Erratum to: On the DJL conjecture for order 6

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Abstract

In this note an erratum is provided to the article "On the DJL conjecture for order 6" by Naomi Shaked-Monderer, published in Operators and Matrices 11(1), 2017, 71–88. We will demonstrate and correct two errors in this article. The first error is in the statement of a proposition, which omits a certain category of extreme matrices. The second error is in the proof of a lemma. Fortunately the lemma itself is correct, and in this note we will in fact show that a stronger result holds.

This note looks at two errors in the paper [1]. The same terms and notations are used in this note as in the original paper, and so it has been decided not to reintroduce them here.

In Section 1 of this note we look at the first error which is in [1, Proposition 2.11], where a certain category of extremal matrix is omitted. This error is briefly explained and a corrected lemma is given.

In Section 2 we look at the second error in the proof of [1, Lemma 3.3]. Fortunately it is only the proof that is incorrect and not the result. We provide a counter example to an important aspect of the proof, and then go on the prove a stronger result from which [1, Lemma 3.3] directly follows.

1 First Error

The first error in [1] is Proposition 2.11. In this it states that if $\mathsf{A} \in \mathscr{COP}_n$ is an extremal copositive matrix with $a_{ii} = 0$ for some i, then $a_{ij} = 0$ for all j. However this is incorrect as it omits the case of extremal copositive matrices which are nonnegative but not positive semidefinite [2]. A summary of known results on the extremal copositive matrices is provided by [3, Theorem 8.20], and using this summary we get the following corrected form of Proposition 2.11:

Lemma 1. *Let $\mathsf{A} \in \mathscr{COP}_n$ be an extremal copositive matrix with at least one negative entry.*

1. *If $a_{ii} = 0$, then $a_{ij} = 0$ for every $i \neq j$, and $\mathsf{A}(i) \in \mathscr{COP}_{n-1}$ is extremal.*
2. *If $\operatorname{diag}(\mathsf{A}) = \mathbf{1}$, then $a_{ij} \in [-1, 1]$ for every $i \neq j$.*

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2 Second Error

The second error in [1] is in the proof of Lemma 3.3, which comes from a misinterpretation of the results from [4]. This error is slightly more difficult to demonstrate and correct, and we will break doing this down into two parts. First, in Section 2.1, we will give a counter example demonstrating that an aspect of the proof is incorrect. Then, in Section 2.2, we will prove a new (stronger) result from which [1, Lemma 3.3] directly follows.

2.1 Counter example

The incorrect claim in the proof of [1, Lemma 3.3] can be summed up as follows:

Claim 2. *Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n_+$ be minimal zeros of $\mathsf{M} \in \mathscr{COP}_n$ with supports $\sigma_\mathbf{v}, \sigma_\mathbf{w}$ respectively, and let $\mathbf{x}$ be a zero of M with support $\sigma_\mathbf{v} \cup \sigma_\mathbf{w}$. Then $\mathbf{x} = a\mathbf{v} + b\mathbf{w}$ for some $\mathbf{v}, \mathbf{w} \in \mathbb{R}_+$.*

This incorrect claim comes from a misinterpretation of the following (correct) result:

Lemma 3 ([4, Corollary 3.4],[1, Proposition 2.18])**.** *For $\mathsf{A} \in \mathscr{COP}_n$ and a zero of this given by $\mathbf{u} \in \mathbb{R}^n_+$, there exists minimal zeros $\mathbf{w}_1, \ldots, \mathbf{w}_m \in \mathbb{R}^n_+$ such that $\mathbf{u} = \sum_{i=1}^m a_i \mathbf{w}_i$ for some $\mathbf{a} \in \mathbb{R}^m$ entrywise strictly positive. Letting σ be the support of $\mathbf{u}$ and σ_i the support of $\mathbf{w}_i$ for all i we then have $\sigma = \bigcup_{i=1}^m \sigma_i$.*

In spite of this (correct) lemma, and in contradiction to the claim, if a zero support is equal to the union of some minimal zero supports, then the corresponding zero is not necessarily a nonnegative combination of the corresponding minimal zeros. Instead it can be a nonnegative combination of a different set of minimal zeros. We now give an example of this occurring.

example 4. Consider the matrix

$$\mathsf{M} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}^\mathsf{T} \in \mathscr{COP}_n.$$

Up to multiplication by a positive scalar, the minimal zeros of this are given uniquely by the vectors:

$$\mathbf{t} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

whose supports are respectively

$$\sigma_\mathbf{t} = \{1, 3\}, \qquad \sigma_\mathbf{u} = \{2, 4\}, \qquad \sigma_\mathbf{v} = \{2, 3\}, \qquad \sigma_\mathbf{w} = \{1, 4\}.$$

Now considering the vector $\mathbf{x} = \mathbf{t} + 2\mathbf{u} = \begin{pmatrix} 1 & 2 & 1 & 2 \end{pmatrix}^\mathsf{T}$, we have that $\mathbf{x}$ is a zero of M with support equal to $\sigma_\mathbf{v} \cup \sigma_\mathbf{w} = \{1, 2, 3, 4\}$, however clearly $\mathbf{x} \neq a\mathbf{v} + b\mathbf{w}$ for all $a, b \in \mathbb{R}$.

2.2 New Result

We will now prove the following new result, from which [1, Lemma 3.3] directly follows:

Theorem 5. *Let* $\mathsf{M} \in \mathscr{COP}_n$ *and for* $m \geq 3$ *let* $\sigma_1, \ldots, \sigma_m$ *be zero supports of* M *such that* $\sigma_{i,j} := \sigma_i \cup \sigma_j$ *is also a zero support of* M *for all* $1 \leq i < j \leq m$. *Then* $\bigcup_{i=1}^m \sigma_i$ *is a zero support of* M.

Proof. For $1 \leq i \leq m$ let $\mathbf{w}_i \in \mathbb{R}^n_+$ be a zero of M with support σ_i.

For all $1 \leq i, j \leq m$ we have that the support of σ_i is contained in the $\sigma_{i,j}$, and thus we trivially have

$$\mathbf{w}_i[\sigma_{i,j}]^\mathsf{T} \mathsf{M}[\sigma_{i,j}] \mathbf{w}_i[\sigma_{i,j}] = \mathbf{w}_i^\mathsf{T} \mathsf{M} \mathbf{w}_i = 0. \tag{1}$$

Therefore, using [3, Theorems 6.3 and 6.4] and the fact that $\sigma_{i,j}$ is a zero support of M, we have that $\mathsf{M}[\sigma_{i,j}]$ is a positive semidefinite matrix and $\mathsf{M}[\sigma_{i,j}]\mathbf{w}_i[\sigma_{i,j}] = \mathbf{0}$.

Therefore, for all $1 \leq i, j \leq m$ we have

$$\mathbf{w}_j^\mathsf{T} \mathsf{M} \mathbf{w}_i = \mathbf{w}_j[\sigma_{i,j}]^\mathsf{T} \mathsf{M}[\sigma_{i,j}] \mathbf{w}_i[\sigma_{i,j}] = \mathbf{w}_j[\sigma_{i,j}]^\mathsf{T} \mathbf{0} = 0.$$

Now letting $\mathbf{z} = \sum_{i=1}^m \mathbf{w}_i \in \mathbb{R}^n_+$, we have that the support of $\mathbf{z}$ is equal to $\bigcup_{i=1}^m \sigma_i$ and $\mathbf{z}^\mathsf{T} \mathsf{M} \mathbf{z} = \left(\sum_{j=1}^m \mathbf{w}_j \right)^\mathsf{T} \mathsf{M} \left(\sum_{i=1}^m \mathbf{w}_i \right) = \sum_{i,j=1}^m \mathbf{w}_j^\mathsf{T} \mathsf{M} \mathbf{w}_i = 0.$ □

An immediate corollary of this result is [1, Lemma 3.3], which for the sake of completeness is included below:

Corollary 6 ([1, Lemma 3.3])**.** *Let* $\mathsf{M} \in \mathscr{COP}_n$, *and let* $\sigma_1, \sigma_2, \sigma_3$ *be three minimal supports of* M, *such that* $\sigma_i \cup \sigma_j$ *is a zero support for every* $1 \leq i \neq j \leq 3$. *Then* $\sigma_1 \cup \sigma_2 \cup \sigma_3$ *is a zero support of* M.

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