On the complexity of the Unit Commitment Problem

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Abstract This article analyzes how the Unit Commitment Problem (UCP) complexity evolves with respect to the number n of units and T of time periods.A classical reduction from the knapsack problem shows that the UCP is NP-hard in the ordinary sense even for $T = 1$. The UCP is proved to be strongly NP-hard.When either a unitary cost or amount of power is considered, the UCP is polynomial for $T = 1$ and NP-hard for arbitrary T. When the constraints are restricted to minimum up and down times, the UCP is shown to be polynomial for a fixed n. The pricing subproblem commonly used in a UCP decomposition scheme is also shown to be strongly NP-hard for a subset of units.

Keywords Unit Commitment Problem · Complexity · Dynamic Programming · Subproblem of decomposition scheme

1 Introduction

Given a discrete time horizon $\mathcal{T} = \{1, ..., T\}$, a demand for electric power D_t is to be met at each time period $t \in \mathcal{T}$. Power is provided by a set $\mathcal{N}$ of n production units. At each time period, unit $i \in \mathcal{N}$ must be either down or up, and in the latter case, its production is within $[P^i_{min}, P^i_{max}]$. Each unit must satisfy minimum up-time (resp. down-time) constraints, denoted by *min-up/min-down* constraints, *i.e.* each unit i must remain up (resp. down) during at least L^i (resp. ℓ^i) periods after start up (resp. shut down). Each unit i has *initial conditions* (e^i, τ^i), indicating unit i has commitment status e^i before time 1, and it should remain in the same status up to time τ^i (if $\tau^i = 0$, unit i is free to shut down or start up at time 1). Each unit i also features three different costs: a fixed cost c^i_f, incurred each time period the unit is up; a start-up cost c^i_0, incurred each time the unit starts up; and a cost c^i_p proportional to its production. The Min-up/min-down Unit Commitment Problem (MUCP) is to find a production plan minimizing the total cost while satisfying the demand and the minimum up/down time constraints (Bendotti et al. (2016)).

When unit set $\mathcal{N}$ is partitioned into k sites Σ_1, ..., Σ_k, each site Σ must satisfy *intra-site* constraints, *i.e.* at each time period t, at most one unit in site Σ is allowed to start up. The Intra-site Min-up/min-down Unit

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Commitment Problem (IMUCP) is a generalization of the MUCP where the intra-site constraints must be satisfied. From a combinatorial point of view, the IMUCP is the core structure of the UCP, which is the real-world electricity production problem daily considered by EDF (Électricité de France). Some more technical constraints have also to be taken into account, *e.g.* ramp constraints or reserve requirement constraints, and the start-up costs are an exponential function of the unit downtime (Renaud (1993)).

In the particular case when there is only one production unit, *i.e.* $n = 1$, several articles provide complexity results for the MUCP. Rajan and Takriti (2005) study the 1-unit polytope associated to the min-up/min-down constraints. They provide a compact characterization of this polytope, thus showing the corresponding problem can be solved in polynomial time for any cost values. Gentile et al. (2016) extend this result to provide a compact characterization of the 1-unit polytope associated with the min-up/min-down constraints and *start-up/shut-down generation limits*. More precisely, if the unit starts up at time t (resp. shuts down at time $t+1$), its power output at time t is limited to P^U (resp. P^D). Morales-Espana et al. (2015) provide a compact characterization of a polytope associated to the same problem with additional *start-up/shut-down trajectories*. In such a case, the start-up (resp. shut-down) of the unit does not take only one time period but t_u (resp. t_d) time periods, during which the unit follows a specific power trajectory P_1^U, ..., $P_{t_u}^U$ (resp. P_1^D, ..., $P_{t_d}^D$). Pan and Guan (2016) provide a compact characterization, for $T = 3$, of the polytope associated to the min-up/min-down constraints, start-up/shut-down generation limits, and *ramp-up/down constraints*, *i.e.* the increase (resp. decrease) in generated power from time period t to time period $t+1$ is limited to P^{RU} (resp. P^{RD}). A compact extended formulation is proposed by Knueven et al. (2016). For the latter problem with an arbitrary convex cost function, a polynomial time dynamic programming algorithm is proposed by Frangioni and Gentile (2006). This algorithm is to find a shortest path in a state graph, where each vertex represents a possible time interval during which the unit is up. Variants of this algorithm are studied by Pan et al. (2016), alongside with compact extended formulations.

In perspective of a decomposition scheme for the MUCP, a Lagrangian relaxation or a column generation is commonly used. Indeed, given the large number and variety of units, the daily production planning problem is solved at EDF using a Lagrangian relaxation (Renaud (1993)), commonly referred as price decomposition (Cohen (1978); Frangioni (2005)). In a Lagrangian relaxation approach for the IMUCP, the only coupling inequalities which are dualized are the demand constraints. The pricing problem is then divided into k subproblems, one for each site. Each subproblem contains all the IMUCP constraints, but the demand satisfaction. Instead, the fixed and proportional production costs are modified to take into account the dualization of the demand constraint, leading to the so-called *reduced costs*. For each unit i and time t, the reduced cost related to the fixed cost of unit i is denoted by $\pi^{i,t}$ and that of proportional production cost by $\rho^{i,t}$. In this case, these two costs can be negative and depend on t and i. The Pricing subproblem of the IMUCP is called P-IMUCP. Furthermore the only coupling constraints of the P-IMUCP are the intra-site constraints, which ensure there is at most one start-up per time period for each site. Note that this problem also arises as the pricing subproblem in a column generation setting.

In this article, we present an in-depth complexity analysis of the MUCP, IMUCP and P-IMUCP. When there are an arbitrary number n of units and a demand satisfaction constraint coupling the units, the MUCP is NP-hard, as for $T = 1$ and $P_{min}^i = P_{max}^i$, $i \in \mathcal{N}$, Tseng (1996) pointed out that the 0-1 knapsack problem reduces to the MUCP. However, this reduction is not fully satisfactory, as the knapsack problem is pseudo-polynomial and many instances of the knapsack are quite well-solved (Pisinger (2005)). One can then wonder whether the MUCP is NP-hard in the strong sense. Another question is whether the complexity of the MUCP would change when either the fixed cost c_f^i or the production $P = P_{min}^i = P_{max}^i$ is unitary. Indeed, these two cases correspond to well-known polynomial cases of the knapsack problem. Moreover, we have seen that the problem can be solved in polynomial time whenever $n = 1$ but is already NP-hard when $T = 1$. The question is then for which values of n the problem becomes NP-hard, and for which values of T the problem would become strongly NP-hard.

In this article, we prove the MUCP is strongly NP-hard in the general case, and remains so in cases where the underlying knapsack problem is polynomial. In particular, the case involving a unitary cost and/or a unitary amount of production per unit remains NP-hard. For each case we focus on the relative impact of parameters n

and T on the complexity. In particular we show that the problem is polynomial whenever n is fixed. Finally we study the complexity of the P-IMUCP.

In Section 2, we prove that the Unit Commitment Problem (UCP) is strongly NP-hard by reduction from the 3-Partition problem. We show in Section 3 that the IMUCP can be solved in polynomial time whenever n is fixed. In Sections 4.1 and 4.2, we study some relaxations of the MUCP which are polynomial for $T = 1$, and prove these subproblems are strongly NP-hard for arbitrary T. In Section 5, we prove the P-IMUCP is strongly NP-hard. In Section 6 we present as a conclusion a table which summarizes the results.

2 The UCP is strongly NP-hard

In this section, we prove the following result.

Theorem 1 *The MUCP is strongly NP-hard for* $T = \frac{n}{3}$.

Proof Let us consider an instance of the 3-Partition problem, with a set A of $3m$ integers a_1, ..., a_{3m}, a bound $B \in \mathbb{N}$ such that $\frac{B}{4} < a < \frac{B}{2}$ for all $a \in A$, and such that $\sum_{a\in A} a = mB$. The question is whether A can be partitioned into m triplets A_1, ..., A_m, such that $\sum_{a\in A_i} a = B$. Note that if a such partition of A into m subsets A_1, ..., A_m with sum B exists, then each subset A_i must contain exactly three elements.

Consider now the following instance of the MUCP. Let $T = m$, with $D_t = (m-t)B$, $\forall t \in \{1,...,T\}$. Note that at each time period the demand decreases by B. Let $n = 3m$ the number of units. For each $i \in \{1,...,3m\}$, $P^i_{min} = P^i_{max} = a_i$; $\ell^i = T$; $L^i = 1$; $c^i_f = a_i$, $c^i_0 = c^i_p = 0$; and with initial state such that $e^i =$"up" and $\tau^i = 0$.

Let us suppose there exists a solution to the latter instance, with cost less or equal to $\sum_{i=1}^{m-1} iB = B\frac{m(m-1)}{2}$.

Since $\sum_{t=1}^{T} D_t = B\frac{m(m-1)}{2}$ and the unit cost is equal to the production, the cost of any solution will be at least $B\frac{m(m-1)}{2}$. If at a given time t the production is greater than the demand $D_t = (m-t)B$ then the solution cost will be greater than $B\frac{m(m-1)}{2}$. So for any solution of cost $B\frac{m(m-1)}{2}$, at each time period t, the units produce exactly D_t.

Let A_t be the subset of units which shut down at time t. Since every unit is up at time 1 and $\ell^i = T$, for each unit i, each unit can shut down just once so subsets A_t are disjoint. Since at each time period the units produce exactly the demand D_t, it follows $\sum_{i\in A_t} P^i_{max} = \sum_{i\in A_t} a_i = D_{t-1} - D_t = B$. Hence, the partition A_1, ..., A_T directly gives a solution to the instance of the 3-Partition problem.

Conversely, from a solution to the 3-Partition problem instance, a solution to this MUCP instance can be constructed with cost equal to $B\frac{m(m-1)}{2}$. □

This result shows that the MUCP is strongly NP-hard even when each unit production cost matches its production, *i.e.* $c^i_f = P^i_{min} = P^i_{max}$, $c^i_0 = c^i_p = 0$, $\forall i \in \mathcal{N}$. As the MUCP is a particular case of the UCP, the UCP is thus strongly NP-hard as well.

Note that for $T = 1$, the MUCP is only weakly NP-hard. Indeed, given an MUCP instance, it can be shown that there exists an optimal solution such that at most one unit i has a production $\overline{p}^i$ with $P^i_{min} < \overline{p}^i < P^i_{max}$. From this property, a pseudo-polynomial dynamic programming algorithm, similar to those already existing for the knapsack problem, can be derived.

3 The IMUCP is polynomial when n is fixed

In the following, we consider the IMUCP where the number of units n is fixed, which is equivalent to say that n is not considered as a parameter of the problem. Note that the IMUCP is a generalization of the MUCP.

For each unit $i \in \mathcal{N}$ and time period $t \in \mathcal{T}$, the possible states for unit i at time t are given by the *unit-state* set $E^i_t = \{-\ell^i,...,-1,1,...,L^i\}$:

- either unit i is up at time t and must remain up for at least ε^i_t time periods (including t), which corresponds to the unit-state $\varepsilon^i_t \in \{1,...,L^i\}$,

- or unit i is down at t and must remain down for at least $|\varepsilon_t^i|$ time periods (including t), which corresponds to the unit-state $\varepsilon_t^i \in \{-\ell^i, ..., -1\}$.

Given $\varepsilon_t^i \in E_t^i$, the set of the next possible unit-states for i are given by $\Gamma(\varepsilon^i)$:

$$\Gamma(\varepsilon_t^i) = \begin{cases} \{\varepsilon^i - 1\} & \text{if } \varepsilon_t^i > 1 \\ \{-\ell^i, 1\} & \text{if } \varepsilon_t^i = 1 \\ \{-1, L^i\} & \text{if } \varepsilon_t^i = -1 \\ \{\varepsilon^i + 1\} & \text{if } \varepsilon_t^i < -1 \end{cases}$$

For instance, if $\varepsilon_t^i = 1$, unit i is up at time t and can at time $t+1$ either stay up or shut down for at least ℓ^i time periods.

We introduce a graph $G = (V, A)$, whose vertices are possible states of the whole n-unit system for a given time period t. An arc will be drawn between a possible state at time t and a reachable state at time $t+1$. The length of this arc will be given by the cost of the state at time $t+1$. We will show that an optimal solution to the MUCP can be obtained by finding a shortest path in this graph.

Let us define the state of the n-unit system at time t as a tuple $v_t = (\varepsilon_t^1, \varepsilon_t^2, ..., \varepsilon_t^n)$ where $\varepsilon_t^i \in E_t^i$. Let V_t be the set of all tuples v_t which correspond to possible states, *i.e.* states that both fulfill the demand and intra-site constraints: the demand D_t has to be satisfied by the subset of units which are up in v_t, assuming they produce at P_{max}; at most one unit per site can start up at time t. We then set $V = \cup_{t=1}^T V_t \cup \{v_0, v_{T+1}\}$ where v_0 is a source vertex corresponding to the initial condition $(e^1, ..., e^n)$ and v_{T+1} is a sink vertex.

For any $t \in \{0, ..., T-1\}$, there is an arc between a state $v_t = (\varepsilon_t^1, \varepsilon_t^2, ..., \varepsilon_t^n) \in V_t$ and $v_{t+1} = (\varepsilon_{t+1}^1, \varepsilon_{t+1}^2, ..., \varepsilon_{t+1}^n) \in V_{t+1}$ if and only if for all $i \in \mathcal{N}$, $\varepsilon_{t+1}^i \in \Gamma(\varepsilon_t^i)$. Moreover, A contains an arc (v_T, v_{T+1}) for any $v_T \in V_T$.

The length of an arc is given by $\lambda : A \to \mathbb{R}_+$. For each $\varepsilon \in V$, $\lambda(\varepsilon, v_{T+1}) = 0$. For each arc (v_t, v_{t+1}), $\{0, ..., T-1\}$, the length is given by:

$$\lambda(v_t, v_{t+1}) = \sum_{i \text{ up in } v_{t+1}} c_f^i + \sum_{i \text{ starts up in } v_{t+1}} c_0^i + \sum_{i \text{ up in } v_{t+1}} c_p^i \overline{p}_{t+1}^i$$

where for each t, $(\overline{p}_t^i)_{i \text{ up in } v_t}$ is the optimal solution to the following production dispatch LP:

$$\begin{aligned} \min_p \quad & \sum_{i \in X_t} c_p^i p_t^i \\ \text{s. t.} \quad & \sum_{i \in X_t} p_t^i \geq D_t \\ & P_{min}^i \leq p_t^i \leq P_{max}^i \quad && \forall i \text{ up in } v_t \\ & p_t^i \in \mathbb{R} \quad && \forall i \text{ up in } v_t \end{aligned}$$

This problem can be solved in linear time, provided that the units are sorted in non-decreasing order of c_p^i. Indeed, the optimal solution can be constructed by generating the maximum quantity of power possible with the lowest cost units, until the demand is met. Note that, by the construction of v_t, this production dispatch LP is always feasible.

Considering graph $G = (V, A)$ and length λ, a solution to the fixed-n MUCP is exactly a solution to the shortest path problem from v_0 to v_{T+1} in graph G. This leads to the following theorem.

Theorem 2 *The fixed-n IMUCP can be polynomially solved in* $O(T^2 \prod_{i=1}^n (L^i + \ell^i)^2)$.

Proof For each unit i and t, $|E_t^i| \leq L^i + \ell^i$. Thus, $|V_t| \leq \Pi_{i=1}^n (L^i + \ell^i)$. It follows that the number of vertices in G is bounded by $2 + T\Pi_{i=1}^n (L^i + \ell^i)$. As n is fixed, this corresponds to a polynomial number of vertices. Since graph G is acyclic, a shortest path can be computed using Bellman algorithm. □

4 NP-hard special cases of the MUCP

When $T = 1$, the MUCP is NP-hard by reduction from the knapsack problem (Tseng (1996)). However the knapsack problem becomes polynomial if either the item cost or the item weight is unitary. Interestingly, it is shown in this section the corresponding MUCP cases are NP-hard. This highlights that the difficulty of the MUCP does not only lie in the knapsack reduction at $T = 1$, but also in the combinatorial aspects introduced by the min-up/min-down time constraints. Contrary to the general case of Theorem 1, we have only proved that these easier cases are strongly NP-hard for T much greater than n.

4.1 The unit-cost MUCP is NP-hard

We define the *unit-cost* MUCP, as a particular case of the MUCP where $c_0^i = c_p^i = 0$ and $c_f^i = 1$, for each unit i. This problem can be solved in polynomial time when $T = 1$, by sorting the units in decreasing order w.r.t. their P_{max}^i.

Theorem 3 *The unit-cost MUCP is NP-hard for* $T = n+1$ *and strongly NP-hard for* $T = \frac{1}{3}n^2 + 1$.

Proof Let us consider an instance of the Partition problem, with a set A of n positive integers $a_1, ..., a_n$. The question is whether A can be partitioned into two subsets A_1 and A_2 such that $\sum_{i\in A_1} a_i = \sum_{i\in A_2} a_i$. Note that if such a partition exists, then $\sum_{i\in A_1} a_i = \sum_{i\in A_2} a_i = B$ where $B = \frac{1}{2}\sum_{i\in A} a_i$.

Consider now the following instance of the unit-cost MUCP: let $T = n+1$ with $D_1 = D_T = B$ and $D_t = 0$, for all $t \in [2, T-1]$. Let us define n units such that $P_{max}^i = P_{min}^i = a_i$, $\ell^i = T$, $L^i = 1$, with initial condition $e^i =$ "down", $\tau^i = 0$ and $i \in \{1, ..., n\}$. Assume there exists a solution to the latter instance with cost less than or equal to n. In such a solution, each unit up at time 1 must be down at time T. Indeed, $\ell^i = T$ so when a unit shuts down it can never start up again. However, if a unit i remains up from time 1 to time T, then the cost of solution S is at least $n+1$, which is a contradiction. Thus, let A_1 be the set of units up at time 1, and A_2 be the set of units up at time T. The claim is that (A_1, A_2) gives a solution to the instance of the Partition problem. Indeed, A_1 and A_2 are disjoint, as all units up at time 1 are down at time T. Moreover, the units in A_1 satisfy the demand at time 1, so $\sum_{i\in A_1} a_i \geq B$. Similarly, $\sum_{i\in A_2} a_i \geq B$. As A_1 and A_2 are disjoint, $2B \leq \sum_{i\in A_1} a_i + \sum_{i\in A_2} a_i \leq \sum_{i\in A} a_i = 2B$, we get $\sum_{i\in A_1} a_i = B$, $\sum_{i\in A_2} a_i = B$ and $A_1 \cup A_2 = A$.

Conversely, any solution to the instance of the Partition problem can similarly be used to construct a solution to this unit-cost MUCP instance with cost n.

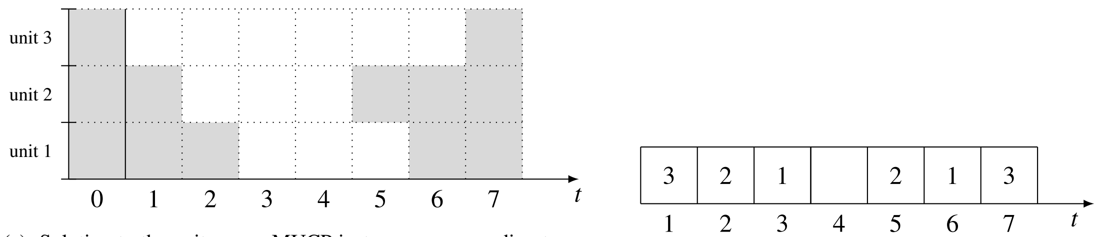
This transformation can be slightly modified to show that the unit-cost MUCP is strongly NP-hard, by reduction from the 3-Partition problem. Let us consider a 3-Partition problem instance with a set A of $3m$ integers a_1, ..., a_{3m}, a bound $B \in \mathbb{N}$ such that $B/4 < a < B/2$ for all $a \in A$, and such that $\sum_{a\in A} a = mB$. We consider an instance of the unit-cost MUCP with $n = 3m$ units, time horizon $T = mn+1$ and demand $D_{kn+1} = B$ for each $k \in \{0, ..., m\}$ and $D_t = 0$ otherwise. The units have the same characteristics as those of the reduction from Partition. We can similarly prove that there is a solution to the 3-Partition problem if and only if there is a solution to this instance of the unit-cost MUCP. □

4.2 The unit-power MUCP is NP-hard

We define the *unit-power* MUCP as a particular case of the MUCP where all units which are up produce the same amount of power P such that $P_{min}^i = P_{max}^i = P$, $i \in \mathcal{N}$. The unit-power MUCP can be solved in polynomial time when $T = 1$, by sorting the units by increasing order w.r.t. their costs.

The unit-power MUCP is shown to be NP-hard for arbitrary T, by reduction from the single machine Flow-Shop Problem with minimum delays and unit-time operations (FSP). This problem, proved strongly NP-hard by Yu et al. (2004), is defined as follows.

INSTANCE: Consider a set of jobs J, such that each job consists of two unit-time operations. The two operations of a given job j must be scheduled with an intermediate delay p_j. Let Δ be an integer.



(a) Solution to the unit-power MUCP instance corresponding to I_{ex}

(b) Solution to the FSP instance I_{ex}

Fig. 1: Instance I_{ex} of the FSP: $|J| = 3$, $p = [3,3,6]$ and $\Delta = 7$

QUESTION: A schedule (σ_1, σ_2) is defined by function $\sigma_1 : J \to \mathbb{Z}_0^+$ (resp. $\sigma_2 : J \to \mathbb{Z}_0^+$) that gives the schedule of operation 1 (resp. 2) for each job. For all $t \geq 1$, there is at most one job $j \in J$ such that $\exists i \in \{1,2\}$, $\sigma_i(j) = t$. Moreover, $\sigma_1(j) + p_j \leq \sigma_2(j)$ for all $j \in J$. Is there a schedule (σ_1, σ_2) with a makespan less than or equal to Δ, *i.e.* $\sigma_2(j) \leq \Delta$, for all $j \in J$?

Figure 1b shows a solution to the following FSP instance, denoted by I_{ex}: $|J| = 3$, $p = [3,3,6]$ and $\Delta = 6$. The first operation of job 3 is executed at time 1, and the second operation of job 3 at time 7. The intermediate delay $p_3 = 6$ is thus respected. Similarly, first and second operations of jobs 1 and 2 are executed before the deadline while satisfying their intermediate delays.

Theorem 4 *The unit-power MUCP is NP-hard.*

Proof Let consider an instance I_{FSP} of the FSP problem. We construct an instance I of the unit-power MUCP as follows: let $n = |J|$ units and a time horizon $T = \Delta$. For each unit $j \in J$, let $L^j = T$ and $\ell^j = p^j$; $c_0^j = c_p^j = 0$ and $c_f^j = 1$; $e^j =$ "up" and $\tau^j = 0$. Note that since there is a single machine, and two unit-time operations per job, if $\Delta < 2n$ then there is no solution to I_{FSP}. We thus suppose $\Delta \geq 2n$. The demand D_t is given by:

$$D_t = \begin{cases} n - t & \text{if } t \in \{1, ..., n\} \\ 0 & \text{if } t \in \{n+1, ..., T-n\} \\ t - (T-n) & \text{if } t \in \{T-n+1, ..., T\} \end{cases}$$

The claim is that if we can find a solution S of cost at most n^2 for instance I then we can find a schedule for instance I_{FSP}. Let us consider a solution S with cost at most n^2. First note that $\sum_{t=1}^{T} D_t = n^2$ and then any solution to I is with cost at least n^2, since each up unit costs 1 per time period.

In solution S, exactly D_t units are then up at time t, for each $t \in \{1, ..., T\}$. As $L^j = T$ for each $j \in \{1, ..., n\}$, each unit shuts down at most once on the time horizon T. As $e^j =$ "up" and $D_n = 0$, each unit must shut down exactly once on the time horizon. Similarly, as the demand at time T is n, each unit starts up exactly once. For a unit j, let $\sigma_1(j)$ (resp. $\sigma_2(j)$) be the shut-down (resp. start-up) time period of unit j in solution S.

For a solution S to instance I, we construct a solution S_{FSP} to instance I_{FSP} in which the first operation of job j is processed at time $\sigma_1(j)$ and the second operation of job j is processed at time $\sigma_2(j)$.

Figure 1 depicts a reduction from the FSP instance I_{ex} to a unit-power MUCP instance. Figure 1b shows the solution to the FSP instance I_{ex}, while Figure 1a shows the solution to the corresponding MUCP instance.

Since unit j has minimum down-time p_j, for all $j \in J$, $\sigma_1(j) + p_j \leq \sigma_2(j)$ holds. Moreover, as $D_T = n$, $\sigma_2(j) \leq \Delta$ holds. The claim is there is at most one shut-down (resp. start-up) per time period. Indeed, since $D_n = 0$ (resp. $D_{T-n} = 0$) and each unit shuts down (resp. starts up) exactly once, all the shut-downs (resp. start-ups) happen between times 1 and n (resp. $T-n+1$ and T). If there were two units shutting-down (resp. starting-up) at the same time period $t \in \{1, ..., n\}$ (resp. $\{T-n+1, ..., T\}$), then, as $D_t = D_{t-1} - 1$ (resp. $D_t = D_{t-1} + 1$), either the demand would not be satisfied at time t (resp. $t-1$), or there would be more than D_{t-1} (resp. D_t) units up at time $t-1$ (resp. t), which would be a contradiction. Consequently, at most one operation is executed per time period.

Conversely, if there is a solution S_{FSP} to instance I_{FSP}, a solution S to instance I of cost at most n^2 can be constructed. Let S_{FSP} be a solution of the FSP instance. It can be supposed that in S_{FSP}, all first operations are executed before all second operations. Indeed, if there was a first operation executed after a second operation, then the two operations could be permuted without reducing the delay between the first and second operations of any job.

The claim is that in S_{FSP} all first operations are executed at times $\{1,...,n\}$, and all second operations are executed at times $\{T-n+1,...,n\}$. Indeed, suppose the first (resp. second) operation of a given job j is executed after time n (resp. before time $T-n+1$). Since all first operations precede all second operations then there is an idle time period $t \in \{1,...,n\}$ (resp. $\{T-n+1,...,T\}$) at which no operation is executed. Thus the execution of the first (resp. second) operation of j can be scheduled at time t without increasing the delay between the first and the second operation of job j.

From this solution S_{FSP}, we compute a solution of MUCP instance S, by shutting-down (resp. starting-up) unit j at time $\sigma_1(j)$ (resp. $\sigma_2(j)$).

This shows that the FSP problem can be polynomially transformed to the unit-power MUCP. □

This reduction holds in the case where both power and cost are unitary. This proves that the corresponding problem, denoted by unit-(power+cost) MUCP, is strongly NP-hard. Contrary to Theorem 3 for the unit-cost MUCP, this result does not provide a real measure of the respective role of parameters n and T toward the problem's complexity.

5 The P-IMUCP is NP-hard

In this section, the P-IMUCP is considered. Recall the demand is no longer to be satisfied, while the fixed production cost $\pi^{i,t}$ and the proportional production cost $\rho^{i,t}$ can be negative and depend on the time period. For a given value $K \in \mathbb{R}$, the P-IMUCP is to decide whether there is a plan satisfying minimum up and down time constraints, with cost at most K and such that there is at most one start-up per time t. We show this problem is strongly NP-hard by reduction from a restricted version of the Satisfiability problem, denoted by R3-SAT. Problem R3-SAT is such that there are at most three variables per clause, and each variable is restricted to appear once negatively and once or twice positively overall in the set of clauses C. This problem has been proved to be strongly NP-hard (Papadimitriou and Steiglitz (1982)).

Theorem 5 *The P-IMUCP is strongly NP-hard.*

Proof Consider an instance of R3-SAT with clauses c_1, ..., c_q and variables x_1, ..., x_p. Consider the following instance of the P-IMUCP, with time horizon $T = 6p$ and n units where $n = p+q$. Each unit $i \in \{1,...,p\}$ is associated to variable x_i while each unit $p+k$, $k \in \{1,...,q\}$, is associated to clause c_k. For each unit $i \in \{1,...,p+q\}$, $L^i = 1$ and $c_0^i = \rho^{i,t} = 0$.
For each unit $i \in \{1,...,p\}$, $\ell^i = 2p$ and

$$\pi^{i,t} = \begin{cases} -\frac{1}{2} & \text{if } t = 2i-1 \text{ or } t = 4p+2i-1 \\ -1 & \text{if } t = 2p+2i-1 \\ 2 & \text{otherwise.} \end{cases}$$

For each unit $p+k$, $k \in \{1,...,q\}$, $\ell^{p+k} = T$. Recall that a given variable x_s, $s \in \{1,...,p\}$ appears once or twice (resp. once) positively (resp. negatively) from c_1 to c_q. To compute $\pi^{p+k,t}$, we construct an auxiliary time value $\mu_q(X)$ associated to every literal $X \in c_k$:

$$\mu_q(X) = \begin{cases} 2s-1 & \text{if } X = x_s \text{ appearing positively} \\ & \text{for the first time} \\ 4p+2s-1 & \text{if } X = x_s \text{ appearing positively} \\ & \text{for the second time} \\ 2p+2s-1 & \text{if } X = \bar{x}_s. \end{cases}$$

unit c_1	3	3	-1	3	-1	3	-1	3	3	3	3	3	3	3	3	3	3	3
unit c_2	-1	3	3	3	3	3	3	3	3	3	-1	3	3	3	-1	3	3	3
unit x_3	2	2	2	2	-1/2	2	2	2	2	2	-1	2	2	2	2	2	-1/2	2
unit x_2	2	2	-1/2	2	2	2	2	2	-1	2	2	2	2	2	-1/2	2	2	2
unit x_1	-1/2	2	2	2	2	2	-1	2	2	2	2	2	-1/2	2	2	2	2	2
	1		3		5		7		9		11		13		15		17	

Fig. 2: Cost $\pi^{i,t}$ for the P-IMUCP instance corresponding to R3-SAT instance I_R

$$\pi^{p+k,t} = \begin{cases} -1 & \text{if } \exists\, X \in c_k \text{ such that } \mu_q(X) = t \\ 3 & \text{otherwise.} \end{cases}$$

To illustrate, consider an R3-SAT instance I_R with $p = 3$, $q = 2$, $c_1 = \bar{x}_1 \vee x_2 \vee x_3$, $c_2 = x_1 \vee x_2 \vee \bar{x}_3$. Figure 2 shows costs $\pi^{i,t}$ in the corresponding instance of the P-IMUCP.

We will prove that there is a solution to R3-SAT if and only if there is a solution with cost at most $-n$ to this instance of the P-IMUCP. First, consider solution S with cost at most $-n$ to this P-IMUCP. The claim is the total cost of a unit $i \in \{1, ..., p\}$ is at least -1. Indeed, each unit $i \in \{1, ..., p\}$ has a negative cost at times $2i-1$, $2p+2i-1$ and $4p+2i-1$, and a positive cost of 2 at any other time. If unit i were up only at times where its cost is negative, unit i would contribute $-1/2-1-1/2 = -2$ to the solution cost. If unit i is up at any time where the cost is positive, its contribution to the solution cost is at least 0. Because of the minimum down time $\ell^i = 2p$, unit i cannot start up at time $2i-1$, shut down at time $2i$, and start up again at $2p+2i-1$. Therefore unit i contributes at least -1 to the solution cost, and contributes exactly -1 in one of the following two cases:
(i) Unit i is up at times $2i-1$ and $4p+2i-1$, and down at all other times
(ii) Unit i is up at time $2p+2i-1$ and down at all other times.

Similarly, the claim is the total cost of a unit $p+k$, $k \in \{1, ..., q\}$, is at least -1. Indeed, each unit $p+k$, $k \in \{1, ..., q\}$, has cost -1 at times $\mu_k(X)$, for each literal X in c_k. Since there are at most three variables per clause, there are at most three time periods where the cost of unit $p+k$ is -1. Since at all other times, the cost of unit $p+k$ is 3, if unit $p+k$ is up at one given time where its cost is positive, its contribution to the cost would be 0 at least. Moreover, unit $p+k$ can start up just once on the time horizon because of the minimum down time $\ell^{p+k} = T$. Since all times $\mu_k(X)$, $X \in c_k$, are odd, if unit $p+k$ is up only at times $\mu_k(X)$ it has to be up at one given time $\mu_k(X)$ and down at all other times. In this case its cost contribution is -1.

Since solution S has cost at most $-n$, each unit contributes exactly -1 to the solution total cost. We construct a solution to the R3-SAT instance such that variable x_i has truth value “false” in case (i), and truth value “true” in case (ii). Each unit $p+k \in \{p+1, ..., p+q\}$ starts up once, meaning there is at least one true literal X in clause c_k. Otherwise the unit associated to X would have started up at the same time, contradicting the intra-site constraints.

Conversely, any solution to R3-SAT can be transformed to a solution to this P-IMUCP with cost $-n$. □

6 Conclusion

The complexity results are summarized in Table 1. Each entry row-wise is associated to one of the problem studied. In the first entry column-wise, some polynomial cases of the problem are listed, the second (resp. third) entry column-wise lists cases in which the problem is NP-hard in the ordinary (resp. strong) sense. First, we have proved that the UCP is strongly NP-hard by reduction from the 3-Partition Problem. We have proved that the MUCP can be solved in polynomial time whenever n is fixed, regardless of parameters T, L, ℓ. This highlights the major impact of parameter n, compared to other parameters, with respect to the problem's complexity. We have also considered NP-hard special cases, polynomial when $T = 1$, to show that the complexity of the MUCP does not only lie in the classical knapsack reduction at $T = 1$. Finally we have shown that the P-IMUCP – the subproblem arising in UCP decomposition methods – is strongly NP-hard for a subset of units.

<table>
<tr><th></th><th>Polynomial</th><th>NP-hard</th><th>Strongly NP-hard</th></tr>
<tr><td>$T = 1$ MUCP</td><td>(fixed n, $T = 1$)</td><td rowspan="2">(n, $T = 1$)</td><td>$\varnothing$</td></tr>
<tr><td>MUCP or IMUCP</td><td rowspan="4">(n, $T = 1$)
(fixed n, T)</td><td>(n, $T = \frac{1}{3}n$)</td></tr>
<tr><td>unit-cost MUCP</td><td>(n, $T = n+1$)</td><td>(n, $T = \frac{1}{3}n^2 + 1$)</td></tr>
<tr><td>unit-(power+cost) MUCP</td><td colspan="2">$(n = |J|, T = \Delta)^{\circ}$</td></tr>
<tr><td>P-IMUCP</td><td colspan="2">$(n = p+k, T = 6p)^{\diamondsuit}$</td></tr>
</table>

Table 1: Summary of the complexity results
$\circ$: where $|J|$ is the number of jobs and Δ the deadline of the FSP
$\diamondsuit$: where p is the number of variables and k the number of clauses of R3-SAT.

Given these complexity results, some perspectives for future work would be to determine which instances will be hard to solve in practice and to assess the impact of parameters n and T over the computational performances. Another perspective would be to consider the complexity of the P-IMUCP when revenues come from trading the site's production on electricity market. The problem is then to find a production plan for a unit, and a subset of so-called power products to trade with delivery patterns. It would be interesting to analyze the complexity of this problem even in the 1-unit case.

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References

Bendotti P, Fouilhoux P, Rottner C (2016) The min-up/min-down unit commitment polytope. Optimization Online URL `http://www.optimization-online.org/DB_HTML/2016/12/5750.html`
Cohen G (1978) Optimization by decomposition and coordination: A unified approach. IEEE Transactions on Automatic Control 23:222–232
Frangioni A (2005) About Lagrangian methods in integer optimization. Annals of Operations Research 139:163–193
Frangioni A, Gentile C (2006) Solving nonlinear single-unit commitment problems with ramping constraints. Operations Research 54(4):767–775
Gentile C, Morales-Espana G, Ramos A (2016) A tight MIP formulation of the unit commitment problem with start-up and shut-down constraints. EURO Journal on Computational Optimization pp 1 – 25
Knueven B, Ostrowski J, Wang J (2016) Generating cuts from the ramping polytope for the unit commitment problem. Optimization Online URL `http://www.optimization-online.org/DB_HTML/2015/09/5099.html`
Morales-Espana G, Gentile C, Ramos A (2015) Tight MIP formulations of the power-based unit commitment problem. OR Spectrum 37(4):929 – 950
Pan K, Guan Y (2016) A polyhedral study of the integrated minimum-up/-down time and ramping polytope. Optimization Online URL `http://www.optimization-online.org/DB_HTML/2015/08/5070.html`
Pan K, Zhou K, Guan Y (2016) Polynomial Time Algorithms and Extended Formulations for Unit Commitment Problems. ArXiv e-prints (2016)
Papadimitriou C, Steiglitz K (1982) Combinatorial Optimization: Algorithms and Complexity. Prentice-Hall, Englewood Cliffs, NJ
Pisinger D (2005) Where are the hard knapsack problems? Comput Oper Res 32(9):2271–2284
Rajan D, Takriti S (2005) Minimum up/down polytopes of the unit commitment problem with start-up costs. IBM Research Report (2005)
Renaud A (1993) Daily generation management at Electricité de France: From planning towards real time. IEEE Transactions on Automatic Control 38:1080–1093
Tseng C (1996) On power system generation unit commitment problems. PhD thesis, UC Berkeley
Yu W, Hoogeveen H, Lenstra JK (2004) Minimizing makespan in a two-machine flow shop with delays and unit-time operations is NP-hard. Journal of Scheduling 7(5):333–348