brevity)

$$\frac{\mathrm{d}|V_i|}{\mathrm{d}t} = -\frac{a(e_i - at) + b(f_i - bt)}{\sqrt{(e_i - at)^2 + (f_i - bt)^2}}. \tag{13}$$

Then, by Cauchy-Schwarz inequality we have $|\mathrm{d}|V_i|/\mathrm{d}t| \leq \sqrt{a^2 + b^2}$. On the other hand, the derivative of $\sum_{j=1}^{n} |Z_{ij} I_j|$ is $\mathrm{d}/\mathrm{d}t \sum_{j=1}^{n} |Z_{ij} I_j| = \sum_{j=1}^{n} \sqrt{a_{ij}^2 + b_{ij}^2} \geq \sqrt{a^2 + b^2}$, where the inequality is due to successive application of trigonometric inequality. Comparing the two derivatives confirms the claim that the derivative of $|V_i|$ has smaller magnitude than the derivative of $\sum_{j=1}^{n} |Z_{ij} I_j|$ for all $i \in \mathcal{N}_L$. Since the function $|V_i| - \sum_{j=1}^{n} |Z_{ij} I_j|$ is monotonically decreasing, we see the line segment connecting v_0 and v_1 lies in $\mathcal{S}_0$. □

4.1.2 SOC representation of voltage stability constraint

The voltage stability constraint (10b) is not directly a convex constraint in the voltage variable V_i, however, we show that it can be reformulated as a convex constraint, more specifically, an SOC constraint in squared voltage magnitude $|V_i|^2$ providing S_L is fixed. This SOC reformulation will be utilized in the following section for SOCP relaxation of VSC-OPF.

Proposition 1. *Constraint* (10b) *is SOC representable in the squared voltage magnitude* $|V_i|^2$*'s, i.e.* (10b) *can be reformulated using SOC constraints in* $|V_i|^2$*'s.*

Proof. First of all, introduce variable $c_{ii} := |V_i|^2$, and x_i, y_i for each bus $i \in \mathcal{N}_L$ such that

$$x_i \leq \sqrt{c_{ii}}, \tag{14}$$
$$x_i y_i \geq 1, \tag{15}$$

$$x_i \geq 0.$$

Then we see both (14) and (15) can be rewritten as the following SOC constraints

$$\sqrt{x_i^2 + \frac{(c_{ii}-1)^2}{4}} \leq \frac{c_{ii}+1}{2},$$
$$\sqrt{1 + \frac{(x_i-y_i)^2}{4}} \leq \frac{x_i+y_i}{2}.$$

Therefore, by defining $A_{ij} = |Z_{ij}S_j|$, ((10b)) can be equivalently represented as

$$x_i - \sum_{j=1}^{n} A_{ij}y_j \geq \underline{t}_i, \tag{16a}$$
$$\left\| [x_i, (c_{ii}-1)/2]^T \right\|_2 \leq (c_{ii}+1)/2, \tag{16b}$$
$$\left\| [1, (x_i-y_i)/2]^T \right\|_2 \leq (x_i+y_i)/2, \tag{16c}$$
$$x_i \geq 0, \tag{16d}$$

for every bus $i \in \mathcal{N}_L$, which are SOCP constraints. □

4.2 SOCP relaxation of VSC-OPF

By Proposition 1, the voltage stability condition (5) is reformulated as SOCP constraints (16). However, the power flow equations (4b)-(4c) are still nonconvex. In the following, we propose an SOCP relaxation of the proposed VSC-OPF model (10) by combining the SOC reformulation of the voltage stability constraint (16) with the recent development of SOCP relaxation of standard AC-OPF [27]. In particular, for each line $(i, j) \in \mathcal{E}$, define

$$c_{ij} = e_ie_j + f_if_j \tag{17a}$$
$$s_{ij} = e_if_j - e_jf_i. \tag{17b}$$

An implied constraint of (17a)-(17b) is the following:

$$c_{ij}^2 + s_{ij}^2 = c_{ii}c_{jj}. \tag{18}$$

Now we can introduce the following SOCP relaxation of the VSC-OPF model (10) in the new variables c_{ii}, c_{ij}, and s_{ij} as follows

$$\min \quad \sum_{i \in \mathcal{N}_G} f_i(P_{G_i})$$

$$
\text{s.t.} \quad P_{G_i} - P_{D_i} = G_{ii}c_{ii} + \sum_{j \in N(i)} P_{ij}, \ i \in \mathcal{N} \tag{19a}
$$
$$
Q_{G_i} - Q_{D_i} = -B_{ii}c_{ii} + \sum_{j \in N(i)} Q_{ij}, i \in \mathcal{N} \tag{19b}
$$
$$
\underline{V}_i^2 \leq c_{ii} \leq \overline{V}_i^2, \quad i \in \mathcal{N} \tag{19c}
$$
$$
c_{ij} = c_{ji}, \ s_{ij} = -s_{ji}, \quad (i,j) \in \mathcal{E} \tag{19d}
$$
$$
c_{ij}^2 + s_{ij}^2 \leq c_{ii}c_{jj} \quad (i,j) \in \mathcal{E} \tag{19e}
$$
$$
(4d), (4e), (16),
$$

where the power flow equations (4b)-(4c) are rewritten in the c, s variables as (19a) and (19b). $N(i)$ denotes the set of buses adjacent to bus i. The line real and reactive powers are $P_{ij} = G_{ij}c_{ij} - B_{ij}s_{ij}$ and $Q_{ij} = -G_{ij}s_{ij} - B_{ij}c_{ij}$. The nonconvex constraint (18) is relaxed as (19e), which can be easily written as an SOCP constraint as $\|[c_{ij}, s_{ij}, (c_{ii} - c_{jj})/2]^T\|_2 \leq (c_{ii} + c_{jj})/2$. (19c) is a linear constraint in the square voltage magnitude c_{ii}. Notice that the SOCP formulation of the voltage stability constraint (16) is not a relaxation, but an exact formulation of the original voltage stability condition (10b), and it fits nicely into the overall SOCP relaxation of the VSC-OPF model (19).

4.3 Sparse approximation of SOCP relaxation

It is observed through experiments that the computation times of the VSC-OPF formulation (19) are significantly longer than normal OPF especially for larger instances, which is primarily due to the density of stability condition (16a). To speed up computation, we approximate the coeffcient matrix A of the stability constraints by a sparse matrix $\tilde{A}$. To illustrate our approach of sparse approximation, we rewrite the linear constraint (16a) in matrix-vector form as

$$
x - Ay \geq \underline{t}. \tag{20}
$$

Then the approach to construct the sparse approximate matrix $\tilde{A}$ can be summarized as follows.

Simply put, for each row of matrix A, Algorithm 1 constructs the corresponding row of the approximate matrix $\tilde{A}$ by ignoring all elements except the largest ones whose sum amounts to more than γ of the total row sum. We notice that the element Z_{ij} of the impedance matrix can be understood as the coupling intensity measure between buses i and j. Thanks to the sparsity of practical power systems, each bus is only strongly coupled with its neighboring buses and weakly coupled with most other buses. Therefore, the matrix $\tilde{A}$ is generally sparse. We notice a similar approximation has been applied to the L-index in the

Algorithm 1 Sparse approximation of A

$\gamma \leftarrow 0.98$ ▷ initialize tunable sparsity parameter
$\tilde{A} \leftarrow \mathbf{0}_{n\times n}$ ▷ initialize $\tilde{A}$
for $1 \le i \le n$ **do**
  $RS \leftarrow \sum_j A_{ij}$ ▷ compute ith row sum of matrix A
  while $\sum_j \tilde{A}_{ij} < \gamma RS$ **do**
    $j_{\max} \leftarrow \arg\max a_i$
    $\tilde{A}_{i,j_{\max}} \leftarrow A_{i,j_{\max}}$
    $A_{i,j_{\max}} \leftarrow 0$
  end while
end for

context of PMU allocation [28]. The connection between L-index and the proposed stability condition has been discussed in [22].

Then (20) can be approximated by

$$x - \tilde{A}y \ge \underline{t} + \Delta a/\overline{V}, \tag{21}$$

where $\Delta a \in \mathbb{R}^n$ is the row sum difference between A and $\tilde{A}$ that is defined as $\Delta a_i = \sum(a_i - \tilde{a}_i)$ and $\overline{V} = \max\{\overline{V}_i \mid i \in \mathcal{N}_L\}$. We have thus obtained the sparse VSC-OPF formulation which is identical to (19) except the stability constraint (16a) is replaced by (21). The new formulation is presented as

$$\begin{aligned}
\min \quad & \sum_{i\in\mathcal{N}_G} f_i(P_{G_i}) \\
\text{s.t.} \quad & P_{G_i} - P_{D_i} = G_{ii}c_{ii} + \sum_{j\in N(i)} P_{ij},\ i \in \mathcal{N} && \text{(22a)}\\
& Q_{G_i} - Q_{D_i} = -B_{ii}c_{ii} + \sum_{j\in N(i)} Q_{ij}, i \in \mathcal{N} && \text{(22b)}\\
& \underline{V}_i^2 \le c_{ii} \le \overline{V}_i^2, \quad i \in \mathcal{N} && \text{(22c)}\\
& c_{ij} = c_{ji},\ s_{ij} = -s_{ji}, \quad (i,j) \in \mathcal{E} && \text{(22d)}\\
& c_{ij}^2 + s_{ij}^2 \le c_{ii}c_{jj} \quad (i,j) \in \mathcal{E} && \text{(22e)}\\
& \text{(4d)}, \text{(4e)}, \text{(16b)–(16d)}, \text{(21)}
\end{aligned}$$

We notice that feasibility of problem (22) is implied by the feasibility of the original problem (19). To see this we only need to focus on (21) and (16a), from which we have

$$x - Ay \le x - \tilde{A}y - (\Delta a)y_{\min}$$

$$\leq x - \tilde{A}y - \Delta a/\overline{V},$$

where the last inequality comes from (14), (15), (19c).

5 Computational Experiments

In this section, we present extensive computational results on the proposed VSC-OPF model (10), its SOCP relaxation (19), and the sparse approximation (22) tested on standard IEEE instances available from Matpower [29] and instances from the NESTA 0.6.0 archive [30]. The code is written in Matlab. For all experiments, we used a 64-bit computer with Intel Core i7 CPU 2.60GHz processor and 4 GB RAM. We study the effectiveness of the proposed VSC-OPF on achieving voltage stability, the tightness of the SOCP relaxation for the VSC-OPF, as well as the speed-up and accuracy of the sparse approximation.

Two different solvers are used for VSC-OPF:

- Nonlinear interior point solver IPOPT [31] is used to find local optimal solutions to VSC-OPF.
- Conic interior point solver MOSEK 7.1 [32] is used to solve the SOCP relaxation of VSC-OPF.

5.1 Method

For each test case, we choose the margin $\underline{t}_i$ in constraint (16a) by running a base case power flow based on the given system loading and dispatch information. We obtain the stability margin for each load bus as

$$t_i = |V_i| - \sum_{j \in \mathcal{N}_L} \frac{A_{ij}}{|V_j|}, \quad i \in \mathcal{N}_L.$$

Then, the thresholds $\underline{t}_i$ are obtained as the minimum of t_i,

$$\underline{t}_i = \min\{t_i, i \in \mathcal{N}_L\}, \quad i \in \mathcal{N}_L.$$

That is, we use the stability margin given by the base case power flow result as the stability threshold for all load buses.

Table 1: Results Summary for Standard IEEE Instances.

Test Case	Cost ($/h)		OG (%)	t_b	t_a^{AC}	MI (%)	t_a^{SOCP}	DT (%)
	AC	SOCP						
case5	14997.04	14997.04	0.00	0.9642	1.0631	10.26	1.0631	0.00
case6ww	3189.39	3124.03	2.05	0.8924	0.8924	0.00	0.8925	0.01
case9	5296.69	5296.67	0.00	0.7967	0.9276	16.43	0.9277	0.01
case14	——	8920.68	—	0.9553	—	—	0.9553	—
case24_ieee_rts	63352.20	63344.53	0.01	0.8355	0.8589	2.80	0.8591	0.02
case30	574.52	573.57	0.17	0.9066	0.9617	6.08	0.9626	0.09
case_ieee30	8907.27	8902.85	0.05	0.8483	0.8483	0.00	0.8483	0.00
case39	41864.18	41854.60	0.02	0.7974	0.8269	3.70	0.8270	0.01
case57	41737.79	41710.79	0.06	0.6411	0.6598	2.92	0.6658	0.91
case89pegase	5817.60	5810.13	0.13	0.6732	0.7033	4.47	0.7257	3.18
case118	129660.69	129341.60	0.25	0.9076	0.9656	6.39	0.9733	0.80
case300	719725.10	718654.33	0.15	0.2296	0.2572	12.02	0.2304	10.42
case1354pegase	74060.41	73998.93	0.08	0.5928	0.6321	6.63	0.6329	0.13
case2383wp	1857927.67	1846921.88	0.59	0.6222	0.7711	23.93	0.7809	1.27
average	——	——	**0.27**	**0.7467**	**0.7975**	**6.80**	**0.7992**	**1.30**

Table 2: Results Summary for NESTA Instances From Congested Operating Conditions.

Test Case	Cost ($/h)		OG (%)	t_b	t_a^{AC}	MI (%)	t_a^{SOCP}	DT (%)
	AC	SOCP						
nesta_case3_lmbd__api	362.57	355.38	1.98	—	—	—	—	—
nesta_case4_gs__api	743.14	741.67	0.20	0.9492	0.9492	0.00	0.9492	0.00
nesta_case5_pjm__api	2984.91	2984.91	0.00	1.0133	1.0268	1.33	1.0268	0.00
nesta_case6_c__api	811.62	811.58	0.00	0.6519	0.6882	5.57	0.6888	0.08
nesta_case6_ww__api	266.01	237.10	10.87	0.8834	0.8834	0.00	0.8838	0.05
nesta_case9_wscc__api	633.21	632.78	0.07	0.6710	0.6710	0.00	0.6710	0.00
nesta_case14_ieee__api	321.32	320.72	0.19	0.8346	0.8479	1.59	0.8484	0.07
nesta_case24_ieee_rts__api	5006.49	5005.73	0.02	0.5963	0.6201	3.99	0.6207	0.09
nesta_case29_edin__api	294620.42	294446.04	0.06	0.9218	0.9252	0.36	0.9289	0.40
nesta_case30_as__api	539.08	538.37	0.13	0.7944	0.7944	0.00	0.7944	0.00
nesta_case30_fsr__api	200.85	200.66	0.09	0.9078	0.9345	2.94	0.9369	0.26
nesta_case30_ieee__api	408.81	408.44	0.09	0.7289	0.7289	0.00	0.7289	0.00
nesta_case39_epri__api	6750.98	6727.85	0.34	0.6802	0.6802	0.00	0.6802	0.00
nesta_case57_ieee__api	1430.65	1427.70	0.21	0.6096	0.6096	0.00	0.6099	0.05
nesta_case73_ieee_rts__api	17010.76	16990.38	0.12	0.5964	0.6117	2.57	0.5999	1.93
nesta_case89_pegase__api	——	3411.40	—	0.6769	—	—	0.6865	—
nesta_case118_ieee__api	5548.61	5524.12	0.44	0.8579	0.9061	5.62	0.9167	1.17
nesta_case162_ieee_dtc__api	5185.20	5154.86	0.59	-0.1207	-0.1047	13.21	-0.1092	4.28
nesta_case189_edin__api	1836.68	1820.78	0.87	0.4631	0.4631	0.00	0.4631	0.00
nesta_case300_ieee__api	22342.36	22290.27	0.23	0.2009	0.2151	7.09	0.2119	1.50
nesta_case1354_pegase__api	——	59544.01	—	0.5748	—	—	0.6005	—
nesta_case1394sop_eir__api	3012.54	2999.02	0.45	0.5412	0.7137	31.87	0.7325	2.64
nesta_case1397sp_eir__api	5662.22	5638.48	0.42	0.4964	0.6174	24.38	0.7489	21.31
nesta_case1460wp_eir__api	5956.71	5924.47	0.54	0.6051	0.6781	12.06	0.7185	5.95
nesta_case2224_edin__api	43280.89	43247.42	0.08	0.6116	0.6116	0.00	0.6118	0.03
nesta_case2383wp_mp__api	23247.15	23184.81	0.27	0.7434	0.7434	0.00	0.7530	1.28
nesta_case2736sp_mp__api	25075.37	24980.89	0.38	0.7957	0.7969	0.14	0.8067	1.23
nesta_case2737sop_mp__api	20751.56	20677.25	0.36	0.8555	0.8657	1.19	0.8784	1.46
average	——	——	**0.73**	**0.6756**	**0.6991**	**4.56**	**0.7080**	**1.75**

Table 3: Results Summary of Sparse Approximation for NESTA Instances From Congested Operating Conditions.

Test Case	Time (sec)		DCT (%)	Cost ($/h)	DC (%)	t_a^s	DT (%)
	Normal	Sparse					
nesta_case3_lmbd__api	0.70	0.68	1.98	355.38	0.00	—	—
nesta_case4_gs__api	0.73	0.74	-1.54	741.67	0.00	0.9492	0.00
nesta_case5_pjm__api	0.70	0.70	0.79	2984.91	0.00	1.0268	0.00
nesta_case6_c__api	0.72	0.72	0.74	811.58	0.00	0.6888	0.00
nesta_case6_ww__api	0.76	0.76	0.03	237.10	0.00	0.8838	0.00
nesta_case9_wscc__api	0.91	0.80	12.59	626.86	0.94	0.6694	0.25
nesta_case14_ieee__api	0.75	0.71	4.97	320.72	0.00	0.8484	0.00
nesta_case24_ieee_rts__api	0.78	0.78	0.53	5005.73	0.00	0.6207	0.00
nesta_case29_edin__api	0.85	0.85	0.40	294446.03	0.00	0.9289	0.00
nesta_case30_as__api	0.70	0.72	-2.69	538.37	0.00	0.7940	0.05
nesta_case30_fsr__api	0.70	0.71	-1.21	200.66	0.00	0.9369	0.00
nesta_case30_ieee__api	0.71	0.69	2.56	408.44	0.00	0.7287	0.03
nesta_case39_epri__api	0.86	0.84	1.62	6727.02	0.01	0.6800	0.02
nesta_case57_ieee__api	0.75	0.74	0.50	1427.70	0.00	0.6099	0.00
nesta_case73_ieee_rts__api	0.76	0.81	-5.87	16990.51	0.00	0.5999	0.00
nesta_case89_pegase__api	0.91	0.94	-3.71	3411.40	0.00	0.6865	0.00
nesta_case118_ieee__api	0.83	0.83	-0.23	5524.12	0.00	0.9167	0.00
nesta_case162_ieee_dtc__api	0.97	0.94	3.31	5154.86	0.00	-0.1092	0.00
nesta_case189_edin__api	1.04	1.03	0.96	1817.51	0.18	0.4626	0.10
nesta_case300_ieee__api	1.80	1.20	33.39	22292.03	0.01	0.2023	4.51
nesta_case1354_pegase__api	25.04	4.24	83.05	59543.72	0.00	0.6005	0.00
nesta_case1394sop_eir__api	39.44	9.75	75.27	2999.03	0.00	0.7325	0.00
nesta_case1397sp_eir__api	40.42	10.34	74.42	5638.34	0.00	0.7489	0.00
nesta_case1460wp_eir__api	39.95	10.54	73.61	5924.42	0.00	0.7184	0.00
nesta_case2224_edin__api	274.68	22.23	91.91	43247.11	0.00	0.6116	0.03
nesta_case2383wp_mp__api	90.54	6.92	92.35	23184.84	0.00	0.7529	0.00
nesta_case2736sp_mp__api	496.59	14.92	97.00	24980.71	0.00	0.8066	0.00
nesta_case2737sop_mp__api	1190.98	35.29	97.04	20676.94	0.00	0.8784	0.00
average	**79.09**	**4.66**	**26.20**	—	**0.04**	**0.7027**	**0.19**

5.2 Results and discussions

The results of our computational experiments on VSC-OPF and its SOCP relaxation are presented in Tables 1 and 2 for standard IEEE and NESTA instances, respectively. The "Cost" columns in both tables show the objective values of the VSC-OPF model (10) and its SOCP relaxation (19). In addition, six sets of information are provided in Tables 1 and 2:

- OG(%) is the percentage optimality gap between the lower bound LB obtained from the SOCP relaxation of VSC-OPF (19) and an upper bound UB obtained by IPOPT. It is calculated as $100\% \times (1 - LB/UB)$.
- t_b is the minimum value of $|V_i| - \sum_{j=1}^{n} a_{ij}/|V_j|$ for all load bus i calculated in the based case *before* solving VSC-OPF.
- t_a^{AC} is the minimum value of $|V_i| - \sum_{j=1}^{n} a_{ij}/|V_j|$ for all load bus i calculated *after* solving VSC-OPF (10).
- MI(%) is the percent increase of stability margin measured as $100\% \times |t_a^{AC}/t_b - 1|$.
- t_a^{SOCP} is the minimum value of $|V_i| - \sum_{j=1}^{n} a_{ij}/|V_j|$ for all load bus i calculated *after* solving SOCP relaxation of VSC-OPF.
- DT(%) is the percent difference between t_a^{AC} and t_a^{SOCP} calculated as $100\% \times |t_a^{SOCP}/t_a^{AC} - 1|$.

5.2.1 Stability margin improvement

As shown by Tables 1 and 2, on average, the proposed VSC-OPF improves the voltage stability margin by 6.80% for the IEEE instances and 3.48% for the NESTA instances over the standard OPF shown in t_b column. We also see that several test instances have significantly larger improvement. For example, the 300-bus system and the 2383-bus system in Table 1 have 12.02% and 23.93% improvement; the 162-bus, 1394-bus, 1397-bus, and 1406-bus systems in the NESTA archive all have more than 10% to 30% improved stability margin than the standard OPF model.

As a more specific example, we observe that for NESTA 162-bus system, the sufficient voltage stability condition (5) is violated at the base operating condition by at least one bus. The condition being violated signifies a severe system stress level and a low margin. In fact, the NESTA 162-bus system is indeed quite close to its solvability boundary in the base case, even though all voltages are within 6% of the nominal value. The result confirms our discussion in Section 1 that conventional security constraints such as voltage magnitude limits are not always sufficient in reflecting true system margin and a more systematic measure should be enforced.

5.2.2 Tightness of SOCP relaxation

Tables 1-2 both show that the optimality gap between the SOCP relaxation (19) and a local solution of the non-convex VSC-OPF (10) is very small: 0.27% and 0.73% for the IEEE and NESTA instances, respectively. It is curious to compare to the optimality gap of the SOCP relaxation of the standard OPF model. In fact, the gaps are much smaller compared to that for the conventional OPF problem for the NESTA instances, which presents more computational challenge than the standard IEEE instances. In particular, the average optimality gap of SOCP relaxation for standard OPF is 6.19%, compare to 0.73% for the VSC-OPF problem shown in Table 2. Specifically, only two instances have optimality gaps larger than 1% in Table 2, while the number for the SOCP relaxation of standard OPF problem is 18 instances. For IEEE instances, the average optimality gaps are similar: 0.25% for standard OPF versus 0.27% for VSC-OPF. This can be attributed to the fact that the flow limits for IEEE instances are high and most of them are not binding.

5.2.3 Effect of sparse approximation

The results of our computational experiments on the sparse approximation of VSC-OPF for NESTA instances are presented in Table 3. The sparsity parameter in Algorithm 1 is chosen to be 0.98. The "Time" columns in the table show the computation time of the VSC-OPF model (19) and the sparse approximation (22). In addition, the table reports five sets of data as described below:

- DCT(%) is the percentage time difference between the computation time ct_n of (19) and ct_s of (22). It is calculated as $100\% \times (1 - ct_s/ct_n)$.
- Cost($/h) is the objective value of the sparse approximation (22).
- DC(%) is the percentage difference between the objective value c^{SOCP} of the model (19) and c^s of (22) calculated as $100\% \times |c^s/c^{SOCP} - 1|$.
- t_a^s is the minimum value of $|V_i| - \sum_{j=1}^{n} a_{ij}/|V_j|$ for all load bus i calculated after solving (22).
- DT(%) is the percentage difference between t_a^{SOCP} and t_a^s calculated as $100\% \times |t_a^s/t_a^{SOCP} - 1|$.

For systems with less than five buses, the sparse approximation (22) and the original SOCP relaxation model (19) are exactly the same. For system sizes ranging between 6-bus and 300-bus, the computation times of model (19) are sufficiently short (less than 2 seconds), which render the sparse approximation unnecessary. However, for systems with more than 1000 buses, the sparse approximation brings about significant speed-up. In fact, the speed-ups are above 90% for all instances with more than 2000 buses and the optimal solutions are obtained in less than 36 seconds for all instances. As can be seen from the columns of DC(%) and DT(%), the solution accuracies are extremely high. For larger systems with more than 1000 buses, the differences of cost and stability margin between (19) and (22) are all less than 0.01% except the margin difference for 2224-bus system, which is 0.03%.

6 Conclusions

We have presented a sufficient condition for power flow Jacobian nonsingularity and have given a new proof for the condition. We have also shown that the condition characterizes a set of voltage stable solutions. A new VSC-OPF model has been proposed based on the sufficient condition. By using the fact that the load powers are constant in an OPF problem, we reformulate the voltage stability condition to a set of second-order conic constraints in a transformed variable space. Furthermore, in the new variable space, we have formulated an SOCP relaxation of the VSC-OPF problem as well as its sparse approximation. Simulation results show that the proposed VSC-OPF and its SOCP relaxation can effectively restrain the stability stress of the system; the optimality gap of the SOCP relaxation is much smaller compared to SOCP relaxation of the standard OPF problem on difficult instances; and the sparse approximation yields significant speed-up on larger instances with hardly any accuracy compromise.

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