

# Facets for Single Module and Multi-Module Capacitated Lot-Sizing Problems without Backlogging

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**Abstract.** In this paper, we consider the well-known constant-batch lot-sizing problem, which we refer to as the single module capacitated lot-sizing (SMLS) problem, and multi-module capacitated lot-sizing (MMLS) problem. We provide sufficient conditions under which the  $(k, l, S, I)$  inequalities of Pochet and Wolsey (Math of OR 18: 767-785, 1993), the mixed  $(k, l, S, I)$  inequalities, derived using mixing procedure, and the paired  $(k, l, S, I)$  inequalities, derived using sequential pairing procedure, are facet-defining for the SMLS problem without backlogging. We also provide conditions under which the inequalities derived using the sequential pairing and the  $n$ -mixing procedures are facet-defining for the MMLS problem without backlogging. All aforementioned inequalities are special cases of  $n$ -step  $(k, l, S, C)$  cycle inequalities of Bansal and Kianfar (Math. Prog. 154(1): 113-144, 2015).

*Keywords.* lot-sizing; multi-module capacities; mixing; sequential pairing; mixed integer programming; cutting planes

## 1. Introduction

Capacitated lot-sizing problem (CLSP) is one of the most widely studied problems in the domain of operations management, mainly because of its numerous applications ranging from production planning to biomass logistics (see [11, 22, 24] and references therein for few examples). In addition, many variants of the CLSP have been considered in the literature which includes two-level multi-item CLSP [26], CLSP with setup times [10], big-bucket CLSP [1], multi-echelon uncapacitated LSP [28], two-stage stochastic CSLP [3, 13], and many more. Over past three decades, several researchers have been studying the polyhedral structure of the feasible region associated with these problems, thereby deriving valid inequalities or extended formulations for them [19, 20, 26]. A promising approach to develop valid inequalities for mixed integer programming problems is using facets (or valid inequalities) of simple mixed integer sets [5, 6, 7, 15, 16, 23, 27]. Using this approach, various families of valid inequalities have been developed for the CLSP and its generalizations [2, 5, 6, 15, 23, 27]. However, not much is known about the facet-defining properties of these inequalities [22]. In this paper, we investigate the facet-defining properties of some of the well-known families of valid inequalities, derived using the aforementioned cut-generation approach, for the constant-batch lot-sizing problem which we refer to as the single module capacitated lot-sizing (SMLS) problem, and one of its generalizations, called multi-module capacitated lot-sizing (MMLS) problem.

In addition to exploring the facet-defining properties (or theoretical strength) of various classes of valid inequalities for the SMLS and MMLS problems, another motivation behind this paper is to build a stepping stone towards gaining insights which will create pathways to new cut-generation

procedures for general mixed integer programs. It is important to note that some well-known cut-generation procedures in the mixed integer programming literature have emanated from the polyhedral studies of lot-sizing or network-design problems. For example, the ideas behind the mixing inequalities [15] and cycle inequalities for continuous mixing set [25] germinated from the seminal paper of Pochet and Wolsey [20] on polyhedral study of lot-sizing problems with Wagner-Whitin costs. Similarly, the traces of the origin of the 2-step mixed integer rounding (MIR) inequalities [9] can be found in Magnanti and Mirchandani [17] which study polyhedral structure of network-design problem.

## 1.1 Literature review and contributions of this paper

The MMLS without backlogging (MMLS-WB) is defined as follows: Let  $\{\alpha_1, \dots, \alpha_n\}$  be the set of sizes of the  $n$  available capacity modules and the setup cost per module of size  $\alpha_t, t = 1, \dots, n$  in period  $p$  is denoted by  $f_p^t$ . Then the MMLS-WB is formulated as:

$$\begin{aligned} \min \sum_{p \in P} & \left( c_p x_p + h_p s_p + \sum_{t=1}^n f_p^t z_p^t \right) \\ \text{s.t. } & s_{p-1} + x_p = d_p + s_p, \quad p \in P \\ & x_p \leq \sum_{t=1}^n \alpha_t z_p^t, \quad p \in P \\ & z \in \mathbb{Z}_+^{|P| \times n}, x \in \mathbb{R}_+^{|P|}, s \in \mathbb{R}_+^{|P|+1} \end{aligned}$$

where  $x_p$  is the production in period  $p$ ,  $s_p$  is the inventory at the end of period  $p$ ,  $s_0$  is the inventory in the beginning of period 1, and  $z_p^t$  is the number of capacity modules of size  $\alpha_t, t = 1, \dots, n$ , used in period  $p$ . In addition, the parameters  $d_p, c_p$ , and  $h_p$  denote the demand, per unit production cost, and per unit inventory cost in period  $p$ , respectively. Note that in the MMLS-WB, both  $s_0$  and  $s_n$  are decision variables. Also, for  $n = 1$ , MMLS-WB reduces to SMLS without backlogging (SMLS-WB). Pochet and Wolsey [19] consider the SMLS-WB problem where the capacity in each period can be some integer multiple of a single capacity module with a given size. They introduce the so-called  $(k, l, S, I)$  inequalities and show that these inequalities subsume facets of a certain form (described implicitly in [19]) for SMLS-WB. In this paper, we explicitly define a subclass of the  $(k, l, S, I)$  inequalities and show that under certain conditions, only the  $(k, l, S, I)$  inequalities belonging to this subclass can be facet-defining. More specifically, we show that for each  $(k, l, S, I)$  inequality which does not belong to this subclass, there exists a stronger valid inequality belonging to this subclass. Later, Günlük and Pochet [15] generalized the mixed integer rounding (MIR) inequalities [18, 27] by introducing mixed MIR inequalities for a multi-constraint mixed integer set. They demonstrate how these inequalities can be utilized to develop valid inequalities for general mixed integer programs (MIPs) including SMLS-WB and referred to this cut-generation procedure as “mixing” (see Section 2.2 for details). We refer to a class of valid inequalities for SMLS-WB which can be derived using the mixing procedure as mixed  $(k, l, S, I)$  inequalities. In this paper, we provide sufficient conditions under which the mixed  $(k, l, S, I)$  inequalities are facet-defining for SMLS-WB.

In another direction, Guan et al. [14] develop facet-defining inequalities for multi-stage stochastic uncapacitated lot-sizing problem, referred to as the  $(Q, S_Q)$  inequalities. Thereafter, Guan et al. [12] introduce a cut-generation procedure, referred to as “pairing”, for MIPs; see Section 2.4 for details. They use (sequential) pairing to derive valid inequalities for multi-stage stochastic mixed

integer programs including multi-stage stochastic capacitated lot-sizing problem with variable capacities [13]. It is important to note that the  $(Q, S_Q)$  inequalities can be derived using the pairing procedure for multi-stage stochastic uncapacitated lot-sizing problem. Recently in [3], Bansal et al. present globally valid parametric inequalities and tight second stage formulations for four variants of the two-stage stochastic capacitated lot-sizing problems with uncertain demands and costs. In this paper, we present a class of valid inequalities for SMLS-WB derived using pairing procedure, investigate their facet-defining properties, and their relationship with the mixed  $(k, l, S, I)$  inequalities. We refer to this class of inequalities as the paired  $(k, l, S, I)$  inequalities.

Sanjeevi and Kianfar [23] generalize the SMLS-WB to MMLS-WB problem, in which the total production capacity in each period can be the summation of some integer multiples of several capacity modules of different sizes. They show that the mixed  $n$ -step MIR inequalities (which generalize the mixed MIR inequalities [15] and  $n$ -step MIR inequalities [16]) can be used to generate valid inequalities for MMLS-WB. These inequalities are referred to as the multi-module mixed  $(k, l, S, I)$  inequalities as they also generalize the  $(k, l, S, I)$  inequalities [19]. Recently, Bansal and Kianfar [5, 6] further generalize the mixed  $n$ -step MIR inequalities [23] to  $n$ -step cycle inequalities and develop the so-called  $n$ -step  $(k, l, S, C)$  cycle inequalities for both MMLS-WB and MMLS with backlogging problems. The class of  $n$ -step  $(k, l, S, C)$  cycle inequalities subsumes the multi-module mixed  $(k, l, S, I)$  inequalities for MMLS-WB problem. Note that in [2], the  $n$ -step cycle inequalities are also used to generate new classes of valid inequalities for multi-module capacitated facility location (MMFL) and multi-module capacitated network design (MMND) problems. In this paper, we investigate the facet-defining properties of the multi-module mixed  $(k, l, S, I)$  inequalities for the MMLS-WB problem. In addition, we present a new class of valid inequalities for MMLS-WB using pairing procedure [12], investigate their facet-defining properties, and their relationship with the multi-module mixed  $(k, l, S, I)$  inequalities.

## 1.2 Organization of this paper

In Section 2, we briefly review the cut-generation procedures: mixed integer rounding (MIR) [18, 27], mixing [15],  $n$ -mixing [23], and sequential pairing [12]. In Section 3, we provide sufficient conditions under which the  $(k, l, S, I)$  inequalities of Pochet and Wolsey [19], the mixed  $(k, l, S, I)$  inequalities, derived using mixing procedure of Günlük and Pochet [15], and paired  $(k, l, S, I)$  inequalities, derived using pairing procedure of Guan et al. [12], are facet-defining for SMLS-WB. In Section 4, we present conditions under which the inequalities derived using the pairing procedure [12] and the  $n$ -mixing cut-generation procedure of Sanjeevi and Kianfar [23] are facet-defining for the MMLS-WB. Finally in Section 5, we provide concluding remarks and discuss about potential extension of the results presented in this paper. Note that all aforementioned inequalities are special cases of  $n$ -step  $(k, l, S, C)$  cycle inequalities [6].

## 2. Necessary Background

In this section, we briefly review the cut-generation procedures: MIR [18, 27], mixing [15],  $n$ -mixing [23], and sequential pairing [12] which we will use to develop valid inequalities for SMLS-WB and MMLS-WB problems in the subsequent sections.

## 2.1 Mixed Integer Rounding (MIR)

One fundamental procedure to develop cuts for general mixed integer programs is the MIR procedure [18, 27] which utilizes the facet of a single-constraint mixed integer set,

$$Q := \{(y, v) \in \mathbb{Z} \times \mathbb{R}_+ : \alpha_1 y + v \geq \beta\},$$

where  $\alpha_1 > 0$  and  $\beta \in \mathbb{R}$  (page 127 of [27]). This facet is referred to as the MIR facet and is given by

$$v + \left( \beta - \alpha_1 \left\lfloor \frac{\beta}{\alpha_1} \right\rfloor \right) y \geq \left( \beta - \alpha_1 \left\lfloor \frac{\beta}{\alpha_1} \right\rfloor \right) \left\lceil \frac{\beta}{\alpha_1} \right\rceil. \quad (1)$$

In order to avoid trivial case, it is assumed that  $\beta/\alpha_1 \notin \mathbb{Z}$  because then Inequality (1) will reduce to  $v \geq 0$ .

## 2.2 Mixing

Günlük and Pochet [15] studied a mixed integer set, defined by

$$Q_0 := \{(y, v) \in \mathbb{Z}^m \times \mathbb{R}_+ : \alpha_1 y^i + v \geq \beta_i, i = 1, \dots, m\},$$

where  $\alpha_1 \in \mathbb{R}_+$ ,  $\beta \in \mathbb{R}^m$ , and  $\beta_i/\alpha_1 \notin \mathbb{Z}$ ,  $i = 1, \dots, m$ . This set is referred to as the mixing set which is a multi-constraint generalization of the set  $Q$  that leads to the well-known MIR inequality (1). Günlük and Pochet [15] derive the mixed MIR inequalities for the mixing set  $Q_0$  as follows: Define  $\beta_0 := 0$ ,  $\beta_i^{(1)} := \beta_i - \lfloor \beta_i/\alpha_1 \rfloor$ ,  $i \in \{0, \dots, m\}$ , and without loss of generality assume that  $\beta_0^{(1)} = 0 < \beta_{i-1}^{(1)} \leq \beta_i^{(1)}$ ,  $i = 2, \dots, m$ . Let  $K := \{i_1, \dots, i_{|K|}\}$ , where  $i_1 < i_2 < \dots < i_{|K|}$ , be a non-empty subset of  $\{1, \dots, m\}$ . Then the inequalities,

$$v \geq \sum_{p=1}^{|K|} \left( \beta_{i_p}^{(1)} - \beta_{i_{p-1}}^{(1)} \right) \left( \left\lceil \frac{\beta_{i_p}}{\alpha_1} \right\rceil - y^{i_p} \right) \quad (2)$$

$$v \geq \sum_{p=1}^{|K|} \left( \beta_{i_p}^{(1)} - \beta_{i_{p-1}}^{(1)} \right) \left( \left\lceil \frac{\beta_{i_p}}{\alpha_1} \right\rceil - y^{i_p} \right) + \left( \alpha_1 - \beta_{i_{|K|}}^{(1)} \right) \left( \left\lfloor \frac{\beta_{i_1}}{\alpha_1} \right\rfloor - y^{i_1} \right), \quad (3)$$

are valid for  $Q_0$  and sufficient to describe the convex hull of  $Q_0$ . Inequalities (2) and (3) are referred to as the type I and type II mixed MIR inequalities, respectively. As mentioned in Section 1, Günlük and Pochet used these inequalities to generate cuts for linear mixed integer programs and referred to this cut-generation procedure as *mixing*. In Section 3.1, we will show how to generate valid inequalities for SMLS-WB using mixing. In addition, Günlük and Pochet mentioned that in a variant of the mixing set  $Q_0$  where  $y \in \mathbb{Z}_+^m$ , inequality (3) is redundant unless  $y^{i_1} < \lfloor \beta_{i_1}/\alpha_1 \rfloor$  for some feasible  $y$ .

## 2.3 $n$ -mixing

Sanjeevi and Kianfar [23] generalize the mixing procedure [15] to  $n$ -mixing cut-generation procedure by considering the  $n$ -mixing set,

$$Q_0^{m,n} := \{(y^1, \dots, y^m, v) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+ : \sum_{t=1}^n \alpha_t y_t^i + v \geq \beta_i, i = 1, \dots, m\},$$

where  $\alpha_t > 0$ ,  $t = 1, \dots, n$ , and  $\beta \in \mathbb{R}^m$ . Note that  $Q_0^{m,1} = Q_0$ . They develop the mixed  $n$ -step MIR inequalities for  $Q_0^{m,n}$  as follows: Without loss of generality, we assume  $\beta_{i-1}^{(n)} \leq \beta_i^{(n)}$ ,  $i = 2, \dots, m$ , where  $\beta_i^{(t)} := \beta_i^{(t-1)} - \alpha_t \lfloor \beta_i^{(t-1)} / \alpha_t \rfloor$ ,  $t = 1, \dots, n$ , and  $\beta_i^{(0)} := \beta_i$ . It is also assumed that  $\beta_i^{(t-1)} / \alpha_t \notin \mathbb{Z}$ ,  $i = 1, \dots, m$ . Note that  $0 < \beta_i^{(t)} < \alpha_t$  for  $t = 1, \dots, n$ . By definition, we set  $\sum_a^b(\cdot) = 0$  and  $\prod_a^b(\cdot) = 1$  if  $a > b$ . If the  $n$ -step MIR conditions,

$$\alpha_t \left\lceil \beta_i^{(t-1)} / \alpha_t \right\rceil \leq \alpha_{t-1}, \quad t = 2, \dots, n, \quad (4)$$

hold for each constraint  $i \in K \subseteq \{1, \dots, m\}$ , then the inequalities

$$v \geq \sum_{p=1}^{|K|} \left( \beta_{i_p}^{(n)} - \beta_{i_{p-1}}^{(n)} \right) \phi_{i_p}^n(y^{i_p}), \quad (5)$$

$$v \geq \sum_{p=1}^{|K|} \left( \beta_{i_p}^{(n)} - \beta_{i_{p-1}}^{(n)} \right) \phi_{i_p}^n(y^{i_p}) + \left( \alpha_n - \beta_{i_{|K|}}^{(n)} \right) \left( \phi_{i_1}^n(y^{i_1}) - 1 \right), \quad (6)$$

are valid for  $Q_0^{m,n}$ , where  $\beta_{i_0}^{(n)} = 0$  and

$$\phi_i^n(y^i) := \prod_{l=1}^n \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{t=1}^n \prod_{l=t+1}^n \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil y_t^i \quad (7)$$

for  $i \in K$ . Inequalities (5) and (6) are referred to as type I and type II mixed  $n$ -step MIR inequalities, respectively. Inequality (5) is shown to be facet-defining for  $Q_0^{m,n}$ . Inequality (6) also defines a facet for  $Q_0^{m,n}$  if some additional conditions are satisfied; see [23] for details. Sanjeevi and Kianfar [23] used mixed  $n$ -step MIR inequalities to generate valid inequalities for MMLS-WB problem (see Section 4.1 for details) and referred to this cut-generation procedure as  $n$ -mixing. Note that the  $n$ -step MIR inequalities [16] are special cases of the type I mixed  $n$ -step MIR inequalities (5), and both type I and type II mixed  $n$ -step MIR inequalities, (5) and (6), are special cases of the  $n$ -step cycle inequalities of Bansal and Kianfar [5, 6, 7].

**Remark 1.** *The  $n$ -step MIR conditions (4) are automatically satisfied when the coefficients,  $\alpha_1, \dots, \alpha_n$ , are divisible, i.e.  $\alpha_t / \alpha_{t-1} \in \mathbb{Z}_+$ ,  $t = 2, \dots, n$ .*

**Remark 2.** *In the rest of the paper, the notation  $\delta^{(t)}$  will denote (recursive) remainder of  $\delta$  with respect to  $\alpha_1, \alpha_2, \dots, \alpha_t$ . More precisely,  $\delta^{(t)} := \delta^{(t-1)} - \alpha_t \lfloor \delta^{(t-1)} / \alpha_t \rfloor$ ,  $t = 1, \dots, n$ , and  $\delta_i^{(0)} := \delta_i$ .*

## 2.4 Pairing of mixed integer inequalities

Given a mixed integer set  $Z := \{(x, z) \in \mathbb{Z}_+^{n_x} \times \mathbb{R}_+^{n_z} : E_x x + E_z z \geq e_R\}$  and two valid inequalities for  $Z$ :

$$\sum_{j=1}^{n_x} a_j^1 x_j + \sum_{j=1}^{n_z} b_j^1 z_j \geq c_R^1 \quad (8)$$

$$\sum_{j=1}^{n_x} a_j^2 x_j + \sum_{j=1}^{n_z} b_j^2 z_j \geq c_R^2. \quad (9)$$

Guan et al. [12] introduce a cut-generation procedure, referred to as pairing, to develop a new valid inequality (10) for  $Z$ .

**Theorem 1** ([12]). *Assuming  $c_R^2 \geq c_R^1$ , the inequality,*

$$\sum_{j=1}^{n_x} \min \{a_j^1 + c_R^2 - c_R^1, \max \{a_j^1, a_j^2\}\} x_j + \sum_{j=1}^{n_z} \max \{b_j^1, b_j^2\} z_j \geq c_R^2, \quad (10)$$

is valid for  $Z$ .

For the sake of convenience, let Inequality (10) be written as

$$\sum_{j=1}^{n_x} (a_j^1 \circ a_j^2) x_j + \sum_{j=1}^{n_z} (b_j^1 \circ b_j^2) z_j \geq c_R^2,$$

and referred to as  $(1 \circ 2)$  paired inequality. Assume that the inequalities

$$\sum_{j=1}^{n_x} a_j^k x_j + \sum_{j=1}^{n_z} b_j^k z_j \geq c_R^k, \text{ for } k = 3, \dots, \bar{m}, \quad (11)$$

are also valid for  $Z$  where  $c_R^2 \leq c_R^3 \leq \dots \leq c_R^{\bar{m}}$ . Then pairing the  $(1 \circ 2)$  paired inequality and inequality (11), for any  $k \in \{3, \dots, \bar{m}\}$ , gives another valid inequality for  $Z$ , i.e.

$$\begin{aligned} \sum_{j=1}^{n_x} \min \{(a_j^1 \circ a_j^2) + c_R^k - c_R^2, \max \{(a_j^1 \circ a_j^2), a_j^k\}\} x_j \\ + \sum_{j=1}^{n_z} \max \{(b_j^1 \circ b_j^2), b_j^k\} z_j \geq c_R^k, \end{aligned} \quad (12)$$

which is referred to as the  $((1 \circ 2) \circ k)$  sequentially paired inequality. This cut-generation procedure is called sequential pairing. Guan et al. [12] provide the following result for these inequalities.

**Theorem 2** ([12]). *Let  $K := \{i_1, \dots, i_{|K|}\}$  be a subset of  $\{1, \dots, \bar{m}\}$  such that  $i_1 \leq i_2 \leq \dots \leq i_{|K|}$ . For each of the following cases, there exists a subset  $K$  such that the  $((\dots((i_1 \circ i_2) \circ i_3) \circ \dots) \circ i_{|K|})$  sequentially paired inequality is at least as strong as any other sequentially paired inequality obtained by arbitrary sequence of pairing of the inequalities (11) for  $k = 1, \dots, \bar{m}$ :*

1. *Nested case, i.e.  $a_j^k \leq a_j^{k+1}$ , for  $j = 1, \dots, n_x$  and  $k = 1, \dots, \bar{m} - 1$ ,*
2. *Disjoint case, i.e.  $a_j^k a_l^k = 0$  for  $j \neq l$ ,  $j, l \in \{1, \dots, n_x\}$ , and  $k \in \{1, \dots, \bar{m}\}$ .*

### 3. Single Module Lot-Sizing Problem without Backlogging

In this section, we first re-define the single module capacitated lot-sizing (SMLS) problem (referred to as the constant batch lot-sizing problem in [19]). Then we present valid inequalities for this problem and investigate their facet-defining properties. The SMLS without backlogging (SMLS-WB) is defined as follows: Let  $P := \{1, \dots, m\}$  be the set of time periods and  $\alpha_1$  be the size of the available capacity module (or batch). Given the setup cost per module, the demand, the production

per unit cost, and the inventory per unit cost in period  $p$ , denoted by  $f_p^1$ ,  $d_p$ ,  $c_p$ , and  $h_p$ , respectively, SMLS-WB can be formulated as:  $\min \left\{ \sum_{p \in P} (c_p x_p + h_p s_p + f_p^1 z_p^1) : (z, x, s) \in X^{SMLS} \right\}$  where

$$X^{SMLS} := \{(z, x, s) \in \mathbb{Z}_+^m \times \mathbb{R}_+^m \times \mathbb{R}_+^{m+1}\} \quad (13)$$

$$s_{p-1} + x_p = d_p + s_p, \quad p \in P \quad (14)$$

$$x_p \leq \alpha_1 z_p^1, \quad p \in P \}, \quad (15)$$

$x_p$  is the production in period  $p$ ,  $s_p$  is the inventory at the end of period  $p$ , and  $z_p^1$  is the number of capacity modules of size  $\alpha_1$  used in period  $p$ .

### 3.1 Valid Inequalities for SMLS-WB Problem

Similar to Pochet and Wolsey [19], we consider a subnetwork consisting of periods  $k, \dots, l$ , for any  $k, l \in P$  where  $k < l$ . Let  $S \subseteq \{k, \dots, l\}$  such that  $k \in S$ . For  $i \in S$ , let  $S_i := S \cap \{k, \dots, i\}$ ,  $m_i = \min\{p : p \in S \setminus S_i\}$  with  $m_i = l + 1$  if  $S \setminus S_i = \emptyset$ , and  $b_i = \sum_{p=k}^{m_i-1} d_p$ . We define  $I := \{i_1, i_2, \dots, i_{|I|}\} \subseteq S$  such that  $b_{i_0}^{(1)} = 0 < b_{i_1}^{(1)} \leq b_{i_2}^{(1)} \leq \dots \leq b_{i_{|I|}}^{(1)} < \alpha_1$ .

**Theorem 3** ([21]). *The inequality,*

$$s_{k-1} + \sum_{p \in \{k, \dots, l\} \setminus S} x_p \geq \sum_{q=1}^{|I|} (b_{i_q}^{(1)} - b_{i_{q-1}}^{(1)}) \left( \left\lceil \frac{b_{i_q}}{\alpha_1} \right\rceil - \sum_{p \in S_{i_q}} z_p^1 \right), \quad (16)$$

referred to as the  $(k, l, S, I)$  inequality, is valid for  $X^{SMLS}$  where  $s_0 = s_m = 0$ . These inequalities generate all facets for the convex hull of  $X^{SMLS} \cap \{(s_0, s_m) : s_0 = s_m = 0\}$  which are of the form

$$s_{k-1} + \sum_{p \in \{k, \dots, l\} \setminus S} x_p + \sum_{p \in S} \pi_p^1 z_p^1 \geq \pi_0$$

for all  $2 \leq k \leq l \leq m$  and  $S \subseteq \{k, \dots, l\}$ .

**Example 1.** Consider a SMLS-WB problem instance defined over six time periods, i.e.  $m = 6$  and  $P = \{1, \dots, 6\}$ , where  $\alpha_1 = 5$  and demand in each time period are as follows:  $d_1 = 13$ ,  $d_2 = 2$ ,  $d_3 = 2$ ,  $d_4 = 23$ ,  $d_5 = 1$ , and  $d_6 = 8$ . Figure 1 represents a flow diagram for this instance. Let the set of feasible solutions, (13)-(15), for this instance be defined by

$$\begin{aligned} X^E := \{(z, x, s) \in \mathbb{Z}_+^6 \times \mathbb{R}_+^6 \times \mathbb{R}_+^7 : \\ & s_0 + x_1 = s_1 + 13 \\ & s_1 + x_2 = s_2 + 2 \\ & s_2 + x_3 = s_3 + 2 \\ & s_3 + x_4 = s_4 + 23 \\ & s_4 + x_5 = s_5 + 1 \\ & s_5 + x_6 = s_6 + 8 \\ & x_p \leq 5z_p^1, p = 1, \dots, 6\}. \end{aligned}$$

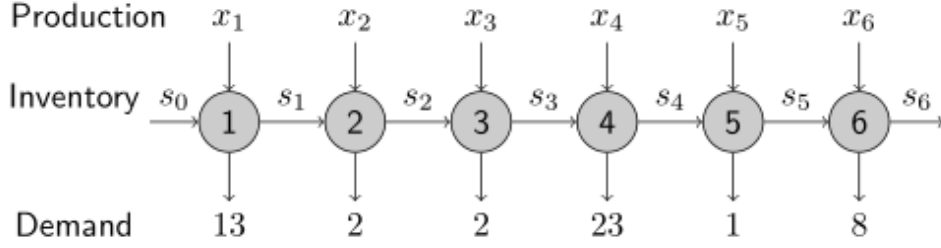


Figure 1: Example of SMLS-WB

Using the above defined notations, we derive valid inequalities for  $X^E$  as follows. Let  $k = 2$ ,  $l = 6$ , and  $S = \{2, 4, 5\}$ . Therefore,  $S_2 = \{2\}$ ,  $S_4 = \{2, 4\}$ ,  $S_5 = \{2, 4, 5\}$ , and  $m_2 = 4$ ,  $m_4 = 5$ ,  $m_5 = 7$ . Also,  $b_2 = d_2 + d_3 = 4$  and  $b_2^{(1)} = 4 - 5 \lfloor 4/5 \rfloor = 4$ . Likewise,  $b_4 = d_2 + d_3 + d_4 = 27$ ,  $b_4^{(1)} = 27 - 5 \lfloor 27/5 \rfloor = 2$ , and  $b_5 = \sum_{p=2}^{m_5-1} d_p = 36$ ,  $b_5^{(1)} = 36 - 5 \lfloor 36/5 \rfloor = 1$ . Notice that  $b_0^{(1)} = 0 < b_5^{(1)} < b_4^{(1)} < b_2^{(1)}$ . Because of Theorem 3, the following  $(k, l, S, I) = (2, 6, \{2, 4, 5\}, \{4, 2\})$  inequality is valid for  $X^E$  where  $s_0 = s_6 = 0$ :

$$\begin{aligned} s_1 + x_3 + x_6 &\geq b_4^{(1)} \left( \left\lceil \frac{b_4}{\alpha_1} \right\rceil - z_2^1 - z_4^1 \right) + (b_2^{(1)} - b_4^{(1)}) \left( \left\lceil \frac{b_2}{\alpha_1} \right\rceil - z_2^1 \right) \\ &= 2(6 - z_2^1 - z_4^1) + 2(1 - z_2^1) = 14 - 4z_2^1 - 2z_4^1. \end{aligned}$$

Similarly, the  $(k, l, S, I) = (2, 6, \{2, 4, 5\}, \{5, 4\})$  inequality,

$$\begin{aligned} s_1 + x_3 + x_6 &\geq b_5^{(1)} \left( \left\lceil \frac{b_5}{\alpha_1} \right\rceil - z_2^1 - z_4^1 - z_5^1 \right) + (b_4^{(1)} - b_5^{(1)}) \left( \left\lceil \frac{b_4}{\alpha_1} \right\rceil - z_2^1 - z_4^1 \right) \\ &= (8 - z_2^1 - z_4^1 - z_5^1) + (6 - z_2^1 - z_4^1) = 14 - 2z_2^1 - 2z_4^1 - z_5^1, \end{aligned}$$

the  $(k, l, S, I) = (2, 6, \{2, 4, 5\}, \{5, 2\})$  inequality,

$$\begin{aligned} s_1 + x_3 + x_6 &\geq b_5^{(1)} \left( \left\lceil \frac{b_5}{\alpha_1} \right\rceil - z_2^1 - z_4^1 - z_5^1 \right) + (b_2^{(1)} - b_5^{(1)}) \left( \left\lceil \frac{b_2}{\alpha_1} \right\rceil - z_2^1 \right) \\ &= (8 - z_2^1 - z_4^1 - z_5^1) + 3(1 - z_2^1) = 11 - 4z_2^1 - z_4^1 - z_5^1, \end{aligned}$$

and the  $(k, l, S, I) = (2, 6, \{2, 4, 5\}, \{5, 4, 2\})$  inequality,

$$\begin{aligned} s_1 + x_3 + x_6 &\geq b_5^{(1)} \left( \left\lceil \frac{b_5}{\alpha_1} \right\rceil - z_2^1 - z_4^1 - z_5^1 \right) + (b_4^{(1)} - b_5^{(1)}) \left( \left\lceil \frac{b_4}{\alpha_1} \right\rceil - z_2^1 - z_4^1 \right) \\ &\quad + (b_2^{(1)} - b_4^{(1)}) \left( \left\lceil \frac{b_2}{\alpha_1} \right\rceil - z_2^1 \right) \\ &= (8 - z_2^1 - z_4^1 - z_5^1) + (6 - z_2^1 - z_4^1) + 2(1 - z_2^1) \\ &= 16 - 4z_2^1 - 2z_4^1 - z_5^1, \end{aligned}$$

are also valid for  $X^E$  where  $s_0 = s_6 = 0$ . □

In order to generate valid inequalities for  $X^{SML}$  using the mixing procedure [15] and the pairing procedure [12], we aggregate equalities (14) from period  $k$  to period  $m_i - 1$ ,  $i \in I \subseteq S$ , and obtain



$$s_{k-1} + \sum_{p=k}^{m_i-1} x_p = b_i + s_{m_i-1}. \quad (17)$$

By definition  $S_i \subseteq \{k, \dots, m_i - 1\}$ . Therefore, relaxing  $x_p, p \in S_i$ , in (17) to its upper bound based on (15) and  $s_{m_i-1}$  in (17) to its lower bound based on (13) gives

$$s_{k-1} + \sum_{p \in \{k, \dots, m_i-1\} \setminus S_i} x_p + \alpha_1 \sum_{p \in S_i} z_p^1 \geq b_i, \quad (18)$$

**Inequalities derived using the mixing procedure.** Let  $i_u = \max\{i : i \in I\}$ . Then we get the following valid inequality for  $X^{SML}$ :

$$s_{k-1} + \sum_{p \in \{k, \dots, m_{i_u}-1\} \setminus S_i} x_p + \alpha_1 \sum_{p \in S_i} z_p^1 \geq b_i, \quad (19)$$

for  $i \in I$ . Setting  $v := s_{k-1} + \sum_{p \in \{k, \dots, m_{i_u}-1\} \setminus S} x_p$  and  $y^i := \sum_{p \in S_i} z_p^1$ , inequality (19) becomes  $v + \alpha_1 y^i \geq b_i$ ,  $i \in I$ , which is of the same form as the defining inequalities of mixing set  $Q_0$ . Notice that  $v \in \mathbb{R}_+$ ,  $y^i \in \mathbb{Z}_+$ . Therefore, the type I mixed MIR inequality (2) written for  $X^{SML}$  with  $K = I$ ,

$$s_{k-1} + \sum_{p \in \{k, \dots, m_{i_u}-1\} \setminus S} x_p \geq \sum_{q=1}^{|I|} \left( b_{i_q}^{(1)} - b_{i_{q-1}}^{(1)} \right) \left( \left\lceil \frac{b_{i_q}}{\alpha_1} \right\rceil - \sum_{p \in S_{i_q}} z_p^1 \right), \quad (20)$$

is valid for  $X^{SML}$ . We refer to inequality (20) as mixed  $(k, l, S, I)$  inequality.

**Example 1** (continued). For  $i \in S$ , inequalities (18) for  $X^E$  are written as:

$$\text{For } i = 2: s_1 + x_3 + 5z_2^1 \geq 4$$

$$\text{For } i = 4: s_1 + x_3 + 5(z_2^1 + z_4^1) \geq 27$$

$$\text{For } i = 5: s_1 + x_3 + x_6 + 5(z_2^1 + z_4^1 + z_5^1) \geq 38.$$

As discussed above, we apply the mixing procedure on these inequalities. For  $I = \{4, 2\}$ , we get the following valid mixed  $(k, l, S, I) = (2, 6, \{2, 4, 5\}, \{4, 2\})$  inequality for  $X^E$  where  $i_u = 4$ :

$$\begin{aligned} s_1 + x_3 &\geq b_4^{(1)} \left( \left\lceil \frac{b_4}{\alpha_1} \right\rceil - z_2^1 - z_4^1 \right) + \left( b_2^{(1)} - b_4^{(1)} \right) \left( \left\lceil \frac{b_2}{\alpha_1} \right\rceil - z_2^1 \right) \\ &= 2(6 - z_2^1 - z_4^1) + 2(1 - z_2^1) = 14 - 4z_2^1 - 2z_4^1. \end{aligned}$$

Likewise, the mixed  $(k, l, S, I) = (2, 6, \{2, 4, 5\}, \{5, 4\})$  inequality where  $i_u = 5$ ,

$$\begin{aligned} s_1 + x_3 + x_6 &\geq b_5^{(1)} \left( \left\lceil \frac{b_5}{\alpha_1} \right\rceil - z_2^1 - z_4^1 - z_5^1 \right) + \left( b_4^{(1)} - b_5^{(1)} \right) \left( \left\lceil \frac{b_5}{\alpha_1} \right\rceil - z_2^1 - z_4^1 \right) \\ &= (8 - z_2^1 - z_4^1 - z_5^1) + (6 - z_2^1 - z_4^1) = 14 - 2z_2^1 - 2z_4^1 - z_5^1, \end{aligned}$$

the mixed  $(k, l, S, I) = (2, 6, \{2, 4, 5\}, \{5, 2\})$  inequality where  $i_u = 5$ ,

$$\begin{aligned} s_1 + x_3 + x_6 &\geq b_5^{(1)} \left( \left\lceil \frac{b_5}{\alpha_1} \right\rceil - z_2^1 - z_4^1 - z_5^1 \right) + \left( b_2^{(1)} - b_5^{(1)} \right) \left( \left\lceil \frac{b_2}{\alpha_1} \right\rceil - z_2^1 \right) \\ &= (8 - z_2^1 - z_4^1 - z_5^1) + 3(1 - z_2^1) = 11 - 4z_2^1 - z_4^1 - z_5^1, \end{aligned}$$

and the mixed  $(k, l, S, I) = (2, 6, \{2, 4, 5\}, \{5, 4, 2\})$  inequality where  $i_u = 5$ ,

$$\begin{aligned} s_1 + x_3 + x_6 &\geq b_5^{(1)} \left( \left\lceil \frac{b_5}{\alpha_1} \right\rceil - z_2^1 - z_4^1 - z_5^1 \right) + (b_4^{(1)} - b_5^{(1)}) \left( \left\lceil \frac{b_4}{\alpha_1} \right\rceil - z_2^1 - z_4^1 \right) \\ &\quad + (b_2^{(1)} - b_4^{(1)}) \left( \left\lceil \frac{b_2}{\alpha_1} \right\rceil - z_2^1 \right) = 16 - 4z_2^1 - 2z_4^1 - z_5^1, \end{aligned}$$

are also valid for  $X^E$ . □

**Inequalities derived using the pairing procedure.** First, for each  $i \in S$ , we apply the MIR procedure on inequalities (18) as follows. By setting  $v := s_{k-1} + \sum_{p \in \{k, \dots, m_i-1\} \setminus S_i} x_p$ ,  $y := \sum_{p \in S_i} z_p^1$ , and  $\beta = b_i$ , Inequality (18) becomes  $v + \alpha_1 y \geq \beta$  which is of the same form as the defining inequalities of set  $Q$ . Notice that  $v \in \mathbb{R}_+$  and  $y \in \mathbb{Z}_+$ . Therefore, the MIR inequality (1) written for inequality (18),

$$s_{k-1} + \sum_{p \in \{k, \dots, m_i-1\} \setminus S_i} x_i + b_i^{(1)} \left( \sum_{p \in S_i} z_p^1 - \left\lfloor \frac{b_i}{\alpha_1} \right\rfloor \right) \geq b_i^{(1)}, \quad (21)$$

is valid for  $X^{SML}$ . Next, we apply sequential pairing procedure (discussed in Section 2.4) on inequalities (21). However, to do so, we have to assume that  $\sum_{p \in S_i} z_p^1 \geq \lfloor b_i/\alpha_1 \rfloor$  for each  $i \in S$ . The sequentially paired inequalities obtained by arbitrary sequence of pairing of the inequalities (21) for  $i \in I$ , in increasing order of  $b_i^{(1)}$ , are valid for  $X^{SML}$  and we refer to thus obtained inequalities as the paired  $(k, l, S, I)$  inequality for SMLS-WB.

### 3.2 Facets for SMLS-WB Problem

We provide sufficient conditions under which the  $(k, l, S, I)$  inequalities (16), the mixed  $(k, l, S, I)$  inequalities (20), and the paired  $(k, l, S, I)$  inequalities are facet-defining for  $X^{SML}$ .

**Observation 1.** *The  $(k, l, S, I)$  inequality (16) is either same as the mixed  $(k, l, S, I)$  inequality (20) or is dominated by the mixed  $(k, l, S, I)$  inequality (20).*

*Proof.* Observe that the mixed  $(k, l, S, I)$  inequality (20) is actually the  $(k, m_{i_u} - 1, S, I)$  inequality (16). This means that if  $l = m_{i_u} - 1$ , then the  $(k, l, S, I)$  inequality is the mixed  $(k, l, S, I)$  inequality. However, in case  $l \neq m_{i_u} - 1$ , the left hand side of inequality (16) is greater than the left hand side of inequality (20), which implies that the  $(k, l, S, I)$  inequality is dominated by the mixed  $(k, l, S, I)$  inequality. □

**Corollary 1.** *The  $(k, l, S, I)$  inequality (16) is valid for  $X^{SML}$  for all  $1 \leq k \leq l \leq m$  and  $S \subseteq \{k, \dots, l\}$ .*

**Corollary 2.** *The  $(k, l, S, I)$  inequality defines a facet for  $\text{conv}(X^{SML})$  if and only if it is a facet-defining mixed  $(k, l, S, I)$  inequality.*

**Example 1** (continued). *Notice that the mixed  $\{2, 6, \{2, 4, 5\}, \{4, 2\}\}$  inequality dominates the  $\{2, 6, \{2, 4, 5\}, \{4, 2\}\}$  inequality because  $l = 6 \neq m_{i_u} - 1 = m_4 - 1 = 4$ ; whereas for  $(k, l, S) = (2, 6, \{2, 4, 5\})$  and  $I \in \{\{5, 4\}, \{5, 2\}, \{5, 4, 2\}\}$ , the mixed  $(k, l, S, I)$  inequalities are same as the  $(k, l, S, I)$  inequalities as  $l = m_{i_u} - 1 = m_5 - 1 = 6$ . □*

**Lemma 1.** Given a mixed  $(k, l, S, I)$  inequality (20), any feasible point  $(\hat{z}, \hat{x}, \hat{s}) \in X^{SML}$  lies on the corresponding face if there exists an  $r \in \{0, 1, \dots, |I|\}$  such that conditions (i)-(iii) hold:

$$(i) \left\lceil \frac{b_{i_q}}{\alpha_1} \right\rceil - \sum_{p \in S_{i_q}} \hat{z}_p^1 = 1, \text{ for } q = 1, \dots, r$$

$$(ii) \left\lceil \frac{b_{i_q}}{\alpha_1} \right\rceil - \sum_{p \in S_{i_q}} \hat{z}_p^1 = 0, \text{ for } q = r + 1, \dots, |I|$$

$$(iii) \hat{s}_{k-1} + \sum_{p \in \{k, \dots, m_{i_u}-1\} \setminus S} \hat{x}_p = b_{i_r}^{(1)}.$$

*Proof.* Given  $(k, l, S, I)$ , the hyperplane corresponding to (20) can be rewritten as

$$s_{k-1} + \sum_{p \in \{k, \dots, m_{i_u}-1\} \setminus S} x_p = \sum_{q=1}^{|I|} (b_{i_q}^{(1)} - b_{i_{q-1}}^{(1)}) \left( \left\lceil \frac{b_{i_q}}{\alpha_1} \right\rceil - \sum_{p \in S_{i_q}} z_p^1 \right) \quad (22)$$

Let  $\Gamma = \{(z, x, s) \in X^{SML} : (22) \text{ holds}\}$  be the face of  $X^{SML}$  defined by hyperplane (22) and  $(\hat{z}, \hat{x}, \hat{s}) \in X^{SML}$  be a point which satisfies conditions (i) – (iii) for an  $r \in \{0, 1, \dots, |I|\}$ . Now, by substituting  $(\hat{z}, \hat{x}, \hat{s})$  in the right-hand side of (22) and using conditions (i) – (ii), we get

$$\begin{aligned} & \sum_{q=1}^r (b_{i_q}^{(1)} - b_{i_{q-1}}^{(1)}) \left( \left\lceil \frac{b_{i_q}}{\alpha_1} \right\rceil - \sum_{p \in S_{i_q}} \hat{z}_p^1 \right) + \sum_{q=r+1}^{|I|} (b_{i_q}^{(1)} - b_{i_{q-1}}^{(1)}) \left( \left\lceil \frac{b_{i_q}}{\alpha_1} \right\rceil - \sum_{p \in S_{i_q}} \hat{z}_p^1 \right) \\ &= \sum_{q=1}^r (b_{i_q}^{(1)} - b_{i_{q-1}}^{(1)}) = b_{i_r}^{(1)}. \end{aligned}$$

Notice that by substituting  $(\hat{z}, \hat{x}, \hat{s})$  in the left-hand side of equation (22), we also get  $b_{i_r}^{(1)}$  because of condition (iii). This shows that  $(\hat{z}, \hat{x}, \hat{s}) \in \Gamma$  as it satisfies (22).  $\square$

**Theorem 4.** The mixed  $(k, l, S, I)$  inequality (20) defines a facet for  $\text{conv}(X^{SML})$  if the following condition holds for each  $i \in I$ :

$$\{j : j \in S, j < i, \lceil b_j/\alpha_1 \rceil = \lceil b_i/\alpha_1 \rceil\} \subset I. \quad (23)$$

**Remark 3.** Notice that Condition (23) in the above results are not very restrictive. However, Theorem 4 shows that not all selections of  $(k, l, S, I)$  provide facet-defining mixed  $(k, l, S, I)$  inequalities for the SMLS-WB (see an example in Section 5).

*Proof.* Let  $\Gamma = \{(z, x, s) \in X^{SML} : (22) \text{ holds}\}$  be the face of  $X^{SML}$  defined by hyperplane (22) corresponding to the mixed  $(k, l, S, I)$  inequality (20). Assuming that for each  $i \in I$ ,  $\{j : j \in S, j < i, \lceil b_j/\alpha_1 \rceil = \lceil b_i/\alpha_1 \rceil\} \subset I$ , we prove that a generic hyperplane passing through  $\Gamma$ ,

$$\nu_0 s_0 + \sum_{p=1}^m (\nu_p s_p + \mu_p x_p + \lambda_p^z z_p^1) = \eta \quad (24)$$

where  $(\lambda_1^z, \dots, \lambda_m^z, \mu_1, \dots, \mu_m, \nu_0, \nu_1, \dots, \nu_m, \eta) \in \mathbb{R}^{3m+2}$ , must be a scalar multiple of (22).

First, we separately add equalities (14) from period  $i \in \{1, \dots, k-1\}$  to period  $k-1$ , and from period  $k$  to period  $i \in \{k, \dots, m\}$  to get

$$s_{i-1} + \sum_{p=i}^{k-1} x_p = \sum_{p=i}^{k-1} d_p + s_{k-1}, \quad \text{for } i = 1, \dots, k-1 \quad (25)$$

$$s_{k-1} + \sum_{p=k}^i x_p = \sum_{p=k}^i d_p + s_i, \quad \text{for } i = k, \dots, m \quad (26)$$

respectively. Since each point belonging to  $\Gamma$  satisfies (25) and (26), we eliminate variables  $s_i$ ,  $i = 0, \dots, k-2, k, \dots, m$ , from (24) by subtracting  $\nu_{i-1}$  times equality (25) for each  $i \in \{1, \dots, k-1\}$  from (24), and adding  $\nu_i$  times equality (26) for each  $i \in \{k, \dots, m\}$  to (24). This gives

$$\lambda_0 s_{k-1} + \sum_{p=1}^m \lambda_p^x x_p + \sum_{p=1}^m \lambda_p^z z_p^1 = \theta \quad (27)$$

where

$$\begin{aligned} \theta &= \eta + \sum_{i=1}^{k-1} \left( \nu_{i-1} \sum_{p=i}^{k-1} d_p \right) - \sum_{i=k}^m \left( \nu_i \sum_{p=k}^i d_p \right), \\ \lambda_0 &= \nu_{k-1} + \sum_{i=1}^{k-1} \nu_{i-1} - \sum_{i=k}^m \nu_i, \\ \lambda_p^x &= \mu_p - \sum_{i=1}^{k-1} \nu_{i-1} \quad \text{for } p = 1, \dots, k-1, \\ \lambda_p^x &= \mu_p + \sum_{i=k}^m \nu_i \quad \text{for } p = k, \dots, m. \end{aligned}$$

It is important to note that to have a point  $(\hat{z}, \hat{x}, \hat{s}) = (\hat{z}_1^1, \dots, \hat{z}_m^1, \hat{x}_1, \dots, \hat{x}_m, \hat{s}_0, \dots, \hat{s}_m) \in \mathbb{Z}_+^m \times \mathbb{R}_+^m \times \mathbb{R}_+^{m+1} \in X^{SML}$ , it is sufficient to know the value of  $\hat{z}_1^1, \dots, \hat{z}_m^1, \hat{x}_1, \dots, \hat{x}_m$ , and  $\hat{s}_{k-1}$  coordinates because the remaining ones can be obtained using equalities (25) and (26). Therefore, in the rest of the proof, we will define a point belonging to  $X^{SML}$  by  $(\hat{z}, \hat{x}, \hat{s}_{k-1}) = (\hat{z}_1^1, \dots, \hat{z}_m^1, \hat{x}_1, \dots, \hat{x}_m, \hat{s}_{k-1}) \in \mathbb{Z}_+^m \times \mathbb{R}_+^m \times \mathbb{R}_+$  (for the sake of convenience).

Next, assuming  $S := \{w_1, \dots, w_{|S|}\}$  where  $w_1 = k$ , consider the point  $\mathcal{A} = (\hat{z}, \hat{x}, \hat{s}_{k-1}) = (\hat{z}_1^1, \dots, \hat{z}_m^1, \hat{x}_1, \dots, \hat{x}_m, \hat{s}_{k-1}) \in \mathbb{Z}_+^m \times \mathbb{R}_+^m \times \mathbb{R}_+$  such that  $\hat{s}_{k-1} = 0$ ,

$$(\hat{z}_p^1, \hat{x}_p) := \begin{cases} (\lceil d_p / \alpha_1 \rceil, d_p) & \text{if } p \in \{1, \dots, m\} \setminus \{k, \dots, l\}, \\ (0, 0) & \text{if } p \in \{k, \dots, l\} \setminus S, \end{cases}$$

for  $p \in \{1, \dots, m\} \setminus S$  and for  $i \in \{1, \dots, |S|\}$ ,

$$(\hat{z}_{w_i}^1, \hat{x}_{w_i}) := \left( \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil - \left\lceil \frac{b_{w_{i-1}}}{\alpha_1} \right\rceil, \alpha_1 \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil - \alpha_1 \left\lceil \frac{b_{w_{i-1}}}{\alpha_1} \right\rceil \right),$$

where  $b_{w_0} = 0$ . As mentioned before, coordinates  $\hat{s}_p$ ,  $p \in \{0, \dots, m\} \setminus \{k-1\}$  can be obtained using (25)-(26), i.e. for  $p \in \{0, \dots, m\} \setminus \{k-1\}$ ,

$$\hat{s}_p := \begin{cases} 0 & \text{if } p \in \{0, \dots, k-2\} \\ \alpha_1 \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil - \sum_{j=k}^p d_j & \text{if } p \in \{m_{w_{i-1}}, \dots, m_{w_i} - 1\} \text{ and } i = 1, \dots, |S|, \\ \alpha_1 \left\lceil \frac{b_{w_{|S|}}}{\alpha_1} \right\rceil - b_{w_{|S|}} & \text{if } p \geq m_{w_{|S|}}, \end{cases}$$

where  $m_{w_0} = k$ . Recall that  $b_{w_i} = \sum_{j=k}^{m_{w_i}-1} d_j$  for  $w_i \in S$  (by definition). It is easy to verify that  $\mathcal{A} \in X^{SML}$  and satisfies conditions (i) – (iii) of Lemma 1. Therefore  $\mathcal{A} \in \Gamma$  and hence must satisfy (27). Substituting  $\mathcal{A}$  into (27) gives

$$\begin{aligned} & \sum_{p \in \{1, \dots, m\} \setminus \{k, \dots, l\}} \lambda_p^x d_p + \sum_{p \in \{1, \dots, m\} \setminus \{k, \dots, l\}} \lambda_p^z \lceil d_p / \alpha_1 \rceil \\ & + \sum_{i=1}^{|S|} (\lambda_{w_i}^x \alpha_1 + \lambda_{w_i}^z) (\lceil b_{w_i} / \alpha_1 \rceil - \lceil b_{w_{i-1}} / \alpha_1 \rceil) = \theta. \end{aligned} \quad (28)$$

Using (28), hyperplane (27) reduces to

$$\begin{aligned} & \lambda_0 s_{k-1} + \sum_{p \in \{1, \dots, m\} \setminus \{k, \dots, l\}} \lambda_p^x (x_p - d_p) + \sum_{p \in \{1, \dots, m\} \setminus \{k, \dots, l\}} \lambda_p^z (z_p^1 - \lceil d_p / \alpha_1 \rceil) \\ & + \sum_{p \in \{k, \dots, l\} \setminus S} (\lambda_p^x x_p + \lambda_p^z z_p^1) + \sum_{i=1}^{|S|} \lambda_{w_i}^x (x_{w_i} - \alpha_1 (\lceil b_{w_i} / \alpha_1 \rceil - \lceil b_{w_{i-1}} / \alpha_1 \rceil)) \\ & = \sum_{i=1}^{|S|} \lambda_{w_i}^z (\lceil b_{w_i} / \alpha_1 \rceil - \lceil b_{w_{i-1}} / \alpha_1 \rceil - z_{w_i}^1). \end{aligned} \quad (29)$$

Now, let  $\mathcal{S}$  be a set of all disjoint subsets of  $S$  such that for each element (or time period)  $p$  of a subset,  $\lceil b_p / \alpha_1 \rceil$  is same and the elements of each subset are arranged in the increasing order of the associated time period or index  $p$ . We denote a set containing only first element of the disjoint subsets of  $\mathcal{S}$  (arranged in increasing order) by  $\Omega := \{\omega_1, \omega_2, \dots, \omega_{|\Omega|}\}$ . Note that  $\lceil b_{\omega_0} / \alpha_1 \rceil = 0 < \lceil b_{\omega_1} / \alpha_1 \rceil < \lceil b_{\omega_2} / \alpha_1 \rceil < \dots < \lceil b_{\omega_{|\Omega|}} / \alpha_1 \rceil$  and  $\Omega \subseteq S$ . For each  $\omega_r \in \Omega$ , we consider the points  $\mathcal{B}^{\omega_r} = (\hat{z}, \hat{x}, \hat{s}_{k-1}) = (\hat{z}_1^1, \dots, \hat{z}_m^1, \hat{x}_1, \dots, \hat{x}_m, \hat{s}_{k-1}) \in \mathbb{Z}_+^m \times \mathbb{R}_+^m \times \mathbb{R}_+$  such that each coordinate of  $\mathcal{B}^{\omega_r}$  is same as the coordinates of the point  $\mathcal{A}$ , except that

$$(\hat{z}_{\omega_r}^1, \hat{x}_{\omega_r}) = \left( \left\lceil \frac{b_{\omega_r}}{\alpha_1} \right\rceil - \left\lceil \frac{b_{\omega_{r-1}}}{\alpha_1} \right\rceil, \alpha_1 \left\lceil \frac{b_{\omega_r}}{\alpha_1} \right\rceil - \alpha_1 \left\lceil \frac{b_{\omega_{r-1}}}{\alpha_1} \right\rceil + b_{\omega}^{(1)} \right),$$

where  $b_{\omega_0} = 0$  and  $b_{\omega}^{(1)} = \max_{i=1,2,\dots} \{b_{w_i}^{(1)}\}$ . It is easy to verify that  $\mathcal{B}^{\omega_r} \in X^{SML}$  and satisfies conditions (i) – (iii) of Lemma 1. Therefore  $\mathcal{B}^{\omega_r} \in \Gamma$  and hence must satisfy (27). Substituting  $\mathcal{B}^{\omega_r}$  into (29) gives

$$\lambda_{\omega_r}^x = 0 \quad \text{for } \omega_r \in \Omega. \quad (30)$$

Next, consider the points  $\mathcal{C}_1^r$  for  $r \in \{1, \dots, k-1\} \cup \{l+1, \dots, m\}$ ,  $\mathcal{C}_2^r$  for  $r \in \{1, \dots, k-2\} \cup \{l+1, \dots, m-1\}$ , and  $\mathcal{C}_3$  such that each coordinate of  $\mathcal{C}_1^r$ ,  $\mathcal{C}_2^r$ , and  $\mathcal{C}_3$  are same as the coordinates of

the point  $\mathcal{A}$ , except that in  $\mathcal{C}_1^r$ ,  $\hat{z}_r^1 = \lceil d_p/\alpha_1 \rceil + 1$ , in  $\mathcal{C}_2^r$ ,

$$(\hat{z}_p^1, \hat{x}_p) := \begin{cases} \left( \left\lceil \frac{\sum_{i=p}^{k-1} d_i}{\alpha_1} \right\rceil, \sum_{i=p}^{k-1} d_i \right) & \text{if } p \in \{1, \dots, k-1\} \cap \{r\}, \\ \left( \left\lceil \frac{\sum_{i=p}^m d_i}{\alpha_1} \right\rceil, \sum_{i=p}^m d_i \right) & \text{if } p \in \{l+1, \dots, m\} \cap \{r\}, \\ (0, 0) & \text{if } p \in \{1, \dots, k-1, l+1, \dots, m\}, p > r, \end{cases}$$

and in  $\mathcal{C}_3$ ,  $\hat{x}_{k-1} = 0$  and  $\hat{s}_p = d_{k-1}$  for  $p = 0, \dots, k-2$ . It is easy to verify that  $\mathcal{C}_1^r$  for  $r \in \{1, \dots, k-1\} \cup \{l+1, \dots, m\}$ ,  $\mathcal{C}_2^r$  for  $r \in \{1, \dots, k-2\} \cup \{l+1, \dots, m-1\}$ , and  $\mathcal{C}_3$  belong to  $X^{SML}$  and satisfy conditions (i) – (iii) of Lemma 1. Therefore  $\mathcal{C}_1^r, \mathcal{C}_2^r, \mathcal{C}_3 \in \Gamma$  and hence must satisfy (27). Substituting  $\mathcal{C}_1^r$  into (29) gives

$$\lambda_r^z = 0 \quad \text{for all } r \in \{1, \dots, k-1\} \cup \{l+1, \dots, m\}. \quad (31)$$

Likewise, one by one substituting the points  $\mathcal{C}_2^1, \dots, \mathcal{C}_2^{k-2}, \mathcal{C}_2^{l+1}, \dots, \mathcal{C}_2^{m-1}$ , and  $\mathcal{C}_3$  into (29) gives

$$\lambda_r^x = 0 \quad \text{for all } r \in \{1, \dots, k-1\} \cup \{l+1, \dots, m-1\}. \quad (32)$$

By definition we know that  $I := \{i_1, \dots, i_{|I|}\} \subseteq S$  such that  $0 < b_{i_1}^{(1)} \leq b_{i_2}^{(1)} \leq \dots \leq b_{i_{|I|}}^{(1)} \leq \alpha_1$ ,  $i_u = \max\{i : i \in I\}$ , and  $S_i = S \cap \{k, \dots, i\}$ . We consider the points  $\mathcal{D}^r = (\hat{z}, \hat{x}, \hat{s}_{k-1}) = (\hat{z}_1^1, \dots, \hat{z}_m^1, \hat{x}_1, \dots, \hat{x}_m, \hat{s}_{k-1}) \in \mathbb{Z}_+^m \times \mathbb{R}_+^m \times \mathbb{R}_+$  for  $r \in \{k, \dots, l\} \setminus S_{i_u}$  such that  $\hat{s}_{k-1} = 0$ ,

$$(\hat{z}_p^1, \hat{x}_p) := \begin{cases} (\lceil d_p/\alpha_1 \rceil, d_p) & \text{if } p \in \{1, \dots, k-1\} \cup \{l+1, \dots, m\}, \\ (0, 0) & \text{if } p \in \{k, \dots, l\} \setminus S \text{ and } p \neq r, \\ (1, 0) & \text{if } p \in \{k, \dots, l\} \setminus S \text{ and } p = r, \end{cases}$$

for  $p \in \{1, \dots, m\} \setminus S$  and

$$(\hat{z}_{w_i}^1, \hat{x}_{w_i}) := \begin{cases} \left( \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil - \left\lceil \frac{b_{w_{i-1}}}{\alpha_1} \right\rceil, \alpha_1 \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil - \alpha_1 \left\lceil \frac{b_{w_{i-1}}}{\alpha_1} \right\rceil \right) & \text{if } w_i \in S \setminus \{r\}, \\ \left( \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil - \left\lceil \frac{b_{w_{i-1}}}{\alpha_1} \right\rceil + 1, \alpha_1 \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil - \alpha_1 \left\lceil \frac{b_{w_{i-1}}}{\alpha_1} \right\rceil \right) & \text{if } w_i \in S \cap \{r\} \\ & \cap \{m_{i_u}, \dots, l\}, \end{cases}$$

for  $i = 1, \dots, |S|$ . It is easy to verify that  $\mathcal{D}^r \in X^{SML}$  and satisfies conditions (i) – (iii) of Lemma 1. Therefore  $\mathcal{D}^r \in \Gamma$  and hence must satisfy (27). Substituting  $\mathcal{D}^r$  into (29) gives

$$\lambda_r^z = 0 \quad \text{for all } r \in \{k, \dots, l\} \setminus S_{i_u}. \quad (33)$$

Using (30), (31), (33), and (32), (29) reduces to

$$\begin{aligned} & \lambda_0 s_{k-1} + \sum_{p \in \{k, \dots, l, m\} \setminus S} \lambda_p^x x_p + \sum_{w_i \in S \setminus \Omega} \lambda_{w_i}^x \left( x_{w_i} - \alpha_1 \left( \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil - \left\lceil \frac{b_{w_{i-1}}}{\alpha_1} \right\rceil \right) \right) \\ & = \sum_{w_i \in S_{i_u}} \lambda_{w_i}^z \left( \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil - \left\lceil \frac{b_{w_{i-1}}}{\alpha_1} \right\rceil - z_{w_i}^1 \right). \end{aligned} \quad (34)$$

Now, consider the points  $\mathcal{E}^r = (\hat{z}, \hat{x}, \hat{s}_{k-1}) = (\hat{z}_1^1, \dots, \hat{z}_m^1, \hat{x}_1, \dots, \hat{x}_m, \hat{s}_{k-1}) \in \mathbb{Z}_+^m \times \mathbb{R}_+^m \times \mathbb{R}_+$  for  $r \in \{m_{i_u}, \dots, l, m\}$  where  $m_{i_u} \leq l \leq m$  such that  $\hat{s}_{k-1} = 0$ ,

$$(\hat{z}_p^1, \hat{x}_p) := \begin{cases} (\lceil d_p/\alpha_1 \rceil, d_p) & \text{if } p \in \{1, \dots, k-1\} \cup \{l+1, \dots, m\} \setminus \{r\}, \\ (\lceil d_p/\alpha_1 \rceil + 1, d_p + \alpha_1) & \text{if } p \in \{r\} \cap \{m\} \text{ and } m \neq l, \\ (0, 0) & \text{if } p \in \{k, \dots, l\} \setminus (S \cup \{r\}), \\ (1, 1) & \text{if } p \in \{r\} \cap (\{m_{i_u}, \dots, l\} \setminus S), \end{cases}$$

for  $p \in \{1, \dots, m\} \setminus S$  and

$$(\hat{z}_{w_i}^1, \hat{x}_{w_i}) := \begin{cases} \left( \left( \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil - \left\lceil \frac{b_{w_{i-1}}}{\alpha_1} \right\rceil, \alpha_1 z_{w_i}^1 \right), & \text{if } w_i \in S \setminus \{r\} \\ \left( \left( \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil - \left\lceil \frac{b_{w_{i-1}}}{\alpha_1} \right\rceil + 1, \alpha_1 z_{w_i}^1 \right), & \text{if } w_i \in \{r\} \cap \{m_{i_u}, \dots, l\} \cap S. \end{cases}$$

for  $i \in \{1, \dots, |S|\}$ . It is easy to verify that  $\mathcal{E}^r \in X^{SM}$  and satisfies conditions (i) – (iii) of Lemma 1. Therefore  $\mathcal{E}^r \in \Gamma$  and hence must satisfy (27). Substituting  $\mathcal{E}^r$  into (29) gives

$$\lambda_r^x = 0 \quad \text{for all } r \in \{m_{i_u}, \dots, l\} \cup \{m\}. \quad (35)$$

Using (35), (34) reduces to

$$\begin{aligned} & \lambda_0 s_{k-1} + \sum_{p \in \{k, \dots, m_{i_u}-1\} \setminus S} \lambda_p^x x_p + \sum_{w_i \in S_{i_u} \setminus \Omega} \lambda_{w_i}^x \left( x_{w_i} - \alpha_1 \left( \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil - \left\lceil \frac{b_{w_{i-1}}}{\alpha_1} \right\rceil \right) \right) \\ &= \sum_{w_i \in S_{i_u}} \lambda_{w_i}^z \left( \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil - \left\lceil \frac{b_{w_{i-1}}}{\alpha_1} \right\rceil - z_{w_i}^1 \right). \end{aligned} \quad (36)$$

Let  $I_a := \{i_{a_1}, \dots, i_{a_{|I|}}\}$  be a set which has same elements as in the set  $I$ , except that  $i_{a_1} < i_{a_2} < \dots < i_{a_{|I|}}$ . In other words, set  $I_a$  is same as the set  $I$ , the only difference is that the elements of  $I_a$  are arranged in the increasing order of time periods. Now, for each  $i_{a_\zeta} \in I$  where  $\zeta \in \{1, \dots, |I|\}$  and  $r \in S_{i_{a_\zeta}} \setminus (S_{i_{a_{\zeta-1}}} \cup \{i_{a_\zeta}\})$  where  $S_{i_{a_0}} = \emptyset$ , consider the points  $\mathcal{F}^{r, a_\zeta} = (\hat{z}, \hat{x}, \hat{s}_{k-1}) = (\hat{z}_1^1, \dots, \hat{z}_m^1, \hat{x}_1, \dots, \hat{x}_m, \hat{s}_{k-1}) \in \mathbb{Z}_+^m \times \mathbb{R}_+^m \times \mathbb{R}_+$  such that  $\hat{s}_{k-1} = 0$ ,

$$(\hat{z}_p^1, \hat{x}_p) := \begin{cases} (\lceil d_p/\alpha_1 \rceil, d_p) & \text{if } p \in \{1, \dots, k-1\} \cup \{l+1, \dots, m\}, \\ (0, 0) & \text{if } p \in \{k, \dots, l\} \setminus S, \end{cases}$$

for  $p \in \{1, \dots, m\} \setminus S$  and

$$(\hat{z}_{w_i}^1, \hat{x}_{w_i}) := \begin{cases} \left( \left( \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil - \left\lceil \frac{b_{w_{i-1}}}{\alpha_1} \right\rceil, \alpha_1 z_{w_i}^1 \right) & \text{if } w_i \in S \setminus \{r, i_{a_\zeta}\}, \\ \left( \left( \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil - \left\lceil \frac{b_{w_{i-1}}}{\alpha_1} \right\rceil + 1, \alpha_1 z_{w_i}^1 \right) & \text{if } w_i = r, \\ \left( \left( \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil - \left\lceil \frac{b_{w_{i-1}}}{\alpha_1} \right\rceil - 1, \alpha_1 z_{w_i}^1 \right) & \text{if } w_i = i_{a_\zeta}, \end{cases}$$

for  $i = 1, \dots, |S|$ . Note that  $\sum_{p \in S_{i_{a_\zeta}}} \hat{z}_p^1 = \lceil b_{i_{a_\zeta}}/\alpha_1 \rceil$  for all  $i_{a_\zeta} \in I$ , and also in case  $\lceil b_{w_i}/\alpha_1 \rceil = \lceil b_{w_{i-1}}/\alpha_1 \rceil$  for  $w_i = i_{a_\zeta} \in I$  and  $w_{i-1} \in S$  then  $w_{i-1} = i_{a_{\zeta-1}}$  where  $i_{a_{\zeta-1}} \in I$  (because of Condition

(23)), and hence  $S_{i_{a_\zeta}} \setminus (S_{i_{a_{\zeta-1}}} \cup \{i_{a_\zeta}\}) = \emptyset$ . It is easy to verify that  $\mathcal{F}^{r, a_\zeta} \in X^{SML}$  under assumptions (23) and satisfies conditions (i)–(iii) of Lemma 1. Therefore  $\mathcal{F}^{r, a_\zeta} \in \Gamma$  and hence must satisfy (27). Next, for  $i_{a_\zeta} \in I$  where  $\zeta \in \{1, \dots, |I|\}$  and  $r \in \left( S_{i_{a_\zeta}} \setminus (S_{i_{a_{\zeta-1}}} \cup \{i_{a_\zeta}\}) \right) \cap (S_{i_u} \setminus \Omega)$  where  $u = a_{|I|}$ , we define another point  $\mathcal{F}_1^{r, a_\zeta} = (\hat{z}, \hat{x}, \hat{s}_{k-1}) = (\hat{z}_1^1, \dots, \hat{z}_m^1, \hat{x}_1, \dots, \hat{x}_m, \hat{s}_{k-1}) \in \mathbb{Z}_+^m \times \mathbb{R}_+^m \times \mathbb{R}_+$  whose coordinates are all exactly same as  $\mathcal{F}^{r, a_\zeta}$  except that for  $w_i = r$ ,

$$\hat{x}_{w_i} = \alpha_1 \left( \lceil b_{w_i} / \alpha_1 \rceil - \lceil b_{w_{i-1}} / \alpha_1 \rceil \right) + b_w^{(1)},$$

where  $b_w^{(1)} = \max_{w_i \in S} \{b_{w_i}^{(1)}\}$ . Again, it is easy to verify that  $\mathcal{F}_1^{r, a_\zeta} \in X^{SML}$  and satisfies conditions (i)–(iii) of Lemma 1. Therefore  $\mathcal{F}_1^{r, a_\zeta} \in \Gamma$  and hence must satisfy (27). For each  $i_{a_\zeta} \in I$  and  $r \in \left( S_{i_{a_\zeta}} \setminus (S_{i_{a_{\zeta-1}}} \cup \{i_{a_\zeta}\}) \right) \cap (S_{i_u} \setminus \Omega)$ , we substitute  $\mathcal{F}^{r, a_\zeta}$  and  $\mathcal{F}_1^{r, a_\zeta}$  into (36), and subtract one equality from the other. This gives

$$\lambda_r^x = 0 \quad \text{for } r \in S_{i_u} \setminus (I \cup \Omega). \quad (37)$$

Again, by substituting the points  $\mathcal{F}^{r, a_\zeta}$ , for  $i_{a_\zeta} \in I$  where  $\zeta \in \{1, \dots, |I|\}$  and  $r \in \left( S_{i_{a_\zeta}} \setminus (S_{i_{a_{\zeta-1}}} \cup \{i_{a_\zeta}\}) \right)$ , into (36) and using (37), we get

$$\lambda_r^z = \lambda_{i_{a_\zeta}}^z \quad \text{for } i_{a_\zeta} \in I, r \in S_{i_{a_\zeta}} \setminus (S_{i_{a_{\zeta-1}}} \cup \{i_{a_\zeta}\}). \quad (38)$$

Substituting (37) and (38) into (36) gives

$$\begin{aligned} & \lambda_0 s_{k-1} + \sum_{p \in \{k, \dots, m_{i_u}-1\} \setminus S} \lambda_p^x x_p + \sum_{w_i \in I \setminus \Omega} \lambda_{w_i}^x \left( x_{w_i} - \alpha_1 \left( \lceil \frac{b_{w_i}}{\alpha_1} \rceil - \lceil \frac{b_{w_{i-1}}}{\alpha_1} \rceil \right) \right) \\ &= \sum_{\zeta=1}^{|I|} \lambda_{i_{a_\zeta}}^z \left( \lceil \frac{b_{i_{a_\zeta}}}{\alpha_1} \rceil - \sum_{p \in S_{i_{a_\zeta}}} z_p^1 - \lceil \frac{b_{i_{a_{\zeta-1}}}}{\alpha_1} \rceil + \sum_{p \in S_{i_{a_{\zeta-1}}}} z_p^1 \right), \end{aligned} \quad (39)$$

where  $b_{i_{a_0}} = 0$  and  $S_{i_{a_0}} = \emptyset$ . By rearranging the right-hand side of (39), we get

$$\begin{aligned} & \lambda_0 s_{k-1} + \sum_{p \in \{k, \dots, m_{i_u}-1\} \setminus S} \lambda_p^x x_p + \sum_{w_i \in I \setminus \Omega} \lambda_{w_i}^x \left( x_{w_i} - \alpha_1 \left( \lceil \frac{b_{w_i}}{\alpha_1} \rceil - \lceil \frac{b_{w_{i-1}}}{\alpha_1} \rceil \right) \right) \\ &= \sum_{\zeta=1}^{|I|-1} \left( \lambda_{i_{a_\zeta}}^z - \lambda_{i_{a_{\zeta+1}}}^z \right) \left( \lceil \frac{b_{i_{a_\zeta}}}{\alpha_1} \rceil - \sum_{p \in S_{i_{a_\zeta}}} z_p^1 \right) + \lambda_{i_{a_{|I|}}}^z \left( \lceil \frac{b_{i_{a_{|I|}}}}{\alpha_1} \rceil - \sum_{p \in S_{i_{a_{|I|}}}} z_p^1 \right). \end{aligned}$$

Furthermore, the last equation can also be simplified to

$$\begin{aligned} & \lambda_0 s_{k-1} + \sum_{p \in \{k, \dots, m_{i_u}-1\} \setminus S} \lambda_p^x x_p + \sum_{w_i \in I \setminus \Omega} \lambda_{w_i}^x \left( x_{w_i} - \alpha_1 \left( \lceil \frac{b_{w_i}}{\alpha_1} \rceil - \lceil \frac{b_{w_{i-1}}}{\alpha_1} \rceil \right) \right) \\ &= \sum_{q=1}^{|I|} \gamma_{i_q}^z \left( \lceil \frac{b_{i_q}}{\alpha_1} \rceil - \sum_{p \in S_{i_q}} z_p^1 \right), \end{aligned} \quad (40)$$

where for  $q = 1, \dots, |I|$ ,  $\gamma_{i_q}^z = \left( \lambda_{i_q}^z - \lambda_{i_{a_{\zeta+1}}}^z \right)$  such that  $a_\zeta = q$  for  $\zeta \in \{1, \dots, |I|\}$  and  $\lambda_{a_{|I|+1}}^z = 0$ .



Next, for  $i_q \in I$ , consider the points  $\mathcal{G}^q = (\hat{z}, \hat{x}, \hat{s}_{k-1}) \in \mathbb{Z}_+^m \times \mathbb{R}_+^m \times \mathbb{R}_+$ , such that  $\hat{s}_{k-1} = b_{i_q}^{(1)}$ ,

$$(\hat{z}_p^1, \hat{x}_p) := \begin{cases} (\lceil d_p/\alpha_1 \rceil, d_p) & \text{if } p \in \{1, \dots, k-1\} \cup \{l+1, \dots, m\}, \\ (0, 0) & \text{if } p \in \{k, \dots, l\} \setminus S, \end{cases}$$

for  $p \in \{1, \dots, m\} \setminus S$ , and for  $i \in \{1, \dots, |S|\}$ ,  $(\hat{z}_{w_i}^1, \hat{x}_{w_i}) := (\chi_{w_i} - \chi_{w_{i-1}}, \alpha_1 \hat{z}_{w_i}^1)$  where  $\chi_{w_0} = 0$  and

$$\chi_{w_i} = \begin{cases} \lceil b_{w_i}/\alpha_1 \rceil & \text{if } w_i \in S \setminus I, \\ \lfloor b_{w_i}/\alpha_1 \rfloor & \text{if } w_i \in \{i_1, \dots, i_q\}, \\ \lceil b_{w_i}/\alpha_1 \rceil & \text{if } w_i \in \{i_{q+1}, \dots, i_{|I|}\}. \end{cases}$$

This implies that

$$\left( \sum_{p \in S_{w_i}} \hat{z}_p^1, \sum_{p \in S_{w_i}} \hat{x}_p \right) = \begin{cases} \left( \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil, \alpha_1 \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil \right) & \text{if } w_i \in S \setminus I, \\ \left( \left\lfloor \frac{b_{w_i}}{\alpha_1} \right\rfloor, \alpha_1 \left\lfloor \frac{b_{w_i}}{\alpha_1} \right\rfloor \right) & \text{if } w_i \in \{i_1, \dots, i_q\}, \\ \left( \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil, \alpha_1 \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil \right) & \text{if } w_i \in \{i_{q+1}, \dots, i_{|I|}\}. \end{cases}$$

Since we assume that  $\{j : j \in S, j < i_q, \lceil b_j/\alpha_1 \rceil = \lfloor b_{i_q}/\alpha_1 \rfloor\} \subset I$  for each  $i_q \in I$  (Condition (23)),

$$\chi_{w_i} \geq \chi_{w_{i-1}} \text{ for } i = 1, \dots, |S|.$$

Now, it is easy to verify that  $\mathcal{G}^q \in X^{SML}$  and satisfies conditions (i) – (iii) of Lemma 1. Therefore  $\mathcal{G}^q \in \Gamma$  and hence must satisfy (27). Furthermore, because of our assumption (23), for  $w_r \in I \setminus \Omega$ ,

$$\left\lfloor \frac{b_{w_r}}{\alpha_1} \right\rfloor = \left\lfloor \frac{b_{w_{r-1}}}{\alpha_1} \right\rfloor, w_{r-1} \in I, \text{ and } b_{w_{r-1}}^{(1)} < b_{w_r}^{(1)}. \quad (41)$$

Let  $\Xi := \{\xi_1, \dots, \xi_{|\Xi|}\} = I \setminus \Omega := \{i_{j_1}, \dots, i_{j_{|\Xi|}}\}$  such that  $\xi_1 = i_{j_1} < \xi_2 = i_{j_2} < \dots < \xi_{|\Xi|} = i_{j_{|\Xi|}}$ . In other words,  $j_r$  provides the positioning of  $\xi_r$  in the set  $I$ . For  $i_{j_r} \in I \setminus \Omega$ , we define another point  $\mathcal{G}_1^r = (\hat{z}, \hat{x}, \hat{s}_{k-1}) \in \mathbb{Z}_+^m \times \mathbb{R}_+^m \times \mathbb{R}_+$ , such that  $\hat{s}_{k-1} = b_{i_{j_v-1}}^{(1)}$  where  $i_{j_v} = \min\{i_{j_{|\Xi|}}, i_{j_{|\Xi|-1}}, \dots, i_{j_r}\}$ ,

$$(\hat{z}_p^1, \hat{x}_p) := \begin{cases} (\lceil d_p/\alpha_1 \rceil, d_p) & \text{if } p \in \{1, \dots, k-1\} \cup \{l+1, \dots, m\}, \\ (0, 0) & \text{if } p \in \{k, \dots, l\} \setminus S, \end{cases}$$

for  $p \in \{1, \dots, m\} \setminus S$ , and for  $i \in \{1, \dots, |S|\}$ ,

$$(\hat{z}_{w_i}^1, \hat{x}_{w_i}) := \begin{cases} (\bar{\chi}_{w_i}^0 - \bar{\chi}_{w_{i-1}}^0, \alpha_1 \hat{z}_{w_i}^1) & \text{if } w_i \neq i_{j_r}, \\ (\bar{\chi}_{w_i}^0 - \bar{\chi}_{w_{i-1}}^0, \max_{w \in S} \{b_w^{(1)}\}) & \text{if } w_i = i_{j_r}, \end{cases}$$

where  $\bar{\chi}_{w_0}^0 = 0$  and

$$\bar{\chi}_{w_i}^0 = \begin{cases} \lceil b_{w_i}/\alpha_1 \rceil & \text{if } w_i \in S \setminus I, \\ \lfloor b_{w_i}/\alpha_1 \rfloor & \text{if } w_i \in \{i_1, \dots, i_{j_v-1}\}, \\ \lceil b_{w_i}/\alpha_1 \rceil & \text{if } w_i \in \{i_{j_v}, \dots, i_{|I|}\}. \end{cases}$$

Notice that for  $i = 1, \dots, |S|$ ,

$$\left( \sum_{p \in S_{w_i}} \hat{z}_p^1, \sum_{p \in S_{w_i}} \hat{x}_p \right) = \begin{cases} \left( \left[ \frac{b_{w_i}}{\alpha_1} \right], \alpha_1 \left[ \frac{b_{w_i}}{\alpha_1} \right] \right) & \text{if } w_i \in S \setminus I, \\ \left( \left[ \frac{b_{w_i}}{\alpha_1} \right], \alpha_1 \left[ \frac{b_{w_i}}{\alpha_1} \right] \right) & \text{if } w_i \in \{i_1, \dots, i_{j_v-1}\}, \\ \left( \left[ \frac{b_{w_i}}{\alpha_1} \right], \alpha_1 \left[ \frac{b_{w_i}}{\alpha_1} \right] \right) & \text{if } w_i \in \{i_{j_v}, \dots, i_{|I|}\} \setminus \{i_{j_r}\}, \\ \left( \left[ \frac{b_{w_i}}{\alpha_1} \right], \alpha_1 \left[ \frac{b_{w_i}}{\alpha_1} \right] + \max_{w \in S} \{b_w^{(1)}\} \right) & \text{if } w_i = i_{j_r}. \end{cases}$$

It is easy to verify that  $\mathcal{G}_1^r \in X^{SML}$  under conditions (23) and satisfies conditions (i) – (iii) of Lemma 1. Therefore  $\mathcal{G}_1^r \in \Gamma$  and hence must satisfy (27). For each  $i_{j_r} \in I \setminus \Omega$ , we substitute  $\mathcal{G}^{j_v-1}, \mathcal{G}_1^r$  into (40) and subtract one equality from the other. This gives

$$\lambda_{i_{j_r}}^x = 0 \quad \text{for } i_{j_r} \in I \setminus \Omega. \quad (42)$$

Using (42), we one by one substitute  $\mathcal{G}^1, \mathcal{G}^2, \dots, \mathcal{G}^{|I|}$  into (40) and get

$$\gamma_{i_1}^z = \lambda_0 b_{i_1}^{(1)}, \quad (43)$$

$$\gamma_{i_2}^z = \lambda_0 b_{i_2}^{(1)} - \gamma_{i_1}^z = \lambda_0 (b_{i_2}^{(1)} - b_{i_1}^{(1)}), \quad (44)$$

⋮

$$\gamma_{i_{|I|}}^z = \lambda_0 b_{i_{|I|}}^{(1)} - (\gamma_{i_1}^z + \dots + \gamma_{i_{|I|-1}}^z) = \lambda_0 (b_{i_{|I|}}^{(1)} - b_{i_{|I|-1}}^{(1)}). \quad (45)$$

This implies that for  $q \in \{1, \dots, |I|\}$ ,

$$\gamma_{i_q}^z = \lambda_0 (b_{i_q}^{(1)} - b_{i_{q-1}}^{(1)}) \quad (46)$$

where  $b_{i_0}^{(1)} = 0$ . For  $r \in \{k, \dots, m_{i_u} - 1\} \setminus S$ , consider the points  $\mathcal{H}^r$  whose coordinates are same as the coordinates of  $\mathcal{G}^u$  except that  $\hat{s}_{k-1} = 0$  and  $(\hat{z}_r^1, \hat{x}_r) = (1, b_{i_u}^{(1)})$ . By substituting the points  $\mathcal{H}^r$  into the (40) and subtracting from equality corresponding to  $\mathcal{G}^u$ , we get

$$\lambda_r^x = \lambda_0 \text{ for } r \in \{k, \dots, m_{i_u} - 1\} \setminus S. \quad (47)$$

Substituting (42), (46), and (47) into the (40) gives

$$\lambda_0 s_{k-1} + \sum_{p \in \{k, \dots, m_{i_u} - 1\} \setminus S} \lambda_0 x_p = \sum_{q=1}^{|I|} \lambda_0 (b_{i_q}^{(1)} - b_{i_{q-1}}^{(1)}) \left( \left[ \frac{b_{i_q}}{\alpha_1} \right] - \sum_{p \in S_{i_q}} z_p^1 \right) \quad (48)$$

The identity (48) is  $\lambda_0 = \mu_{k-1} + \sum_{i=k-1}^m \eta_i$  times (22). Hence  $\Gamma$  defines a facet for  $\text{conv}(X^{SML})$  if conditions (23) hold. This completes the proof.  $\square$

**Example 1** (continued). According to Theorem 4, for  $(k, l, S) = (2, 6, \{2, 4, 5\})$  and  $I \in \{\{5, 4\}, \{5, 2\}, \{4, 2\}, \{5, 4, 2\}\}$ , the mixed  $(k, l, S, I)$  inequalities are facet-defining for  $X^E$  because in these inequalities for each  $i \in I$ ,  $\{j : j \in S, j < i, \lceil b_j/\alpha_1 \rceil = \lceil b_i/\alpha_1 \rceil\} = \emptyset$ .

**Theorem 5.** *Assuming that  $\sum_{p \in S_{i_q}} z_p^1 \geq \lfloor b_{i_q}/\alpha_1 \rfloor$  for each  $i_q \in I$ , the paired  $(k, l, S, I)$  inequality is either dominated by the mixed  $(k, l, S, I)$  inequality (20) or is same as the mixed  $(k, l, S, I)$  inequality (20).*

*Proof.* First we repeatedly apply sequential pairing procedure on inequalities (21) for  $i \in I$ , in increasing order of  $b_i^{(1)}$ , to get  $((\dots((i_1 \circ i_2) \circ i_3) \circ \dots) \circ i_{|I|})$  sequentially paired inequality. The inequality thus obtained, referred to as the  $((\dots((i_1 \circ i_2) \circ i_3) \circ \dots) \circ i_{|I|})$  sequentially paired  $(k, l, S, I)$  inequality, is same as the mixed  $(k, l, S, I)$  inequality (20). Moreover, the set of valid inequalities (21) for  $i \in I$  belong to the nested case. Therefore, according to Theorem 2, the mixed  $(k, l, S, I)$  inequality or the  $((\dots((i_1 \circ i_2) \circ i_3) \circ \dots) \circ i_{|K|})$  sequentially paired  $(k, l, S, I)$  inequality is at least as strong as any other sequentially paired inequality obtained by arbitrary sequence of pairing of the inequalities (21) for  $i \in I$ .  $\square$

**Corollary 3.** *Assuming that  $\sum_{p \in S_{i_q}} z_p^1 \geq \lfloor b_{i_q}/\alpha_1 \rfloor$  for each  $i_q \in I$ , the paired  $(k, l, S, I)$  inequality defines a facet for  $\text{conv}(X^{SML})$  if and only if it is a facet-defining mixed  $(k, l, S, I)$  inequality.*

## 4. Multi-Module Capacitated Lot-Sizing Problem without Backlogging

In this section, we redefine the multi-module capacitated lot-sizing (MMLS) problem (introduced in [23]). Then we present valid inequalities for this problem and investigate their facet-defining properties. The MMLS without backlogging (MMLS-WB) is defined as follows: Let  $\{\alpha_1, \dots, \alpha_n\}$  be the set of sizes of the  $n$  available capacity modules and the setup cost per module of size  $\alpha_t, t = 1, \dots, n$  in period  $p$  is denoted by  $f_p^t$ . We formulate MMLS-WB as:

$$\min \left\{ \sum_{p \in P} \left( c_p x_p + h_p s_p + \sum_{t=1}^n f_p^t z_p^t \right) : (z, x, s) \in X^{MML} \right\}$$

where

$$X^{MML} := \{(z, x, s) \in \mathbb{Z}_+^{m \times n} \times \mathbb{R}_+^m \times \mathbb{R}_+^{m+1}\} \quad (49)$$

$$s_{p-1} + x_p = d_p + s_p, \quad p \in P \quad (50)$$

$$x_p \leq \sum_{t=1}^n \alpha_t z_p^t, \quad p \in P = \{1, \dots, m\}. \quad (51)$$

Here,  $z_p^t$  is the number of capacity modules of size  $\alpha_t, t = 1, \dots, n$ , used in period  $p$  and parameters  $(d_p, c_p, h_p)$  and variables  $(x_p, s_p)$  are same as defined for SMLS-WB. Notice that for  $n = 1$ , MMLS-WB reduces to SMLS-WB.

### 4.1 Valid Inequalities for MMLS-WB Problem

In order to generate valid inequalities for MMLS-WB problem using  $n$ -mixing procedure [23] and the pairing procedure [12], we use notations defined in Section 3.1 except that  $I := \{i_1, i_2, \dots, i_{|I|}\} \subseteq S$

such that  $b_{i_0}^{(n)} = 0 < b_{i_1}^{(n)} \leq b_{i_2}^{(n)} \leq \dots \leq b_{i_{|I|}}^{(n)} < \alpha_n$ . Also, we assume that the  $n$ -step MIR conditions hold, i.e. for each  $i \in I$ ,

$$\alpha_t \left\lceil b_i^{(t-1)} / \alpha_t \right\rceil \leq \alpha_{t-1}, \text{ for } t = 2, \dots, n, \quad (52)$$

which are automatically satisfied when the sizes of the capacity modules are divisible, i.e.  $\alpha_t / \alpha_{t-1} \in \mathbb{Z}_+$ ,  $t = 2, \dots, n$ . First, we aggregate equalities (50) from period  $k$  to period  $m_i - 1$ ,  $i \in S$ , and then relax  $x_p$ ,  $p \in S_i$ , to its upper bound based on (51) and  $s_{m_i} - 1$  to its lower bound based on (49). This gives

$$s_{k-1} + \sum_{p \in \{k, \dots, m_i - 1\} \setminus S_i} x_p + \sum_{t=1}^n \alpha_t \sum_{p \in S_i} z_p^t \geq b_p. \quad (53)$$

**Inequalities derived using the  $n$ -mixing procedure.** Assuming that  $i_u = \max\{i : i \in I\}$ , we get the following valid inequality for  $X^{MML}$ :

$$s_{k-1} + \sum_{p \in \{k, \dots, m_{i_u} - 1\} \setminus S_i} x_p + \sum_{t=1}^n \alpha_t \sum_{p \in S_i} z_p^t \geq b_p. \quad (54)$$

Setting  $v := s_{k-1} + \sum_{p \in \{k, \dots, m_{i_u} - 1\} \setminus S} x_p$  and  $y_t^i := \sum_{p \in S_i} z_p^t$ ,  $t = 1, \dots, n$ , inequality (54) becomes  $v + \sum_{t=1}^n \alpha_t y_t^i \geq b_i$ ,  $i \in I$ , which is of the same form as the defining inequalities of  $n$ -mixing set  $Q_0^{m,n}$ . Notice that  $v \in \mathbb{R}_+$ ,  $y_t^i \in \mathbb{Z}_+$ ,  $t = 1, \dots, n$ . Therefore, the type I mixed  $n$ -step MIR inequality (5) written for  $X^{MML}$  with  $K = I$ ,

$$s_{k-1} + \sum_{p \in \{k, \dots, m_{i_u} - 1\} \setminus S} x_p \geq \sum_{q=1}^{|I|} \left( b_{i_q}^{(n)} - b_{i_{q-1}}^{(n)} \right) \phi_{i_q}^n(z) \quad (55)$$

where

$$\phi_{i_q}^n(z) := \prod_{\sigma=1}^n \left\lceil \frac{b_{i_q}^{(\sigma-1)}}{\alpha_\sigma} \right\rceil - \sum_{p \in S_{i_q}} \sum_{t=1}^n \left( \prod_{\sigma=t+1}^n \left\lceil \frac{b_{i_q}^{(\sigma-1)}}{\alpha_\sigma} \right\rceil \right) z_p^t, \quad (56)$$

is valid for  $X^{MML}$  if the  $n$ -step MIR conditions (52) hold. In the rest of the paper, we will refer to the inequality (55) as the  $n$ -mixed  $(k, l, S, I)$  inequality.

**Inequalities derived using the pairing procedure.** Similar to SMLS-WB, first, we apply the  $n$ -step MIR procedure [16] on inequality (53) for all  $i \in S$ , which gives the following valid inequalities if the  $n$ -step MIR conditions hold:

$$s_{k-1} + \sum_{p \in \{k, \dots, m_i - 1\} \setminus S} x_p \geq b_i^{(n)} \phi_i^n(z) \quad (57)$$

Assuming that  $\phi_i^n(z) \leq 1$  for all  $i \in K$ , we repeatedly apply sequential pairing procedure on inequalities (57) for  $i \in K$ , in increasing order of  $i$ , to get  $((\dots((i_1 \circ i_2) \circ i_3) \circ \dots) \circ i_{|K|})$  sequentially paired inequality, which is same as inequality (55).

## 4.2 Facets for MMLS-WB Problem

We provide conditions under which the  $n$ -mixed  $(k, l, S, I)$  inequality (55) is facet-defining for the  $\text{conv}(X^{MML})$ .

**Lemma 2.** *Given the  $n$ -mixed  $(k, l, S, I)$  inequality (55), any feasible point  $(\hat{z}, \hat{x}, \hat{s}) \in X^{MML}$  lies on the corresponding face if there exists an  $r \in \{0, 1, \dots, |I|\}$  such that conditions (i)-(iii) hold:*

- (i)  $\phi_{i_q}^n(\hat{z}) = 1$ , for  $q = 1, \dots, r$
- (ii)  $\phi_{i_q}^n(\hat{z}) = 0$ , for  $q = r + 1, \dots, |I|$
- (iii)  $\hat{s}_{k-1} + \sum_{p \in \{k, \dots, m_{iu}-1\} \setminus S} \hat{x}_p = b_{i_r}^{(n)}$

*Proof.* Given  $(k, l, S, I)$ , the hyperplane corresponding to (55) can be rewritten as

$$s_{k-1} + \sum_{p \in \{k, \dots, m_{iu}-1\} \setminus S} x_p = \sum_{q=1}^{|I|} \left( b_{i_q}^{(n)} - b_{i_{q-1}}^{(n)} \right) \phi_{i_q}^n(z) \quad (58)$$

Let  $\Gamma = \{(z, x, s) \in X^{MML} : (58) \text{ holds}\}$  be the face of  $X^{MML}$  defined by hyperplane (58), and  $(\hat{z}, \hat{x}, \hat{s}) \in X^{MML}$  be a point which satisfies conditions (i) – (iii) hold for an  $r \in \{0, 1, \dots, |I|\}$ . Now, by substituting  $(\hat{z}, \hat{x}, \hat{s})$  in the right-hand side of (58) and using conditions (i)-(ii), we get

$$\sum_{q=1}^r \left( b_{i_q}^{(n)} - b_{i_{q-1}}^{(n)} \right) \phi_{i_q}^n(\hat{z}) + \sum_{q=r+1}^{|I|} \left( b_{i_q}^{(n)} - b_{i_{q-1}}^{(n)} \right) \phi_{i_q}^n(\hat{z}) = \sum_{q=1}^r \left( b_{i_q}^{(n)} - b_{i_{q-1}}^{(n)} \right) = b_{i_r}^{(n)}.$$

Notice that by substituting  $(\hat{z}, \hat{x}, \hat{s})$  in the left-hand side of equation (58), we also get  $b_{i_r}^{(n)}$  because of condition (iii). This shows that  $(\hat{z}, \hat{x}, \hat{s}) \in \Gamma$  as it satisfies (58).  $\square$

**Theorem 6.** *For  $n \geq 2$ , assuming that the  $n$ -step MIR conditions (52) hold, the  $n$ -mixed  $(k, l, S, I)$  inequality (55) defines a facet for the convex hull of  $X^{MML}$  if the following conditions hold for each  $w_r \in S$ :*

$$\left\lceil \frac{b_{w_r}^{(t-1)}}{\alpha_t} \right\rceil > \left\lceil \frac{b_{w_{r-1}}^{(t-1)}}{\alpha_t} \right\rceil, \quad t = 1, \dots, n. \quad (59)$$

**Remark 4.** *Since inequality (55) is same as the mixed  $(k, l, S, I)$  inequality for  $n = 1$ , in this theorem we only investigate the facet-defining properties of the  $n$ -mixed  $(k, l, S, I)$  inequality (55) for  $n \geq 2$ . Conditions (59) show that not all selections of  $(k, l, S, I)$  provide facet-defining  $n$ -mixed  $(k, l, S, I)$  inequalities for the MMLS-WB.*

*Proof.* Let  $\Gamma = \{(z, x, s) \in X^{MML} : (58) \text{ holds}\}$  be the face of  $X^{MML}$  defined by hyperplane (58) corresponding to the  $n$ -mixed  $(k, l, S, I)$  inequality (55). Assuming that Conditions (59) hold, we prove that a generic hyperplane passing through  $\Gamma$ ,

$$\nu_0 s_0 + \sum_{p=1}^m \left( \nu_p s_p + \mu_p x_p + \sum_{t=1}^n \lambda_p^t z_p^t \right) = \eta \quad (60)$$

where  $(\lambda_1^1, \dots, \lambda_m^1, \dots, \lambda_1^n, \dots, \lambda_m^n, \mu_1, \dots, \mu_m, \nu_0, \nu_1, \dots, \nu_m, \eta) \in \mathbb{R}^{mn+2m+2}$ , must be a scalar multiple of (58).

First, we separately add equalities (50) from period  $i \in \{1, \dots, k-1\}$  to period  $k-1$ , and from period  $k$  to period  $i \in \{k, \dots, m\}$  to get

$$s_{i-1} + \sum_{p=i}^{k-1} x_p = \sum_{p=i}^{k-1} d_p + s_{k-1}, \quad \text{for } i = 1, \dots, k-1 \quad (61)$$

$$s_{k-1} + \sum_{p=k}^i x_p = \sum_{p=k}^i d_p + s_i, \quad \text{for } i = k, \dots, m \quad (62)$$

respectively. Since each point belonging to  $\Gamma$  satisfies (61) and (62), we eliminate variables  $s_0, \dots, s_{k-2}$ ,  $s_k, \dots, s_{m-1}$ , and  $s_m$  from (60) by subtracting  $\nu_{i-1}$  times equality (61) for each  $i \in \{1, \dots, k-1\}$  from (60), and adding  $\nu_i$  times equality (62) for each  $i \in \{k, \dots, m\}$  to (60). This gives

$$\lambda_0 s_{k-1} + \sum_{p=1}^m \lambda_p^x x_p + \sum_{p=1}^m \sum_{t=1}^n \lambda_p^t z_p^t = \theta \quad (63)$$

where

$$\begin{aligned} \theta &= \eta + \sum_{i=1}^{k-1} \left( \nu_{i-1} \sum_{p=i}^{k-1} d_p \right) - \sum_{i=k}^m \left( \nu_i \sum_{p=k}^i d_p \right), \\ \lambda_0 &= \nu_{k-1} + \sum_{i=1}^{k-1} \nu_{i-1} - \sum_{i=k}^m \nu_i, \\ \lambda_p^x &= \mu_p - \sum_{i=1}^{k-1} \nu_{i-1} \quad \text{for } p = 1, \dots, k-1, \\ \lambda_p^x &= \mu_p + \sum_{i=k}^m \nu_i \quad \text{for } p = k, \dots, m. \end{aligned}$$

It is important to note that to have a point  $(\hat{z}, \hat{x}, \hat{s}) = (\hat{z}_1, \dots, \hat{z}_m, \hat{x}_1, \dots, \hat{x}_m, \hat{s}_0, \dots, \hat{s}_m) \in \mathbb{Z}_+^{mn} \times \mathbb{R}_+^m \times \mathbb{R}_+^{m+1} \in X^{MML}$  where  $\hat{z}_p = (\hat{z}_p^1, \dots, \hat{z}_p^n)$  for  $p = 1, \dots, m$ , it is sufficient to know the value of  $\hat{z}_p^t$  and  $\hat{x}_p$  for  $t = 1, \dots, n$  and  $p = 1, \dots, m$ , and  $\hat{s}_{k-1}$  coordinates because the remaining coordinates can be obtained using equalities (61) and (62). Therefore, in the rest of the proof, we will define a point belonging to  $X^{MML}$  by  $(\hat{z}, \hat{x}, \hat{s}_{k-1}) = (\hat{z}_1, \dots, \hat{z}_m, \hat{x}_1, \dots, \hat{x}_m, \hat{s}_{k-1}) \in \mathbb{Z}_+^{mn} \times \mathbb{R}_+^m \times \mathbb{R}_+$ .

Next, assuming  $S := \{w_1, \dots, w_{|S|}\}$  where  $w_1 = k$ , consider the point  $\mathcal{J} = (\hat{z}, \hat{x}, \hat{s}_{k-1}) = (\hat{z}_1, \dots, \hat{z}_m, \hat{x}_1, \dots, \hat{x}_m, \hat{s}_{k-1}) \in \mathbb{Z}_+^{mn} \times \mathbb{R}_+^m \times \mathbb{R}_+$  such that  $\hat{s}_{k-1} = 0$ ,

$$(\hat{z}_p^1, \hat{z}_p^2, \dots, \hat{z}_p^n, \hat{x}_p) := \begin{cases} \left( \left\lceil \frac{d_p}{\alpha_1} \right\rceil, 0, \dots, 0, d_p \right) & \text{if } p \in \{1, \dots, m\} \setminus \{k, \dots, l\}, \\ (0, 0, \dots, 0, 0) & \text{if } p \in \{k, \dots, l\} \setminus S, \end{cases}$$

for  $p \in \{1, \dots, m\} \setminus S$  and for  $i \in \{1, \dots, |S|\}$ ,

$$(\hat{z}_{w_i}^1, \hat{z}_{w_i}^2, \dots, \hat{z}_{w_i}^n, \hat{x}_{w_i}) := \left( \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil - \left\lceil \frac{b_{w_{i-1}}}{\alpha_1} \right\rceil, 0, \dots, 0, \alpha_1 \left\lceil \frac{b_{w_i}}{\alpha_1} \right\rceil - \alpha_1 \left\lceil \frac{b_{w_{i-1}}}{\alpha_1} \right\rceil \right),$$

where  $b_{w_0} = 0$ . Recall that  $b_{w_i} = \sum_{p=k}^{m_{w_i}-1} d_p$  for  $w_i \in S$  (by definition). It is easy to verify that  $\mathcal{J} \in X^{MML}$ . Also, since for all  $w_i \in I \subseteq S$ ,

$$\begin{aligned}\phi_{w_i}^n(\hat{z}) &= \prod_{\sigma=1}^n \left\lfloor \frac{b_{w_i}^{(\sigma-1)}}{\alpha_\sigma} \right\rfloor - \sum_{p \in S_{w_i}} \sum_{t=1}^n \prod_{\sigma=t+1}^n \left\lfloor \frac{b_{w_i}^{(\sigma-1)}}{\alpha_\sigma} \right\rfloor \hat{z}_p^t \\ &= \prod_{\sigma=1}^n \left\lfloor \frac{b_{w_i}^{(\sigma-1)}}{\alpha_\sigma} \right\rfloor - \prod_{\sigma=2}^n \left\lfloor \frac{b_{w_i}^{(\sigma-1)}}{\alpha_\sigma} \right\rfloor \sum_{p \in S_{w_i}} \hat{z}_p^1 = 0,\end{aligned}$$

the point  $\mathcal{J}$  satisfies conditions (i) – (iii) of Lemma 2. Therefore,  $\mathcal{J} \in \Gamma$  and hence must satisfy (63). Substituting  $\mathcal{J}$  into (63) gives

$$\begin{aligned}& \sum_{p \in \{1, \dots, m\} \setminus \{k, \dots, l\}} \left( \lambda_p^x d_p + \lambda_p^1 \left\lfloor \frac{d_p}{\alpha_1} \right\rfloor \right) \\ & + \sum_{i=1}^{|S|} (\lambda_{w_i}^x \alpha_1 + \lambda_{w_i}^1) \left( \left\lfloor \frac{b_{w_i}}{\alpha_1} \right\rfloor - \left\lfloor \frac{b_{w_{i-1}}}{\alpha_1} \right\rfloor \right) = \theta.\end{aligned}\tag{64}$$

Using (64), hyperplane (63) reduces to

$$\begin{aligned}& \lambda_0 s_{k-1} + \sum_{p \in \{1, \dots, m\} \setminus \{k, \dots, l\}} \lambda_p^x (x_p - d_p) + \sum_{p \in \{1, \dots, m\} \setminus \{k, \dots, l\}} \lambda_p^1 \left( z_p^1 - \left\lfloor \frac{d_p}{\alpha_1} \right\rfloor \right) \\ & + \sum_{p \in \{k, \dots, l\} \setminus S} (\lambda_p^x x_p + \lambda_p^1 z_p^1) + \sum_{i=1}^{|S|} \lambda_{w_i}^x \left( x_{w_i} - \alpha_1 \left( \left\lfloor \frac{b_{w_i}}{\alpha_1} \right\rfloor - \left\lfloor \frac{b_{w_{i-1}}}{\alpha_1} \right\rfloor \right) \right) \\ & + \sum_{p=1}^m \sum_{t=2}^n \lambda_p^t z_p^t = \sum_{i=1}^{|S|} \lambda_{w_i}^1 \left( \left\lfloor \frac{b_{w_i}}{\alpha_1} \right\rfloor - \left\lfloor \frac{b_{w_{i-1}}}{\alpha_1} \right\rfloor - z_{w_i}^1 \right).\end{aligned}\tag{65}$$

For each  $w_r \in S$ , we consider the points  $\mathcal{L}^{w_r} = (\hat{z}, \hat{x}, \hat{s}_{k-1}) \in \mathbb{Z}_+^{mn} \times \mathbb{R}_+^m \times \mathbb{R}_+$  such that each coordinate of  $\mathcal{L}^{w_r}$  is same as the coordinates of the point  $\mathcal{J}$ , except that

$$\hat{x}_{w_r} = \alpha_1 \left( \left\lfloor \frac{b_{w_r}}{\alpha_1} \right\rfloor - \left\lfloor \frac{b_{w_{r-1}}}{\alpha_1} \right\rfloor \right) + b_w^{(1)},$$

where  $b_w^{(1)} = \max_{i=1,2,\dots} \{b_{w_i}^{(1)}\}$ . Under assumptions (59) for  $t = 1$ , it is easy to verify that  $\mathcal{L}^{w_r} \in X^{MML}$  and satisfies conditions (i) – (iii) of Lemma 2. Therefore  $\mathcal{L}^{w_r} \in \Gamma$  and hence must satisfy (63). Substituting  $\mathcal{L}^{w_r}$  into (65) gives

$$\lambda_{w_r}^x = 0 \quad \text{for } w_r \in S.\tag{66}$$

Next, consider the points  $\mathcal{M}_1^r$  for  $r \in \{1, \dots, k-1\} \cup \{l+1, \dots, m\}$ ,  $\mathcal{M}_2^r$  for  $r \in \{1, \dots, k-2\} \cup \{l+1, \dots, m-1\}$ , and  $\mathcal{M}_3$  such that each coordinate of  $\mathcal{M}_1^r$ ,  $\mathcal{M}_2^r$ , and  $\mathcal{M}_3$  are same as the coordinates of the point  $\mathcal{J}$ , except that in  $\mathcal{M}_1^r$ ,  $\hat{z}_r^1 = \lceil d_p/\alpha_1 \rceil + 1$ , in  $\mathcal{M}_2^r$ ,

$$(\hat{z}_p^1, \hat{x}_p) := \begin{cases} \left( \left\lfloor \frac{\sum_{i=p}^{k-1} d_i}{\alpha_1} \right\rfloor, \sum_{i=p}^{k-1} d_i \right) & \text{if } p \in \{1, \dots, k-1\} \cap \{r\}, \\ \left( \left\lfloor \frac{\sum_{i=p}^m d_i}{\alpha_1} \right\rfloor, \sum_{i=p}^m d_i \right) & \text{if } p \in \{l+1, \dots, m\} \cap \{r\}, \\ (0, 0) & \text{if } p \in \{1, \dots, k-1, l+1, \dots, m\}, p > r, \end{cases}$$

and in  $\mathcal{M}_3$ ,  $\hat{x}_{k-1} = 0$  and  $\hat{s}_p = d_{k-1}$  for  $p = 0, \dots, k-2$ . It is easy to verify that  $\mathcal{M}_1^r$  for  $r \in \{1, \dots, k-1\} \cup \{l+1, \dots, m\}$ ,  $\mathcal{M}_2^r$  for  $r \in \{1, \dots, k-2\} \cup \{l+1, \dots, m-1\}$ , and  $\mathcal{M}_3$  belong to  $X^{MML}$  and satisfy conditions (i) – (iii) of Lemma 2. Therefore  $\mathcal{M}_1^r, \mathcal{M}_2^r, \mathcal{M}_3 \in \Gamma$  and hence must satisfy (63). Substituting  $\mathcal{M}_1^r$  into (65) gives

$$\lambda_r^1 = 0 \quad \text{for all } r \in \{1, \dots, k-1\} \cup \{l+1, \dots, m\}. \quad (67)$$

Likewise, one by one substituting the points  $\mathcal{M}_2^1, \dots, \mathcal{M}_2^{k-2}, \mathcal{M}_2^{l+1}, \dots, \mathcal{M}_2^{m-1}$ , and  $\mathcal{M}_3$  into (65) gives

$$\lambda_r^x = 0 \quad \text{for all } r \in \{1, \dots, k-1\} \cup \{l+1, \dots, m-1\}. \quad (68)$$

For  $r \in \{1, \dots, k-2\} \cup \{l+1, \dots, m-1\}$  and  $\tau \in \{2, \dots, n\}$ , we consider point  $\mathcal{N}_\tau^r$  such that coordinates of  $\mathcal{N}_\tau^r$  are same as the coordinates of the point  $\mathcal{J}$ , except that

$$(\hat{z}_r^1, \dots, \hat{z}_r^{\tau-1}, \hat{z}_r^\tau, \hat{z}_r^{\tau+1}, \dots, \hat{z}_r^n) := \left( \left\lfloor \frac{d_r}{\alpha_1} \right\rfloor, \dots, \left\lfloor \frac{d_r^{(\tau-2)}}{\alpha_{\tau-1}} \right\rfloor, \left\lfloor \frac{d_r^{(\tau-1)}}{\alpha_\tau} \right\rfloor, 0, \dots, 0 \right).$$

Since

$$\sum_{t=1}^n \alpha_t \hat{z}_r^t = \sum_{t=1}^{\tau} \alpha_t \left\lfloor \frac{d_r^{(t-1)}}{\alpha_t} \right\rfloor + \alpha_\tau = d_r + \alpha_\tau - d_r^{(t)} \geq d_r$$

as  $\alpha_r > d_r^{(t)}$ , it is easy to verify that  $\mathcal{N}_\tau^r$  for  $r \in \{1, \dots, k-2\} \cup \{l+1, \dots, m-1\}$  and  $t \in \{2, \dots, n\}$  belongs to  $X^{MML}$  and satisfy conditions (i) – (iii) of Lemma 2. Therefore  $\mathcal{N}_\tau^r \in \Gamma$  and hence must satisfy (63). Now, one by one substituting the points  $\mathcal{N}_2^r, \dots, \mathcal{N}_n^r$  into (65) and using (67) gives

$$\lambda_r^t = 0 \quad \text{for all } r \in \{1, \dots, k-1\} \cup \{l+1, \dots, m-1\}, t \in \{2, \dots, n\}. \quad (69)$$

By definition of the  $n$ -mixed  $(k, l, S, I)$  inequality, we know that  $I := \{i_1, \dots, i_{|I|}\} \subseteq S$  such that  $b_{i_0}^{(n)} = 0 < b_{i_1}^{(n)} \leq b_{i_2}^{(n)} \leq \dots \leq b_{i_{|I|}}^{(n)} < \alpha_1$ ,  $i_u = \max\{i : i \in I\}$ , and  $S_i = S \cap \{k, \dots, i\}$ . For  $r \in \{k, \dots, l\} \setminus S_{i_u}$ , we consider the points  $\mathcal{O}^r = (\hat{z}, \hat{x}, \hat{s}_{k-1}) = (\hat{z}_1, \dots, \hat{z}_m, \hat{x}_1, \dots, \hat{x}_m, \hat{s}_{k-1}) \in \mathbb{Z}_+^{mn} \times \mathbb{R}_+^m \times \mathbb{R}_+$  such that  $\hat{s}_{k-1} = 0$ ,  $\hat{z}_1^1 = \dots = \hat{z}_m^1 = 0$  for  $t = 2, \dots, n$ ,

$$(\hat{z}_p^1, \hat{x}_p) := \begin{cases} (\lfloor d_p/\alpha_1 \rfloor, d_p) & \text{if } p \in \{1, \dots, k-1\} \cup \{l+1, \dots, m\}, \\ (0, 0) & \text{if } p \in \{k, \dots, l\} \setminus S \text{ and } p \neq r, \\ (1, 0) & \text{if } p \in \{k, \dots, l\} \setminus S \text{ and } p = r, \end{cases}$$

for  $p \in \{1, \dots, m\} \setminus S$  and

$$(\hat{z}_{w_i}^1, \hat{x}_{w_i}) := \begin{cases} \left( \left( \left\lfloor \frac{b_{w_i}}{\alpha_1} \right\rfloor - \left\lfloor \frac{b_{w_{i-1}}}{\alpha_1} \right\rfloor, \alpha_1 \left\lfloor \frac{b_{w_i}}{\alpha_1} \right\rfloor - \alpha_1 \left\lfloor \frac{b_{w_{i-1}}}{\alpha_1} \right\rfloor \right) \right) & \text{if } w_i \in S \setminus \{r\}, \\ \left( \left( \left\lfloor \frac{b_{w_i}}{\alpha_1} \right\rfloor - \left\lfloor \frac{b_{w_{i-1}}}{\alpha_1} \right\rfloor + 1, \alpha_1 \left\lfloor \frac{b_{w_i}}{\alpha_1} \right\rfloor - \alpha_1 \left\lfloor \frac{b_{w_{i-1}}}{\alpha_1} \right\rfloor \right) \right) & \text{if } w_i \in S \cap \{r\} \\ & \cap \{m_{i_u}, \dots, l\}, \end{cases}$$



for  $i = 1, \dots, |S|$ . It is easy to verify that  $\mathcal{O}^r \in X^{MML}$  and satisfies conditions (i)–(iii) of Lemma 2. Therefore  $\mathcal{O}^r \in \Gamma$  and hence must satisfy (63). Substituting  $\mathcal{O}^r$  into (65) gives

$$\lambda_r^1 = 0 \quad \text{for all } r \in \{k, \dots, l\} \setminus S_{i_u}. \quad (70)$$

For  $r \in \{k, \dots, l\} \setminus S_{i_u}$  and  $t \in \{2, \dots, n\}$ , we also consider the point  $\mathcal{O}_t^r = (\hat{z}, \hat{x}, \hat{s}_{k-1}) = (\hat{z}_1, \dots, \hat{z}_m, \hat{x}_1, \dots, \hat{x}_m, \hat{s}_{k-1}) \in \mathbb{Z}_+^{mn} \times \mathbb{R}_+^m \times \mathbb{R}_+$  whose coordinates are same as the coordinates of the point  $\mathcal{J}$ , except that

$$(\hat{z}_r^2, \dots, \hat{z}_r^{t-1}, \hat{z}_r^t, \hat{z}_r^{t+1}, \dots, \hat{z}_r^n) := (0, \dots, 0, 1, 0, \dots, 0).$$

It is easy to verify that  $\mathcal{O}_t^r \in X^{MML}$  and satisfies conditions (i)–(iii) of Lemma 2. Therefore  $\mathcal{O}_t^r \in \Gamma$  and hence must satisfy (63). Substituting  $\mathcal{O}_t^r$  into (65) and using (70) gives

$$\lambda_r^t = 0 \quad \text{for all } r \in \{k, \dots, l\} \setminus S_{i_u} \text{ and } t \in \{2, \dots, n\}. \quad (71)$$

Using (66), (67), (68), (69), (70), and (71), (65) reduces to

$$\begin{aligned} & \lambda_0 s_{k-1} + \sum_{p \in \{k, \dots, l, m\} \setminus S} \lambda_p^x x_p \\ &= \sum_{w_i \in S_{i_u}} \lambda_{w_i}^1 \left( \left\lfloor \frac{b_{w_i}}{\alpha_1} \right\rfloor - \left\lfloor \frac{b_{w_{i-1}}}{\alpha_1} \right\rfloor - z_{w_i}^1 \right) - \sum_{w_i \in S_{i_u}} \sum_{t=2}^n \lambda_{w_i}^t z_{w_i}^t. \end{aligned} \quad (72)$$

Now, consider the points  $\mathcal{P}^r = (\hat{z}, \hat{x}, \hat{s}_{k-1}) = (\hat{z}_1, \dots, \hat{z}_m, \hat{x}_1, \dots, \hat{x}_m, \hat{s}_{k-1}) \in \mathbb{Z}_+^{mn} \times \mathbb{R}_+^m \times \mathbb{R}_+$  for  $r \in \{m_{i_u}, \dots, l, m\}$  where  $m_{i_u} \leq l \leq m$  such that  $\hat{s}_{k-1} = 0$ ,  $\hat{z}_1^t = \dots = \hat{z}_m^t = 0$  for  $t = 2, \dots, n$ ,

$$(\hat{z}_p^1, \hat{x}_p) := \begin{cases} (\lceil d_p / \alpha_1 \rceil, d_p) & \text{if } p \in \{1, \dots, k-1\} \cup \{l+1, \dots, m\} \setminus \{r\}, \\ (\lceil d_p / \alpha_1 \rceil + 1, d_p + \alpha_1) & \text{if } p \in \{r\} \cap \{m\} \text{ and } m \neq l, \\ (0, 0) & \text{if } p \in \{k, \dots, l\} \setminus (S \cup \{r\}), \\ (1, 1) & \text{if } p \in \{r\} \cap (\{m_{i_u}, \dots, l\} \setminus S), \end{cases}$$

for  $p \in \{1, \dots, m\} \setminus S$  and

$$(\hat{z}_{w_i}^1, \hat{x}_{w_i}) := \begin{cases} \left( \left\lfloor \frac{b_{w_i}}{\alpha_1} \right\rfloor - \left\lfloor \frac{b_{w_{i-1}}}{\alpha_1} \right\rfloor, \alpha_1 z_{w_i}^1 \right), & \text{if } w_i \in S \setminus \{r\} \\ \left( \left\lfloor \frac{b_{w_i}}{\alpha_1} \right\rfloor - \left\lfloor \frac{b_{w_{i-1}}}{\alpha_1} \right\rfloor + 1, \alpha_1 z_{w_i}^1 \right), & \text{if } w_i \in \{r\} \cap \{m_{i_u}, \dots, l\} \cap S. \end{cases}$$

for  $i \in \{1, \dots, |S|\}$ . It is easy to verify that  $\mathcal{P}^r \in X^{MML}$  and satisfies conditions (i)–(iii) of Lemma 2. Therefore  $\mathcal{P}^r \in \Gamma$  and hence must satisfy (63). Substituting  $\mathcal{P}^r$  into (65) gives

$$\lambda_r^x = 0 \quad \text{for all } r \in \{m_{i_u}, \dots, l\} \cup \{m\}. \quad (73)$$

Using (73), (72) reduces to

$$\begin{aligned} & \lambda_0 s_{k-1} + \sum_{p \in \{k, \dots, m_{i_u}-1\} \setminus S} \lambda_p^x x_p \\ &= \sum_{w_i \in S_{i_u}} \lambda_{w_i}^1 \left( \left\lfloor \frac{b_{w_i}}{\alpha_1} \right\rfloor - \left\lfloor \frac{b_{w_{i-1}}}{\alpha_1} \right\rfloor - z_{w_i}^1 \right) - \sum_{w_i \in S_{i_u}} \sum_{t=2}^n \lambda_{w_i}^t z_{w_i}^t. \end{aligned} \quad (74)$$

Similar to the proof of Theorem 4, let  $I_a := \{i_{a_1}, \dots, i_{a_{|I|}}\}$  be a set which has same elements as in the set  $I$ , except that  $i_{a_1} < i_{a_2} < \dots < i_{a_{|I|}}$ . In other words, set  $I_a$  is same as the set  $I$ , the only difference is that the elements of  $I_a$  are arranged in the increasing order of time periods. Now, for each  $i_{a_\zeta} \in I$  where  $\zeta \in \{1, \dots, |I|\}$  and  $r \in S_{i_{a_\zeta}} \setminus (S_{i_{a_{\zeta-1}}} \cup \{i_{a_\zeta}\})$  where  $S_{i_{a_0}} = \emptyset$ , we consider the points  $\mathcal{Q}^{r, a_\zeta} = (\hat{z}, \hat{x}, \hat{s}_{k-1}) = (\hat{z}_1, \dots, \hat{z}_m, \hat{x}_1, \dots, \hat{x}_m, \hat{s}_{k-1}) \in \mathbb{Z}_+^{mn} \times \mathbb{R}_+^m \times \mathbb{R}_+$  such that  $\hat{s}_{k-1} = 0$ ,  $\hat{z}_p^t = 0$  for  $p = 1, \dots, m$ , and  $t = 2, \dots, n$ ,

$$(\hat{z}_p^1, \hat{x}_p) := \begin{cases} (\lceil d_p / \alpha_1 \rceil, d_p) & \text{if } p \in \{1, \dots, k-1\} \cup \{l+1, \dots, m\}, \\ (0, 0) & \text{if } p \in \{k, \dots, l\} \setminus S, \end{cases}$$

for  $p \in \{1, \dots, m\} \setminus S$  and

$$(\hat{z}_{w_i}^1, \hat{x}_{w_i}) := \begin{cases} \left( \left( \left\lfloor \frac{b_{w_i}}{\alpha_1} \right\rfloor - \left\lfloor \frac{b_{w_{i-1}}}{\alpha_1} \right\rfloor, \alpha_1 z_{w_i}^1 \right) & \text{if } w_i \in S \setminus \{r, i_{a_\zeta}\}, \\ \left( \left( \left\lfloor \frac{b_{w_i}}{\alpha_1} \right\rfloor - \left\lfloor \frac{b_{w_{i-1}}}{\alpha_1} \right\rfloor + 1, \alpha_1 z_{w_i}^1 \right) & \text{if } w_i = r, \\ \left( \left( \left\lfloor \frac{b_{w_i}}{\alpha_1} \right\rfloor - \left\lfloor \frac{b_{w_{i-1}}}{\alpha_1} \right\rfloor - 1, \alpha_1 z_{w_i}^1 \right) & \text{if } w_i = i_{a_\zeta}, \end{cases}$$

for  $i = 1, \dots, |S|$ . Since  $\lceil b_{w_i} / \alpha_1 \rceil > \lceil b_{w_{i-1}} / \alpha_1 \rceil$  for  $w_i = i_{a_\zeta} \in I \subseteq S$  (Condition (59)), it is easy to verify that  $\mathcal{Q}^{r, a_\zeta} \in X^{MML}$ . Also, this point satisfies conditions (i) – (iii) of Lemma 2 because for all  $i_{a_\zeta} \in I$ ,

$$\begin{aligned} \phi_{i_{a_\zeta}}^n(\hat{z}) &= \prod_{\sigma=1}^n \left\lfloor \frac{b_{i_{a_\zeta}}^{(\sigma-1)}}{\alpha_\sigma} \right\rfloor - \sum_{p \in S_{i_{a_\zeta}}} \sum_{t=1}^n \prod_{\sigma=t+1}^n \left\lfloor \frac{b_{i_{a_\zeta}}^{(\sigma-1)}}{\alpha_\sigma} \right\rfloor \hat{z}_p^t \\ &= \prod_{\sigma=1}^n \left\lfloor \frac{b_{i_{a_\zeta}}^{(\sigma-1)}}{\alpha_\sigma} \right\rfloor - \prod_{\sigma=2}^n \left\lfloor \frac{b_{i_{a_\zeta}}^{(\sigma-1)}}{\alpha_\sigma} \right\rfloor \sum_{p \in S_{i_{a_\zeta}}} \hat{z}_p^1 \\ &= \prod_{\sigma=1}^n \left\lfloor \frac{b_{i_{a_\zeta}}^{(\sigma-1)}}{\alpha_\sigma} \right\rfloor - \prod_{\sigma=2}^n \left\lfloor \frac{b_{i_{a_\zeta}}^{(\sigma-1)}}{\alpha_\sigma} \right\rfloor \left\lfloor \frac{b_{i_{a_\zeta}}}{\alpha_1} \right\rfloor = 0. \end{aligned}$$

Therefore  $\mathcal{Q}^{r, a_\zeta} \in \Gamma$  and hence must satisfy (63). By substituting the points  $\mathcal{Q}^{r, a_\zeta}$ , for  $i_{a_\zeta} \in I$  where  $\zeta \in \{1, \dots, |I|\}$  and  $r \in (S_{i_{a_\zeta}} \setminus (S_{i_{a_{\zeta-1}}} \cup \{i_{a_\zeta}\}))$ , into (74), we get

$$\lambda_r^1 = \lambda_{i_{a_\zeta}}^1 \quad \text{for } i_{a_\zeta} \in I, r \in S_{i_{a_\zeta}} \setminus (S_{i_{a_{\zeta-1}}} \cup \{i_{a_\zeta}\}). \quad (75)$$

Next, we consider a point  $\mathcal{J}_\tau = (\hat{z}, \hat{x}, \hat{s}_{k-1}) = (\hat{z}_1, \dots, \hat{z}_m, \hat{x}_1, \dots, \hat{x}_m, \hat{s}_{k-1}) \in \mathbb{Z}_+^{mn} \times \mathbb{R}_+^m \times \mathbb{R}_+$  for  $\tau \in \{2, \dots, n\}$ , such that  $\hat{s}_{k-1} = 0$ ,

$$(\hat{z}_p^1, \hat{z}_p^2, \dots, \hat{z}_p^n, \hat{x}_p) := \begin{cases} \left( \left\lfloor \frac{d_p}{\alpha_1} \right\rfloor, 0, \dots, 0, d_p \right) & \text{if } p \in \{1, \dots, m\} \setminus \{k, \dots, l\}, \\ (0, 0, \dots, 0, 0) & \text{if } p \in \{k, \dots, l\} \setminus S, \end{cases}$$

for  $p \in \{1, \dots, m\} \setminus S$  and for  $i \in \{1, \dots, |S|\}$ ,  $\hat{x}_{w_i} := \sum_{t=1}^{\tau} \alpha_t \hat{z}_{w_i}^t$  where

$$\hat{z}_{w_i}^t := \begin{cases} \left[ \frac{b_{w_i}^{(t-1)}}{\alpha_t} \right] - \left[ \frac{b_{w_{i-1}}^{(t-1)}}{\alpha_t} \right] & \text{if } t \in \{1, \dots, \tau-1\} \\ \left[ \frac{b_{w_i}^{(t-1)}}{\alpha_t} \right] - \left[ \frac{b_{w_{i-1}}^{(t-1)}}{\alpha_t} \right] & \text{if } t = \tau \\ 0 & \text{if } t \in \{\tau+1, \dots, n\}, \end{cases}$$

for  $t = 1, \dots, n$ . Recall that  $b_{w_0} = 0$  and  $b_{w_i} = \sum_{j=k}^{m_{w_i}-1} d_j$  for  $w_i \in S$  (by definition). Under assumptions (59), it is easy to verify that  $\mathcal{J}_\tau \in X^{MML}$  because for each  $w_i \in S$ ,

$$\sum_{p \in S_{w_i}} \hat{x}_p = \sum_{t=1}^{\tau} \sum_{p \in S_{w_i}} \alpha_t \hat{z}_p^t = \sum_{t=1}^{\tau} \alpha_t \left[ \frac{b_{w_i}^{(t-1)}}{\alpha_t} \right] + \alpha_\tau = b_{w_i} - b_{w_i}^{(\tau)} + \alpha_\tau \geq b_{w_i}.$$

Also, since for all  $i_q \in I$  and  $\tau \in \{2, \dots, n\}$ ,

$$\begin{aligned} \phi_{i_q}^n(\hat{z}) &= \prod_{\sigma=1}^n \left[ \frac{b_{i_q}^{(\sigma-1)}}{\alpha_\sigma} \right] - \sum_{t=1}^{\tau} \prod_{\sigma=t+1}^n \left[ \frac{b_{i_q}^{(\sigma-1)}}{\alpha_\sigma} \right] \sum_{p \in S_{i_q}} \hat{z}_p^t \\ &= \prod_{\sigma=1}^n \left[ \frac{b_{i_q}^{(\sigma-1)}}{\alpha_\sigma} \right] - \sum_{t=1}^{\tau-1} \prod_{\sigma=t+1}^n \left[ \frac{b_{i_q}^{(\sigma-1)}}{\alpha_\sigma} \right] \left[ \frac{b_{i_q}^{(t-1)}}{\alpha_t} \right] - \prod_{\sigma=\tau+1}^n \left[ \frac{b_{i_q}^{(\sigma-1)}}{\alpha_\sigma} \right] \left[ \frac{b_{i_q}^{(\tau-1)}}{\alpha_\tau} \right] \\ &= \prod_{\sigma=1}^n \left[ \frac{b_{i_q}^{(\sigma-1)}}{\alpha_\sigma} \right] - \sum_{t=1}^{\tau-1} \left( \prod_{\sigma=t}^n \left[ \frac{b_{i_q}^{(\sigma-1)}}{\alpha_\sigma} \right] - \prod_{\sigma=t+1}^n \left[ \frac{b_{i_q}^{(\sigma-1)}}{\alpha_\sigma} \right] \right) - \prod_{\sigma=\tau}^n \left[ \frac{b_{i_q}^{(\sigma-1)}}{\alpha_\sigma} \right] \\ &= 0, \end{aligned}$$

the point  $\mathcal{J}_\tau$  satisfies conditions (i) – (iii) of Lemma 2. Therefore,  $\mathcal{J}_\tau \in \Gamma$  and hence must satisfy (63).

Again, for each  $i_{a_\zeta} \in I$  where  $\zeta \in \{1, \dots, |I|\}$ ,  $r \in S_{i_{a_\zeta}} \setminus (S_{i_{a_{\zeta-1}}} \cup \{i_{a_\zeta}\})$  where  $S_{i_{a_0}} = \emptyset$ , and  $\tau \in \{2, \dots, n\}$ , we consider the point  $\mathcal{Q}_\tau^{r, a_\zeta} = (\hat{z}, \hat{x}, \hat{s}_{k-1}) = (z_1, \dots, z_m, x_1, \dots, x_m, s_{k-1}) \in \mathbb{Z}_+^{mn} \times \mathbb{R}_+^m \times \mathbb{R}_+$  whose coordinates are same as the coordinates of point  $\mathcal{J}_\tau$ , except that

$$(\hat{z}_{w_i}^\tau, \hat{x}_{w_i}) := \begin{cases} \left( \left( \left[ \frac{b_{w_i}^{(\tau-1)}}{\alpha_\tau} \right] - \left[ \frac{b_{w_{i-1}}^{(\tau-1)}}{\alpha_\tau} \right] + 1, \sum_{t=1}^{\tau} \alpha_t z_{w_i}^t \right) & \text{for } w_i = r, \\ \left( \left( \left[ \frac{b_{w_i}^{(\tau-1)}}{\alpha_\tau} \right] - \left[ \frac{b_{w_{i-1}}^{(\tau-1)}}{\alpha_\tau} \right] - 1, \sum_{t=1}^{\tau} \alpha_t z_{w_i}^t \right) & \text{for } w_i = i_{a_\zeta}. \end{cases}$$

Since  $\left[ \frac{b_{w_i}^{(\tau-1)}}{\alpha_\tau} \right] > \left[ \frac{b_{w_{i-1}}^{(\tau-1)}}{\alpha_\tau} \right]$  for  $w_i = i_{a_\zeta} \in I \subseteq S$  (Condition (59) for  $t = \tau$ ), it is easy to verify that  $\mathcal{Q}_\tau^{r, a_\zeta} \in X^{MML}$  and satisfies conditions (i) – (iii) of Lemma 2. Therefore  $\mathcal{Q}_\tau^{r, a_\zeta} \in \Gamma$  and hence must satisfy (63). By substituting the points  $\mathcal{Q}_\tau^{r, a_\zeta}$ , for  $i_{a_\zeta} \in I$  where  $\zeta \in \{1, \dots, |I|\}$ ,  $r \in S_{i_{a_\zeta}} \setminus (S_{i_{a_{\zeta-1}}} \cup \{i_{a_\zeta}\})$ , and  $\tau \in \{2, \dots, n\}$ , into (74), we get

$$\lambda_r^\tau = \lambda_{i_{a_\zeta}}^\tau \quad \text{for } i_{a_\zeta} \in I, r \in S_{i_{a_\zeta}} \setminus (S_{i_{a_{\zeta-1}}} \cup \{i_{a_\zeta}\}), \text{ and } \tau = 2, \dots, n. \quad (76)$$

Using (75) and (76), Equation (74) reduces to

$$\begin{aligned} \lambda_0 s_{k-1} + \sum_{p \in \{k, \dots, m_{i_u} - 1\} \setminus S} \lambda_p^x x_p &= \sum_{\zeta=1}^{|I|} \sum_{t=2}^n \lambda_{i_{a_\zeta}}^t \left( \sum_{p \in S_{i_{a_{\zeta-1}}}} z_p^t - \sum_{p \in S_{i_{a_\zeta}}} z_p^t \right) \\ &+ \sum_{\zeta=1}^{|I|} \lambda_{i_{a_\zeta}}^1 \left( \left\lceil \frac{b_{i_{a_\zeta}}}{\alpha_1} \right\rceil - \sum_{p \in S_{i_{a_\zeta}}} z_p^1 - \left\lceil \frac{b_{i_{a_{\zeta-1}}}}{\alpha_1} \right\rceil + \sum_{p \in S_{i_{a_{\zeta-1}}}} z_p^1 \right), \end{aligned} \quad (77)$$

where  $b_{i_{a_0}} = 0$  and  $S_{i_{a_0}} = \emptyset$ . By rearranging the right-hand side of (77), we get

$$\begin{aligned} \lambda_0 s_{k-1} + \sum_{p \in \{k, \dots, m_{i_u} - 1\} \setminus S} \lambda_p^x x_p &= - \sum_{\zeta=1}^{|I|-1} \sum_{t=2}^n (\lambda_{i_{a_\zeta}}^t - \lambda_{i_{a_{\zeta+1}}}^t) \left( \sum_{p \in S_{i_{a_\zeta}}} z_p^t \right) \\ &+ \sum_{\zeta=1}^{|I|-1} (\lambda_{i_{a_\zeta}}^1 - \lambda_{i_{a_{\zeta+1}}}^1) \left( \left\lceil \frac{b_{i_{a_\zeta}}}{\alpha_1} \right\rceil - \sum_{p \in S_{i_{a_\zeta}}} z_p^1 \right) \\ &+ \lambda_{i_{a_{|I|}}^1} \left( \left\lceil \frac{b_{i_{a_{|I|}}}}{\alpha_1} \right\rceil - \sum_{p \in S_{i_{a_{|I|}}}} z_p^1 \right) - \sum_{t=2}^n \lambda_{i_{a_{|I|}}}^t \left( \sum_{p \in S_{i_{a_{|I|}}}} z_p^t \right). \end{aligned} \quad (78)$$

Furthermore, Equation (78) can also be simplified to

$$\begin{aligned} \lambda_0 s_{k-1} + \sum_{p \in \{k, \dots, m_{i_u} - 1\} \setminus S} \lambda_p^x x_p \\ = \sum_{q=1}^{|I|} \left( \gamma_{i_q}^1 \left( \left\lceil \frac{b_{i_q}}{\alpha_1} \right\rceil - \sum_{p \in S_{i_q}} z_p^1 \right) - \sum_{t=2}^n \gamma_{i_q}^t \left( \sum_{p \in S_{i_q}} z_p^t \right) \right) \end{aligned} \quad (79)$$

where for  $q = 1, \dots, |I|$  and  $t = 1, \dots, n$ ,  $\gamma_{i_q}^t = \lambda_{i_q}^t - \lambda_{i_{a_{\zeta+1}}}^t$  such that  $a_\zeta = q$  for  $\zeta \in \{1, \dots, |I|\}$  and  $\lambda_{i_{|I|+1}}^t = 0$ . We substitute the point  $\mathcal{J}_n$  in (79) to get

$$\sum_{q=1}^{|I|} \left( \gamma_{i_q}^1 - \sum_{t=2}^{n-1} \gamma_{i_q}^t \left\lceil \frac{b_{i_q}^{(t-1)}}{\alpha_t} \right\rceil - \gamma_{i_q}^n \left\lceil \frac{b_{i_q}^{(t-1)}}{\alpha_t} \right\rceil \right) = 0. \quad (80)$$

Subtracting (80) from (79) gives

$$\lambda_0 s_{k-1} + \sum_{\substack{p=k \\ p \notin S}}^{m_{i_u} - 1} \lambda_p^x x_p = \sum_{q=1}^{|I|} \left( \sum_{t=1}^n \gamma_{i_q}^t \left( \left\lceil \frac{b_{i_q}^{(t-1)}}{\alpha_t} \right\rceil - \sum_{p \in S_{i_q}} z_p^t \right) + \gamma_{i_q}^n \right). \quad (81)$$

Next, for  $i_r \in I$ , we consider the point  $\mathcal{R}^r = (\hat{z}, \hat{x}, \hat{s}_{k-1}) \in \mathbb{Z}_+^{mn} \times \mathbb{R}_+^m \times \mathbb{R}_+$ , such that  $\hat{s}_{k-1} = b_{i_r}^{(n)}$ ,

$$(\hat{z}_p^1, \hat{z}_p^2, \dots, \hat{z}_p^n, \hat{x}_p) := \begin{cases} (\lceil d_p / \alpha_1 \rceil, 0, \dots, 0, d_p) & \text{if } p \in \{1, \dots, m\} \setminus \{k, \dots, l\}, \\ (0, 0, \dots, 0, 0) & \text{if } p \in \{k, \dots, l\} \setminus S, \end{cases}$$

for  $p \in \{1, \dots, m\} \setminus S$ , and for  $i \in \{1, \dots, |S|\}$ ,

$$(\hat{z}_{w_i}, \hat{x}_{w_i}) := \left( \chi_{w_i} - \chi_{w_{i-1}}, \sum_{t=1}^n \alpha_t \hat{z}_{w_i}^t \right)$$

where  $\chi_{w_0} = (0, \dots, 0) \in \mathbb{R}^n$  and

$$(\chi_{w_i}^1, \dots, \chi_{w_i}^n) = \begin{cases} \left( \left[ \frac{b_{w_i}}{\alpha_1} \right], \dots, \left[ \frac{b_{w_i}^{(n-2)}}{\alpha_{n-1}} \right], \left[ \frac{b_{w_i}^{(n-1)}}{\alpha_n} \right] \right) & \text{if } w_i \in S \setminus I, \\ \left( \left[ \frac{b_{w_i}}{\alpha_1} \right], \dots, \left[ \frac{b_{w_i}^{(n-2)}}{\alpha_{n-1}} \right], \left[ \frac{b_{w_i}^{(n-1)}}{\alpha_n} \right] \right) & \text{if } w_i \in \{i_1, \dots, i_r\}, \\ \left( \left[ \frac{b_{w_i}}{\alpha_1} \right], \dots, \left[ \frac{b_{w_i}^{(n-2)}}{\alpha_{n-1}} \right], \left[ \frac{b_{w_i}^{(n-1)}}{\alpha_n} \right] \right) & \text{if } w_i \in \{i_{r+1}, \dots, i_{|I|}\}. \end{cases}$$

This implies that  $\sum_{p \in S_{w_i}} \hat{z}_p = \chi_{w_i}$ . Because of Conditions (59),

$$\hat{z}_{w_i}^t = \chi_{w_i}^t - \chi_{w_{i-1}}^t \geq 0 \text{ for } i = 1, \dots, |S| \text{ and } t = 1, \dots, n.$$

Now, it is easy to verify that  $\mathcal{R}^r \in X^{MML}$  and satisfies conditions (i) – (iii) of Lemma 1 because for  $w_i \in \{i_1, \dots, i_r\}$ ,

$$\begin{aligned} \phi_{w_i}^n(\hat{z}) &= \prod_{\sigma=1}^n \left[ \frac{b_{w_i}^{(\sigma-1)}}{\alpha_\sigma} \right] - \sum_{t=1}^n \prod_{\sigma=t+1}^n \left[ \frac{b_{w_i}^{(\sigma-1)}}{\alpha_\sigma} \right] \sum_{p \in S_{w_i}} \hat{z}_p^t \\ &= \prod_{\sigma=1}^n \left[ \frac{b_{w_i}^{(\sigma-1)}}{\alpha_\sigma} \right] - \sum_{t=1}^{n-1} \prod_{\sigma=t+1}^n \left[ \frac{b_{w_i}^{(\sigma-1)}}{\alpha_\sigma} \right] \left[ \frac{b_{w_i}^{(t-1)}}{\alpha_t} \right] - \left[ \frac{b_{w_i}^{(n-1)}}{\alpha_n} \right] \\ &= \prod_{\sigma=1}^n \left[ \frac{b_{w_i}^{(\sigma-1)}}{\alpha_\sigma} \right] - \sum_{t=1}^{n-1} \left( \prod_{\sigma=t}^n \left[ \frac{b_{w_i}^{(\sigma-1)}}{\alpha_\sigma} \right] - \prod_{\sigma=t+1}^n \left[ \frac{b_{w_i}^{(\sigma-1)}}{\alpha_\sigma} \right] \right) \\ &\quad - \left[ \frac{b_{w_i}^{(n-1)}}{\alpha_n} \right] + 1 = 1, \end{aligned}$$

and for  $w_i \in \{i_{r+1}, \dots, i_{|I|}\}$ ,

$$\begin{aligned} \phi_{w_i}^n(\hat{z}) &= \prod_{\sigma=1}^n \left[ \frac{b_{w_i}^{(\sigma-1)}}{\alpha_\sigma} \right] - \sum_{t=1}^n \prod_{\sigma=t+1}^n \left[ \frac{b_{w_i}^{(\sigma-1)}}{\alpha_\sigma} \right] \sum_{p \in S_{w_i}} \hat{z}_p^t \\ &= \prod_{\sigma=1}^n \left[ \frac{b_{w_i}^{(\sigma-1)}}{\alpha_\sigma} \right] - \sum_{t=1}^{n-1} \prod_{\sigma=t+1}^n \left[ \frac{b_{w_i}^{(\sigma-1)}}{\alpha_\sigma} \right] \left[ \frac{b_{w_i}^{(t-1)}}{\alpha_t} \right] - \left[ \frac{b_{w_i}^{(n-1)}}{\alpha_n} \right] \\ &= \prod_{\sigma=1}^n \left[ \frac{b_{w_i}^{(\sigma-1)}}{\alpha_\sigma} \right] - \sum_{t=1}^{n-1} \left( \prod_{\sigma=t}^n \left[ \frac{b_{w_i}^{(\sigma-1)}}{\alpha_\sigma} \right] - \prod_{\sigma=t+1}^n \left[ \frac{b_{w_i}^{(\sigma-1)}}{\alpha_\sigma} \right] \right) - \left[ \frac{b_{w_i}^{(n-1)}}{\alpha_n} \right] \\ &= 0, \end{aligned}$$

Therefore  $\mathcal{R}^r \in \Gamma$  and hence must satisfy (63). One by one we substitute  $\mathcal{R}^1, \mathcal{R}^2, \dots, \mathcal{R}^{|I|}$  into (81) and get

$$\begin{aligned}
\gamma_{i_1}^n &= \lambda_0 b_{i_1}^{(n)}, \\
\gamma_{i_2}^n &= \lambda_0 b_{i_2}^{(n)} - \gamma_{i_1}^n = \lambda_0 (b_{i_2}^{(n)} - b_{i_1}^{(n)}), \\
&\vdots \\
\gamma_{i_{|I|}}^n &= \lambda_0 b_{i_{|I|}}^{(n)} - \sum_{q=1}^{|I|-1} \gamma_{i_q}^n = \lambda_0 (b_{i_{|I|}}^{(n)} - b_{i_{|I|-1}}^{(n)}).
\end{aligned}$$

This implies that for  $q \in \{1, \dots, |I|\}$ ,

$$\gamma_{i_q}^n = \lambda_0 (b_{i_q}^{(n)} - b_{i_{q-1}}^{(n)}), \quad (82)$$

where  $b_{i_0}^{(n)} = 0$ . For  $r \in \{k, \dots, m_{i_u} - 1\} \setminus S$ , consider the points  $\mathcal{S}^r$  whose coordinates are same as the coordinates of  $\mathcal{R}^u$ , where  $i_u = \max\{i_q \in I\}$ , except that  $\hat{s}_{k-1} = 0$  and  $(\hat{z}_r^1, \hat{x}_r) = (1, b_{i_u}^{(n)})$ . By substituting the points  $\mathcal{S}^r$  into the (81) and subtracting from equality corresponding to  $\mathcal{R}^u$ , we get

$$\lambda_r^x = \lambda_0 \text{ for } r \in \{k, \dots, m_{i_u} - 1\} \setminus S. \quad (83)$$

For  $i_{a_\zeta} \in I$  where  $\zeta \in \{1, \dots, |I|\}$  and  $\tau \in \{1, \dots, n-1\}$ , we consider the point  $\mathcal{T}^{\zeta, \tau}$  such that  $s_{k-1} = 0$ ,

$$(\hat{z}_p^1, \hat{z}_p^2, \dots, \hat{z}_p^n, \hat{x}_p) := \begin{cases} (\lceil d_p / \alpha_1 \rceil, 0, \dots, 0, d_p) & \text{if } p \in \{1, \dots, m\} \setminus \{k, \dots, l\}, \\ (0, 0, \dots, 0, 0) & \text{if } p \in \{k, \dots, l\} \setminus S, \end{cases}$$

for  $p \in \{1, \dots, m\} \setminus S$ , and for  $i \in \{1, \dots, |S|\}$ ,

$$(\hat{z}_{w_i}, \hat{x}_{w_i}) := \left( \Delta_{w_i} - \Delta_{w_{i-1}}, \sum_{t=1}^n \alpha_t \hat{z}_{w_i}^t \right)$$

where  $\Delta_{w_i} \in \mathbb{R}_+^n$  such that  $\Delta_{w_0} = (0, \dots, 0)$  and

$$\Delta_{w_i} = \begin{cases} \left( \left[ \frac{b_{w_i}}{\alpha_1} \right], 0, \dots, 0 \right) & \text{if } w_i \in \{w_1, \dots, i_{a_\zeta} - 1\}, \\ \left( \left[ \frac{b_{w_i}}{\alpha_1} \right], \dots, \left[ \frac{b_{w_i}^{(\tau-2)}}{\alpha_{\tau-1}} \right], \left[ \frac{b_{w_i}^{(\tau-1)}}{\alpha_\tau} \right], 0, \dots, 0 \right) & \text{if } w_i \in \{i_{a_\zeta}\}, \\ \left( \left[ \frac{b_{w_i}}{\alpha_1} \right], \dots, \left[ \frac{b_{w_i}^{(n-2)}}{\alpha_{n-1}} \right], \left[ \frac{b_{w_i}^{(n-1)}}{\alpha_n} \right] \right) & \text{if } w_i \in \{i_{a_\zeta} + 1, \dots, w_{|S|}\}. \end{cases}$$

Observe that  $\sum_{p \in S_{w_i}} \hat{z}_p = \Delta_{w_i}$  and because of assumptions (59),  $\hat{z}_{w_i} = \Delta_{w_i} - \Delta_{w_{i-1}} \geq \mathbf{0}$  for all  $w_i \in S$ . Also, since  $\sum_{p \in S_{w_i}} \hat{x}_p \geq b_{w_i}$ , it is easy to verify that  $\mathcal{T}^{\zeta, \tau} \in X^{MML}$  and it satisfies conditions (i) – (iii) of Lemma 1 because for all  $w_i \in I$ ,  $\phi_{w_i}^n(\hat{z}) = 0$ . Therefore,  $\mathcal{T}^{\zeta, \tau} \in \Gamma$  and hence must satisfy (63). One by one we substitute  $\mathcal{T}^{1, n-1}, \dots, \mathcal{T}^{1, 1}, \dots, \mathcal{T}^{|I|, n-1}, \dots, \mathcal{T}^{|I|, 1}$  into (81) and

get

$$\begin{aligned}
\gamma_{i_{a_\zeta}}^{n-1} &= \gamma_{i_{a_\zeta}}^n \left[ \frac{b_{i_{a_1}}^{(n-1)}}{\alpha_n} \right], \\
\gamma_{i_{a_\zeta}}^{n-2} &= \gamma_{i_{a_\zeta}}^{n-1} \left[ \frac{b_{i_{a_\zeta}}^{(n-2)}}{\alpha_{n-1}} \right] + \gamma_{i_{a_\zeta}}^n \left[ \frac{b_{i_{a_1}}^{(n-1)}}{\alpha_n} \right] = \left( \prod_{\sigma=n-1}^n \left[ \frac{b_{i_{a_\zeta}}^{(\sigma-1)}}{\alpha_\sigma} \right] \right) \gamma_{i_{a_\zeta}}^n, \\
&\vdots \\
\gamma_{i_{a_\zeta}}^1 &= \left( \prod_{\sigma=2}^n \left[ \frac{b_{i_{a_\zeta}}^{(\sigma-1)}}{\alpha_\sigma} \right] \right) \gamma_{i_{a_\zeta}}^n,
\end{aligned}$$

for  $\zeta = 1, \dots, |I|$ . This implies for  $\tau = 1, \dots, n-1$ ,

$$\gamma_{i_{a_\zeta}}^\tau = \left( \prod_{\sigma=\tau+1}^n \left[ \frac{b_{i_{a_\zeta}}^{(\sigma-1)}}{\alpha_\sigma} \right] \right) \gamma_{i_{a_\zeta}}^n, \tag{84}$$

Substituting (82), (83), and (84) into Equation (81) gives

$$\lambda_0 s_{k-1} + \sum_{\substack{p=k \\ p \notin S}}^{m_{i_u}-1} \lambda_0 x_p = \sum_{q=1}^{|I|} \lambda_0 \left( b_{i_q}^{(n)} - b_{i_{q-1}}^{(n)} \right) \phi_{i_q}^n(z) \tag{85}$$

where

$$\phi_{i_q}^n(z) := \prod_{\sigma=1}^n \left[ \frac{b_{i_q}^{(\sigma-1)}}{\alpha_\sigma} \right] - \sum_{p \in S_{i_q}} \sum_{t=1}^n \prod_{\sigma=t+1}^n \left[ \frac{b_{i_q}^{(\sigma-1)}}{\alpha_\sigma} \right] z_p^t.$$

The identity (85) is  $\lambda_0$  times (58). Hence  $\Gamma$  defines a facet for  $\text{conv}(X^{MML})$  if conditions (59) hold. This completes the proof.  $\square$

## 5. Conclusion and Future Work

We provided sufficient conditions under which the  $(k, l, S, I)$  inequalities of Pochet and Wolsey [19], the mixed  $(k, l, S, I)$  inequalities, derived using mixing procedure of Günlük and Pochet [15], and the paired  $(k, l, S, I)$  inequalities, derived using sequential pairing procedure of Guan et al. [12], are facet-defining for the single module (or constant batch) capacitated lot-sizing problem without backlogging (SMLS-WB). We also investigated the facet-defining properties of the inequalities derived using the sequential pairing and the  $n$ -mixing procedure of Sanjeevi and Kianfar [23] for the multi-module capacitated lot-sizing problem without backlogging.

One potential extension would be to provide necessary conditions under which the mixed  $(k, l, S, I)$  inequalities for SMLS-WB problem are facet-defining. In order to proceed in this direction, the following example can be helpful as it showcases that in case the given  $k, l, S$ , and  $I$  do not satisfy the sufficient condition (23) then the associated mixed  $(k, l, S, I)$  inequality is dominated by another facet-defining inequality which satisfies the condition (23).

**Example 1** (continued). Next, we consider  $k = 2$ ,  $l = 6$ , and  $S = \{2, 4, 5, 6\}$ . Therefore,  $S_2 = \{2\}$ ,  $S_4 = \{2, 4\}$ ,  $S_5 = \{2, 4, 5\}$ ,  $S_6 = \{2, 4, 5, 6\}$ ,  $m_2 = 4$ ,  $m_4 = 5$ ,  $m_5 = 5$ ,  $m_6 = 7$ ,  $b_2 = 4$ ,  $b_4 = 27$ ,  $b_5 = 28$ ,  $b_6 = 36$ ,  $b_2^{(1)} = 4$ ,  $b_4^{(1)} = 3$ ,  $b_5^{(1)} = 4$ , and  $b_6^{(1)} = 1$ . Notice that  $b_0^{(1)} = 0 < b_6^{(1)} < b_4^{(1)} < b_2^{(1)} = b_5^{(1)}$  and more importantly,  $\lceil b_4/\alpha_1 \rceil = \lceil b_5/\alpha_1 \rceil = 6$ . The mixed  $(2, 6, \{2, 4, 5, 6\}, \{5, 2\})$  inequality where  $i_u = 5$ ,

$$\begin{aligned} s_1 + x_3 &\geq b_5^{(1)} \left( \left\lceil \frac{b_5}{\alpha_1} \right\rceil - z_2^1 - z_4^1 - z_5^1 \right) + (b_2^{(1)} - b_5^{(1)}) \left( \left\lceil \frac{b_2}{\alpha_1} \right\rceil - z_2^1 \right) \\ &= 24 - 4z_2^1 - 4z_4^1 - 4z_5^1, \end{aligned}$$

does not satisfy the conditions (23) because for  $i = 5$ , there exist a  $j = 4$  such that  $j \in S$ ,  $j < i$ ,  $\lceil b_j/\alpha_1 \rceil = \lceil b_i/\alpha_1 \rceil$ , and  $j \notin I$ . In addition, the facet-defining mixed  $(2, 6, \{2, 4, 5, 6\}, \{4, 5, 2\})$  inequality where  $i_u = 5$ ,

$$\begin{aligned} s_1 + x_3 &\geq b_4^{(1)} \left( \left\lceil \frac{b_4}{\alpha_1} \right\rceil - z_2^1 - z_4^1 \right) + (b_5^{(1)} - b_4^{(1)}) \left( \left\lceil \frac{b_5}{\alpha_1} \right\rceil - z_2^1 - z_4^1 - z_5^1 \right) \\ &\quad + (b_2^{(1)} - b_5^{(1)}) \left( \left\lceil \frac{b_2}{\alpha_1} \right\rceil - z_2^1 \right) \\ &= 3(6 - z_2^1 - z_4^1) + (6 - z_2^1 - z_4^1 - z_5^1) = 24 - 4z_2^1 - 4z_4^1 - z_5^1, \end{aligned}$$

dominates the the mixed  $(2, 6, \{2, 4, 5, 6\}, \{5, 2\})$  inequality because  $24 - 4z_2^1 - 4z_4^1 - z_5^1 > 24 - 4z_2^1 - 4z_4^1 - 4z_5^1$  as  $z_5^1 \in \mathbb{Z}_+$ .  $\square$

Another future direction would be to perform computational study on the performance of the  $n$ -mixed  $(k, l, S, I)$  inequalities for  $n \geq 1$ , which are at least as strong as the inequalities derived using the sequential pairing cut-generation procedure, for solving SMLS-(W)B, MMLS-(W)B, and their two-stage stochastic or distributionally robust optimization variants [3, 4, 8] by using them within Benders' decomposition or distributionally robust L-shaped algorithms [4]. Since there are exponential number of the foregoing inequalities, it would be useful to utilize efficient separation algorithms associated with them. Also, for  $n \geq 2$ , these experiments will involve consideration of various strategies in selecting parameters  $(\alpha_1, \dots, \alpha_n)$  for separation algorithms for these inequalities; for an example, readers can refer to [2, 6] in which 2-step  $(k, l, S, C)$  cycle inequalities have been utilized for solving two-module capacitated lot-sizing problem with and without backlogging.

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