Robust Optimization for the Vehicle Routing Problem with Multiple Deliverymen

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Abstract

This paper studies the vehicle routing problem with time windows and multiple deliverymen in which customer demands are uncertain and belong to a predetermined polytope. In addition to the routing decisions, this problem aims to define the number of deliverymen used to provide the service to the customers on each route. A new mathematical formulation is presented for the deterministic counterpart based on auxiliary variables that define the assignment of customers to routes. Building on this formulation, we apply a static robust optimization approach to obtain a robust counterpart formulation that captures the random nature of customer demand. Due to the difficulty of solving this formulation, we propose a constructive heuristic to generate a robust solution that is used as an initial solution for solving the robust counterpart formulation. The heuristic is an extension of Solomon's heuristic I1. Computational results using problem instances from the literature and risk analysis via Monte-Carlo simulation indicate the potential of this static robust optimization to deal with trade-off between cost and risk. The results also reveal that the proposed approach provides good results even without having exact knowledge of some probabilistic measure of the customer demand.

Keywords: Vehicle Routing Problem with Multiple Deliverymen, Static Robust Linear Optimization, Uncertain Demand, Heuristics.

1 Introduction

In this paper, we study the vehicle routing problem with time windows and multiple deliverymen (VRPTWMD). This problem arises in different applications of collecting or distributing products in logistics settings where the service times at customer sites are long when compared with travel times and depend on the number of deliverymen (servicemen) in the vehicle cab. This variant is relevant in cases in which customers should be served in congested urban areas on the same day, and not serving these customers within their time windows are highly undesirable (Pureza et al., 2012; Senarclens de Grancy and Reimann, 2016; Álvarez and Munari, 2017). In this context, the VRPTWMD allows each route to be assigned a number of deliverymen in order to reduce service times and minimize costs.

In practical environments, an important aspect to be considered is the uncertainty of some parameters of the problem. For instance, customer demand may become known only when the vehicle arrives at the customer site (Bertsimas, 1992; Laporte et al., 2002; Jabali et al., 2014). Ignoring uncertainty in customer demands can lead to inefficient and/or non-implementable solutions in practice. In a delivery situation, if the load of a vehicle leaving the depot is not sufficient to meet the total demand of the route, some customers are no longer served or are served by some additional service, e.g., using another vehicle or returning and reloading the vehicle at the depot, increasing transport costs still further. Ignoring uncertainty in customer demand can lead to some of the situations mentioned above, which in turn may lead to additional costs and deterioration of customer service level. In this context, using methods that appropriately address the inherent uncertainties of the customer demand is relevant.

Stochastic Programming (SP) and Robust Optimization (RO) are the most commonly used methods in the mathematical programming literature to deal with uncertain parameters of optimization problems. In SP, two approaches are often adopted: with recourse and with chance constraints. The basic idea of SP with recourse is to take some decisions in the absence of uncertainties (here-and-now decisions) and then, when the uncertainties are revealed, to make recourse decisions or contingency actions (wait-and-see decisions) in order to mitigate the impact of uncertainties of the here-and-now decisions (Birge and Louveaux, 2011; De La Vega and Alem, 2015). In SP with chance constraints, recourse decisions are avoided, i.e., only here-and-now decisions are defined. Instead, the constraints involving uncertain parameters are expressed in terms of probability statements about the here-and-now decisions (Prékopa, 1995; Birge and Louveaux, 2011; Shapiro et al., 2014). For this reason, in the SP approach it is necessary to have knowledge about some probabilistic measure of the random parameters, e.g., the probability distribution. On the other hand, RO aims at determining here-and-now decisions with minimal cost that preserves feasibility for any realization of the uncertain parameters under some prespecified set, such as polyhedral, ellipsoidal or intersections of these sets (Bertsimas and Sim, 2004; Ben-Tal et al., 2009). This approach of RO is known in the literature as static RO, since the wait-and-see decisions are not allowed (Ben-Tal and Nemirovski, 2000; Ben-Tal et al., 2004, 2009; Bertsimas and Goyal, 2012; Bertsimas et al., 2015).

We are not aware of other studies in the literature addressing the VPRTWMD under uncertainty. Classical variants of the vehicle routing problem (VRP), such as the capacited VRP (CVRP) and the VRP with time windows (VRPTW), have been studied in uncertain contexts. Typically, in the SP approach with recourse in VRPs with uncertain demand, the here-and-now decisions correspond to the determination of the routes. For a given delivery route, there is possibility of failure, i.e., the load of the vehicle cannot meet the demand of the customers belonging to the route. Or, similarly, for the pickup situation, the vehicle capacity is not sufficient for loading all customer demands of the route. In these situations, recourse decisions may be necessary. One of the recourses widely used in the literature is to return to the depot from the customer site where the failure occurred, reload/unload the vehicle and continue the route (Bertsimas, 1992). This resource has been used in different studies, e.g., Laporte et al. (2002); Christiansen and Lysgaard (2007); Gauvin et al. (2014).

In Laporte et al. (2002), the CVRP with uncertain demand is addressed and solved using the integer L-shaped method. This same problem is also addressed in Christiansen and Lysgaard

(2007), but the solution method is a branch-and-price algorithm. Gauvin et al. (2014) proposed an efficient method based on a branch-price-and-cut algorithm to solve the CVRP with uncertain demand. This method was able to solve optimally some instances not solved in Christiansen and Lysgaard (2007) and reduced the computational time of the instances solved.

RO has also been used to deal with VRP variants with uncertain parameters. The first study to use the term "robust vehicle routing" was Sungur et al. (2008). However, the proposed robust formulations for the CVRP provide extremely conservative solutions given that the uncertain parameters assume their respective worst-case values. In Ordóñez (2010), different uncertain sets and based on the two- and three-index formulations for the deterministic VRPTW were presented and the conservatism level of the resultant static robust linear programs was discussed. Gounaris et al. (2013) addressed the CVRP with uncertain demand and proposed robust models based on the deterministic problem counterpart. Lee et al. (2012) studied the CVRP with deadlines and travel time/demand uncertainty within a polytope. The problem was formulated as a set partitioning problem and solved using a branch-and-price algorithm.

The present paper uses a static RO approach to address customer demand uncertainty in the VRPTWMD, which leads to the robust VRPTWMD (RVRPTWMD). In this approach, it is convenient to have the whole set of uncertain parameters in the vehicle capacity constraint in order to obtain a robust formulation capable of better controlling the trade-off between performance and conservatism (Bertsimas and Sim, 2004). The deterministic formulation presented in Pureza et al. (2012) (called three-index formulation) is not convenient for this approach as it does not have all these parameters in the same constraint. It is possible to extend this formulation to consider the routes in the flow variables yielding a four-index formulation. However, the number of binary variables in the resulting formulation would be too large. Furthermore, these two formulations (three- and four-index) have weak linear programming (LP) bounds, preventing the effective use of general-purpose solution methods, such as branch-and-bound and branch-and-cut. For this reason, we propose a novel deterministic formulation, which is more appropriate for using static RO.

The contributions of this study are threefold: (1) the extension of the deterministic mathematical formulation of the VRPTWMD and the presentation of its robust counterpart formulation for the RVRPTWMD under static RO; (2) the extension of the heuristic I1, originally presented in Solomon (1987) for the VRPTW, for finding a good robust feasible solution for the RVRPTWMD; and (3) the analysis of the potential of this static RO approach to deal with the RVRPTWMD in terms of the behavior of the objective function, the routing decisions and the risk of solution infeasibility as the problem uncertainty level increases.

The remainder of this paper is organized as follows. In Section 2, we briefly review the problem definition and a mathematical formulation of the VRPTWMD and present another deterministic formulation. In Section 3, we obtain the robust counterpart RVRPTWMD of this formulation. In Section 4, we describe the extension of heuristic I1 to solve the RVRPTWMD. The computational results obtained by applying a branch-and-cut method to solve the RVRPTWMD, using the solution obtained by the heuristic as an initial feasible solution for the problem, are analyzed in Section 5. Finally, some concluding remarks and perspectives for future research are discussed in Section 6.

2 Definition and formulations of the VRPTWMD

In addition to the routing and scheduling decisions, the VRPTWMD aims to define the number of deliverymen used to perform the service on the customers of each route. As this decision results in different service times for the customers, it is possible that more customers per route can be served, leading to a reduction in the number of routes generated. Thus, the VPRTWMD deals with the existing trade-off of the costs incurred in the decisions related to the total number of routes generated and deliverymen used.

Few studies have addressed the deterministic VRPTWMD. Pureza et al. (2012) proposed a three-index formulation to represent the problem and two meta-heuristic, Tabu-Search and Ant-Colony, to obtain routes of minimum cost. The three-index formulation is difficult to solve, even for instances with 25 customers. For this reason, most of the papers that have addressed the VRPTWMD have used heuristic and meta-heuristic algorithms (Pureza et al., 2012; Ferreira and Pureza, 2012; Senarclens de Grancy and Reimann, 2016; Álvarez and Munari, 2016). Exact methods have also been used to solve this problem. Munari and Morabito (2016) used an algorithm based on a branch-price-and-cut method to solve the problem more effectively. Álvarez and Munari (2017) used meta-heuristics to accelerate the convergence of the branch-price-and-cut method proposed in Munari and Morabito (2016).

The VRPTWMD has been also extended to consider the decision of customer clustering and determines the visit order of the customers within each cluster. In this case, the visit order of clusters and customers in each cluster are called primary and secondary routes, respectively. The primary routes are traveled by the vehicles, while the secondary routes by the deliverymen. Senarclens de Grancy and Reimann (2015) developed a heuristic algorithm based on the Ant-Colony meta-heuristic to deal with the decisions involved in the extended VRPTWMD. Senarclens de Grancy (2015) proposed a two stage solution approach for the problem: in the first, an insertion heuristic is used for determining the clusters, while in the second stage, the heuristic I1 of Solomon has been adapted to determine the visit order of the clusters.

In the absence of uncertainty, the parameters of the optimization problem are represented as a single value which corresponds to their nominal and/or expected value. Based on this, the mathematical models outlined in this section assume that the nominal value of the parameters involved in the problem are either known or may be estimated. This indicates that the models correspond to the deterministic counterpart of VRPTWMD.

2.1 Vehicle flow formulation

The VRPTWMD can be defined by a graph $\mathscr{G}(\mathscr{V}, \mathscr{A})$, where $\mathscr{V} = \{0, 1, \ldots, n, n+1\}$ and $\mathscr{A} = \{(i, j) : i, j \in \mathscr{V}, i \neq j\}$ represent, respectively, the sets of nodes and arcs of the graph. Nodes 0 and $n{+}1$ of set $\mathscr{V}$ represent the depot, whereas the remaining nodes define the customers, which are indexed by set $\mathscr{N} = \mathscr{V} \setminus \{0, n+1\}$. Additionally, $\mathcal{L}$ defines the maximum number of deliverymen that can be assigned to a single vehicle ($l = 1, \ldots, \mathcal{L}$). In practical situations, in general, $\mathcal{L}$ is small due to size limitations of the vehicle cab. If the number of deliverymen assigned to a vehicle is l, we say that this vehicle travels in mode l. Each customer $i \in \mathscr{N}$ has a non-negative demand q_i and a non-negative service time s_i^l dependent on the number of deliverymen or mode l. Each node $i \in \mathscr{V}$, is given a time windows $[w_i^a, w_i^b]$ describing the time interval in which customer i is available to receive the service. Moreover, associated to each arc $(i,j) \in \mathscr{A}$, there is a distance d_{ij} and a travel time t_{ij}, which satisfy the triangle inequality. A fleet of identical vehicles is available in the depot, which are indexed by set $\mathscr{K}$, where $|\mathscr{K}| = m$. There is a maximum number of deliverymen that can be used in the depot, denoted by $\mathcal{E}$. The VRPTWMD consists of determining a set of routes of minimum cost with their correspondent number of deliverymen, denoted by $\mathrm{R} = \{R_1, \ldots, R_{m'}\}$, where $m' \leq m$, such that (i) each route starts and finishes at the depot, (ii) each customer is visited exactly once and the service must be started within its time window, (iii) the number of deliverymen on board the vehicle must respect the maximum number allowed; (iv) the total demand for a vehicle's route must not exceed its capacity Q and (v) the total number of vehicles and deliverymen used must not exceed the quantity available in the depot.

The mathematical model presented in this section corresponds to the formulation proposed in Pureza et al. (2012). This formulation will be referred to here as the three-index formulation. Let x_{ij}^l be a binary variable that assumes value 1 if and only if there is a route in mode l that visits customer j immediately after visiting customer i. Variable u_i^l denotes the load in the vehicle in mode l just after visiting customer i. Variable w_i^l defines the instant of time when the service in customer i starts in a mode l. The mathematical model that determines the routes of minimum cost can be described as follows:

$$\text{minimize} \quad p_1 \sum_{l=1}^{\mathcal{L}} \sum_{j \in \mathscr{N}} x_{0j}^l + p_2 \sum_{l=1}^{\mathcal{L}} \sum_{j \in \mathscr{N}} l x_{0j}^l + p_3 \sum_{l=1}^{\mathcal{L}} \sum_{(i,j) \in \mathscr{A}} d_{ij} x_{ij}^l. \tag{2.1}$$

$$\text{subject to:} \quad \sum_{i \in \mathscr{V} \setminus \{n+1\}} \sum_{l=1}^{\mathcal{L}} x_{ij}^l = 1, \quad \forall j \in \mathscr{N}. \tag{2.2}$$

$$\sum_{j \in \mathscr{V} \setminus \{0\}} \sum_{l=1}^{\mathcal{L}} x_{ij}^l = 1, \quad \forall i \in \mathscr{N}. \tag{2.3}$$

$$\sum_{j \in \mathscr{N} \setminus \{0\}} x_{ij}^l = \sum_{j \in \mathscr{V} \setminus \{n+1\}} x_{ji}^l, \quad \forall i \in \mathscr{N}, \quad l = 1, \ldots, \mathcal{L}. \tag{2.4}$$

$$u_j^l \geq u_i^l + q_j x_{ij}^l - Q(1 - x_{ij}^l), \quad \forall (i,j) \in \mathscr{A}, \quad l = 1, \ldots, \mathcal{L}. \tag{2.5}$$

$$q_i \leq u_i^l \leq Q, \quad \forall i \in \mathscr{N}, \quad l = 1, \ldots, \mathcal{L}. \tag{2.6}$$

$$w_j^l \geq w_i^l + (t_{ij} + s_i^l) x_{ij}^l - M_{ij}^l (1 - x_{ij}^l), \quad \forall (i,j) \in \mathscr{A}, \quad l = 1, \ldots, \mathcal{L}. \tag{2.7}$$

$$w_i^a \leq w_i^l \leq w_i^b, \quad \forall i \in \mathscr{V}, \quad l = 1, \ldots, \mathcal{L}. \tag{2.8}$$

$$\sum_{j \in \mathscr{N}} \sum_{l=1}^{\mathcal{L}} l x_{0j}^l \leq \mathcal{E}, \tag{2.9}$$

$$\sum_{j \in \mathscr{N}} \sum_{l=1}^{\mathcal{L}} x_{0j}^l \leq |\mathscr{K}|, \tag{2.10}$$

$$x_{ij}^l \in \{0,1\}, \quad \forall (i,j) \in \mathscr{A}, \quad l = 1, \ldots, \mathcal{L}. \tag{2.11}$$

In the above formulation, the objective function (2.1) minimizes the costs related to the routes number, the deliverymen number and the transportation costs incurred by the total distance traveled. Parameters p_1, p_2 and p_3 define the unit costs of route (vehicle), deliveryman and distance, respectively, or a prioritization of these objectives when these costs are difficult to be estimated in practice. Constraints (2.2) and (2.3) define that each customer must be visited by a single route in mode l. Constraints (2.4) define the flow constraints and ensure that a route in mode l reaching customer site i must necessarily leave this site with the same mode either going to another customer or the depot (together with constraints (2.4), either constraints (2.2) or (2.3) are redundant). Constraints (2.5) ensure that if there is a route visiting customer j immediately after visiting customer i, then the load in the vehicle with l deliverymen when leaving j must be at least equal to the total load collected up to i plus the additional load (demand) collected at customer j. Constraints (2.6) ensure that the total load collected when a vehicle leaves a customer i must not exceed the capacity Q of the vehicle and that the minimum load on the vehicle is the load (demand) collected at customer i.

Since the service at customer sites must start within the time windows (constraints (2.8)), constraints (2.7) ensure that the service start time in customer j must not be less than the service start time in customer i added to the service time with this customer and the travel time from i to j. M^l_{ij} in constraints (2.7) is a parameter large enough to make the problem feasible, which can be determined as $M^l_{ij} = \max\left(w^b_i - w^a_j, 0\right)$. Constraints (2.9) impose that the sum of deliverymen number used in all routes does not exceed the total number available. Similarly, constraints (2.10) ensure that the number of vehicles used cannot be larger than the total number available in the depot. Finally, constraints (2.11) indicate that variables x^l_{ij} can only assume values 0 or 1.

As mentioned before and discussed in the next section, in static RO, it is convenient to have all the uncertain parameters in a single constraint, because this leads to a reformulation that takes into account the trade-off between performance and conservatism (robustness or risk). The performance is measured in terms of the optimal value of the objective function and the level of conservatism (robustness or risk) on the violation probability of the solution. The formulation (2.1)-(2.11) is not convenient for the use of static RO, since its robust counterpart is equivalent to an instance of the deterministic problem in which all the uncertain parameters assume its worst-case value, resulting in an entirely conservative solution. Besides that, at least for the instances tested in this paper, the LP bound of formulation (2.1)-(2.11) is weak, preventing the effective use of general purpose solution methods such as branch-and-bound and branch-and-cut. Consequently, we propose a new formulation for the VRPTWMD, which is more convenient for static RO application and with a tighter LP bound. The proposed formulation is based on the formulation of Pureza et al. (2012) and Gounaris et al. (2013) with the addition of variables that inform the assignment of customers to routes.

2.2 Vehicle assignment formulation

To reformulate the deterministic counterpart of the VRPTWMD, we define the following additional variables: (i) y^l_k: binary variable that assumes value 1 if and only if route k is in mode l

and (ii) z_{ik}^l: binary variable that assumes value 1 if and only if customer i is assigned to route k in mode l. We use $\boldsymbol{x}$, $\boldsymbol{z}$ and $\boldsymbol{y}$ to denote vectors whose components correspond to the decision variables x_{ij}^l, z_{ik}^l and y_k^l, respectively. Thus, the new mathematical model for the VRPTWMD can be described as follows:

$$\text{minimize} \quad p_1 \sum_{l=1}^{\mathcal{L}} \sum_{k \in \mathscr{K}} y_k^l + p_2 \sum_{l=1}^{\mathcal{L}} \sum_{k \in \mathscr{K}} l y_k^l + p_3 \sum_{l=1}^{\mathcal{L}} \sum_{i \in \mathscr{V}} \sum_{j \in \mathscr{V}} d_{ij} x_{ij}^l. \tag{2.12}$$

$$\text{subject to:} \quad \text{constraints: (2.2)-(2.4), (2.7)-(2.8) and (2.11).} \tag{2.13}$$

$$\sum_{k \in \mathscr{K}} \sum_{l=1}^{\mathcal{L}} y_k^l \leq |\mathscr{K}|. \tag{2.14}$$

$$\sum_{k \in \mathscr{K}} \sum_{l=1}^{\mathcal{L}} l y_k^l \leq \mathcal{E}. \tag{2.15}$$

$$\sum_{j \in \mathscr{N}} x_{0j}^l = \sum_{k \in \mathscr{K}} y_k^l, \quad l = 1, \ldots, \mathcal{L}. \tag{2.16}$$

$$\sum_{l=1}^{\mathcal{L}} y_k^l \leq 1, \quad \forall k \in \mathscr{K}. \tag{2.17}$$

$$\sum_{i \in \mathscr{N}} z_{ik}^l \geq y_k^l, \quad \forall k \in \mathscr{K}, \quad l = 1, \ldots, \mathcal{L}. \tag{2.18}$$

$$\sum_{i \in \mathscr{N}} z_{ik}^l \leq |\mathscr{N}| y_k^l, \quad \forall k \in \mathscr{K}, \quad l = 1, \ldots, \mathcal{L}. \tag{2.19}$$

$$\sum_{k \in \mathscr{K}} z_{ik}^l \leq \sum_{j \in \mathscr{V} \setminus \{0\}} x_{ij}^l, \quad \forall i \in \mathscr{N}, \quad l = 1, \ldots, \mathcal{L}. \tag{2.20}$$

$$1 - x_{ij}^l - x_{ji}^l \geq z_{ik}^l - z_{jk}^l, \forall i \in \mathscr{N}, \forall j \in \mathscr{N}, \quad i \neq j, \quad \forall k \in \mathscr{K}, \quad l = 1, \ldots, \mathcal{L}. \tag{2.21}$$

$$\sum_{k \in \mathscr{K}} \sum_{l=1}^{\mathcal{L}} z_{ik}^l = 1, \quad \forall i \in \mathscr{N}. \tag{2.22}$$

$$\sum_{i \in \mathscr{N}} q_i z_{ik}^l \leq Q y_k^l, \quad \forall k \in, \mathscr{K} \quad l = 1, \ldots, \mathcal{L}. \tag{2.23}$$

$$z_{ik}^l \in \{0, 1\}, \quad \forall i \in \mathscr{N}, \quad \forall k \in \mathscr{K}, \quad l = 1, \ldots, \mathcal{L}. \tag{2.24}$$

$$y_k^l \in \{0, 1\}, \quad \forall k \in \mathscr{K}, \quad l = 1, \ldots, \mathcal{L}. \tag{2.25}$$

The objective function (2.12) of model (2.12)-(2.24) also minimizes the costs associated to the number of routes, the number of deliverymen and the total distance traveled. Note that, in this case, the number of routes and deliverymen is determined using variable vector $\boldsymbol{y}$. Constraints (2.13) have already been explained in Subsection 2.1. Since in the depot there are $m = |\mathscr{K}|$ vehicles and $\mathcal{E}$ deliverymen available, constraints (2.14) and (2.15) express that these availabilities must be respected. Constraints (2.16) couple variables x_{ij}^l and y_k^l, indicating that all customers visited immediately after the vehicle leaves depot 0 in mode l must be part of distinct routes, but in the same mode. Constraints (2.17) ensure that a route k has a single mode l. Constraints (2.18) and (2.19) ensure that if there is a route k in mode l, it must have at least one customer and at most $n = |\mathscr{N}|$ customers, respectively. Constraints (2.20) relate variables z_{ik}^l and x_{ij}^l and impose that if customer i is visited by a route in mode l in terms of z_{ik}^l, it belongs to a single route k and this route must be in mode in terms of x_{ij}^l. Constraints

(2.21) also relate these two variables and ensure that if customers i and j are visited by the same route k and in same mode l, then they belong to the same route in terms of x^l_{ij}. Thus, these constraints are responsible for the assignment of customers to routes. Constraints (2.22) impose that each customer i belongs to exactly one route k in a single mode l.

The maximum load of each route is imposed by constraints (2.23). Finally, constraints (2.24)-(2.25) indicate that the vectors of decision variables $\boldsymbol{z}$ and $\boldsymbol{y}$ are binary. Concerning the model presented in Subsection 2.1, the number of binary variables is increased by $|\mathscr{N}|\times|\mathscr{K}|\times\mathcal{L}+|\mathscr{K}|\times\mathcal{L}$. However, the number of binary decision variables of formulation (2.12)-(2.25) can be reduced through the fixation of the flow variable x^l_{ij} at 0 if the inequalities $w^a_i+s^l_i+t_{ij}>w^b_j$ and $q_i+q_j>Q$ are true. Moreover, Proposition 2.1 shows that the binary requirement of decision variable vector $\boldsymbol{y}$ can be relaxed.

Proposition 2.1. *Assume that the binary vectors* $\boldsymbol{x}$ *and* $\boldsymbol{z}$ *satisfy constraints* (2.13), (2.21), (2.22) *and* (2.24). *Then, constraints* (2.14)-(2.19) *are satisfied for the variable vector* $\boldsymbol{y}\in\{\mathbf{0},\mathbf{1}\}^{|\mathscr{K}|\times\mathcal{L}}$ *if and only if they are satisfied by the variable vector* $\boldsymbol{y}\in[\mathbf{0},\mathbf{1}]^{|\mathscr{K}|\times\mathcal{L}}$.

Proof. Let $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ be a solution that satisfies constraints (2.13), (2.21), (2.22) and (2.24). For each l in $\{1,\ldots,\mathcal{L}\}$, let $\mathcal{K}^l$ be the set of routes in mode l in the solution, given by all routes k in $\mathscr{K}$ such that $z^l_{ik}=1$ for a given i in $\mathscr{N}$. Let v_k be the first customer visited by route $k\in\mathcal{K}^l$. From constraints (2.16), we have for any l in $\{1,\ldots,\mathcal{L}\}$:

$$\sum_{k\in\mathcal{K}^l} x^l_{0,v_k}=y^l_1+y^l_2+\ldots+y^l_{|\mathscr{K}|} \tag{2.26}$$

By hypothesis, $\boldsymbol{x}$ is a binary vector, so the summation on the lefthand side of (2.26) is equal to $|\mathcal{K}^l|$. Let $\widetilde{\mathcal{K}}^l$ be the set of all indices k in $\mathscr{K}$ such that $y^k_l>0$. When all components of $\boldsymbol{y}$ are binary, then $|\widetilde{\mathcal{K}}^l|=|\mathcal{K}^l|$. Otherwise, we have $|\widetilde{\mathcal{K}}^l|>|\mathcal{K}^l|$. Assume by contradiction that there is a positive number of indices in $\widetilde{\mathcal{K}}^l$ for which the corresponding variables y^l_k assume fractional values. Then, we can partition set $\widetilde{\mathcal{K}}^l$ as $\widetilde{\mathcal{K}}^l_{frac}$ and $\widetilde{\mathcal{K}}^l_{int}$ such that $\widetilde{\mathcal{K}}^l=\widetilde{\mathcal{K}}^l_{frac}\cup\widetilde{\mathcal{K}}^l_{int}$, $\widetilde{\mathcal{K}}^l_{frac}\cap\widetilde{\mathcal{K}}^l_{int}=\emptyset$ and $|\widetilde{\mathcal{K}}^l_{frac}|+|\widetilde{\mathcal{K}}^l_{int}|>|\mathcal{K}^l|$. Thus, $\exists k'\in\widetilde{\mathcal{K}}^l$ such that $k'\notin\mathcal{K}^l$. We have that $\sum_{i\in\mathscr{N}}z^l_{ik}\geq 1$ for all $k\in\mathcal{K}^l$ and $\sum_{i\in\mathscr{N}}z^l_{ik}=0$ for all $k\in\widetilde{\mathcal{K}}^l\backslash\mathcal{K}^l$. However, $\exists k'\in\widetilde{\mathcal{K}}^l\backslash\mathcal{K}^l$ such that $y^l_{k'}>0$, which is absurd because it was assumed that constraints (2.18) are satisfied. On the other hand, as $\boldsymbol{y}\in\{0,1\}^{|\mathscr{K}|\times\mathcal{L}}\subseteq\boldsymbol{y}\in[0,1]^{|\mathscr{K}|\times\mathcal{L}}$ then the proposition is proved. □

As shown in Proposition 2.1, the integrality of decision variables vector $\boldsymbol{y}$ of the mathematical model (2.12)-(2.25) can be relaxed, which decreases the number of binary variables of the model. Additionally, the LP bound of the proposed model is, on average, approximately 271 and 6 times greater than the LP bound of formulation (2.1)-(2.11) for classes C1 and R1 of the Solomon's instances considered in the computational experiments described in Section 5, respectively. Therefore, we deem the mathematical formulation proposed in this paper more effective than formulation (2.1)-(2.11) for applying static RO for the RVRPTWMD. It is important to point out that the model proposed, after some minor adjustments, is also valid for the case in which the vehicle fleet is heterogeneous, if we assume that index k denotes a vehicle instead of a route (Belfiore and Fávero, 2007). Thereby, the robust formulation described in the next section is also valid for this problem version.

3 Formulation of the RVRPTWMD with uncertain demand

This section introduces a static robust linear program that captures the random nature of customer demand in the RVRPTWMD.

3.1 Uncertainty set and robust counterpart of capacity constraints (2.23)

We model uncertain demand $\tilde{q}_i$, for all $i \in \mathscr{N}$, as an independent, bounded and symmetric random variable taking values in the range $[\bar{q}_i - \hat{q}_i, \bar{q}_i + \hat{q}_i]$, where $\bar{q}_i$ and $\hat{q}_i$ define the nominal value (or expected value) and the maximum deviation allowed of the random variable $\tilde{q}_i$, respectively. For each random variable $\tilde{q}_i$, we associate another random variable ξ_i (typically called as primitive uncertainty variable) that assumes values in $[-1, 1]$ such that $\tilde{q}_i = \bar{q}_i + \hat{q}_i \xi_i$, $\forall i \in \mathscr{N}$. Formulating the RO model for the RVRPTWMD under the static approach using this representation of the random variables may lead to the generation of the totally conservative solutions. This is because the resulting robust formulation is equivalent to the deterministic counterpart with the random variables $\tilde{q}_i$ assuming their worst case values (Gounaris et al., 2013). This formulation is known in the literature as Soyster's method. Let $\boldsymbol{\xi}$ be a vector with $|\mathscr{N}|$ components, one for each primitive uncertainty variable ξ_i. To avoid this conservatism, in this paper, we restrict the primitive uncertainty vector $\boldsymbol{\xi}$ to the following polyhedral uncertainty set of cardinality constrained (Bertsimas et al., 2011).

$$\mathcal{Q}_k^l(\boldsymbol{z}, \boldsymbol{\Gamma}) = \left\{ \boldsymbol{\xi} \in \mathbb{R}_+^{|\mathscr{N}|} : \xi_i \leq 1, \sum_{i \in \mathcal{R}_k^l(\boldsymbol{z})} \xi_i \leq \Gamma_k^l \right\}, \quad \forall k \in \mathscr{K}, \quad l = 1, \ldots, \mathcal{L}. \tag{3.1}$$

For a given vector $\boldsymbol{z}$ of assignment of customers to routes, $\mathcal{R}_k^l(\boldsymbol{z})$ describes the set of all customers belonging to route k in mode l. Parameter Γ_k^l, called the uncertainty budget, defines the maximum number of customers belonging to route k in mode l whose demand can deviate from their nominal value. For a given vector $\boldsymbol{z}$, if $\Gamma_k^l = |\mathcal{R}_k^l(z)|$, then the demand of all customers in route k will assume the worst case value (Soyster's method) (Soyster, 1973). In this case, the resulting formulation is deemed too conservative and defines the maximum protection against the uncertainties. On the other hand, if $\Gamma_k^l = 0$, the resulting problem is equivalent to the nominal problem and the uncertainties are completely ignored. The choice of a value of parameter Γ_k^l between 0 and $|R_k^l(z)|$ controls the trade-off between the level of robustness/conservativeness and the performance of the solution in terms of the objective function (Bertsimas and Sim, 2004).

The uncertainty set depends on the value of the budget of uncertainty, the routes chosen and the customers belonging to those routes. This is the reason for the indexation of the set with vector $\boldsymbol{z}$. Using the uncertainty set (3.1), we build the robust counterpart of constraints (2.23) as:

$$\sum_{i \in \mathscr{N}} (\bar{q}_i + \hat{q}_i \xi_i) z_{ik}^l \leq Q y_k^l, \; \forall k \in \mathscr{K}, \; l = 1, \ldots, \mathcal{L}, \; \forall \boldsymbol{\xi} \in \mathcal{Q}_k^l(\boldsymbol{z}, \boldsymbol{\Gamma}). \tag{3.2}$$

An integer linear program with infinite constraints results if constraints (2.23) are replaced by (3.2) in model (2.12)-(2.25), since constraints (3.2) have to be satisfied for all possible realizations of the primitive uncertainty vector $\boldsymbol{\xi}$ within the set $\mathcal{Q}_k^l(\boldsymbol{z},\boldsymbol{\Gamma})$. In this context, the resulting integer linear program cannot be solved by a general purpose optimization software. However, Proposition 3.1 shows an equivalent way to represent constraints (3.2) through a finite number of constraints.

Proposition 3.1. *Constraints* (3.2) *are equivalent to the following pair of constraints:*

$$\sum_{i\,\in\mathcal{N}} \bar{q}_i z_{ik}^l + \sum_{i\,\in\mathcal{N}} \alpha_{ik}^l + \Gamma_k^l \beta_k^l \leq Q y_k^l, \quad \forall k \in \mathcal{K}, \quad l = 1,\ldots,\mathcal{L}. \tag{3.3a}$$

$$\alpha_{ik}^l + \beta_k^l \geq \hat{q}_i z_{ik}^l, \forall i \in \mathcal{N}, \quad \forall k \in \mathcal{K}, \quad l = 1,\ldots,\mathcal{L}. \tag{3.3b}$$

where α_{ik}^l and β_k^l correspond to the dual variables of the constraints of the optimization problem associated to the primal protection level of constraints (3.2)*.*

Proof. Let $\boldsymbol{y}^\star$ and $\boldsymbol{z}^\star$ be two vectors satisfying the constraints of model (2.12)-(2.25). Constraints (3.2) are equivalent to the following worst case reformulation:

$$\sum_{i\,\in\mathcal{N}} \bar{q}_i z_{ik}^{l\star} + \max_{\boldsymbol{\xi}\,\in\mathcal{Q}_k^l(\boldsymbol{z},\boldsymbol{\Gamma})} \left\{ \sum_{i\in\mathcal{R}_k^l(\boldsymbol{z})} \hat{q}_i z_{ik}^{l\star} \xi_i \right\} \leq Q y_k^{l\star}, \quad \forall k \in \mathcal{K}, \quad l = 1,\ldots,\mathcal{L}. \tag{3.4}$$

By replacing constraints (2.23) by (3.4) in model (2.12)-(2.24), a nonlinear programming model is obtained, since the primitive uncertainty vector $\boldsymbol{\xi}$ would also be a decision variable vector. We use the dualization technique of protection function $\mathcal{B}_k^l(\boldsymbol{z}) = \max_{\boldsymbol{\xi}\in\mathcal{Q}_k^l(\boldsymbol{z},\boldsymbol{\Gamma})} \left\{ \sum_{i\in\mathcal{R}_k^l(\boldsymbol{z})} \hat{q}_i z_{ik}^{l\star} \xi_i \right\}$ to linearize constraints (3.4). This function is equivalent to the following optimization problem.

$$\mathcal{B}_k^l(\boldsymbol{z}) = \max_{\boldsymbol{\xi}\geq 0} \left\{ \sum_{i\in\mathcal{R}_k^l(\boldsymbol{z})} \hat{q}_i z_{ik}^{l\star} \xi_i : \xi_i \leq 1, \forall i \in \mathcal{R}_k^l(\boldsymbol{z}), \sum_{i\in\mathcal{R}_k^l(\boldsymbol{z})} \xi_i \leq \Gamma_k^l \right\} \tag{3.5}$$

In the optimization problem (3.5), the primitive uncertainty vector $\boldsymbol{\xi}$ corresponds to the continuous decision variable vector, whereas $\boldsymbol{z}^\star$ is a parameter vector. Let α_{ik}^l and β_k^l be the dual variables associated to problem constraints (3.5). The dualization of protection function $\mathcal{B}_k^l(\boldsymbol{z})$ leads to:

$$\mathcal{B}_k^l(\boldsymbol{z}) = \min_{\boldsymbol{\alpha}\geq 0,\,\boldsymbol{\beta}\geq 0} \left\{ \sum_{i\in\mathcal{R}_k^l(\boldsymbol{z})} \alpha_{ik}^l + \Gamma_k^l \beta_k^l : \alpha_{ik}^l + \beta_k^l \geq \hat{q}_i z_{ik}^{\star l}, \forall i \in \mathcal{R}_k^l(\boldsymbol{z}) \right\} \tag{3.6}$$

By replacing the equivalent form of protection function $\mathcal{B}_k^l(\boldsymbol{z})$ given in equation (3.6) in constraints (3.4), we obtain:

$$\sum_{i\,\in\mathcal{N}} \bar{q}_i z_{ik}^{l\star} + \min_{\boldsymbol{\alpha}\,\geq 0,\,\boldsymbol{\beta}\geq 0} \left\{ \sum_{i\in\mathcal{R}_k^l(\boldsymbol{z})} \alpha_{ik}^l + \Gamma_k^l \beta_k^l : \alpha_{ik}^l + \beta_k^l \geq \hat{q}_i z_{ik}^{\star l}, \quad \forall i \in \mathcal{R}_k^l(\boldsymbol{z}) \right\} \leq Q y_k^{l\star}, \quad \forall k \in \mathcal{K}, \quad l = 1,\ldots,\mathcal{L}. \tag{3.7}$$

We can omit the minimization term in (3.7), since it is sufficient that the constraint holds for at least one value of α_{ik}^l and β_k^l. Because $\boldsymbol{z}$ is a decision variable vector of the original problem, the customers belonging to set $\mathcal{R}_k^l(\boldsymbol{z})$ cannot be known a priori. Therefore, we assume that $\mathcal{R}_k^l(\boldsymbol{z}) = \mathcal{N}$. As a result, we obtain constraints (3.3a) and (3.3b), thus completing the proof. □

3.2 Robust formulation for the RVRPTWMD with uncertain demand

The resulting model, replacing constraints (2.23) by constraints (3.3a) and (3.3b) in the optimization model (2.12)-(2.24), represents the robust formulation of the RVRPTWMD with uncertain demand. Thus, the RO model for the RVRPTWMD with uncertain demand can be written as:

$$\text{minimize} \quad p_1 \sum_{l=1}^{\mathcal{L}} \sum_{k \in \mathscr{K}} y_k^l + p_2 \sum_{l=1}^{\mathcal{L}} \sum_{k \in \mathscr{K}} l y_k^l + p_3 \sum_{l=1}^{\mathcal{L}} \sum_{i \in \mathscr{V}} \sum_{j \in \mathscr{V}} d_{ij} x_{ij}^l. \tag{3.8}$$

$$\text{subject to:} \quad \text{constraints: (2.13)-(2.22) and (2.24)-(2.25).} \tag{3.9}$$

$$\sum_{i \in \mathscr{N}} \bar{q}_i z_{ik}^l + \sum_{i \in \mathscr{N}} \alpha_{ik}^l + \Gamma_k^l \beta_k^l \leq Q y_k^l, \quad \forall k \in \mathscr{K}, \quad l = 1, \ldots, \mathcal{L}. \tag{3.10}$$

$$\alpha_{ik}^l + \beta_k^l \geq \hat{q}_i z_{ik}^l, \quad \forall i \in \mathscr{N}, \quad \forall k \in \mathscr{K}, \quad l = 1, \ldots, \mathcal{L}. \tag{3.11}$$

$$\alpha_{ik}^l \geq 0, \quad \forall i \in \mathscr{N}, \quad \forall k \in \mathscr{K}, \quad l = 1, \ldots, \mathcal{L}. \tag{3.12}$$

$$\beta_k^l \geq 0, \quad \forall k \in \mathscr{K}, \quad l = 1, \ldots, \mathcal{L}. \tag{3.13}$$

The decision variable vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in model (3.8)-(3.13) are used to quantify the protection level against possible variations of the uncertain parameter vector $\tilde{\boldsymbol{q}}$. From constraints (3.10)-(3.11), it can be inferred that the total protection level is given by the sum of individual and global protection levels. The decision vector $\boldsymbol{\alpha}$ defines the individual protection level of the routes and quantifies the marginal protection against any possible variation of the uncertain parameter vector $\tilde{\boldsymbol{q}}$. On the other hand, the decision vector $\boldsymbol{\beta}$ defines the global protection level of the routes and expresses the marginal protection against the assumption that at most $\boldsymbol{\Gamma}$ times, the nature will behave unfavorably.

Given a value of uncertainty budget Γ_k^l, the model (3.8)-(3.13) provides solutions that ensure 100% of feasibility if at most Γ_k^l random parameters change. However, if more than Γ_k^l random parameters change, the solution provided will be feasible with high probability (Bertsimas and Sim, 2004). Bertsimas and Sim (2004) presented different ways to determine upper bounds associated to the violation probability of the problem constraints. One of the ways discussed, requiring little computational effort, is given by:

$$Pr\left(\sum_{i \in \mathscr{N}} \tilde{q}_i z_{ik}^{l\star} > Q y_k^{l\star}\right) \leq (1-\psi) C(n, \lfloor \nu \rfloor) + \sum_{t=\lfloor \nu \rfloor + 1}^{n} C(n,t) \tag{3.14}$$

where $n = |\mathscr{N}|$, $\nu = (\Gamma_k^l + n)/2$, $\psi = \nu - \lfloor \nu \rfloor$ and:

$$C(n,t) = \begin{cases} \frac{1}{2^n}, & \text{if } t = 0 \text{ or } t = n \\ \frac{1}{\sqrt{2\pi}} \sqrt{\frac{n}{(n-t)t}}\, exp\left(n\, log\left(\frac{n}{2(n-t)}\right) + t\, log\left(\frac{n-t}{t}\right)\right), & \text{otherwise} \end{cases}$$

4 Solution Method

Although both models (2.12)-(2.25) and (3.8)-(3.13) have the same complexity regarding the computational analysis, numerical experiments using a general purpose optimization software (IBM CPLEX v.12.6) have revealed that solving model (3.8)-(3.13) requires a greater computational effort than model (2.12)-(2.25) in general. In fact, using model (3.8)-(3.13), it was not possible to find even a feasible solution for some instances with $n = 25$ customers, within the time limit of 3600 seconds. For this reason, we propose a heuristic for finding a good feasible solution to be used as an initial point in the general purpose software.

4.1 Heuristic I1 for the VRPTWMD

The heuristic I1 proposed by Solomon (1987) constructs the routes one at a time. For a particular iteration of the heuristic, let $\mathscr{R}^l = (i_0, i_1, i_2, \ldots, i_m)$ be the partial route in mode l, which was initialized following the criterion of the farthest or most urgent customer. Each customer u not visited should be tested on all feasible positions in route $\mathscr{R}^l$. Next, it determines an indicator of how good it is to put customer u in each of the feasible positions of $\mathscr{R}^l$. This indicator computed as the linear convex combination of the variations of distance and time caused by inserting customer u between positions i and j of route $\mathscr{R}^l$. Let ϕ_1 and ϕ_2 be the parameters that define the distance and time weights, respectively. If $\phi_1 = 1$, the priority in generating the routes is in the distance. On the other hand, when $\phi_2 = 1$, the priority is in the total duration. Let $w^l_{j_u}$ be the service start time in mode l given that customer u was tested between positions i and j of the partial route $\mathscr{R}^l$, and μ be a parameter that sets the priority level of the route to visit customer j immediately after visiting i. The indicator of how good it is to insert customer u between two positions i and j of the partial route $\mathscr{R}^l$ is given by:

$$c_1(i,u,j) = \phi_1 c_{11}(i,u,j) + \phi_2 c_{12}(i,u,j), \quad \phi_1 + \phi_2 = 1, \quad \phi_1 \geq 0, \quad \phi_2 \geq 0 \tag{4.1}$$

where

$$c_{11}(i,u,j) = d_{iu} + d_{uj} - \mu d_{ij}, \quad \mu \geq 0 \quad \text{e} \quad c_{12}(i,u,j) = w^l_{j_u} - w^l_j$$

The best feasible position of customer u in partial route $\mathscr{R}$ is where $c_1(i,u,j)$ is minimal, given by equation (4.1). The customer to be inserted in the partial route is the one that maximizes the benefit of serving it in the partial route rather than serving it directly, i.e., through a unique route to it. This is represented by $c_2(i,u,j) = \lambda d_{0u} - c_1(i,u,j)$, $\lambda \geq 0$, where parameter λ corresponds to the weight given to meet customer u via a unique route.

Let $\mathcal{ND}$ be the number of final deliverymen of the route. Route $\mathscr{R}^l$ starts with a single deliveryman, i.e., $l = 1$ and $\mathcal{ND} = 1$. When it is no longer possible to make another feasible insertion with mode l, it is increased by one unit, i.e., $l = l+1$. If at least one feasible insertion is made with updated mode l, we also update the number of deliverymen of partial route $\mathscr{R}^l$, i.e., $\mathcal{ND} = l$. A new route is created when it is no longer possible to make another feasible insertion, or until the maximum number of deliverymen per route is reached. The heuristic stops when all customers have been visited. In this study, the maximum number of deliverymen allowed per route is three, i.e., $\mathcal{L} = 3$.

4.2 Robust heuristic I1 for the RVRPTWMD with uncertain demand

Heuristic I1 has been adapted to the RVRPTWMD with uncertain demand. Unlike deterministic heuristic I1, in the robust version, every customer u not visited must be tested on all robust feasible positions of the partial route $\mathscr{R}^l$. Using the same criteria to initialize a route, it starts with a single deliveryman and robust feasible insertions are made until they are no longer possible, or until the maximum number of deliverymen per route is reached. The concept of a robust feasible route $\mathscr{R}^l$ is given in Definition 4.1. The set $\mathcal{S}^l$ is a subset of $\mathscr{R}^l$ that stores the indices of customers that have the $|\mathcal{S}^l| \leq \Gamma^l_{\mathscr{R}^l}$ worst-case deviations. Each time a customer is inserted into the partial path $\mathscr{R}^l$, set $\mathcal{S}^l$ is updated and the equation of Definition 4.1 is used to verify the robust feasibility of the insertion. The pseudocode of the robust heuristic I1 (RHI1) is shown in algorithm 1. The solution found by the RHI1 is used as an initial point in a general purpose optimization solver with the aim of improving its computational performance.

Definition 4.1 (Robust feasible route)**.** *A partial route $\mathscr{R}^l$ is robust feasible if in addition to satisfying the time window constraints, the following equation holds:*

$$\sum_{i \in \mathscr{R}^l} \bar{q}_i + \max_{\left\{\mathcal{S}^l | \mathcal{S}^l \subseteq \mathscr{R}^l, |\mathcal{S}^l| \leq \Gamma^l_{\mathscr{R}^l}\right\}} \left\{ \sum_{i \in \mathcal{S}^l} \hat{q}_i \right\} \leq Q.$$

Algorithm 1: Robust heuristic I1 for the RVRPTWMD with uncertain demand

Input: Instance and parameters.

Output: Solution.

1 Choose some criteria to initialize a route.

2 $l = 1$.

3 For all $u \in \mathscr{N}$, determine the best robust feasible position of this customer on the partial route $\mathscr{R}^l = (i_0, i_1, i_2, \ldots, i_m, i_{m+1})$ using the following criteria.
(I) $c_1(i, u, j) = \min[c_1(i_{p-1}, u, i_p)], \quad p = 1, \ldots, m.$

4 Choose the best customer $u^\star$ to be inserted in the partial route $\mathscr{R}^l$ from the following criteria,
(I) $c_2(i, u^\star, j) = \max[c_2(i, u, j)]$, where u was not visited before and the insertion is robust feasible (see Definition 4.1).

5 If $u^\star$ exists, then $\mathscr{N} = \mathscr{N} \backslash u^\star$, $\mathcal{ND} = l$ and go back to step 3.

6 If $\mathscr{N} = \emptyset$, stop. Return solution.

7 If $l > \mathcal{L}$, then go back to step 1, otherwise do $l = l + 1$ and go back to step 3.

5 Computational Results

We present the results of the computational experiments performed: to assess the impact of parameter Γ on routing decisions and on the performance of the objective function; to determine, through a Monte-Carlo simulation, the risk related to the violation probability of the model and its respective comparison with the theoretical violation probability bound in (3.14); and to evaluate the performance of the proposed solution approach based on the RHI1 to solve the RVRPTWMD with uncertain demand. The mathematical model was coded in the C++ language and solved by the solver CPLEX 12.6 using the ILOG technology CONCERT. The RHI1 was implemented in the C++ language. An Intel(R) Core(TM) i7-2600 3.4 GHz processor and 16 GB of RAM was used to perform the computational experiments.

5.1 Data Description

The C1 and R1 classes and parameter settings of Solomon's instances (Solomon, 1987) were used in the computational experiments. The instances in class R1 have a short scheduling horizon, which allows only a few customers per route. However, the instances of class C1 are able to cover a larger number of customers per route since their planning horizons are longer than the one of class R1 instances. The customers of R1 instances are randomly scattered in a quadratic area with dimensions 100x100, whereas customers of C1 instances are clustered. Due to the difficulty of solving large instances for the RVRPTWMD, we report the results for the first 25 customers ($n = 25$). When solving problems with $n = 50$ and $n = 100$, the runtimes required to optimally solve the problem were much more than 36000 seconds. To better evaluate the potential of the static RO methodology, the capacity, Q, of the vehicles was reduced to 80 in class C1 and 50 in class R1. Therefore, in this paper, these classes are hereafter referenced as C1.n25.Q80 and R1.n25.Q50. The parameter values $\bar{q}_i, t_{ij}, d_{ij}, w_i^a$ and w_i^b correspond to the data provided in the original instances, except for the service time, which was determined following Pureza et al. (2012):

$$s_i^l = \frac{\min\{q_i^o * rs, D - \max\{w_i^a, t_{0i}\} - t_{i,n+1}\}}{l}$$

where $D = w_{n+1}^b$ and $rs = 2$ represent the maximum route duration and rate of service, respectively.

The cost related to the number of routes, the number of deliverymen and the traveled distance were set as $p_1 = 1$, $p_2 = 0.1$ and $p_3 = 0.0001$, respectively, as in Pureza et al. (2012). The numbers of vehicles and deliverymen available in the depot were set as $|\mathscr{K}| = 15$ and $\mathcal{E} = 20$, respectively. We also run the experiments with unlimited numbers of vehicles and deliverymen, e.g., with $|\mathscr{K}| = 25$ (at least one vehicle per customer) and $\mathcal{E} = 75$ (three deliverymen per vehicle), in order to compare the performance of the solution approach for the limited and unlimited cases. Regarding the data of the robust model, the deviation $\hat{q}_i$, for all $i \in \mathscr{N}$, corresponds to a percentage $uld\%$ (15%, 20% and 30%) of nominal demand $\bar{q}_i$, i.e., $\hat{q}_i = uld\% * \bar{q}_i/100$. In this paper, this percentage $uld\%$ is called the uncertainty level of demand. For simplicity, the budget of uncertainty for each capacity constraint is equal, i.e., $\Gamma_1^1 = \Gamma_2^1 = \cdots = \Gamma_{|\mathscr{K}|}^1 = \cdots = \Gamma_1^{|\mathscr{L}|} = \Gamma_2^{|\mathscr{L}|} = \cdots = \Gamma_{|\mathscr{K}|}^{|\mathscr{L}|} = \Gamma$. The robust heuristic I1 of Section 4 was initialized following the criterion of the farthest customer. According to preliminary computational experiments, this criterion obtained better results than the most urgent customer. Moreover, from preliminary computational experiments, the best values for the RHI1 parameters were $\phi_1 = 0.6$, $\phi_2 = 0.4$, $\mu = 1$ and $\lambda = 1$.

5.2 Impact of parameter Γ

As mentioned before, the robust model (3.8)-(3.13) for the RVRPTWMD is difficult to solve to optimality, depending on the instance size. Typically, the analysis of the impact of uncertainty budget Γ is done in relation to the optimal values of the objective function. For most of the instances used here in computational experiments with $\Gamma > 0$, the CPLEX solver could not prove optimality for some of them within a running time of 3600 seconds. For this reason, in the experiments reported here, we allowed the solver to run for up to 36000 seconds to solve instances with $\Gamma > 0$ of model (3.8)-(3.13) to do the robust analysis.

Figure 1 shows the optimal solution of instance R101.n25.Q50 for $|\mathcal{K}| = 15$, $\mathcal{E} = 20$, parameter values Γ equal to 0, 1, 2 and 5 and for an uncertainty level of demand $uld = 15\%$. For each value of Γ, the figure gives the value of objective function ($z^\star$), number of routes generated (NR), total number of deliverymen used (ND), total distance traveled ($dist$), sequence of the customers in each route and number of deliverymen per route. Note that for $\Gamma = 0$, there are 8 distinct routes which represent a totally unprotected solution against the uncertainties, since it corresponds to the solution of the nominal problem. The total number of deliverymen in the solution is 15 and the total distance traveled is 653.028. When $\Gamma = 1$, the model allows the realization of the demand of one customer per route to assume its respective worst case value. When this is incorporated in the robust model, there is at least one route in the optimal solution of the nominal problem (problem with $\Gamma = 0$), whose total demand served exceeds the vehicle capacity. Hence, the model leads to new routes that ensure the feasibility in relation to the vehicle capacity. For this reason, some routes of the model with $\Gamma = 1$ changed with respect to the deterministic solution, increasing the total distance traveled to 661.604, which represents approximately 1.3%. Note that the number of routes and deliverymen were maintained in relation to the nominal problem solution. In that case, it suffices to reassign customers that cause infeasibility to other routes, as this simply incurs costs related to the total distance traveled, which corresponds to the cheapest alternative. For this reason, a low increase in the objective function is obtained in this problem compared to the nominal problem objective, approximately 0.009% (increase of 0% in the vehicle cost, 0% in the deliverymen cost and 1.3% in the travel cost).

For $\Gamma = 2$, the model allows that the demand realization of two customers per route to assume the worst case value. In this case, the number of routes is maintained in relation to the problem solution with $\Gamma = 1$. However, to ensure feasibility in the solution, one additional deliveryman (from 15 to 16) should be used. That is because it is not possible to determine routes with 15 deliverymen in total in which the service for all customers starts within the time windows. Thus, in this solution, in addition to incurring additional costs related to the total distance traveled, there is also the cost related to the additional deliveryman used. This leads to an increase in the objective function by approximately 1.1% (increase of 0% in the vehicle cost, 6.7% in the deliverymen cost and 5.2% in the travel cost) over the nominal problem. Note that the routes 0-5-18-6-26, 0-7-8-13-26, 0-11-19-10-26, 0-12-9-20-26 and 0-14-16-17-26 of the nominal problem are kept in the optimal solution with $\Gamma = 2$. This is because there is sufficient slack on these routes to absorb the additional load of up to 2 customers with the worst deviations.

Finally, for $\Gamma = 5$, Soyster's solution is reached, as all random parameters assume the worst case value in each routes. The routes have also changed, since at least one route among the ones obtained when solving the model with $\Gamma = 2$ cannot absorb the additional load of more 3 customers having the next 3 largest deviations. Note that there is an increase in the number of routes (from 8 to 9). For this reason, the largest increase in the objective function, around 8.5% (increase of 12.5% in the vehicle cost, 15.3% in the travel cost and decrease of 13.3% in the deliverymen cost), was obtained in this problem when compared to the deterministic optimal solution. In this case, reallocating customers between the routes generated in $\Gamma = 2$, not even assigning the maximum number of deliverymen per route, ensures feasibility in the constraints related to the time windows. For this reason, an additional route has to be generated in order to avoid infeasibility. These results clearly show how the RO methodology can redesign the routing decisions as the uncertainty in the problem increases.

5.3 Risk analysis via Monte-Carlo simulation

In this study, risk denotes that the probability of the solution of the robust model should be infeasible in practice. This probability is estimated via the Monte-Carlo simulation, as well as the theoretical bounds in (3.14). Two simulations were performed, one in the half-interval $[\bar{q}_i, \bar{q}_i + \hat{q}_i]$ and the other in the interval $[\bar{q}_i - \hat{q}_i, \bar{q}_i + \hat{q}_i]$. To perform the simulation, we generated 10000 different uniformly distributed samples from the corresponding interval and, for each iteration of the simulation, we verified if at least one realization of each sample (out of the 10000 generated) violated capacity constraints in the model solution, i.e., if at least one of the solution routes violates the vehicle capacity. After performing all simulation iterations, we counted the number of samples that violated vehicle capacity, which was divided by the total number of samples, 10000, to obtain the estimated risk. For different values of uncertainty budget (Γ) and the uncertainty level of demand ($uld\%$), Table 1 illustrates for instance C1.n25.Q200 with $|\mathscr{K}| = 15$ and $\mathcal{E} = 20$, the optimal value of the objective function $z^\star$, the price of robustness $PR\%$, the proportion of samples generated with respect to the half-interval (HF) and full-interval (FI), respectively, that violated the vehicle capacity constraints, and the theoretical bound (TB) of the risk as presented in Bertsimas and Sim (2004), calculated using equation (3.14). The price of robustness was determined by $PR\% = (z^\star_{rob} - z^\star_{det})/z^\star_{det} \times 100$, where $z^\star_{rob}$ and $z^\star_{det}$ represent the optimal values of the robust and deterministic problems, respectively. Thus, the price of robustness ($PR\%$) measures the relative increase in the optimal value of the robust model in relation to the optimal value of the nominal/deterministic problem.

As expected, regardless of the uncertainty level, $uld\%$, the violation probability values are smaller when the Monte-Carlo simulation is performed in the whole interval rather than in the half-interval. Note that the TB satisfies the probability values in the simulation performed at interval $[q^o_i - \hat{q}_i, q^o_i + \hat{q}_i]$, but this is not tight enough. Furthermore, we observed in both simulations the existence of the trade-off between the optimal value and the risk as the parameter Γ increases. Regardless of the uncertainty level of demand, $uld\%$, the line of Table 1 related to $\Gamma = 0$ corresponds to the nominal problem solution, which represents a totally unprotected solution against the demand uncertainty for the half-interval simulations. Note that the risk reaches its maximum value. On the other hand, when $\Gamma = 10$, the risk of the solution found by formulation (3.8)-(3.13) drops to zero. However, the price of robustness reaches its maximum values, 0.132%, 0.135% and 0.237% for $uld\%$ equal to 15, 20, and 30, respectively. For this reason, regardless of the interval used to perform the simulation, it can be seen that the parameter Γ defines the level of risk aversion of the decision maker. When $\Gamma = 0$, it is said that the decision maker is neutral or indifferent to risk, whereas for $\Gamma > 0$, the decision maker is risk averse. Then, according to the decision maker's preference, the RO methodology is able to provide solutions that are worse in relation to the objective function value, but which, in general, remain feasible in practice.

insert table 1

5.4 Performance of the heuristic approach

As mentioned earlier, the robust model was initially tested for classes C1.n25.Q80 and R1.n25.Q50 with $|\mathscr{K}| = 15$ and $\mathcal{E} = 20$ created from Solomon's instances. The solver CPLEX, with a time limit of 3600 seconds, was unable to solve to optimality the robust model (3.8)-(3.13) for the instances considered here, except for a few instances. For some instances, no feasible solution was found within this time limit. This reveals the difficulty of solving the robust problems and motivates the proposition of new methodologies to solve this model effectively. We propose a heuristic method to find a good quality robust feasible solution for model (3.8)-(3.13) (see Section 4) to be used as an initial point in the CPLEX solver.

Tables 2 and 3 present, respectively, the results for the instances of classes C1.n25.Q80 and R1.n25.Q50 in terms of: (1) objective value (z); (2) number of routes (NR); (3) number of deliverymen (ND); (4) total distance traveled ($dist$); and (5) *gap* with respect to the best solution estimated by the solver *CPLEX* after 3600 seconds. These results correspond to an uncertainty level of demand $uld\% = 20$. Note that these tables present the results found by the RHI1 alone (columns referring to RHI1) and also by combination of the RHI1 with the CPLEX solver (columns referring to RHI1 with CPLEX). The combination of the RHI1 with CPLEX is called combined strategy here. In Table 2, it can be observed that the combined strategy was able to improve the solution out of the 27 instances resolved of class C1.n25.Q80. It should be noted that these improvements are associated to the decrease in the total distance traveled, since in all these instances, the number of routes and the deliverymen did not vary. The average of the gains in relation to the difference of the *gaps* given by the combined strategy and the RHI1 was approximately 0.1%, which can be considered of little significance since the combined strategy was run for 3600 seconds and the RHI1 found the solution in less than 1 second, on average. In relation to instances of class R1.n25.Q50 of Solomon, the combined strategy improved 100% of the instances. Here, a more significant gain was obtained by the combined strategy in relation to the RHI1, since the average of the *gaps* in terms of the difference where there was gain was 8.6%. This gain is because CPLEX has been able to reduce the total number of routes generated and deliverymen used. Due to the lexicographic classification of costs in the model objective function, any reduction in the number of routes and deliverymen can significantly reduce the total cost.

Tables 4, 5, 6 and 7 present the average results for the limited ($|\mathscr{K}| = 15$ and $\mathcal{E} = 20$) and unlimited ($|\mathscr{K}| = 25$ and $\mathcal{E} = 75$) cases, respectively. For each uncertainty level of demand, $uld\%$, and for each solution strategy (CPLEX, RHI1, RHI1 with CPLEX), these average results related to the objective function value (z), execution time in seconds (t) and the optimality *gap* estimated as the relative difference between the objective function value and the best solution estimated by CPLEX within 3600 seconds. The results of these tables correspond to values of uncertain budget Γ equal to 0, 2 and 5. They corroborate that, regardless of the uncertainty level of demand ($uld\%$), the combined strategy can not significantly improve the quality of the solutions found by RHI1 in the instances of class C1.n25.Q80. In most instances, the RHI1 determined the optimal number of routes. The capacity of instances C1.n25.Q80 restrict the visits to a few customers per route. In addition, instances of this class have a long planning horizon, which indicates that the waiting contributes significantly to the duration of the routes. In these cases, to improve the solution given by the RHI1, it suffices that CPLEX reallocates customers between the routes without needing to use more than one deliveryman to meet the time windows. For this reason, the gain in the problems of class C1.n25.Q80 is insignificant, since only the total distance traveled is decreased, and by the objective lexicographic classification, any improvement in the distance does not significantly affect the objective value. Moreover, for the unlimited case, for the instances in class C1.n25.Q80, it can be observed that the RHI1 developed in this paper outperforms the stand-alone CPLEX solver used in terms of quality and computational efficiency.

In relation to class R1.n25.Q50, regardless of the uncertainty level of demand ($uld\%$) and for both limited and unlimited cases, the RHI1 outperforms the stand-alone CPLEX solver used in computational efficiency. However, in relation to the quality, the RHI1 only overcomes the stand-alone CPLEX solver used in instances with $uld = 15\%$ for the unlimited case. Although the stand-alone CPLEX solver used obtained better results on average than the RHI1 for the problems with $uld = 30\%$, $|\mathscr{K}| = 25$ and $\mathcal{E} = 75$, the average *gap* of RHI1 is smaller. This is because CPLEX was unable to determine, within 3600 seconds, a feasible solution for some of the instances. The respective *gap* of these instances was 100%, which increases the average *gap* of CPLEX. On the other hand, regardless of the uncertainty level of demand ($uld\%$), for the instances of this class, more significant gains are obtained in the combined strategy when compared to RHI1. In these instances, the RHI1 overestimates the number of routes and deliverymen in the optimal solution. Thus, in instances of class R1.n25.Q50, CPLEX decreased the total distance traveled as well as the number of routes and deliverymen, leading to a more significant gain. It should also be noted in Tables 4, 5, 6 and 7 that the gaps of the limited cases are lower than the ones of the unlimited cases, indicating that the first case is easier to solve than the second as expected, as the first involves smaller numbers of variables and constraints.

insert table 2

insert table 3

6 Concluding remarks

In this paper, we studied the VRPTWMD, a variant of the VRP involving the additional decision of assigning deliverymen to each route. Uncertainty in customer demand was incorporated into a MIP model for the RVRPTWMD adopting a static RO approach. The formulation is based on the addition of decision variables that indicate the assignment of customers to routes, as weel as the routing variables. As indicated by the computational results, it can be deduced how the variation of parameter Γ impacts the generation of the routes, cost and risk. Thus, according to the decision maker's preference, the RO methodology is able to provide solutions that address the trade-off between cost and risk. We proposed a heuristic method based on constructive heuristic I1 of Solomon to generate an initial solution for CPLEX. The resulting combined strategy outperforms the solution quality provided by using stand-alone CPLEX, but it still cannot prove the solution optimality for most of the instances considered in this paper. For this reason, interesting future research would be to develop effective exact solution methods exploring the structure of the RVRPTWMD, for instance, an ad-hoc branch-and-price method.

The goal in static RO is to determine the best value of the problem decision variables against all possible realizations of the uncertain parameters belonging to an uncertainty set. For this reason, some authors consider this approach conservative. An alternative line of research to avoid conservative solutions for the RVRPTWMD would be to apply an adjustable robust optimization approach (Ben-Tal et al., 2004). Moreover, RO with recourse could be used to mitigate the conservatism of the static approach (Thiele et al., 2009; Hanasusanto et al., 2015). These methodologies lead to an increase in the problem size in terms of the numbers of variables and constraints and therefore, appropriate solution methods should also be developed, for instance, L-shaped, branch-and-price, among others (Laporte et al., 2002; Christiansen and Lysgaard, 2007; Zeng and Zhao, 2013; Gauvin et al., 2014). Finally, future steps for research would be to consider uncertainty in travel and service times, as well as in customer demands.

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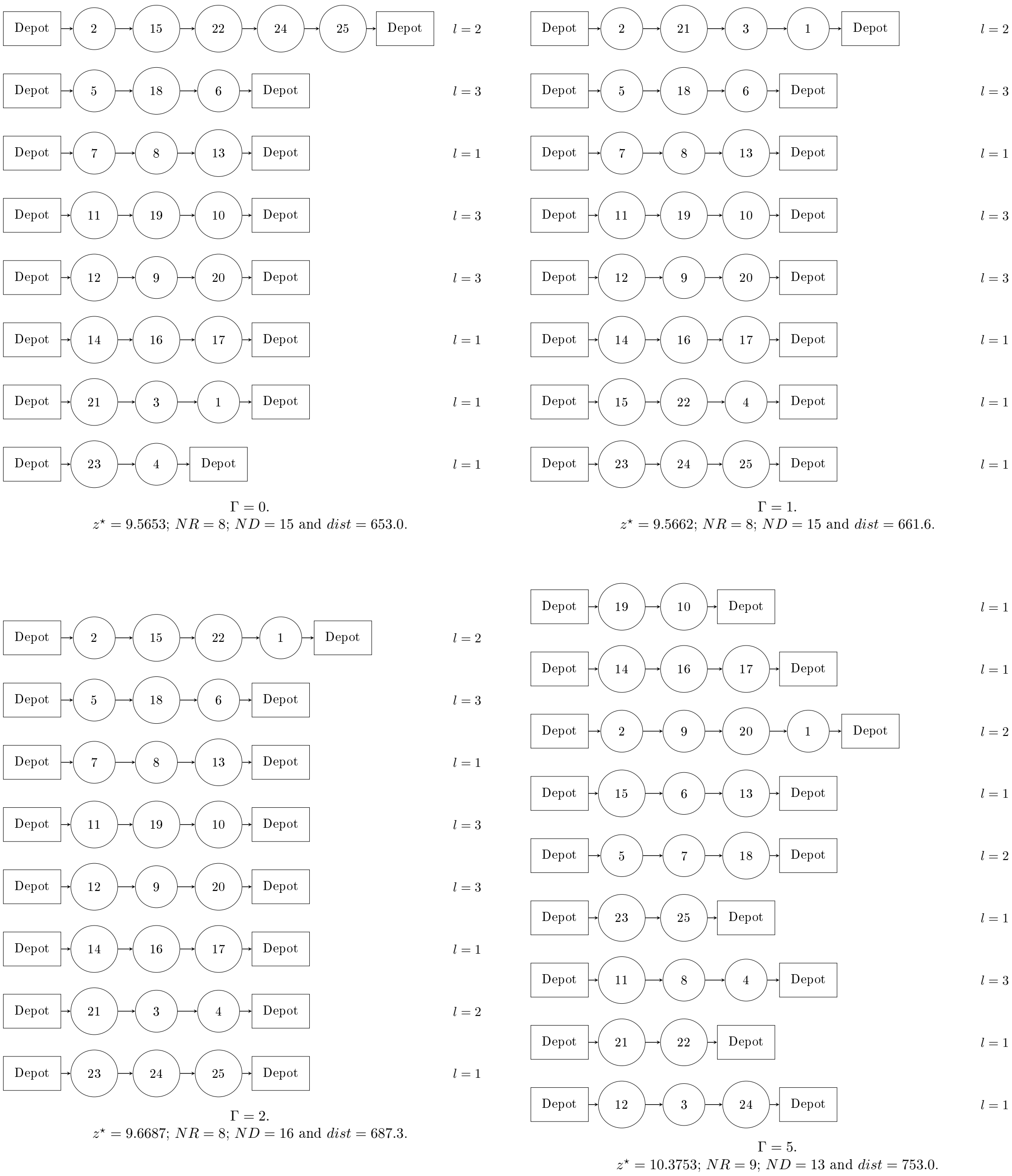

Fig. 1: Impact of Γ in the optimal routes obtained for the instance R101 of class R1.n25.Q50 with $uld = 15\%$, $|\mathscr{K}| = 10$ and $\mathcal{E} = 20$.

Tab. 1: Results of the Monte-Carlo simulation (instance C101 of class C1.n25.Q200 with $|\mathcal{K}|=$ 15 and $\mathcal{E}=20$).

	$uld=15\%$				$uld=20\%$				$uld=30\%$				
Γ	$z^\star$	$PR(\%)$	HI(%)	FI(%)	$z^\star$	$PR\%$	HI(%)	FI(%)	$z^\star$	$PR\%$	HI(%)	FI(%)	TB(%)
0	3.3192	0.000	90.2	6.3	3.3192	0.000	100.0	0.2	3.3192	0.0	99.9	22.8	100.0
1	3.3192	0.000	91.0	6.1	3.3192	0.000	99.7	0.1	3.3227	0.106	93.0	4.8	66.4
2	3.3227	0.106	1.3	0.0	3.3227	0.106	69.0	0.0	3.3229	0.113	89.8	3.0	52.3
3	3.3227	0.106	1.2	0.0	3.3227	0.106	72.1	0.0	3.3235	0.132	22.9	0.4	38.1
4	3.3227	0.106	1.2	0.0	3.3229	0.113	3.8	0.0	3.3237	0.135	0.2	0.0	23.9
5	3.3228	0.109	1.2	0.0	3.3236	0.132	0.0	0.0	3.3239	0.138	0.1	0.0	15.3
6	3.3230	0.113	0.7	0.0	3.3236	0.132	0.0	0.0	3.3234	0.143	0.0	0.0	6.7
7	3.3234	0.127	0.5	0.0	3.3237	0.135	0.0	0.0	3.3234	0.143	0.0	0.0	0.8
8	3.3236	0.132	0.3	0.0	3.3237	0.135	0.0	0.0	3.3234	0.143	0.0	0.0	0.0
9	3.3236	0.132	0.0	0.0	3.3237	0.135	0.0	0.0	3.3271	0.237	0.0	0.0	0.0
10	3.3236	0.132	0.0	0.0	3.3237	0.135	0.0	0.0	3.3271	0.237	0.0	0.0	0.0

Tab. 2: Results of the method proposed for class C1.n25.Q80 with $uld=20\%$, $|\mathcal{K}|=15$ and $\mathcal{E}=20$.

		RHI1					RHI1 with CPLEX				
Instance	Γ	z	NR	ND	$dist$	$GAP(\%)$	z	NR	ND	$dist$	$GAP(\%)$
	0	6.6465	6	6	465.2	0.17	6.6355	6	6	355.3	**0.00**
C101	2	7.7431	7	7	430.9	11.69	7.7419	7	7	419.1	**11.67**
	5	8.8520	8	8	520.4	23.04	8.8466	8	8	466.4	**22.96**
	0	6.6450	6	6	450.3	0.16	6.6355	6	6	354.6	**0.01**
C102	2	7.7485	7	7	485.3	14.30	7.7432	7	7	432.2	**14.22**
	5	8.8563	8	8	563.4	25.28	8.8469	8	8	468.9	**25.14**
	0	6.6440	6	6	439.9	0.25	6.6359	6	6	359.1	**0.13**
C103	2	7.7541	7	7	541.1	15.41	7.7472	7	7	471.6	**15.31**
	5	8.8565	8	8	564.9	25.23	8.8473	8	8	473.0	**25.10**
	0	6.6446	6	6	445.8	0.27	6.6353	6	6	353.4	**0.13**
C104	2	7.7476	7	7	475.5	15.21	7.7464	7	7	464.2	**15.20**
	5	8.8557	8	8	557.0	24.75	8.8469	8	8	469.1	**24.62**
	0	6.6467	6	6	467.1	0.17	6.6355	6	6	355.3	**0.00**
C105	2	7.7431	7	7	430.9	10.92	7.7411	7	7	411.1	**10.89**
	5	8.8520	8	8	520.4	24.37	8.8462	8	8	461.7	**24.29**
	0	6.6471	6	6	471.2	0.17	6.6355	6	6	355.3	**0.00**
C106	2	7.7431	7	7	430.9	10.89	7.7418	7	7	418.3	**10.87**
	5	8.8520	8	8	520.4	23.43	8.8466	8	8	465.5	**23.35**
	0	6.6456	6	6	455.5	0.15	6.6355	6	6	354.9	**0.00**
C107	2	7.7437	7	7	437.4	12.82	7.7416	7	7	415.8	**12.79**
	5	8.8527	8	8	527.2	24.25	8.8466	8	8	466.1	**24.16**
	0	6.6437	6	6	436.5	0.22	6.6355	6	6	354.9	**0.10**
C108	2	7.7431	7	7	430.9	13.49	7.7414	7	7	413.9	**13.47**
	5	8.8492	8	8	492.1	24.38	8.8459	8	8	459.1	**24.33**
	0	6.6425	6	6	425.2	0.24	6.6354	6	6	354.1	**0.13**
C109	2	7.7436	7	7	435.7	14.80	7.7413	7	7	413.0	**14.76**
	5	8.8492	8	8	492.1	23.92	8.8476	8	8	476.1	**23.90**
Average		**7.7478**				**12.59**	**7.7417**				**12.50**

Tab. 3: Results of the method proposed for the class R1.n25.Q50 with $uld = 20\%$, $|\mathscr{K}| = 15$ and $\mathcal{E} = 20$.

		RHI1					RHI1 with CPLEX				
Instance	Γ	z	NR	ND	$dist$	$GAP(\%)$	z	NR	ND	$dist$	$GAP(\%)$
	0	11.5761	10	15	761.2	21.02	9.5653	8	15	653.0	**0.00**
R101	2	11.3792	10	13	792.4	18.24	10.4750	9	14	750.1	**8.84**
	5	11.3793	10	13	793.4	17.76	10.5728	9	15	727.9	**9.41**
	0	9.27306	8	12	730.6	19.44	9.0637	8	10	636.7	**16.75**
R102	2	10.3748	9	13	748.0	33.21	9.9692	9	9	691.9	**28.00**
	5	10.1783	9	11	783.3	22.72	10.0697	9	10	697.3	**21.41**
	0	10.1753	9	11	753.3	31.18	7.7579	7	7	579.0	**0.02**
R103	2	10.1782	9	11	782.4	30.66	9.9639	9	9	638.7	**27.91**
	5	11.1785	10	11	785.2	36.68	9.9657	9	9	656.6	**21.85**
	0	7.97006	7	9	700.6	2.79	7.7534	7	7	534.2	**0.00**
R104	2	10.0783	9	10	783.0	30.04	9.9633	9	9	633.2	**28.56**
	5	10.0733	9	10	733.4	22.07	10.0730	9	10	730.5	**22.07**
	0	9.47793	8	14	779.3	18.74	8.0648	7	10	648.0	**1.03**
R105	2	10.3826	9	13	825.5	31.62	10.1791	9	11	791.5	**29.04**
	5	11.1859	10	11	859.5	32.54	10.0763	9	10	763.4	**19.40**
	0	9.07468	8	10	746.8	16.98	7.7590	7	7	590.2	**0.02**
R106	2	10.2824	9	12	824.5	32.57	9.9667	9	9	666.5	**28.50**
	5	11.2747	10	12	747.1	38.15	10.1659	9	11	658.6	**24.56**
	0	8.17159	7	11	715.9	5.41	7.7552	7	7	552.4	**0.04**
R107	2	11.0766	10	10	765.8	42.66	9.9650	9	9	650.1	**28.34**
	5	11.172	10	11	720.0	35.28	9.9654	9	9	653.7	**20.67**
	0	7.96911	7	9	691.1	2.79	7.7529	7	7	528.5	**0.00**
R108	2	10.0782	9	10	782.4	28.08	9.9675	9	9	675.2	**26.68**
	5	10.0749	9	10	748.5	20.05	10.0650	9	10	649.6	**19.93**
	0	9.17435	8	11	743.5	18.26	8.8595	8	8	594.7	**14.20**
R109	2	10.2792	9	12	791.8	29.75	9.9653	9	9	652.6	**25.79**
	5	10.385	9	13	850.3	25.93	9.9694	9	9	694.2	**20.89**
	0	8.37306	7	13	730.6	8.00	7.8608	7	8	607.7	**1.39**
R110	2	10.0772	9	10	772.0	26.68	9.9638	9	9	638.0	**25.25**
	5	11.1713	10	11	713.2	36.57	9.9650	9	9	650.2	**21.82**
	0	8.17177	7	11	717.7	5.39	7.7541	7	7	540.9	**0.00**
R111	2	11.076	10	10	760.3	38.04	9.9633	9	9	632.5	**24.17**
	5	10.1749	9	11	749.0	21.82	9.9638	9	9	637.8	**19.30**
	0	8.96295	8	9	629.5	15.65	7.7524	7	7	524.1	**0.03**
R112	2	10.0759	9	10	759.4	30.06	8.8683	8	8	682.9	**14.47**
	5	11.1723	10	11	723.4	35.31	9.9671	9	9	670.6	**20.72**
Average		**10.0869**				**24.50**	**9.3813**				**15.86**

Tab. 4: Results of instance C1.n25.Q80 with $|\mathscr{K}| = 15$ and $\mathcal{E} = 20$

	$uld = 15\%$			$uld = 20\%$			$uld = 30\%$		
Algorithm	z	t (s)	gap (%)	z	t (s)	gap (%)	z	t (s)	gap (%)
CPLEX	7.7435	3112	12.65	7.7436	3221	12.53	8.1118	3113	13.22
RHI1	7.7478	< 1	12.25	7.7478	< 1	12.59	8.1168	< 1	13.30
RHI1 with CPLEX	7.7415	2944	12.21	7.7417	2930	12.50	8.1110	2950	13.20

Tab. 5: Results of instance R1.n25.Q50 with $|\mathscr{K}| = 15$ and $\mathcal{E} = 20$

	$uld = 15\%$			$uld = 20\%$			$uld = 30\%$		
Algorithm	z	t (s)	gap (%)	z	t (s)	gap (%)	z	t (s)	gap (%)
CPLEX	9.5726	3246	20.05	9.6925	3311	19.75	10.2610	3376	22.63
RHI1	9.7632	< 1	22.14	10.0869	< 1	24.50	10.5970	< 1	26.57
RHI1 with CPLEX	9.1677	3084	14.85	9.3813	3110	15.86	10.0049	3313	19.58

Tab. 6: Results of instance C1.n25.Q80 with $|\mathscr{K}| = 25$ and $\mathcal{E} = 75$

	$uld = 15\%$			$uld = 20\%$			$uld = 30\%$		
Algorithm	z	t (s)	gap (%)	z	t (s)	gap (%)	z	t (s)	gap (%)
CPLEX	7.7507	3503	14.80	7.7564	3396	14.96	8.1535	3502	19.19
RHI1	7.7479	< 1	14.72	7.7478	< 1	14.83	8.1167	< 1	18.67
RHI1 with CPLEX	7.7434	3600	14.62	7.7452	3440	14.78	8.1137	3603	18.63

Tab. 7: Results of instance R1.n25.Q50 with $|\mathscr{K}| = 25$ and $\mathcal{E} = 75$

	$uld = 15\%$			$uld = 20\%$			$uld = 30\%$		
Algorithm	z	t (s)	gap (%)	z	t (s)	gap (%)	z	t (s)	gap (%)
CPLEX	9.7752	3499	36.22	9.9568	3502	25.82	10.4282	3496	33.45
RHI1	9.7632	< 1	24.09	10.0869	< 1	27.28	10.5970	< 1	31.80
RHI1 with CPLEX	9.4723	3506	20.61	9.6732	3503	22.17	10.2476	3504	27.58