

Robust Utility Maximization of Terminal Wealth with Drift and Volatility Uncertainty

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Abstract

We give explicit solutions for utility maximization of terminal wealth problem $u(X_T)$ in the presence of Knightian uncertainty $\sup_{\pi \in \Pi_{[0,T]}^{\text{ad}}} \inf_{\theta \in \Theta_{[0,T]}} \mathbb{E}[u(X_T^{\pi, \theta})]$ in continuous time $[0, T]$. We assume there is uncertainty on both drift and volatility of the underlying stocks, which induce nonequivalent measures on canonical space of continuous paths Ω . We take that the uncertainty set resides in compact sets that are time dependent. In this framework, we solve the robust optimization problem with logarithmic, power and exponential utility functions, explicitly.

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1 Introduction

Starting with the pioneering works of [28, 1, 2, 27, 4], the underlying risky assets are modelled as Markovian diffusions, where there exists a fixed underlying reference probability measure \mathbb{P} that is retrieved from historical data of the price movements. Our main motivation to develop our proposed methodology is it is impossible to precisely identify \mathbb{P} . Hence, as a result, *model ambiguity*, also called *Knightian uncertainty*, in utility maximization is inevitably taken into consideration. Namely, the investor is diffident about the odds, and takes a robust approach to the utility maximization problem, where she minimizes over the priors, corresponding to different scenarios, and then maximizes over the investment strategies.

The literature on robust utility maximization in mathematical finance, (see e.g. [3, 7, 8, 9, 13, 10, 11, 18, 21, 22, 19, 14, 15, 16, 17, 26, 29, 30, 32, 42, 44, 45, 46, 47, 48, 49] among others), mostly assumes that the set of priors is dominated by a reference measure \mathbb{P} . Hence, it presumes a setting where volatility of risky assets are perfectly known, but drifts are uncertain. Namely, these approaches assume the equivalence of priors. In particular, they assume the equivalence of probability measures P with a dominating reference prior \mathbb{P} .

A more general direction is the case, where the uncertainty on both mean and volatility is taken into consideration. Here, the set of priors is nondominated, and there exists no dominating reference prior \mathbb{P} . This approach started with the seminal works of [33, 35] in option pricing framework. In a more recent work, [41] studied robust optimal stopping using nondominated measures, and its applications to subhedging of American options under volatility uncertainty. Regarding utility maximization, [47] studied the case, where uncertainty in the volatility is due to an unobservable factor. [24] works in a jump-diffusion context, with ambiguity on drift, volatility and jump intensity. A minimax result and the existence of a worst-case measure in a setup are established in [25], where prices have continuous paths and the utility function is bounded. Uncertainty is modelled by allowing drift and volatility to vary in two constant order intervals in [5]. Here, the optimization using power utility of the form $U(x) = x^\gamma$ for $0 < \gamma < 1$ is performed via a robust control (G-Brownian motion) technique, which requires the uncertain volatility matrix is diagonal. We refer the reader to [20] for a detailed exposure on G-Brownian motion and its applications. A utility maximization problem with power utility, where there is an ellipsoidal uncertainty for drift and volatility uncertainty that reside in a fixed compact set is studied in [31]. In [12], under the assumption that stock prices are discontinuous semi-martingales and strategies are compact, power utility maximization is studied, and semi-explicit solutions are given. In [36], robust utility maximization in an incomplete market is studied, where there exists a fixed compact uncertainty set for volatility and drift. They prove the existence of optimal strategies with power and utility functions using backward stochastic differential equations theory. In [40], a general robust utility maximization problem is studied, where it proposes to model a way to model drift and volatility. In [37], the mean variance optimization in a diffusion setting is studied, where it is assumed that the drift of the stock is known with certainty, whereas the volatility is assumed to be in some compact set. [34] shows the existence of optimal strategies in the robust exponential utility maximization problem in discrete time.

On the other hand, we are studying a utility maximization problem in finite continuous time horizon in a diffusion setting, where there is time-dependent uncertainty on both drift and volatility residing in a compact set. Contrary to the usual stream that the compact set containing the differential characteristics is fixed throughout $[0, T]$, we assume that the set of

priors is time dependent. There can be at least two arguments to support this construction. First, in an intraday movement of a stock, it is not reasonable to assume that drift and volatility uncertainty reside in a fixed compact set throughout $[0, T]$. Second, with time drift and volatility of the stock can be *learned* (see e.g. [38]) and hence the corresponding compact sets might change, as time proceeds. This more general approach entails additional technical problems. In particular, depending on the confidence set, the optimal value function might not be $C^{1,2}$, hence the classical Hamilton-Jacobi-Bellman-Ishii (HJBI) or the martingale optimality principle approach can not be used at the first place (see e.g. Theorem 1.1 [28]) and it requires a more careful analysis to overcome this hurdle.

The rest of the paper is as follows. In Section 2, we describe the model dynamics of the problem and state our general main problem and propose the solution methodology. In Section 3, we solve our utility maximization problem explicitly using logarithmic, power and exponential utility functions. In Section 4, we discuss our results and conclude the paper.

2 Model Dynamics and Investor's Value Function

2.1 Framework for Model Uncertainty and Model Dynamics

We fix the dimension $d \in \mathbb{N}$ and time horizon $T \in (0, \infty)$. We let $\Omega = C_0([0, T])$ be the space of continuous paths $\omega = (\omega_t)_{0 \leq t \leq T}$ starting at $0 \in \mathbb{R}^d$. We define the coordinate functional for $\omega \in \Omega$ as $W_t(\omega) \triangleq \omega_t$ and take the corresponding Borel σ -algebra by $\mathcal{F}_t \triangleq \sigma(W_s(\omega) : 0 \leq s \leq t)$. We denote \mathbb{P}_0 as the Wiener measure on Ω such that W_t is the (Ω, \mathcal{F}_t) Wiener process and take \mathbb{P}_0 as the *reference* measure. We consider a market consisting of d risky assets $S_t^\theta = (S_t^{\theta,1}, \dots, S_t^{\theta,d})$ and one riskless asset R_t . We assume S_t^θ and R_t satisfy the following dynamics

$$\begin{aligned} dR_t &= rR_t dt \\ S_0 &= s_0 \\ dS_t^\theta &= \text{Diag}(S_t^\theta)(\mu_t dt + \sigma_t dW_t), \mathbb{P}_0\text{-a.s.} \end{aligned} \tag{2.1}$$

Here, $S_0 = s_0$, with $s_0 \in \mathbb{R}^d$ and with positive coordinates is the present value of d stocks. Furthermore, $r > 0$ is the risk-free interest rate. $\text{Diag}(S_t^\theta)$ is a $d \times d$ diagonal matrix with $(S_t^{\theta,1}, \dots, S_t^{\theta,d})$ its diagonal entries. We take that $(\mu_t)_{t \in [0, T]}$ is a $\mathcal{B}([0, T]) \otimes \mathcal{F}_T$ progressively measurable \mathbb{R}^d -valued mapping, whereas $(\sigma_t)_{t \in [0, T]}$ is $d \times d$ matrix valued and $\mathcal{B}([0, T]) \otimes \mathcal{F}_T$ progressively measurable. We further denote by $(\Sigma_t)_{t \in [0, T]} \triangleq (\sigma_t \sigma_t^\top)_{t \in [0, T]}$ the covariance matrix of d stocks.

Assumption 2.1. We assume $0 \leq \|\mu_t\| \leq C_{t_i}^\mu$, \mathbb{P}_0 -a.s. and $0 < c_{t_i} \leq \|\Sigma_t\| \leq C_{t_i}$ for $t \in [t_i, t_{i+1})$, $i = 0, \dots, n-1$ and $0 \leq \|\mu_t\| \leq C_{t_n}^\mu$ and $0 < c_{t_n} \leq \|\Sigma_t\| \leq C_{t_n}$, \mathbb{P}_0 -a.s. for $t \in [t_n, T]$, where $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = T$. We denote by

$$\Theta_{[t, t_{i+1})} \subset \mathbb{R}^d \times \mathbb{S}_+^d,$$

the compact set containing the differential characteristics $\theta_t \triangleq (\mu_t, \sigma_t)$ for $t \in [t_i, t_{i+1})$ and

$$\Theta_{[t, T]} \subset \mathbb{R} \times \mathbb{S}_+^d$$

for $t \in [t_n, T]$. We denote the corresponding compact set of uncertainty set on $[0, T]$ as

$$\Theta_{[0, T]} \subset \mathbb{R} \times \mathbb{S}_+^d.$$

We note that the uncertainty sets $\Theta_{[t, t_{i+1})} = \Theta_{[t_i, t_{i+1})}$ for $t \in [t_i, t_{i+1})$ and $\Theta_{[t, T]} = \Theta_{[t_n, T]}$ for $t \in [t_n, T]$. We further assume that there exists a strong solution to (2.1) for any given $(\theta_t)_{t \in [0, T]} \in \Theta_{[0, T]}$ on $(\Omega, \mathcal{F}, \mathbb{P}_0)$. Namely, denoting $C_0[0, T] = \Omega$, we take that there exists an \mathcal{F}_T measurable mapping $G : \Omega \rightarrow \Omega$ such that $S^\theta \equiv G(x_t, W)(\cdot)$ solves (2.1) on $(\Omega, \mathcal{F}_T, \mathbb{P}_0)$, as in Definition 10.9 in [48].

Note that for different $(\theta_t)_{t \in [0, T]} \in \Theta_{[0, T]}$, different probability measures are induced on Ω , which is defined as

$$Q^\theta \triangleq \mathbb{P}_0 \circ (\log(S^\theta))^{-1}, \quad (2.2)$$

where S^θ has the differential characteristics as in Equation (2.1) with $(\theta_t)_{t \in [0, T]} = (\mu_t, \sigma_t)_{t \in [0, T]}$. Further, different σ_1, σ_2 induce nonequivalent probability measures. Indeed, for $\theta_1 = (\mu_1, \sigma_1)$ and $\theta_2 = (\mu_2, \sigma_2)$, where $(\mu_i, \sigma_i)_{i=1,2}$ are constants in \mathbb{R}^d and \mathbb{S}_+^d , respectively, we have

$$\begin{aligned} Q^{\theta_1}(\langle \log(S^{\theta_1}) \rangle = \sigma_1 \sigma_1^\top) &= 1 \\ Q^{\theta_2}(\langle \log(S^{\theta_2}) \rangle = \sigma_2 \sigma_2^\top) &= 1. \end{aligned}$$

Here $\langle \cdot \rangle$ stands for the quadratic variation of $\log(S^\theta)$. However, the dynamics of differential characteristics are given with respect to \mathbb{P}_0 , in particular, *we look through the lenses of the Wiener measure \mathbb{P}_0* . This is possible, since we consider only strong solutions in Equation (2.1).

2.2 Financial Scenario

We consider the problem of an agent investing in d risky assets S and one riskless asset R . For a given initial endowment $x_0 > 0$, the investor trades in a self financing way.

We denote $(\hat{\pi}_t)_{t \in [0, T]}$ as a d -dimensional $\mathcal{B}([0, T]) \otimes \mathcal{F}_T$ progressively measurable stochastic process, which stands for the total amount of money invested in d risky assets S_t at time $t \in [0, T]$. Then, we have for $X_0 = x_0 > 0$

$$\begin{aligned} d\hat{X}_t^{\hat{\pi}, \theta} &= \hat{\pi}_t^\top S_t^{-1} \cdot dS_t + (\hat{X}_t^{\hat{\pi}, \theta} - \hat{\pi}_t^\top \mathbf{1})r dt, \\ d\hat{X}_t^{\hat{\pi}, \theta} &= \hat{\pi}_t^\top (\mu_t dt + \sigma_t \cdot dW_t) + (\hat{X}_t^{\hat{\pi}, \theta} - \hat{\pi}_t^\top \mathbf{1})r dt \quad \mathbb{P}_0\text{-a.s.} \end{aligned} \quad (2.3)$$

We further represent the amount of money invested in d risky assets as a fraction of current wealth via $\hat{\pi}_t = \hat{X}_t^{\pi, \theta} \pi_t$ for $t \in [0, T]$, where π_t stands for the corresponding fraction at time $t \in [0, T]$ and take the discounted wealth $X_t^{\pi, \theta} = e^{-rt} \hat{X}_t^{\pi, \theta} \pi_t$. Hence, for $X_0 = x_0$, the dynamics of wealth in this setting are given by

$$\begin{aligned} dX_t^{\pi, \theta} &= X_t^{\pi, \theta} \pi_t^\top ((\mu_t - r\mathbf{1})dt + \sigma_t dW_t) \\ X_t^{\pi, \theta} &= x_0 \exp \int_0^t \pi_u^\top (\mu_u - r\mathbf{1}) - \frac{1}{2} \pi_u^\top \Sigma_u \pi_u du + \int_0^t \pi_u^\top \sigma_u dW_u \quad \mathbb{P}_0\text{-a.s.}, \end{aligned} \quad (2.4)$$

where $\mathbf{1}$ stands for d dimensional vector $(1, \dots, 1)$. We further denote $X^{\pi, \theta}$ as the wealth process with dynamics $(\theta_t)_{0 \leq t \leq T} = (\mu_t, \sigma_t)_{t \in [0, T]}$ as in Equation (2.4).

Definition 2.1. Let $(\pi_t)_{t \in [0, T]}$ denote the $\mathcal{B}([0, T]) \otimes \mathcal{F}_T$ progressively measurable process representing the cash-value allocated in d risky assets. We call $(\pi_t)_{t \in [0, T]}$ admissible and denote it by $\pi \in \Pi_{[0, T]}$, if it satisfies

$$X_t^\pi > 0, \quad t \in [0, T], \quad \mathbb{P}_0\text{-a.s.}$$

Analogously, we denote by for $0 = t_0 < t_1 < t_2 < \dots < t_{i+1} \leq t_n < T$,

$$\Pi_{[t_i, t_{i+1}]} \quad t \in [t_i, t_{i+1}]$$

and

$$\Pi_{[t_n, T]} \quad \text{for } t \in [t_n, T]$$

the restrictions of $\Pi_{[0, T]}$ on to the corresponding time subintervals, respectively.

2.3 Investor's Problem

The investor utilizes the classical Merton problem, but he is also diffident about the underlying dynamics of the stocks both in terms of drift μ_t and covariance matrix Σ_t . Let $\theta = (\theta_t)_{t \in [0, T]}$ and $\Theta_{[0, T]}$ be as in Assumption 2.1, and assume that the prior belief of the investor on $(\theta_t)_{t \in [0, T]}$ is as in Assumption 2.1. We take that the investor reevaluates his priors

$(\theta_t)_{t \in [0, T]}$ on some prespecified times $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = T$. At time $t \in [0, T]$, we write the optimization problem of the investor for $X_t^\pi = x$ as

$$V(t, x) \triangleq \sup_{\pi \in \Pi_{[t, T]}} \inf_{\theta \in \Theta_{[t, T]}} \mathbb{E}_t[u(X_T^{\pi, \theta})], \quad (2.5)$$

where $\mathbb{E}_t[\cdot] \triangleq \mathbb{E}[\cdot | \mathcal{F}_t]$. Here, $u(\cdot)$ is a $C^2((0, \infty))$ function that is concave, and increasing. Hence, at time $t_0 = 0$ and $x > 0$, the optimization problem reads via

$$V(t_0, x) \triangleq \sup_{\pi \in \Pi_{[0, T]}} \inf_{\theta \in \Theta_{[0, T]}} \mathbb{E}[u(X_T^{\pi, \theta})]. \quad (2.6)$$

By Bellman's principle of optimality (see e.g. [49]), for $t \in [0, t_1)$, we have

$$\begin{aligned} V(t, x) = & \sup_{\pi \in \Pi_{[t, t_1)}} \inf_{\theta \in \Theta_{[t_0, t_1)}} \mathbb{E} \left[\sup_{\pi \in \Pi_{[t_1, t_2)}} \inf_{\theta \in \Theta_{[t_1, t_2)}} \mathbb{E}_{t_1} [\right. \\ & \left. \dots \sup_{\pi \in \Pi_{[t_n, T]}} \inf_{\theta \in \Theta_{[t_n, T]}} \mathbb{E}_t [u(X_T^{\pi, \theta})] \dots \right] \end{aligned}$$

As described in [47], the robust optimal control problem (2.5) can be perceived as a non-cooperative game between two agents, the investor and the fictitious agent, nature. The investor tries to maximize his expected utility of terminal wealth by choosing policy $\pi \in \Pi_{[0, T]}$ judiciously, while nature is competing with the investor by choosing the parameters $\theta \in \Theta_{[0, T]}$ of the underlying dynamics of to minimize the terminal wealth.

Remark 2.1. • *We can consider to include a penalty function for $\theta \in \Theta_{[0, T]}$ in (2.6) that constraints the choice of θ . However, since we aim to get explicit solutions, we follow the worst case evaluation approach.*

- *We note that in (2.6) can well be infinity, since, as in [50], we don't put any integrability assumption on the admissible cash-values. However, we will see that for the specific utility functions $u(\cdot)$ we are going to work on, (2.6) turns out to be, indeed, finite.*

We continue with the following variant of so called Martingale Optimality Principle (see also Theorem 1.1 of [28]).

Theorem 2.1. *Suppose that the objective is (2.5) for $t = t_n$. Assume the followings hold:*

(A1) *There exists a function $v : [t_n, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}$, that is $C^{1,2}([t_n, T] \times \mathbb{R}^+)$ with*

$$v(T, \cdot) = u(\cdot) \quad (2.7)$$

(A2) Let $t \in [t_n, T]$. For any $\pi \in \Pi_{[t_n, T]}$, there exists an optimal solution $(\theta_t^*)_{t \in [t_n, T]} \in \Theta_{[t_n, T]}$ of

$$\mathbb{E}_{t_n} [u(X_T^{\pi, \theta^*})] = \inf_{\theta \in \Theta_{[t_n, T]}} \mathbb{E}_{t_n} [u(X_T^{\pi, \theta})]$$

and define for $x > 0$

$$Y_{t_n}^\pi \triangleq v(t_n, x). \quad (2.8)$$

Assume further Y^π satisfies

$$\mathbb{E}_s [Y_t^\pi] \leq Y_s^\pi, \mathbb{P}_0\text{-a.s.} \quad (2.9)$$

for $t_n \leq s \leq t \leq T$.

(A3) There exists some $(\pi_t^*)_{t \in [s, T]} \in \Pi_{[s, T]}$ such that for $t_n \leq s \leq t \leq T$

$$\mathbb{E}_s [Y_t^{\pi^*}] = Y_s^{\pi^*}, \mathbb{P}_0\text{-a.s.} \quad (2.10)$$

Then, $(\pi_t^*)_{t \in [t_n, T]} \in \Pi_{[t_n, T]}$ is optimal for the problem (2.5) such that

$$\begin{aligned} V(t_n, x_{t_n}) &= \sup_{\pi \in \Pi_{[t_n, T]}} \inf_{\theta \in \Theta_{[t_n, T]}} \mathbb{E}_{t_n} [u(X_T^{\pi, \theta})] \\ &= \inf_{\theta \in \Theta_{[t_n, T]}} \mathbb{E}_{t_n} [u(X_T^{\pi^*, \theta})] \\ &= \mathbb{E}_{t_n} [u(X_T^{\pi^*, \theta^*})] \\ &= v(t_n, x_{t_n}) \end{aligned}$$

Proof. By (A1) and (A2), we have

$$\begin{aligned} \mathbb{E}_{t_n} [Y_T^\pi] &= \inf_{\theta \in \Theta_{[t_n, T]}} \mathbb{E}_{t_n} [u(X_T^{\pi, \theta})] \leq Y_{t_n}^\pi, \mathbb{P}_0\text{-a.s.} \\ Y_{t_n}^\pi &= v(t_n, x) \end{aligned}$$

Taking supremum over $\Pi_{[t_n, T]}$, we get

$$\begin{aligned} V(t_n, x_{t_n}) &= \sup_{\pi \in \Pi_{[t_n, T]}} \inf_{\theta \in \Theta_{[t_n, T]}} \mathbb{E}_t [u(X_T^{\pi, \theta})] \\ &\leq v(t_n, x_{t_n}), \end{aligned}$$

where the last inequality follows by (2.7) and (2.9). By (A3), we have $\mathbb{E}_s [Y_t^{\pi^*}] = Y_s^{\pi^*}$ \mathbb{P}_0 -a.s. for $t_n \leq s \leq t \leq T$ for some $(\pi_t^*)_{t \in [t_n, T]} \in \Pi_{[t_n, T]}$. Then,

$$\begin{aligned} V(t_n, x_{t_n}) &= \mathbb{E}_{t_n} [Y_T^{\pi^*}] \\ &= Y_{t_n}^{\pi^*} \\ &= v(t_n, x_{t_n}). \end{aligned}$$

We emphasize here that there is no convexity assumption imposed on $u(\cdot)$ for the theorem to hold. Hence, we conclude the proof. \square

Applying Ito lemma for $t \in [t_n, T]$ to Y_t^π in (2.8), we have by (2.4)

$$dY_t^\pi = \left(v_t + v_x X_t^\pi (\pi_t^\top (\mu_t - r\mathbf{1})) + (X_t^\pi)^2 \frac{1}{2} v_{xx} \pi_t^\top \Sigma_t \pi_t \right) dt + X_t^\pi v_x \pi_t^\top \sigma dW_t, \quad \mathbb{P}_0\text{-a.s.}$$

By Theorem 2.1, for $x > 0$, using (2.10), (2.5) satisfies the following HJBI PDE:

$$\begin{aligned} & \sup_{\pi \in \Pi_{[t_n, T]}} \inf_{\theta \in \Theta_{[t_n, T]}} \left[v_t + v_x x \pi_t^\top (\mu_t - r\mathbf{1}) + \frac{1}{2} x^2 v_{xx} \pi_t^\top \Sigma_t \pi_t \right] \\ &= v_t + x \sup_{\pi \in \Pi_{[t_n, T]}} \inf_{\theta \in \Theta_{[t_n, T]}} \left[v_x \pi_t^\top (\mu_t - r\mathbf{1}) + x \frac{1}{2} v_{xx} \pi_t^\top \Sigma_t \pi_t \right] \\ &= 0 \end{aligned} \tag{2.11}$$

Similarly, by (2.3), we have

$$\begin{aligned} & \sup_{\pi \in \Pi_{[t_n, T]}} \inf_{\theta \in \Theta_{[t_n, T]}} \left[v_t + v_x \pi_t^\top (\mu_t - r\mathbf{1}) + \frac{1}{2} v_{xx} \pi_t^\top \Sigma_t \pi_t \right] \\ &= v_t + \sup_{\pi \in \Pi_{[t_n, T]}} \inf_{\theta \in \Theta_{[t_n, T]}} \left[v_x \pi_t^\top (\mu_t - r\mathbf{1}) + \frac{1}{2} v_{xx} \pi_t^\top \Sigma_t \pi_t \right] \\ &= 0 \end{aligned}$$

Lemma 2.1. *The value function $V(t, x)$ as defined in Equation (2.5) is increasing and concave in $x > 0$ for all $t \in [t_n, T]$.*

Proof. Recall that by assumption, the utility function $u(\cdot)$ is increasing and concave and by (2.4)

$$X_T^{\pi, \theta} = x \exp \left[\int_t^T (\pi_u^\top (\mu_u - r\mathbf{1}) - \frac{1}{2} \pi_u^\top \Sigma_u \pi_u) du + \int_t^T \pi_u^\top \sigma_u dW_u \right], \quad \mathbb{P}_0\text{-a.s.}$$

In particular, for any $x^1 \leq x^2$ with fixed $\Pi_{[t, T]}$ and fixed $\theta \in \Theta_{[t, T]}$, by monotonicity of $u(\cdot)$, we have

$$\mathbb{E}_t[u(X_T^{\pi, \theta} - x^1 + x^1)] \leq \mathbb{E}_t[u(X_T^{\pi, \theta} - x^2 + x^2)].$$

Since this holds for any $\pi \in \Pi_{[t, T]}$ and $\theta \in \Theta_{[t, T]}$, taking first infimum over $\theta \in \Theta_{[t, T]}$ for fixed $\pi \in \Pi_{[t, T]}$ and then supremum over $\pi \in \Pi_{[t, T]}$, we have

$$V(t, x^1) \leq V(t, x^2).$$

Next, we show concavity of $V(t, x)$. Let $0 < \alpha < 1$ and denote

$$x^3 \triangleq \alpha x^1 + (1 - \alpha)x^2$$

Then, we have

$$V(t, x^3) = \sup_{\pi_1 \in \Pi_{[t, T]}, \pi_2 \in \Pi_{[t, T]}} \inf_{\theta_1 \in \Theta_{[t, T]}, \theta_2 \in \Theta_{[t, T]}} \mathbb{E}_t [u(\alpha X_T^{\pi_1, \theta_1} + (1 - \alpha)X_T^{\pi_2, \theta_2})]$$

Since u is concave by assumption, we have

$$\begin{aligned} V(t, x^3) \geq & \sup_{\pi^1 \in \Pi_{[t, T]}, \pi^2 \in \Pi_{[t, T]}} \left\{ \alpha \inf_{\theta \in \Theta_{[t_n, T]}} \mathbb{E}_t [u(X_T^{\pi^1, \theta})] \right. \\ & \left. + (1 - \alpha) \inf_{\theta \in \Theta_{[t, T]}} \mathbb{E}_t [u(X_T^{\pi^2, \theta})] \right\}. \end{aligned}$$

Since the last expression is sum of two suprema, we conclude that

$$V(t, x^3) \geq \alpha V(t, x^1) + (1 - \alpha)V(t, x^2).$$

Hence, we conclude the proof. \square

3 Explicit Solutions with Specific Utility Functions

We will be working with the logarithmic, power and exponential utility functions. These are of the form $\log(x)$, x^γ for $0 < \gamma < 1$, $-\beta e^{-\beta x}$ with $\beta > 0$ for $x > 0$, respectively, and give *explicit* solutions in our robust setting. We denote

$$\begin{aligned} \mu_{t_n}^* &\triangleq \arg \min_{\mu_t \in \Theta_{[t_n, T]}} (\|\mu_t - r\mathbf{1}\|) \\ \Sigma_{t_n}^* &\triangleq \arg \max_{\Sigma_t \in \Theta_{[t_n, T]}} (\|\pi_t^\top \Sigma_t \pi_t\|) \\ &= C_{t_n} * I_{d \times d} \\ \theta_{t_n}^* &\triangleq (\mu_{t_n}^*, \Sigma_{t_n}^*), \end{aligned} \tag{3.1}$$

where $I_{d \times d}$ stands for the d -dimensional identity matrix.

Lemma 3.1. *Let $t \in [t_n, T]$, $x > 0$ and $v(t, x)$ be an increasing and concave $C^{1,2}([t_n, T] \times \mathbb{R}_+)$ function such that $\frac{v_x}{-v_{xx}} = c$ for some nonnegative scalar c for all $x > 0$. Then, for all $t \in [t_n, T]$, the infimum in the HJBI equation in (2.11) are attained for $\theta_{t_n}^*$.*

Proof. We have for every $t \in [t_n, T]$

$$\begin{aligned} 0 &= \sup_{\pi \in \Pi_{[t_n, T]}} \inf_{\theta \in \Theta_{[t_n, T]}} \left[v_t + v_x(x\pi_t^\top(\mu_t - r\mathbf{1})) + x^2 \frac{1}{2} \pi_t^\top \Sigma_t \pi_t v_{xx} \right] \\ &= v_t + \sup_{\pi \in \Pi_{[t_n, T]}} \inf_{\theta \in \Theta_{[t_n, T]}} \left[xv_x \pi_t^\top(\mu_t - r\mathbf{1}) + x^2 \frac{1}{2} \pi_t^\top \Sigma_t \pi_t v_{xx} \right] \end{aligned}$$

Since v is $C^{1,2}([t_n, T] \times \mathbb{R}^+)$ and increasing and concave, the result follows by inner minimization for any fixed $\pi \in \Pi_{[t_n, T]}$. Since Σ_t is positive definite and v is concave, to minimize $\frac{1}{2} \pi_t^\top \Sigma_t \pi_t$, we need to choose $\Sigma_t^* = C_{t_n} * I_{d \times d}$ for any fixed $(\pi_t)_{t \in [t_n, T]} \in \Pi_{[t_n, T]}$. Next, to find the minimizing $(\mu_t)_{t \in [t_n, T]}$ for any fixed $\pi \in \Pi_{[t_n, T]}$ at each $t \in [t_n, T]$, we proceed as follows. We choose specifically $(\tilde{\pi}_t)_{t \in [t_n, T]}$ as

$$\tilde{\pi}_t \triangleq \frac{1}{C_{t_n}} \frac{v_x(\mu_{t_n}^* - r\mathbf{1})}{-v_{xx}x}.$$

The reason for choosing $(\tilde{\pi}_t)_{t \in [t_n, T]}$ in this specific form is that it is the optimal policy for (2.5) with the optimal $(\theta_t^*)_{t \in [t_n, T]} \in \Theta_{[t_n, T]}$ as in (3.1). Indeed, first note that $(\tilde{\pi}_t)_{t \in [t_n, T]}$ is constant on $[t_n, T]$, deterministic and is an element of $\Pi_{[t_n, T]}$. Since v is increasing with $v_x \geq 0$, for that $(\tilde{\pi}_t)_{t \in [t_n, T]}$ to minimize the expression

$$v_x \tilde{\pi}_t^\top(\mu_t - r\mathbf{1}) \quad \text{for } t \in [t_n, T],$$

over μ_t , we must choose

$$\arg \min_{\mu_t \in \Theta_{[t_n, T]}} \|\mu_t - r\mathbf{1}\|.$$

Furthermore, we have by classical minmax inequality (see e.g. [43]) at each $t \in [t_n, T]$

$$\begin{aligned} &\sup_{\pi \in \Pi_{[t_n, T]}} \inf_{\theta \in \Theta_{[t_n, T]}} \left[v_x x \pi_t^\top(\mu_t - r\mathbf{1}) + \frac{1}{2} x^2 \pi_t^\top \Sigma_t \pi_t v_{xx} \right] \\ &\leq \inf_{\theta \in \Theta_{[t_n, T]}} \sup_{\pi \in \Pi_{[t_n, T]}} \left[xv_x \pi_t^\top(\mu_t - r\mathbf{1}) + \frac{1}{2} x^2 \pi_t^\top \Sigma_t \pi_t v_{xx} \right]. \end{aligned} \quad (3.2)$$

Next, for the right hand side of the inequality (3.2), for a fixed $(\theta_t)_{t \in [t_n, T]} \in \Theta_{[t_n, T]}$ with $v_{xx} \leq 0$ and Σ_t being positive definite, we must have

$$\arg \max_{\pi \in \Pi_{[t_n, T]}} \left[xv_x \pi_t^\top(\mu_t - r\mathbf{1}) + \frac{1}{2} x^2 \pi_t^\top \Sigma_t \pi_t v_{xx} \right] = (\Sigma_t)^{-1} \frac{v_x(\mu_t - r\mathbf{1})}{-v_{xx}x}$$

Here, noting that $\frac{v_x}{-xv_{xx}}$ being a nonnegative constant by assumption and plugging

$$(\Sigma_t)^{-1} \frac{v_x(\mu_t - r\mathbf{1})}{-v_{xx}x} \quad (3.3)$$

to the right hand side of the inequality (3.2), we conclude for all $t \in [t_n, T]$

$$\arg \min_{\theta \in \Theta_{[t, T]}} \left(\frac{1}{2} (\Sigma_t)^{-1} \frac{v_x \|\mu_t - r\mathbf{1}\|^2}{-v_{xx}x} \right) = \left(\mu_{t_n}^*, C_{t_n} * I_{d \times d} \right). \quad (3.4)$$

But (3.3) and (3.4) are the values that we have plugged in and found for the left hand side of Equation (3.2). Hence, again by inequality (3.2), we conclude that the HJBI equation are attained for the values as in (3.4), i.e. $\theta_{t_n}^*$ for all $t \in [t_n, T]$. Hence, we conclude the proof. \square

Based on Lemma 3.1, our solution methodology is, as follows. We assume first $V(t, x)$ is $C^{1,2}([t_n, T] \times \mathbb{R}_+)$ for $t \in [t_n, T]$ for $x > 0$. Then, by Lemma 3.1, we plug in the corresponding parameters for $\theta \in \Theta_{[t_n, T]}$ and solve the classical Merton problem. We verify that the resulting value function $V(t, x)$ is indeed $C^{1,2}([t_n, T] \times \mathbb{R}_+)$ and $\frac{V_x}{-xV_{xx}}$ is a non-negative constant for $x > 0$. Hence, we will have solved the problem for $[t_n, T]$. Then, we will solve the problem for $[t_{n-1}, t_n)$, and we proceed backwards up to $[0, t_1)$ via the same methodology. We emphasize here that the resulting value function $V : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is not necessarily $C^{1,2}([0, T] \times \mathbb{R}_+)$, but is a concatenation of $C^{1,2}$ functions on $[0, t_1) \times \mathbb{R}_+, [t_1, t_2) \times \mathbb{R}_+, \dots, [t_n, T] \times \mathbb{R}_+$.

3.1 Logarithmic Utility Case

First, we are going to solve the robust optimization problem with logarithmic utility $\log(x)$ with $x > 0$. Let $t \in [t_n, T]$ and consider

$$V(t, x) = \sup_{\pi \in \Pi_{[t, T]}} \inf_{\theta \in \Theta_{[t, T]}} \mathbb{E}_{t_n} [\log(X_T^{\pi, \theta})].$$

We assume that $V(t, x)$ is $C^{1,2}([t_n, T] \times \mathbb{R}_+)$ and by Lemma 3.1, we let

$$\begin{aligned} \mu_{t_n}^* &= \arg \min_{\mu_t \in \Theta_{[t_n, T]}} \|\mu_t - r\| \\ \Sigma_{t_n}^* &= C_{t_n} * I_{d \times d} \end{aligned}$$

and let

$$dX_t^\pi = X_t^\pi \pi_t^\top ((\mu_{t_n}^* - r\mathbf{1})dt + \sigma_{t_n}^* dW_t), \text{ for } t \in [t_n, T].$$

The optimization problem reads as

$$\sup_{\pi \in \Pi_{[t, T]}} \mathbb{E}_t [\log(X_T^{\pi, \theta})].$$

By Ito lemma, we have

$$\begin{aligned} & \sup_{\pi \in \Pi_{[t,T]}} \mathbb{E}_t [\log(X_T^{\pi, \theta})] \\ &= \log(x) + \sup_{\pi \in \Pi_{[t,T]}} \mathbb{E}_t \left[\int_t^T (\pi_u^\top (\mu_{t_n}^* - r \cdot \mathbf{1}) \right. \\ & \quad \left. - \frac{1}{2} \pi_u^\top \Sigma_{t_n}^* \pi_u) du \right]. \end{aligned}$$

Hence, by concavity on π inside the integral, we conclude that checking first order condition inside the expectation on π is sufficient and get that for $t \in [t_n, T]$

$$(\mu_{t_n}^* - r \mathbf{1}) - \Sigma^* \pi_t = 0,$$

Thus, the optimal cash flow is

$$\begin{aligned} \pi_t^* &\triangleq (\Sigma_{t_n}^*)^{-1} (\mu_{t_n}^* - r \mathbf{1}) \\ &= \frac{1}{C_{t_n}} (\mu_{t_n}^* - r \mathbf{1}) \end{aligned}$$

for $t \in [t_n, T]$. Furthermore, since $(\mu_t, \Sigma_t) = (\mu_{t_n}^*, \Sigma_{t_n}^*)$ for all $t \in [t_n, T]$, the optimal $\pi_t^* = \frac{1}{C_{t_n}} (\mu_{t_n}^* - r \mathbf{1})$ is constant and deterministic and is in $\Pi_{[t_n, T]}$, as well. Hence, the optimal value function for any $t \in [t_n, T]$ and for $X_t^\pi = x > 0$ reads as

$$V(t, x) = \log(x) + \frac{1}{2C_{t_n}} \|\mu_{t_n}^* - r \cdot \mathbf{1}\|^2 (T - t). \quad (3.5)$$

Hence, we verify that $V(t, x)$ is indeed $C^{1,2}([t_n, T] \times \mathbb{R}^+)$ and $-\frac{V_x}{xV_{xx}} = 1$. Next, we go one time step backwards and examine the following optimization problem for $t \in [t_{n-1}, t_n)$ and $x > 0$

$$\begin{aligned} V(t, x) &= \sup_{\pi \in \Pi_{[t,T]}} \inf_{\theta \in \Theta_{[t,T]}} \mathbb{E}_t [\log(X_T^\pi)] \\ &= \log(x) + \sup_{\pi \in \Pi_{[t,T]}} \inf_{\theta \in \Theta_{[t,T]}} \mathbb{E}_t \left[\int_t^T (\pi_u^\top (\mu_u - r \cdot \mathbf{1}) \right. \\ & \quad \left. - \frac{1}{2} \pi_u^\top \Sigma_u \pi_u) du \right] \\ &= \log(x) + \sup_{\Pi_{[t,t_n)}} \inf_{\theta \in \Theta_{[t,t_n)}} \mathbb{E}_t \left[\int_t^{t_n} (\pi_u^\top (\mu_u - r \cdot \mathbf{1}) \right. \\ & \quad \left. - \frac{1}{2} \pi_u^\top \Sigma_u \pi_u) du \right] + \sup_{\pi \in \Pi_{[t_n,T]}} \inf_{\theta \in \Theta_{[t_n,T]}} \mathbb{E}_t \left[\int_{t_n}^T (\pi_u^\top (\mu_u - r \cdot \mathbf{1}) \right. \\ & \quad \left. - \frac{1}{2} \pi_u^\top \Sigma_u \pi_u) du \right] \end{aligned}$$

By Equation (3.5) for $[t_n, T]$, we have

$$\begin{aligned}
V(t, x) &= \sup_{\pi \in \Pi_{[t, T]}} \inf_{\theta \in \Theta_{[t, T]}} \mathbb{E}_t [\log(X_T^{\pi, \theta})] \\
&= \left(\frac{1}{2C_{t_n}} \|\mu_{t_n}^* - r \cdot \mathbf{1}\|^2 \right) (T - t_n) \\
&\quad + \sup_{\pi \in \Pi_{[t, t_n)}} \inf_{\theta \in \Theta_{[t, t_n)}} \mathbb{E}_t \left[\int_t^{t_n} (\pi_u^\top (\mu_u - r \cdot \mathbf{1}) \right. \\
&\quad \left. - \frac{1}{2} \pi_u^\top \Sigma_u \pi_u) du \right].
\end{aligned}$$

Here, we apply Lemma 3.1 and Theorem 2.1 on the interval $[t, t_n)$ with t_n in place of T and $\log(X_{t_n}^\pi)$ in place of $\log(X_T^\pi)$ to the expression,

$$\begin{aligned}
&\sup_{\pi \in \Pi_{[t, t_n)}} \inf_{\theta \in \Theta_{[t, t_n)}} \mathbb{E}_t \left[\log(X_{t_n}^\pi) \right] \\
&= \log(x) + \sup_{\pi \in \Pi_{[t, t_n)}} \inf_{\theta \in \Theta_{[t, t_n)}} \mathbb{E}_t \left[\int_t^{t_n} (\pi_u^\top (\mu_u - r \cdot \mathbf{1}) - \frac{1}{2} \pi_u^\top \Sigma_u \pi_u) du \right].
\end{aligned}$$

Hence, for $t \in [t_{n-1}, t_n)$, we conclude that the optimal parameters in $\Theta_{[t, t_n)}$ are

$$\begin{aligned}
\mu_{t_{n-1}}^* &\triangleq \arg \min_{\mu_t \in \Theta_{[t, t_n)}} (\|\mu_t - r \mathbf{1}\|) \\
&= \arg \min_{\mu_t \in \Theta_{[t_{n-1}, t_n)}} (\|\mu_t - r \mathbf{1}\|) \\
\Sigma_{t_{n-1}}^* &\triangleq \arg \max_{\Sigma_t \in \Theta_{[t, t_n)}} (\|\pi_t^\top \Sigma_t \pi_t\|) \\
&= C_{t_{n-1}} * I_{d \times d} \\
\pi_{t_{n-1}}^* &\triangleq \frac{1}{C_{t_{n-1}}} (\mu_{t_{n-1}}^* - r \mathbf{1})
\end{aligned}$$

Thus, for $t \in [t_{n-1}, t_n)$ and $x > 0$, we have

$$\begin{aligned}
V(t, x) &= \log(x) + \frac{1}{2} (\mu_{t_{n-1}}^* - r \cdot \mathbf{1})^\top (\Sigma_{t_{n-1}}^*)^{-1} (\mu_{t_{n-1}}^* - r \cdot \mathbf{1}) (t_n - t) \\
&\quad + \frac{1}{2} (\mu_{t_n}^* - r \cdot \mathbf{1})^\top (\Sigma_{t_n}^*)^{-1} (\mu_{t_n}^* - r \cdot \mathbf{1}) (T - t_n)
\end{aligned}$$

Iterating backwards this way to $t \in [t_0, t_1)$ and $x > 0$, we have

$$\begin{aligned}
V(t, x) &= \log(x) + \sum_{i=1}^n \frac{1}{2C_{t_i}} \|\mu_{t_i}^* - r \cdot \mathbf{1}\|^2 (t_{i+1} - t_i) \\
&\quad + \frac{1}{2C_{t_0}} \|\mu_{t_0}^* - r \cdot \mathbf{1}\|^2 (t_1 - t),
\end{aligned}$$

and the corresponding optimal parameters $(\theta_t)_{t \in [t_i, t_{i+1})}$ and the optimal policy $(\pi_t^*)_{t \in [t_i, t_{i+1})}$ are

$$\begin{aligned}\mu_{t_i}^* &= \arg \min_{\mu_t \in \Theta_{[t_i, t_{i+1})}} (\|\mu_t - r\mathbf{1}\|) \\ \Sigma_{t_i}^* &= \arg \max_{\Sigma_t \in \Theta_{[t_i, t_{i+1})}} (\|\pi_t^\top \Sigma_t \pi_t\|) \\ &= C_{t_i} * I_{d \times d} \\ \pi_{t_i}^* &= \frac{1}{C_{t_i}} (\mu_{t_i}^* - r\mathbf{1})\end{aligned}\tag{3.6}$$

Here, note that the derived optimal dynamics $\theta_t^* = (\mu_t^*, \sigma_t^*)_{t \in [0, T]}$ and optimal policy $(\pi_t^*)_{t \in [0, T]}$ in (3.6) are indeed admissible being in $\Theta_{[0, T]}$ and in $\Pi_{[0, T]}$, respectively.

3.2 Power Utility Case

We proceed to solve the robust optimization problem in power utility case. As in logarithmic utility function, following the above recipe, we assume that $V(t, x)$ is $C^{1,2}([t_n, T] \times \mathbb{R}_+)$ and pick the corresponding $\theta_{t_n}^* \in \Theta_{[t_n, T]}$ as in (3.1), and solve the classical nonrobust problem for $t \in [t_n, T]$ and $x > 0$

$$V(t, x) = \sup_{\pi \in \Pi_{[t, T]}} \mathbb{E}_t[(\hat{X}_T^{\hat{\pi}, \hat{\theta}^*})^\gamma],\tag{3.7}$$

for $0 < \gamma < 1$. The equation for (3.7) on $[t_n, T]$ retrieved from Lemma 3.1 assuming $V(t, x)$ is $C^{1,2}([t_n, T] \times \mathbb{R}_+)$ reads as

$$\begin{aligned}V_t + \sup_{\pi \in \Pi_{[t_n, T]}} \left\{ x \pi_t^\top (\mu_{t_n}^* - r\mathbf{1}) V_x + x^2 \frac{1}{2C_{t_n}} \pi_t^\top \pi_t V_{xx} \right\} &= 0 \\ V(T, x) &= x^\gamma\end{aligned}\tag{3.8}$$

We make the Ansatz to (3.8) on $[t_n, T]$ for $V(t, x)$ along with the optimal policy

$$V(t, x) = x^\gamma \exp\left(\frac{\gamma \|\mu_{t_n}^* - r\mathbf{1}\|^2}{2(1-\gamma)C_{t_n}}(T-t)\right)\tag{3.9}$$

for $t \in [t_n, T]$, which is $C^{1,2}([t_n, T] \times \mathbb{R}_+)$ and satisfies the conditions in Lemma 3.1 and fulfills (3.9). Hence, as in logarithmic case, we conclude that

$$V(t, x) = x^\gamma \mathbb{E} \left[\exp \left(\sum_{i=1}^n \frac{\gamma \|\mu_{t_i}^* - r\mathbf{1}\|^2}{2(1-\gamma)C_{t_i}} (t_{i+1} - t_i) + \frac{\gamma \|\mu_{t_0}^* - r\mathbf{1}\|^2}{2(1-\gamma)C_{t_0}} (t_1 - t) \right) \right].$$

The optimal $\theta \in \Theta_{[t, t_{i+1})}$ and cashflow for $t \in [t_i, t_{i+1})$ are constant and deterministic with

$$\begin{aligned}\mu_t^* &= \arg \min_{\mu_t \in \Theta_{[t_i, t_{i+1})}} (\|\mu_t - r\mathbf{1}\|) \\ \Sigma_t^* &= \arg \max_{\Sigma_t \in \Theta_{[t_i, t_{i+1})}} (\|\pi_t^\top \Sigma_t \pi_t\|) = C_{t_i} * I_{d \times d} \\ \pi_t^* &= \frac{\mu_{t_i}^* - r\mathbf{1}}{C_{t_i}(1 - \gamma)},\end{aligned}\tag{3.10}$$

for all $t \in [t_i, t_{i+1})$. As in logarithmic utility case, the derived optimal dynamics as in (3.10), $\theta_t^* = (\mu_t^*, \sigma_t^*)_{t \in [0, T]}$ and optimal policy $(\pi_t^*)_{t \in [0, T]}$ are indeed admissible being in $\Theta_{[0, T]}$ and in $\Pi_{[0, T]}$, respectively.

3.3 Exponential Utility Case

We next analyze the robust utility optimization problem for the exponential utility case

$$u(x) = -\beta e^{-\beta x}$$

for $x > 0$ and $\beta > 0$. We take $\hat{\pi}_t = X_t^{\pi, \theta^*} \pi_t$ in (2.4) such that

$$d\hat{X}_T^{\hat{\pi}, \hat{\theta}} = \hat{\pi}_t^\top (\mu_t - r \cdot \mathbf{1}) dt + \hat{\pi}_t^\top \sigma_t dW_t, \quad \mathbb{P}_0\text{-a.s.}\tag{3.11}$$

At $t \in [t_n, T]$, the optimization problem reads as

$$V(t, x) = \sup_{\pi \in \Pi_{[t, T]}} \inf_{\theta \in \Theta_{[t, T]}} \mathbb{E}_t[-\beta e^{-\beta \hat{X}_T^{\hat{\pi}, \hat{\theta}}}]$$

Next, we state the following result analogous to Lemma 3.1.

Lemma 3.2. *Let $v(t, x)$ be an increasing and concave $C^{1,2}([t_n, T] \times \mathbb{R}_+)$ function such that $\frac{v_x}{-v_{xx}} = c$ for some nonnegative scalar c . Then, for $t \in [t_n, T]$, the infimum in*

$$\begin{aligned}v_t + \sup_{\pi \in \Pi_{[t_n, T]}} \inf_{\theta \in \Theta_{[t_n, T]}} \left[v_x \pi_t^\top (\mu_t - r\mathbf{1}) + \frac{1}{2} v_{xx} \pi_t^\top \Sigma_t \pi_t \right] &= 0 \\ v(T, x) &= -\beta e^{-\beta x}\end{aligned}$$

are attained for $\theta_{t_n}^*$ as in (3.1).

Proof. The proof is a simple modification of Lemma 3.1. □

Based on Lemma 3.1 and Theorem 2.1, for $t \in [t_n, T]$ and $x > 0$, we proceed to solve

$$\begin{aligned}V(t, x) &= \sup_{\hat{\pi} \in \Pi_{[t, T]}} \mathbb{E}_t[-\beta e^{-\beta \hat{X}_T^{\hat{\pi}, \theta}}] \\ &= -\beta \inf_{\hat{\pi} \in \Pi_{[t, T]}} \mathbb{E}_t[e^{-\beta \hat{X}_T^{\hat{\pi}, \theta}}],\end{aligned}$$

with $\hat{X}_T^{\hat{\pi}, \hat{\theta}}$ having the dynamics Equation (3.11) with $\theta_{t_n}^*$ as in (3.1). As in the previous two cases, we find $V(t, x)$ and verify that it is $C^{1,2}([t_n, T] \times \mathbb{R}_+)$. Indeed,

$$\begin{aligned} V(t, x) &= -\beta e^{-\beta x} \inf_{\hat{\pi} \in \Pi_{[t, T]}} \mathbb{E}_t \left[\exp \left(\int_t^T -\beta \hat{\pi}_u^\top (\mu_{t_n}^* - r \cdot \mathbf{1}) du + \int_t^T \hat{\pi}_u^\top \sigma_{t_n}^* dW_u \right) \right] \\ &= -\beta e^{-\beta x} \inf_{\hat{\pi} \in \Pi_{[t, T]}} \mathbb{E}_t \left[\exp \left(\int_t^T -\beta \hat{\pi}_u^\top (\mu_{t_n}^* - r \cdot \mathbf{1}) + \frac{1}{2} \beta^2 \hat{\pi}_u^\top \Sigma_{t_n}^* \hat{\pi}_u \right) du \right] \end{aligned}$$

We note that

$$-\beta \hat{\pi}_u^\top (\mu_{t_n}^* - r \cdot \mathbf{1}) + \frac{1}{2} \beta^2 \hat{\pi}_u^\top \Sigma_{t_n}^* \hat{\pi}_u$$

is convex in $\hat{\pi}$. Hence, by pointwise minimisation, we get that for $t \in [t_n, T]$

$$\begin{aligned} \hat{\pi}_t^* &= \frac{1}{C_{t_n}} \frac{1}{\beta} (\mu_{t_n}^* - r \mathbf{1}) \\ V(t, x) &= -\beta e^{-\beta X_t^{\pi_t^*, \theta^*}} \exp \left(-\frac{\|\mu_{t_n}^* - r \cdot \mathbf{1}\|^2}{2C_{t_n}} (T - t) \right) \end{aligned}$$

We see that $V(t, x)$ is $C^{1,2}([t_n, T] \times \mathbb{R}_+)$ and $-\frac{V_x}{V_{xx}} = 1/\beta$ being the nonnegative constant for $x > 0$. Hence, the verification is complete. Going backwards by repeating the above verification procedure for $[t_{n-1}, t_n), [t_{n-2}, t_{n-1}), \dots, [t_0, t_1)$, we conclude that the optimal parameters for $t \in [t_i, t_{i+1})$ are

$$\begin{aligned} \mu_{t_i}^* &= \arg \min_{\mu_t \in \Theta_{[t_i, t_{i+1})}} (\|\mu_t - r \mathbf{1}\|) \\ \Sigma_{t_i}^* &= \arg \max_{\Sigma_t \in \Theta_{[t_i, t_{i+1})}} (\|\pi_t^\top \Sigma_t \pi_t\|) = C_{t_i} * I_{d \times d}, \\ \hat{\pi}_{t_i}^* &= \frac{1}{C_{t_i}} \frac{1}{\beta} (\mu_{t_i}^* - r \mathbf{1}), \end{aligned} \tag{3.12}$$

and the value function at $t \in [t_0, t_1)$ and $x > 0$ reads as

$$V(t, x) = -\beta e^{-\beta x} \exp \left(-\sum_{i=1}^n \frac{\|\mu_{t_i}^* - r \cdot \mathbf{1}\|^2}{2C_{t_i}} (t_{i+1} - t_i) + \frac{\|\mu_{t_i}^* - r \cdot \mathbf{1}\|^2}{2C_{t_0}} (t_{i+1} - t) \right)$$

As in logarithmic and power utility cases, the derived optimal dynamics $\theta_t^* = (\mu_t^*, \sigma_t^*)_{t \in [0, T]}$ and optimal policy $(\pi_t^*)_{t \in [0, T]}$ in (3.12) are admissible being in $\Theta_{[0, T]}$ and in $\Pi_{[0, T]}$, respectively.

4 Discussion and Concluding Remarks

4.1 Numerical Study

For illustrative purposes, we take that the investor has logarithmic utility and has 3 risky assets and 1 riskless asset on $[0, T]$ with $T = 10$ with $t_i = i$ for $i = 0, \dots, 9$. For simplicity,

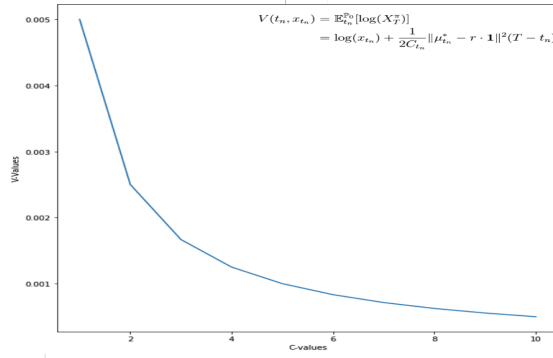


Figure 1: Value Function with Logarithmic Utility

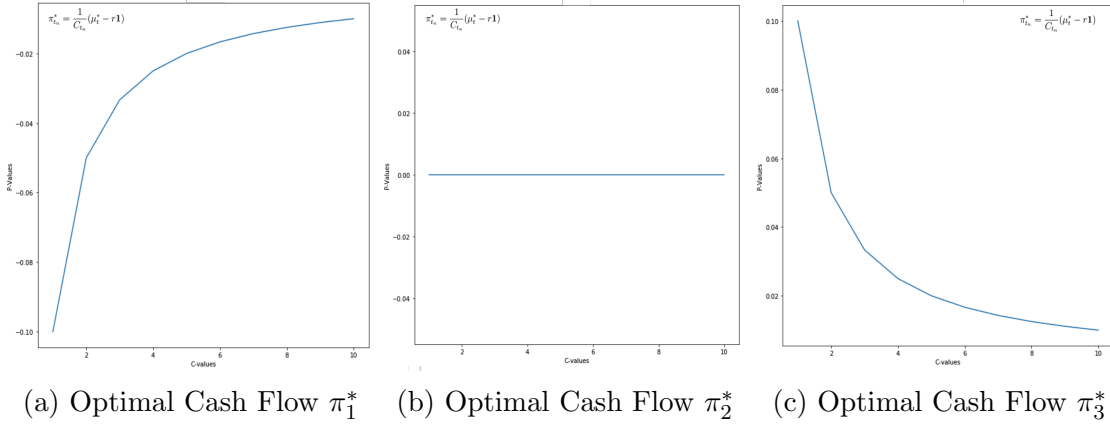


Figure 2: Portion Invested into Risky Assets

we assume that the uncertainty intervals are fixed and time independent. The uncertainty intervals for $t \in [0, T]$ and $r > 0$ are given as follows

$$\begin{aligned} \mu_t &= [[0.2, 0.3], [0.3, 0.4], [0.4, 0.5]] \\ r &= 0.4 \\ C &\in [1, 2, \dots, 10] \end{aligned}$$

The value function and the cash flow at each risky asset at $t_n = 9$ with respect to C are shown in the figures above.

We see that our long and short positions in risky assets decrease proportional to the uncertainty of Σ^* , $C = i$ for $i = 1, \dots, 10$, which is consistent with the intuition of the investor being risk-averse. Furthermore, if the uncertainty set includes the risk free interest rate r , the uncertainty set does not play a role anymore. The investor puts all his wealth in to the riskless asset. The value function is either constant, in case the uncertainty of (μ_1, μ_2, μ_3)

includes r , or the portion put into risky assets converges to zero as the uncertainty on μ decreases. As a result, as the uncertainty increases, the value function $V(t, x)$ converges to $\log(x)$ for initial wealth $x > 0$.

4.2 Concluding Remarks

In this paper, we have solved robust utility maximization problem, where there is uncertainty on both mean and covariance matrix of the risky assets. We see that the robust approach in three classical utility functions necessitates to choose the volatility of the largest magnitude with $\Sigma_{t_i}^* = C_{t_i} * I_{d \times d}$ for $t \in [t_i, t_{i+1})$ for $i = 0, \dots, n-1$ and $\Sigma_{t_n}^* = C_{t_n} \times I_{d \times d}$ for $t \in [t_n, T]$, whereas the drift term is to be chosen closest to the risk free interest rate with $\mu_{t_i}^* = \arg \min_{\mu_t \in \Theta_{[t_i, t_{i+1})}} \|\mu_t - r\mathbf{1}\|$, respectively.

Furthermore, an important limiting argument of the uncertainty sets is also immediate by our framework. In particular, given that the uncertainty interval denoted by Θ_t changes at each time $t \in [0, T]$ rather than at prespecified times $0 = t_0 < t_1 < \dots < t_n < T$, letting the mesh $\Delta t_i \triangleq t_{i+1} - t_i \rightarrow 0$, we have the optimal parameters along with the value function in exponential utility case for $t \in [0, T]$

$$\begin{aligned} \mu_t^* &= \arg \min_{\mu \in \Theta_t} (\|\mu_t - r\mathbf{1}\|) \\ \hat{\pi}_t^* &= \frac{1}{C_t} \frac{1}{\beta} (\mu_t^* - r\mathbf{1}) \\ \Sigma_t^* &= \arg \max_{\Sigma \in \Theta_t} (\|\pi_t^\top \Sigma \pi_t\|) = C_t * I_{d \times d} \end{aligned}$$

and the value function at (t_0, x_0) reads as

$$V(t_0, x_0) = -\beta e^{-\beta x_0} \exp \left(- \int_0^T \frac{\|\mu_t^* - r \cdot \mathbf{1}\|^2}{2C_t} dt \right). \quad (4.1)$$

An important observation here is that we do not lose generality of the problem, when we use this limiting argument to reach to (4.1). To clarify, the limiting value in (4.1) equals

$$\inf_{\theta \in \Theta_{[0, T]}} \sup_{\pi \in \Pi_{[0, T]}^{\text{ad}}} -\beta \mathbb{E}^{\mathbb{P}_0} [e^{-\beta X_T^{\pi, \theta}}], \quad (4.2)$$

which is by minmax inequality greater than or equal to

$$\sup_{\pi \in \Pi_{[0, T]}^{\text{ad}}} \inf_{\theta \in \Theta_{[0, T]}} -\beta \mathbb{E}^{\mathbb{P}_0} [e^{-\beta X_T^{\pi, \theta}}],$$

Hence, the limiting argument of the uncertainty sets for drift and volatility reaches to the upper bound of the problem (4.2). The power and utility cases have the analogous optimal parameters and optimal values, accordingly.

For future outlook, we leave to investigate model uncertainty with respect to different models, as well as to investigate the noncompact uncertainty sets for mean and covariance uncertainty for future research. Another research direction would be to investigate in a more general jump diffusion framework optimization under model uncertainty.

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