New algorithms for discrete vector optimization based on the Graef-Younes method and cone-monotone sorting functions

Christian Günther[a] and Nicolae Popovici[b]

[a]Martin Luther University Halle-Wittenberg, Faculty of Natural Sciences II, Institute for Mathematics, 06099 Halle (Saale), Germany;
[b]Babeş-Bolyai University, Faculty of Mathematics and Computer Science, 400084 Cluj-Napoca, Romania

ARTICLE HISTORY
Compiled January 13, 2018

ABSTRACT
The well-known Jahn-Graef-Younes algorithm, proposed by Jahn in 2006, generates all minimal elements of a finite set with respect to an ordering cone. It consists of two Graef-Younes procedures, namely the forward iteration, which eliminates a part of the non-minimal elements, followed by the backward iteration, which is applied to the reduced set generated by the previous iteration. Without using the backward iteration, we develop new algorithms that also compute all minimal elements of the initial set, by combining the forward iteration with certain sorting procedures based on cone-monotone functions. In particular, when the ordering cone is polyhedral, computational results obtained in MATLAB allow us to compare our algorithms with the Jahn-Graef-Younes algorithm, within a bi-objective optimization problem.

KEYWORDS
Partially ordered space; Minimal element; Domination property; Graef-Younes reduction method; Cone-monotone function; Multiobjective subdivision technique

Mathematics Subject Classification: 90C29; 90C26; 90C56; 68T20

1. Introduction

In this paper we develop new methods for computing all minimal elements of a finite set A with respect to a nontrivial pointed convex cone K in a real linear space E. To this aim we combine the Graef-Younes method with certain sorting procedures based on K-monotone real-valued functions.

The Graef-Younes method is an iterative procedure that generates a reduced set $B \subseteq A$, by eliminating a part of the non-minimal elements of A. As mentioned by Jahn [1, Sec. 12.4], this method was originally proposed by Younes [2], being based on an algorithmic conception by Graef; it is a self learning method, which becomes better and better step by step, leading to a drastic reduction of large sets in some concrete instances (from 5×10^6 points of A to around 3×10^3 points of B). However, B may still contain a significant number of non-minimal elements of A.

Dedicated to Professor Johannes Jahn in honor of his 65th birthday

Corresponding author: Christian Günther. Email: Christian.Guenther@mathematik.uni-halle.de

A very interesting method to determine precisely all minimal elements of A is the Jahn-Graef-Younes algorithm (also called "Graef-Younes method with backward iteration" in the monograph of Jahn [1, Alg. 12.20] as in the early paper by Jahn and Rathje [3]). It performs two Graef-Younes reduction procedures, namely the "forward iteration," which generates the reduced set B out of A, as described above, and the "backward iteration," which is applied to the set B, by reverting the enumeration of its points. The new reduced set is precisely the set of all minimal elements of A.

Without using the backward iteration, in this paper we develop new algorithms that also compute all minimal elements of the initial set A, by combining the standard Graef-Younes method (seen as forward iteration) with certain sorting procedures based on real-valued K-monotone functions. Our approach mainly relies on Theorem 3.1, which shows that whenever the points of A are itemized by rising the values of a strongly K-increasing function, the set B generated by the Graef-Younes method is exactly the set of all minimal elements of A.

The paper is organized as follows. In Section 2 we highlight the role of cone-monotone scalar functions in the context of vector optimization. In particular we establish some theoretical results that play a key role for developing our algorithms in the next section, especially Theorem 2.11.

After recalling the classical pairwise comparison method (Algorithm 1), the Graef-Younes method (Algorithm 2) and the Jahn-Graef-Younes method (Algorithm 3), in Section 3 we introduce our new methods (Algorithm 4 and Algorithm 5), that are essentially based on Theorem 3.1. More precisely, Algorithm 4 is a combination of the Graef-Younes reduction procedure with a pre-sorting procedure, while Algorithm 5 consists of two Graef-Younes reduction procedures along with an intermediate sorting procedure. Both methods produce the whole set of minimal elements of A.

In order to derive implementable versions of our new algorithms, in Section 4 we identify appropriate cone-monotone sorting functions, that possess two key features: their values are computable efficiently and they allow us to test whether two points are comparable with respect to the ordering cone. Actually, within two distinct subsections, we present some useful properties of the original Tammer-Weidner functions within topological linear spaces and certain polyhedral convex functions within the Euclidean space $\mathbb{R}^n$.

In Section 5 we derive the implementable versions (Algorithms 6 and 7) of our new algorithms introduced in Section 3, in the particular framework where the Euclidean space $\mathbb{R}^n$ is partially ordered by polyhedral cone while the strongly cone-monotone sorting functions are linear.

Among many possible applications, our algorithms can be used to approximate the set of minimal outcomes for certain continuous multiobjective optimization problems, via different discretization approaches known in the literature. In Section 6 we consider a concrete continuous bi-objective test problem, known in the literature as being very difficult to solve, via the "Multiobjective search algorithm with subdivision technique" (MOSAST), proposed by Jahn [4]. A detailed comparative analysis of our algorithms and other classical methods is provided for this concrete case, based on computational experiments in MATLAB.

Finally, in Section 7 we point out possible directions for further research.

2. The role of cone-monotone scalar functions in vector optimization

Consider a real linear space E. Recall that a set $K \subseteq E$ is said to be a cone, if $0 \in K = \mathbb{R}_+ \cdot K$, where 0 is the origin of E and $\mathbb{R}_+$ is the set of all nonnegative real numbers. A cone $K \subseteq E$ is called: nontrivial, if $\{0\} \neq K \neq E$; pointed, if $K \cap (-K) = \{0\}$; convex, if $K = \operatorname{conv} K$, i.e., $K + K = K$.

In what follows we assume that $K \subseteq E$ is a nontrivial pointed convex cone. It induces an ordering (i.e., a reflexive, transitive and antisymmetric binary relation) defined for any $x, y \in E$ by

$$x \leqq_K y :\iff y \in x + K.$$

Its irreflexive part is defined for any $x, y \in E$ by

$$x \lneqq_K y :\iff y \in x + (K \setminus \{0\}).$$

Definition 2.1. Let A be a nonempty subset of E. The elements of the set

$$\mathrm{MIN}(A \mid K) := \{x^0 \in A \mid \nexists x \in A :\ x \lneqq_K x^0\}$$

are called minimal elements of A (with respect to K).

Remark 1. By Definition 2.1, we actually have

$$\begin{aligned}\mathrm{MIN}(A \mid K) &= \{x^0 \in A \mid x^0 \notin (A \setminus \{x^0\}) + K\} \\ &= \{x^0 \in A \mid (x^0 - K) \cap A = \{x^0\}\}.\end{aligned}$$

Remark 2. It is easy to check that for any set $A \subseteq E$ the following relation holds:

$$\mathrm{MIN}(A \mid K) = \mathrm{MIN}(A + K \mid K).$$

In general, the numerical methods of vector optimization are aimed to compute the entire set $\mathrm{MIN}(A \mid K)$ or to produce an inner/outer approximation of it. In what follows we highlight the role of cone-monotone functions in developing such methods. The next definition recalls some concepts which are known in the literature under different names (see, e.g., Jahn [1, Def. 5.1]).

Definition 2.2. A function $\varphi : D \to \mathbb{R}$, defined on a nonempty set $D \subseteq E$, is called:

- K-increasing, if for any $x, y \in D$,

$$x \leqq_K y \implies \varphi(x) \leq \varphi(y);$$

- strongly K-increasing, if for any $x, y \in D$,

$$x \lneqq_K y \implies \varphi(x) < \varphi(y).$$

- (strongly) K-decreasing if $-\varphi$ is (strongly) K-increasing.

Obviously, strongly K-increasing/decreasing functions are K-increasing/decreasing.

Linear functionals are known to play an important role in vector optimization. Let E' be the algebraic dual space of E. Recall (see, e.g., Jahn [1] and Luc [5]) that the algebraic polar cone of K is

$$K'_+ := \{\varphi \in E' \mid \forall\, x \in K : \ \varphi(x) \geq 0\},$$

while its quasi-interior is defined as

$$K'_{++} := \{\varphi \in E' \mid \forall\, x \in K \setminus \{0\} : \ \varphi(x) > 0\}.$$

The cone-monotonicity of linear functionals can be characterized as follows.

Proposition 2.3. *For any $\varphi \in E'$, the following assertions hold:*

1°. *φ is K-increasing if and only if $\varphi \in K'_+$.*
2°. *φ is strongly K-increasing if and only if $\varphi \in K'_{++}$.*

The next result (see, e.g., Jahn [1, Lem. 5.14]) shows that cone-monotone functions can be used for developing scalarization methods that provide an inner approximation of $\mathrm{MIN}(A \mid K)$, that is, a set

$$S \subseteq \mathrm{MIN}(A \mid K).$$

Proposition 2.4. *Let $\varphi : D \to \mathbb{R}$ be a function defined on a nonempty set $D \subseteq E$ and let A be a nonempty subset of D. If either φ is strongly K-increasing or φ is K-increasing and $|\operatorname{argmin}_{x\in A}\ \varphi(x)| = 1$, then*

$$\operatorname*{argmin}_{x\in A}\ \varphi(x) \subseteq \mathrm{MIN}(A \mid K).$$

As a direct consequence of Propositions 2.3 and 2.4 we obtain the following result.

Corollary 2.5. *Let $A \subseteq E$ be a nonempty set and let $\varphi \in E'$. Then, for any minimizer $x^0 \in \operatorname{argmin}_{x\in A}\ \varphi(x)$, the following assertions hold:*

1°. *If $\varphi \in K'_{++}$, then $x^0 \in \mathrm{MIN}(A \mid K)$.*
2°. *If $\varphi \in K'_+$ and $|\operatorname{argmin}_{x\in A}\ \varphi(x)| = 1$, then $x^0 \in \mathrm{MIN}(A \mid K)$.*

Other numerical methods of vector optimization are rather conceived to provide an outer approximation of $\mathrm{MIN}(A \mid K)$, that is, a set $B \subseteq E$ such that

$$\mathrm{MIN}(A \mid K) \subseteq B \subseteq A.$$

In other words, an outer approximation B of $\mathrm{MIN}(A \mid K)$ is obtained from A by removing a part of its non-minimal elements.

The following results are relevant for this approach. In preparation we recall a basic concept of vector optimization (see, e.g. Göpfert et al. [6] and references therein).

Definition 2.6. A set $A \subseteq E$ satisfies the domination property with respect to K if

$$A \subseteq \mathrm{MIN}(A \mid K) + K,$$

i.e., for any $x \in A$ there exists $x^0 \in \mathrm{MIN}(A \mid K)$ such that $x^0 \leqq_K x$.

Remark 3. It is easily seen that A satisfies the domination property if and only if

$$A + K = \mathrm{MIN}(A \mid K) + K.$$

Remark 4. Every finite set $A \subseteq E$ satisfies the domination property.

Lemma 2.7. *Let $(A_i)_{i \in I}$ be a family of subsets of E, each of them satisfying the domination property. Then we have*

$$\mathrm{MIN}\left(\cup_{i \in I} A_i \mid K\right) = \mathrm{MIN}\left(\cup_{i \in I} \mathrm{MIN}(A_i \mid K) \mid K\right).$$

Moreover, if I is finite, then $\cup_{i \in I} A_i$ satisfies the domination property.

Proof. In view of Remarks 2 and 3, we have

$$\begin{aligned}
\mathrm{MIN}\left(\cup_{i \in I} A_i \mid K\right) &= \mathrm{MIN}\left((\cup_{i \in I} A_i) + K \mid K\right) \\
&= \mathrm{MIN}\left(\cup_{i \in I} (A_i + K) \mid K\right) \\
&= \mathrm{MIN}\left(\cup_{i \in I} (\mathrm{MIN}(A_i \mid K) + K) \mid K\right) \\
&= \mathrm{MIN}\left((\cup_{i \in I} \mathrm{MIN}(A_i \mid K)) + K \mid K\right) \\
&= \mathrm{MIN}\left(\cup_{i \in I} \mathrm{MIN}(A_i \mid K) \mid K\right).
\end{aligned}$$

For proving the second assertion, we can assume without loss of generality that $\operatorname{card} I = 2$. Let $A_1, A_2 \subseteq E$ be two sets satisfying the domination property. We are going to prove that for any $x \in A_1 \cup A_2$ there exists $x^0 \in \mathrm{MIN}(A_1 \cup A_2 \mid K)$ such that $x^0 \leqq_K x$. For simplicity assume that $x \in A_1$. Due to the domination property of A_1 we can find a point $y \in \mathrm{MIN}(A_1 \mid K)$ such that $y \leqq_K x$. Two cases may occur. If $y \in \mathrm{MIN}(A_1 \cup A_2 \mid K)$, then we can choose $x^0 := y$. If $y \notin \mathrm{MIN}(A_1 \cup A_2 \mid K)$, then there exists $z \in A_1 \cup A_2$ such that $z \lneqq_K y$. Since $y \in \mathrm{MIN}(A_1 \mid K)$, it follows that $z \notin A_1$, hence $z \in A_2$. Now, by the domination property of A_2 there exists $u \in \mathrm{MIN}(A_2 \mid K)$ such that $u \leqq_K z$. We claim that $u \in \mathrm{MIN}(A_1 \cup A_2 \mid K)$. Indeed, otherwise it would exist some $v \in A_1 \cup A_2$ such that $v \lneqq_K u$. Then, since $u \in \mathrm{MIN}(A_2 \mid K)$, we should have $v \notin A_2$, hence $v \in A_1$. But this contradicts the fact that $y \in \mathrm{MIN}(A_1 \mid K)$, because $v \lneqq_K u \leqq_K z \lneqq_K y$, hence $v \lneqq_K y$. Thus $u \in \mathrm{MIN}(A_1 \cup A_2 \mid K)$ and we can choose $x^0 := u$. □

Lemma 2.8. *Let $A \subseteq E$ be a set that satisfies the domination property. Then, for any set $B \subseteq A$ the following assertions are equivalent:*

1°. $\mathrm{MIN}(A \mid K) \subseteq B$.
2°. $\mathrm{MIN}(A \mid K) = \mathrm{MIN}(B \mid K)$.

Proof. Obviously (even in the absence of the domination property) 2° implies 1°, since $\mathrm{MIN}(B \mid K) \subseteq B$.

Conversely, assume that 1° holds. Then, for any $x \in \mathrm{MIN}(A \mid K)$ we have $x \in B$ and $(x - K) \cap A = \{x\}$. Since $B \subseteq A$, we infer that $(x - K) \cap B = \{x\}$, i.e., $x \in \mathrm{MIN}(B \mid K)$. Thus inclusion $\mathrm{MIN}(A \mid K) \subseteq \mathrm{MIN}(B \mid K)$ holds. In order to prove the converse inclusion, let $x' \in \mathrm{MIN}(B \mid K)$. Then we have $x' \in B \subseteq A$ and also $x' \in \mathrm{MIN}(B + K \mid K) = \mathrm{MIN}(B \mid K)$. By the domination property of A and 1°, it follows that $x' \in (x' - K) \cap A \subseteq (x' - K) \cap (\mathrm{MIN}(A \mid K) + K) \subseteq (x' - K) \cap (B + K) = \{x'\}$. Therefore we have $(x' - K) \cap A = \{x'\}$, i.e., $x' \in \mathrm{MIN}(A \mid K)$. Thus the inclusion $\mathrm{MIN}(B \mid K) \subseteq \mathrm{MIN}(A \mid K)$ in 2° also holds. □

Remark 5. The domination property assumption imposed on A in Lemma 2.8 is essential for the implication $1^\circ \Rightarrow 2^\circ$, as shown by the following example.

Example 2.9. Let the Euclidean plane $\mathbb{R}^2$ be endowed with the standard ordering cone $K := \mathbb{R}^2_+$. Consider the sets

$$A := \{(0,1)\} \cup \big(\,]0,1] \times \{0\}\big) \quad \text{and} \quad B := \{(0,1);(1,0)\}.$$

It is easily seen that $B \subseteq A$ and

$$\mathrm{MIN}(A \mid K) = \{(0,1)\} \subseteq B = \mathrm{MIN}(B \mid K) \neq \mathrm{MIN}(A \mid K).$$

Next we show that strongly cone-monotone functions can be used to generate the whole set $\mathrm{MIN}(A \mid K)$.

Lemma 2.10. *Let $\varphi : D \to \mathbb{R}$ be a strongly K-increasing function, defined on a nonempty set $D \subseteq E$. For any set $B \subseteq D$ the following assertions are equivalent:*

- 1°. $B = \mathrm{MIN}(B \mid K)$.
- 2°. *For any points $b, b' \in B$ with $\varphi(b) < \varphi(b')$ we have $b' \notin b + K$ (i.e., $b \nleqq_K b'$).*
- 3°. *For any points $b, b' \in B$ with $b \leqq_K b'$ we have $\varphi(b) \geq \varphi(b')$, i.e., the restriction of φ to B is K-decreasing.*

Proof. The equivalence $2^\circ \iff 3^\circ$ is obvious.

Assume that 1° holds and let $b, b' \in B$ be such that $\varphi(b) < \varphi(b')$. Supposing by the contrary that $b' \in b + K$ we would have $b \in (b' - K) \cap B$. Since $b' \in B = \mathrm{MIN}(B \mid K)$, we infer $b = b'$ hence $\varphi(b) = \varphi(b')$, a contradiction. Thus 1° implies 2°.

Conversely, assume that 2° holds. In order to prove 1°, we just have to show that $B \subseteq \mathrm{MIN}(B \mid K)$. Suppose by the contrary that there is $b' \in B \setminus \mathrm{MIN}(B \mid K)$. Then it would exist $b \in B$ such that $b' \in b + (K \setminus \{0\})$. Since φ is strongly K-increasing we infer $\varphi(b) < \varphi(b')$. This contradicts 2°. $\square$

Theorem 2.11. *Let $A \subseteq E$ be a nonempty set satisfying the domination property and let $B \subseteq A$ be such that $\mathrm{MIN}(A \mid K) \subseteq B$. If $\varphi : D \to \mathbb{R}$ is a strongly K-increasing function, with $A \subseteq D$, then the following assertions are equivalent:*

- 1°. $B = \mathrm{MIN}(A \mid K)$.
- 2°. *For any points $b, b' \in B$ with $\varphi(b) < \varphi(b')$ we have $b' \notin b + K$ (i.e., $b \nleqq_K b'$).*
- 3°. *For any points $b, b' \in B$ with $b \leqq_K b'$ we have $\varphi(b) \geq \varphi(b')$, i.e., the restriction of φ to B is K-decreasing.*

Proof. Follows from Lemmas 2.8 and 2.10. $\square$

Remark 4 suggests to use Theorem 2.11 in order to develop new numerical methods for solving discrete vector optimization problems.

3. New algorithms based on the Graef-Younes method

As stated in the previous section, in what follows $K \subseteq E$ is a nontrivial pointed convex cone. Moreover, since in the sequel we focus on discrete vector optimization problems, throughout this section the notation A will represent a nonempty finite set

$$A := \{a^1, \ldots, a^p\} \subseteq E,$$

whose cardinality is $|A| := \operatorname{card} A = p \in \mathbb{N}$, i.e., $a^1, \ldots, a^p$ are pairwise distinct.

In this section we will develop new numerical methods for computing the set $\mathrm{MIN}(A \mid K)$. For evaluating their complexity, we will assume that there is a finite bound for the number of operations needed to evaluate whether two points of E are comparable with respect to the order relation $\leqq_K$.

We start by recalling three well-known methods. The simplest one (yet naive) is Algorithm 1, which only requires Definition 2.1.

Algorithm 1: COMPLETE PAIRWISE COMPARISON (NAIVE METHOD)

Input: The set $A = \{a^1, \ldots, a^p\}$.
$T \leftarrow \emptyset$;
for $j \leftarrow 1$ **to** p **do**
  if $a^j \notin (A \setminus \{a^j\}) + K$ **then**
    $T \leftarrow T \cup \{a^j\}$;
  end
end
Output: The set T (representing the set of all minimal elements of A).

Remark 6. The computational complexity of Algorithm 1 is $\mathcal{O}(|A|^2)$. Since in practice the cardinality of A might be very high, it is important to consider some reduction procedures that generate a smaller set $B \subseteq A$, whose minimal elements are the same as those of the initial set A, i.e.,

$$\mathrm{MIN}(B \mid K) = \mathrm{MIN}(A \mid K). \tag{1}$$

The next Algorithm 2 represents a reduction procedure, known in the literature as the Graef-Younes method. It was proposed by Younes [2] and presented by Jahn [1, Sec. 12.4] within the finite-dimensional space $E = \mathbb{R}^n$, being still valid in any real linear space E.

Algorithm 2: REDUCTION PROCEDURE (GRAEF-YOUNES METHOD)

Input: The set $A = \{a^1, \ldots, a^p\}$.
$B \leftarrow \{a^1\}$;
for $j \leftarrow 2$ **to** p **do**
  if $a^j \notin B + K$ **then**
    $B \leftarrow B \cup \{a^j\}$;
  end
end
Output: The set B (that satisfies $\mathrm{MIN}(B \mid K) = \mathrm{MIN}(A \mid K) \subseteq B \subseteq A$).

Remark 7. Since A is finite, the domination property holds in view of Remark 4. Therefore, according to Lemma 2.8, the set B generated by Algorithm 2 satisfies the property (1). As pointed out by Jahn [1, Ex. 12.19], in some particular instance the Graef-Younes method can reduce a set A containing 5 000 000 points to a set B containing 3 067 points, among which 1 497 are minimal elements of A. However, simple examples in $\mathbb{R}^2$ show that the set B generated by the Graef-Younes method may be very large (sometimes $B = A$), hence the computation of $\mathrm{MIN}(B \mid K)$ is not easier than the computation of $\mathrm{MIN}(A \mid K)$. In the worst-case scenario when $B = A$ we need to perform at most

$$1 + 2 + \ldots + (p-1) = \frac{(|A|-1) \cdot |A|}{2} \tag{2}$$

pairwise comparisons of points with respect to the ordering induced by K.

A very interesting approach to overcome the drawback of B to exceed the set $\mathrm{MIN}(A \mid K)$ is the so-called "Graef-Younes method with backward iteration," presented within the finite-dimensional space $E = \mathbb{R}^n$ by Jahn [1, Alg. 12.20], following the early paper by Jahn and Rathje [3]. Actually, this method is still valid in general real linear spaces, being nowadays known in the literature as the Jahn-Graef-Younes method (cf. Eichfelder [7]). Its formulation is given in Algorithm 3.

Algorithm 3: JAHN-GRAEF-YOUNES METHOD

```
Input: The set A = {a^1, ..., a^p}.
/* Forward iteration                                   */
i ← 1;
b^i ← a^i;
B ← {b^i};
for j ← 2 to p do
    if a^j ∉ B + K then
        i ← i + 1;
        b^i ← a^j;
        B ← B ∪ {b^i};
    end
end
/* Backward iteration                                  */
T ← {b^i};
for j ← 1 to i − 1 do
    if b^(i−j) ∉ T + K then
        T ← T ∪ {b^(i−j)};
    end
end
Output: The set T (representing the set of all minimal elements of A).
```

Remark 8. The worst-case computational complexity of Algorithm 3 is $\mathcal{O}(|A|^2)$. Actually, due to the domination property, this algorithm becomes more efficient when the enumeration of the input set A (that is, $a^1, a^2, \ldots$) starts with some minimal elements of A. Therefore a natural idea arises, namely to sort the elements of A in an appropriate way, before the forward iteration.

Given any function $\varphi : D \to \mathbb{R}$ with $A \subseteq D$ it is easy to find an enumeration $\{a^{j_1}, \dots, a^{j_p}\}$ of the a priori given set $A = \{a^1, \dots, a^p\}$ such that

$$\varphi(a^{j_1}) \leq \varphi(a^{j_2}) \leq \cdots \leq \varphi(a^{j_p}). \tag{3}$$

Thus we can always pre-sort the elements of A according to (3) and thereafter apply the algorithms stated above to $\{a^{j_1}, \dots, a^{j_p}\}$ instead of the original enumeration of A. When φ is a strongly K-increasing function we obtain an interesting property, highlighted by the next theorem.

Theorem 3.1. *Let $\varphi : D \to \mathbb{R}$ be a strongly K-increasing function with $A \subseteq D$. If*

$$\varphi(a^1) \leq \varphi(a^2) \leq \cdots \leq \varphi(a^p), \tag{4}$$

then Algorithm 2 (Graef-Younes reduction method) generates the set $B = \mathrm{MIN}(A \mid K)$.

Proof. It is a direct consequence of Theorem 2.11. Indeed, the set B is constructed within Algorithm 2 by eliminating a part of the non-minimal elements of A, hence $\mathrm{MIN}(A \mid K) \subseteq B \subseteq A$. Therefore, it suffices to check that the set B satisfies the property 2° of Theorem 2.11. To this end, let any $b, b' \in B$ with $\varphi(b) < \varphi(b')$. Since $b, b' \in A$, there exist $i, i' \in \{1, \dots, p\}$ such that $b = a^i$ and $b' = a^{i'}$, hence $\varphi(a^i) < \varphi(a^{i'})$. By (4) it follows that $i < i'$, which shows that during Algorithm 2, the point $b' = a^{i'}$ is added to B after $b = a^i$, hence $b' \notin b + K$. □

Remark 9. If the pre-sorting (3) is given by an arbitrary scalar function φ, then the "reduced" set B generated by applying Algorithm 2 to $\{a^{j_1}, \dots, a^{j_p}\}$ instead of the original enumeration of A may be very large. In particular, if the function φ is strongly K-decreasing and (4) holds, then the output set coincides with the initial one, i.e., $B = A$ (no reduction occurs). Indeed, by construction of B we have $a^1 \in B$. Moreover, for any $j \in \{2, \dots, p\}$ we have $a^j \notin a^i + K$ for all $i \in \{1, \dots, p\}$ with $i < j$ (otherwise we should have $a^i \lneq_K a^j$, hence $\varphi(a^i) > \varphi(a^j)$, a contradiction). This means that $a^j \in B$.

Remark 10. If φ is K-increasing and all inequalities in (4) are strict, then Algorithm 2 generates the set $B = \mathrm{MIN}(A \mid K)$, which actually means that $B \subseteq \mathrm{MIN}(B \mid K)$ in view of (1). Indeed, assuming by the contrary that $a^j \in B \setminus \mathrm{MIN}(B \mid K)$ for some $j \in \{1, \dots, p\}$, we can deduce (by the domination property) the existence of a point $a^i \in \mathrm{MIN}(B \mid K)$ with $i \in \{1, \dots, p\}$ such that $a^i \lneq_K a^j$. Since φ is K-increasing and all inequalities in (4) are strict, we infer that $i < j$, contradicting the construction of B (because $a^j \in B$ and $a^j \in a^i + K$ with $a^i \in B$).

Theorem 3.1 allows us to develop a new method (Algorithm 4) that generates the set $\mathrm{MIN}(A \mid K)$, by pre-sorting the elements of A before applying the classical Graef-Younes method.

Algorithm 4: PRE-SORTING PROCEDURE & REDUCTION PROCEDURE

Input: The set $A = \{a^1, \dots, a^p\}$ and the strongly K-increasing function $\varphi : E \to \mathbb{R}$.
/* Phase 1 (Pre-sorting procedure) */
Compute an enumeration $A = \{a^{j_1}, \dots, a^{j_p}\}$ which satisfies (3).
/* Phase 2 (Reduction procedure) */
$B \leftarrow \{a^{j_1}\}$;
for $k \leftarrow 2$ **to** p **do**
  if $a^{j_k} \notin B + K$ **then**
    $B \leftarrow B \cup \{a^{j_k}\}$;
  end
end
$T \leftarrow B$;
Output: The set T (representing the set of all minimal elements of A).

Remark 11. In Phase 1 of Algorithm 4 we generate an enumeration of A by rising the values of the sorting function φ. In Phase 2 of Algorithm 4 the Graef-Younes method (Algorithm 2) is applied to the sorted set $\{a^{j_1}, \dots, a^{j_p}\}$ in the role of A.

Theorem 3.2. *Algorithm 4 has a worst-case computational complexity of*

$$\mathcal{O}(|A| \cdot \log(|A|) + |T| \cdot (2|A| - |T| - 1)/2), \tag{5}$$

where $T = \mathrm{MIN}(A \mid K)$.

Proof. Phase 1 can be performed by means of effective algorithms known in the literature for sorting $p = |A|$ real numbers in ascending order, namely $\varphi(a^1), \dots, \varphi(a^p)$, with worst-case computational complexity $\mathcal{O}(p \log(p))$, i.e., $\mathcal{O}(|A| \cdot \log(|A|))$. On the other hand, since $B \subseteq \mathrm{MIN}(A \mid K) = T$ at every iteration, the worst-case scenario in Phase 2 occurs when $a^{j_1}, \cdots, a^{j_{|T|}} \in T$ and $a^{j_{|T|+1}}, \cdots, a^{j_{|A|}} \notin T$, requesting to perform

$$\frac{(|T| - 1) \cdot |T|}{2} + (|A| - |T|) \cdot |T| = \frac{|T| \cdot (2|A| - |T| - 1)}{2} \tag{6}$$

pairwise comparisons. $\square$

Remark 12. Algorithm 4 is an output-sensitive method, because its runtime depends on the size of the output set T. In particular, if $|T| \leq \log(|A|)$, then its complexity does not exceed $\mathcal{O}(|A| \cdot \log(|A|))$, since $|A| \cdot |T|$ is an upper bound for (6). On the other hand, even if $|T| > \log(|A|)$, the number of pairwise comparisons given by (6) for Algorithm 4 may be significantly smaller than the number of pairwise comparisons given by (2) for Algorithm 2 (for instance, when $|A| = 500\,000$ and $|T| = 471$ as in Jahn [1, Ex. 12.19], (6) is bounded by 2.4×10^8, while (2) exceeds 1.2×10^{11}).

In what follows we propose another method (Algorithm 5) for computing the set $\mathrm{MIN}(A \mid K)$, which consists of two Graef-Younes reduction procedures along with an intermediate sorting procedure based on a strongly K-increasing function $\varphi : E \to \mathbb{R}$.

Algorithm 5: REDUCTION PROCEDURE & INTERMEDIATE SORTING PROCEDURE & REDUCTION PROCEDURE

Input: The set $A = \{a^1, \dots, a^p\}$ and the strongly K-increasing function $\varphi : E \to \mathbb{R}$.

/* Phase 1 (Reduction procedure) */

Apply Algorithm 2 for the set A as input to get the set B.

/* Phase 2 (Intermediate sorting procedure & reduction procedure) */

Apply Algorithm 4 for the set B as input to get the set T.

Output: The set T (representing the set of all minimal elements of A).

Remark 13. In Phase 1 of Algorithm 5 we apply the classical Graef-Younes reduction procedure, which has a complexity of $\mathcal{O}(|A|^2)$, in view of Remark 7. Then, in Phase 2 the elements of the reduced set B (generated in the previous phase) are sorted by the strongly K-increasing function φ. Moreover, in Phase 2 the Graef-Younes reduction procedure is applied to the sorted set $\{b^{j_1}, \dots, b^{j_i}\}$ in the role of B in order to generate all minimal elements of A with respect to K, resulting in a complexity given by (5) with B in the role of A. So, we conclude that Algorithm 5 has a complexity of $\mathcal{O}(|A|^2)$.

4. Special classes of cone-monotone sorting functions

We are going to study some special classes of K-monotone functions, currently used in scalarization methods for vector optimization. We will use them as sorting functions within Phase 1 of Algorithm 4 and Phase 2 of Algorithm 5. Moreover, we will show that these functions can be used to compare points with respect to the ordering induced by K. Therefore, they allows us to implement the procedures described in Phase 2 of Algorithm 4 and in Phases 1 and 3 of Algorithm 5.

4.1. *Tammer-Weidner sorting functions within topological linear spaces*

Throughout this section we assume that E is a real topological linear space, partially ordered by a nontrivial pointed convex cone K. Following Tammer and Weidner [8], for any set $X \subseteq E$ and any point $c^0 \in E$ we define an extended real-valued function $\phi_{X,c^0} : E \to \mathbb{R} \cup \{-\infty, +\infty\}$ for all $x \in E$ by

$$\phi_{X,c^0}(x) := \inf\{s \in \mathbb{R} \mid x \in sc^0 - X\},$$

where $\inf \emptyset = +\infty$. The next result collects some important properties of this function (see Göpfert et al. [6, Th. 2.3.1 and Prop. 2.3.4]):

Proposition 4.1. *Let X be a closed proper subset of E. Assume that there exists a cone $C \subseteq E$ with $\operatorname{int} C \neq \emptyset$, such that*

$$X + \operatorname{int} C \subseteq X, \tag{7}$$

and let $c^0 \in \operatorname{int} C$. Then, the following assertions hold:

1°. *ϕ_{X,c^0} is finite-valued and continuous.*
2°. $\operatorname{epi} \phi_{X,c^0} := \{(x, s) \in E \times \mathbb{R} \mid \phi_{X,c^0}(x) \le s\} = \{(x, s) \in E \times \mathbb{R} \mid x \in sc^0 - X\}$.

3°. *ϕ_{X,c^0} is convex if and only if X is convex.*
4°. *ϕ_{X,c^0} is K-increasing if and only if $X + K \subseteq X$.*
5°. *ϕ_{X,c^0} is strongly K-increasing if and only if $X + (K \setminus \{0\}) \subseteq \operatorname{int} X$.*

As a direct consequence of Propositions 2.4 and 4.1 we get the following result that motivates the use of ϕ_{X,c^0} in nonlinear scalarization methods for solving vector optimization problems.

Corollary 4.2. *Assume that the hypotheses of Proposition 4.1 hold and let $A \subseteq E$ be a nonempty set. For any $x^0 \in \operatorname{argmin}_{x\in A} \phi_{X,c^0}(x)$ the following assertions hold:*

1°. *If $X + (K \setminus \{0\}) \subseteq \operatorname{int} X$, then $x^0 \in \operatorname{MIN}(A \mid K)$.*
2°. *If $X + K \subseteq X$ and $|\operatorname{argmin}_{x\in A} \phi_{X,c^0}(y)| = 1$, then $x^0 \in \operatorname{MIN}(A \mid K)$.*

In preparation of the next Corollary 4.4 we present an auxiliary result.

Lemma 4.3. *Let $C \subseteq E$ be a closed convex cone with nonempty interior. Then, for any $c^0 \in \operatorname{int} C$ and any nonempty sets $X, Y, Z \subseteq E$, the following assertions hold:*

1°. $\operatorname{int} C = \bigcup_{\alpha>0}(\alpha c^0 + \operatorname{int} C)$.
2°. *If X is convex, then $X + \operatorname{int} C = \operatorname{int}(X + C)$.*
3°. *If Y is bounded and $Y + Z \subseteq Y + C$, then $Z \subseteq C$.*
4°. *If Y is compact and $Y + Z \subseteq Y + \operatorname{int} C$, then $Z \subseteq \operatorname{int} C$.*

Proof. The assertions 1°, 2° and 3° are direct consequences of some known results, obtained by Popovici [9, Lem. 2.1], Tanaka and Kuroiwa [10, Th. 2.2], and Urbański [11, Prop. 2.1 (extension of Rådström's cancellation law)], respectively.

In order to prove 4°, assume that Y is compact and $Y + Z \subseteq Y + \operatorname{int} C$. Let $z \in Z$ be an arbitrary point. By 1° it follows that, for any fixed $c^0 \in \operatorname{int} C$, we have $Y + z \subseteq Y + \bigcup_{\alpha>0}(\alpha c^0 + \operatorname{int} C)$, i.e. $Y \subseteq \bigcup_{\alpha>0}(\alpha c^0 - z + Y + \operatorname{int} C)$. Since Y is compact and all the sets $\alpha c^0 - z + Y + \operatorname{int} C$ with $\alpha > 0$ are open, we can find a finite selection of positive real numbers, say $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_p$, such that

$$Y \subseteq \bigcup_{i=1}^{p}(\alpha_i c^0 - z + Y + \operatorname{int} C). \tag{8}$$

Observe that for every $i \in \{2, \ldots, p\}$ we have $(\alpha_i - \alpha_1)c^0 \in]0, \infty[\cdot \operatorname{int} C = \operatorname{int} C$, i.e., $\alpha_i c^0 \in \alpha_1 c^0 + \operatorname{int} C$, hence $\alpha_i c^0 - z + Y + \operatorname{int} C \subseteq (\alpha_1 c^0 + \operatorname{int} C) - z + Y + \operatorname{int} C = \alpha_1 c^0 - z + Y + \operatorname{int} C$. Therefore (8) reduces to $Y \subseteq \alpha_1 c^0 - z + Y + \operatorname{int} C$, which entails

$$Y + z - \alpha_1 c^0 \subseteq Y + C. \tag{9}$$

By applying 3° for the singleton $\{z - \alpha_1 c^0\}$ in the role of Z, we deduce from (9) that $\{z - \alpha_1 c^0\} \subseteq C$, i.e., $z \in \alpha_1 c^0 + C$. Finally, in view of 1°, we infer that $z \in \operatorname{int} C$. Since z has been arbitrarily chosen in Z, we conclude that $Z \subseteq \operatorname{int} C$. □

Corollary 4.4. *Assume that $X = Y + C$, where $Y \subseteq E$ is a nonempty compact convex set and $C \subseteq E$ is a nontrivial closed convex cone with nonempty interior. Let $c^0 \in \operatorname{int} C$. Then the following assertions hold:*

1°. *ϕ_{X,c^0} is finite-valued, continuous and convex.*
2°. *$K \subseteq C$ if and only if ϕ_{X,c^0} is K-increasing.*
3°. *$K \setminus \{0\} \subseteq \operatorname{int} C$ if and only if ϕ_{X,c^0} is strongly K-increasing.*

Proof. First observe that the hypotheses of Proposition 4.1 hold. Indeed, the set $X = Y + C$ is closed, as the sum of a compact convex set and a closed convex one, and it is proper, since $\emptyset \neq C \neq E$. Moreover, X is convex, since it is the sum of two convex sets. Therefore, by Lemma 4.3 (2°) and the fact that $X+C = Y+C+C = Y+C = X$, it follows that

$$X + \operatorname{int} C = \operatorname{int} X, \tag{10}$$

which in particular shows that X satisfies (7).

Now it is easy to see that assertion 1° follows from Proposition 4.1 (1° and 3°).

Also, by Proposition 4.1 (4°), in order to prove assertion 2° it suffices to show that

$$K \subseteq C \iff X + K \subseteq X.$$

Clearly, if $K \subseteq C$, then we have $X + K \subseteq X + C = X$. Conversely, assume that $X + K \subseteq X$, i.e., $Y + C + K \subseteq Y + C$. Recalling that Y is compact, hence bounded, we deduce by Lemma 4.3 (3°) applied for $Z := C + K$, that $C + K \subseteq C$, hence $K \subseteq C$.

Finally, in view of Proposition 4.1 (5°), for proving 3° we just have to show that

$$K \setminus \{0\} \subseteq \operatorname{int} C \iff X + (K \setminus \{0\}) \subseteq \operatorname{int} X.$$

Indeed, if $K \setminus \{0\} \subseteq \operatorname{int} C$, then (10) entails $X + (K \setminus \{0\}) \subseteq \operatorname{int} X$. Conversely, assume that $X + (K \setminus \{0\}) \subseteq \operatorname{int} X$, i.e., $Y + C + (K \setminus \{0\}) \subseteq \operatorname{int}(Y + C)$. Since $Y \subseteq Y + C$, it follows that $Y + (K \setminus \{0\}) \subseteq \operatorname{int}(Y + C)$, which in view of Lemma 4.3 (2°) applied for Y in the role of X, actually means that $Y + (K \setminus \{0\}) \subseteq Y + \operatorname{int} C$. By applying Lemma 4.3 (4°) for $Z := K \setminus \{0\}$, we conclude that $K \setminus \{0\} \subseteq \operatorname{int} C$. □

Corollary 4.4 shows that we can construct various sorting functions for particular choices of Y and C. In the next two examples we present two of them.

Example 4.5. The non-zero linear continuous functionals can be recovered as particular instances of Tammer-Weidner functions. Indeed, let E^* be the topological dual space of E. For any $\varphi \in E^* \setminus \{0\}$, consider $C := \{x \in E \mid \varphi(x) \geq 0\}$. Obviously, C is a nontrivial closed convex cone with nonempty interior, $\operatorname{int} C = \{x \in E \mid \varphi(x) > 0\}$. Consider a point $c^0 \in E$ such that $\varphi(c^0) = 1$ (hence $c^0 \in \operatorname{int} C$) and let $X := C$ (i.e., $Y = \{0\}$ in Corollary 4.4). It is easy to check that for all $x \in E$ we have

$$\phi_{X,c^0}(x) = \varphi(x).$$

Therefore, assertions 2° and 3° of Corollary 4.4 can be refined in view of Proposition 2.3, as follows. Consider the topological polar cone of K and its quasi-interior:

$$K_+^* := \{\varphi \in E^* \mid \forall\, x \in K : \ \varphi(x) \geq 0\};$$
$$K_{++}^* := \{\varphi \in E^* \mid \forall\, x \in K \setminus \{0\} : \ \varphi(x) > 0\}.$$

Then, we have $K \subseteq C$ if and only if $\phi_{X,c^0} = \varphi \in K_+^*$ (i.e., it is K-increasing), and $K \setminus \{0\} \subseteq \operatorname{int} C$ if and only if $\phi_{X,c^0} = \varphi \in K_{++}^*$ (i.e., it is strongly K-increasing).

Example 4.6. Assume that the nontrivial pointed convex cone $K \subseteq E$ is closed and has nonempty interior. Let $c^0 \in \operatorname{int} K$. Then, by choosing $X := K$ (i.e., $Y := \{0\}$ and $C := K$ in Corollary 4.4), the sorting function ϕ_{K,c^0} can be used to characterize the order induced by K (see Göpfert et al. [6, Corollary 2.3.5]). More precisely, for any $x, y \in E$ we have

$$x \leqq_K y \iff \phi_{K,c^0}(x - y) \leq 0. \tag{11}$$

Remark 14. In the particular framework when E is a real locally convex space, the linear continuous functionals can be also used to evaluate the ordering relation induced by a nontrivial pointed closed convex cone $K \subseteq E$, due to the Bipolar Theorem. More precisely, for any points $x, y \in E$ we have

$$x \leqq_K y \iff \forall \varphi \in K_+^* : \varphi(x) \leq \varphi(y).$$

4.2. *Polyhedral convex sorting functions within the Euclidean space* $\mathbb{R}^n$

In this section, we restrict our attention to the particular finite-dimensional Euclidean space $E = \mathbb{R}^n$ ($n \in \mathbb{N}$), endowed with the usual inner product $\langle \cdot, \cdot \rangle$. As usual (see, e.g., Aliprantis and Tourky [12]) we define the polar cone and the annihilator of a nonempty set $S \subseteq \mathbb{R}^n$, respectively, by

$$\begin{aligned} S^+ &:= \{x \in \mathbb{R}^n \mid \forall\, y \in S : \langle x, y \rangle \geq 0\}, \\ S^\perp &:= \{x \in \mathbb{R}^n \mid \forall\, y \in S : \langle x, y \rangle = 0\} = S^+ \cap (-S^+). \end{aligned}$$

Notice that S^+ is a closed convex cone, whose linear part is $S^\perp$.

A set $C \subseteq \mathbb{R}^n$ is said to be a polyhedral cone if there exists a nonempty finite set $U \subseteq \mathbb{R}^n \setminus \{0\}$ such that $C = U^+$. Next we recall some properties of polyhedral cones.

Proposition 4.7. *Let $U \subseteq \mathbb{R}^n \setminus \{0\}$ be a nonempty finite set. The polyhedral cone $C := U^+$ satisfies the following properties:*

1°. *$C \neq \mathbb{R}^n$, hence C is nontrivial if and only if $0 \notin \operatorname{int} \operatorname{conv} U$.*
2°. *C is pointed if and only if $U^\perp = \{0\}$.*
3°. *The interior of C, i.e., $\operatorname{int} C = \{x \in \mathbb{R}^n \mid \forall\, u \in U : \langle u, x \rangle > 0\}$, is nonempty if and only if $0 \notin \operatorname{conv} U$.*
4°. *The polar of C is a polyhedral cone, namely $C^+ = (U^+)^+ = \mathbb{R}_+ \cdot \operatorname{conv} U$, whose interior is $\operatorname{int} C^+ = \{\lambda \in \mathbb{R}^n \mid \forall\, x \in C \setminus \{0\} : \langle \lambda, x \rangle > 0\}$.*

We say that $Y \subseteq \mathbb{R}^n$ is a polytope if it is the convex hull of a nonempty finite set $P \subseteq \mathbb{R}^n$, i.e., $Y = \operatorname{conv} P$. A set $X \subseteq \mathbb{R}^n$ is said to be polyhedral if it can be written as the sum of a polytope and a polyhedral cone, i.e., $X = Y + C$, where $Y = \operatorname{conv} P$ and $C = U^+$ for some nonempty finite sets $P \subseteq \mathbb{R}^n$ and $U \subseteq \mathbb{R}^n \setminus \{0\}$.

According to Borwein and Lewis [13], we say that $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a polyhedral convex function if its epigraph is a polyhedral convex set in $\mathbb{R}^{n+1}$.

Proposition 4.8. *Assume that the hypotheses of Proposition 4.1 hold within $E = \mathbb{R}^n$. Then, the function ϕ_{X,c^0} is polyhedral convex if and only if X is a polyhedral set.*

Proof. Consider the linear operator $L : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ defined for all $(x, s) \in \mathbb{R}^n \times \mathbb{R}$ by $L(x, s) := sc^0 - x$. It is easily seen that L is surjective. By Proposition 4.1 (2°) we have $\operatorname{epi} \phi_{X,c^0} = \{(x, s) \in \mathbb{R}^n \times \mathbb{R} \,|\, L(x, s) \in X\} = L^{-1}(X)$ and, consequently, $X = L(L^{-1}(X)) = L(\operatorname{epi} \phi_{X,c^0})$. By a classical result in Convex Analysis (see, e.g., Rockafellar [14, Th. 19.3]) we deduce that X is polyhedral if and only if $\operatorname{epi} \phi_{X,c^0}$ is polyhedral. $\square$

Proposition 4.9. *Let $X = Y + C$ be a polyhedral set, where $Y = \operatorname{conv} P$ and $C = U^+$ for some nonempty finite sets $P \subseteq \mathbb{R}^n$ and $U \subseteq \mathbb{R}^n \setminus \{0\}$, such that $0 \notin \operatorname{conv} U$, i.e., $\operatorname{int} C \neq \emptyset$. Consider a point $c^0 \in \operatorname{int} C$. Let $K \subseteq \mathbb{R}^n$ be a nontrivial closed pointed convex cone (not necessarily polyhedral). Then the following assertions hold:*

1°. *ϕ_{X,c^0} is polyhedral convex.*
2°. *For all $x \in \mathbb{R}^n$ we have*

$$\phi_{X,c^0}(x) = \min_{p \in P} \max_{u \in U} \frac{\langle u, x \rangle + \langle u, p \rangle}{\langle u, c^0 \rangle}.$$

3°. *$K \subseteq C$ if and only if ϕ_{X,c^0} is K-increasing.*
4°. *$K \setminus \{0\} \subseteq \operatorname{int} C$ if and only if ϕ_{X,c^0} is strongly K-increasing.*

Proof. Assertion 1° follows by Proposition 4.8, 3° follows by Corollary 4.4 (2°) and 4° follows by Corollary 4.4 (3°). Moreover, 2° holds, since for all $x \in \mathbb{R}^n$ we have

$$\begin{aligned}
\phi_{X,c^0}(x) &= \inf\{s \in \mathbb{R} \mid sc^0 - x \in Y + C\} \\
&= \inf\{s \in \mathbb{R} \mid \exists\, y \in Y\ \forall\, u \in U :\ \langle u, sc^0 - x - y \rangle \geq 0\} \\
&= \inf\{s \in \mathbb{R} \mid \exists\, y \in Y :\ \max_{u \in U} \frac{\langle u, x \rangle + \langle u, y \rangle}{\langle u, c^0 \rangle} \leq s\} \\
&= \min_{y \in Y} \max_{u \in U} \frac{\langle u, x \rangle + \langle u, y \rangle}{\langle u, c^0 \rangle} \\
&= \min_{p \in P} \max_{u \in U} \frac{\langle u, x \rangle + \langle u, p \rangle}{\langle u, c^0 \rangle},
\end{aligned}$$

where $\langle u, c^0 \rangle > 0$ for all $u \in U$, in view of Proposition 4.7 (3°). Notice that the last equality holds, since we minimize a concave function (namely the maximum of a finite number of affine functions) over the polytope S, and therefore the minimum is attained at some extreme point of S hence an element of P, by a classical argument in convex analysis (see, e.g., Rockafellar [14, Cor. 32.3.4]). $\square$

Remark 15. By choosing $X = C = K = U^+$ (i.e., $P = \{0\}$) in Proposition 4.9 we get $\phi_{X,c^0}(x) = \max_{u \in U} \frac{\langle u, x \rangle}{\langle u, c^0 \rangle}$. Recalling that $\langle u, c^0 \rangle > 0$ for all $u \in U$, it follows by (11) that for any points $x, y \in \mathbb{R}^n$ we have

$$x \leqq_K y \iff \forall u \in U :\ \langle u, x \rangle \leq \langle u, y \rangle. \tag{12}$$

Of course, this also follows by the definition of $K = U^+$.

Now we are going to consider the particular framework where $K = \mathbb{R}^n_+$ is the standard ordering cone and $X \subseteq \mathbb{R}^n$ is a special polyhedral set, defined by means of a certain block norm, following an approach developed by Tammer and Winkler [15]. Since we will operate with different index sets, we denote

$$I_k := \{1, \ldots, k\} \quad \text{for any } k \in \mathbb{N}.$$

Let $\gamma : \mathbb{R}^n \to \mathbb{R}$ be a norm and let

$$B_\gamma := \{x \in \mathbb{R}^n \,|\, \gamma(x) \le 1\}$$

be the corresponding unit ball. Notice that B_γ is convex, compact and symmetric with respect to the origin. We suppose that γ is a block norm, meaning that

$$B_\gamma = \{x \in \mathbb{R}^n \,|\, \forall i \in I_q : \ \langle v^i, x\rangle \le 1\},$$

where $v^1, \ldots, v^q \in \mathbb{R}^n \setminus \{0\}$, $q \in \mathbb{N}$, with $\mathbb{R}_+ \cdot v^i \ne \mathbb{R}_+ \cdot v^j$ for all $i, j \in I_q$, $i \ne j$. Let

$$I^{\text{act}} := \left\{ i \in I_q \,\middle|\, \{x \in \mathbb{R}^n \,|\, \langle v^i, x\rangle = 1\} \cap B_\gamma \cap \operatorname{int} \mathbb{R}^n_+ \ne \emptyset \right\}. \tag{13}$$

Notice that I^{act} is nonempty. By means of the polyhedral set

$$X_\gamma := \{x \in \mathbb{R}^n \,|\, \forall i \in I^{\text{act}} : \langle v^i, x\rangle \le 1\},$$

we define for any point $w^0 \in \mathbb{R}^n$ the (polyhedral) set

$$X := -X_\gamma - w^0.$$

In order to give an easier representation for the index set I^{act}, in the sequel we assume that the block norm γ is absolute, which means that for every $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n) \in \mathbb{R}^n$ we have $\gamma(x) = \gamma(\overline{x})$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $|x_i| = |\overline{x}_i|$, $i \in I_n$ (see, e.g., Schandl et al. [16]).

The next lemma presents equivalent characterizations for the index set I^{act} defined by (13) (see Tammer and Winkler [15, Lem. 3.2]). Recall that the block norm γ is said to be oblique if it is absolute and $(x - \mathbb{R}^n_+) \cap \mathbb{R}^n_+ \cap \operatorname{bd} B_\gamma = \{x\}$ for every $x \in \mathbb{R}^n_+ \cap \operatorname{bd} B_\gamma$.

Lemma 4.10. *The following assertions hold:*

1°. $I^{\text{act}} = \{i \in I_q \,|\, v^i \in \mathbb{R}^n_+ \setminus \{0\}\}$.
2°. *If γ is oblique, then* $I^{\text{act}} = \{i \in I_q \,|\, v^i \in \operatorname{int} \mathbb{R}^n_+\}$.

Consider the set $V := \{v^i \,|\, i \in I^{\text{act}}\}$. Then, for a given $c^0 \in \operatorname{int} \mathbb{R}^n_+$, we define a function $\varphi_{V,c^0,w^0} : \mathbb{R}^n \to \mathbb{R}$ for all $x \in \mathbb{R}^n$ by

$$\varphi_{V,c^0,w^0}(x) := \max_{i \in I^{\text{act}}} \frac{\langle v^i, x\rangle - \langle v^i, w^0\rangle - 1}{\langle v^i, c^0\rangle}. \tag{14}$$

Remark 16. For every $c^0 \in \operatorname{int} \mathbb{R}^n_+$, we have $\langle v^i, c^0\rangle > 0$ for all $i \in I^{\text{act}}$, since $v^i \in \mathbb{R}^n_+ \setminus \{0\}$ by Lemma 4.10.

The next result is based on Tammer and Winkler [15, Cor. 3.1, 3.2 and 3.3].

Proposition 4.11. *The following assertions hold:*

1°. *φ_{V,c^0,w^0} is polyhedral convex and $\mathbb{R}^n_+$-increasing.*
2°. *For all $x \in \mathbb{R}^n$ we have*

$$\varphi_{V,c^0,w^0}(x) = \phi_{X,c^0} = \inf\{s \in \mathbb{R} \,|\, x \in sc^0 + X_\gamma + w^0\}.$$

3°. *If γ is oblique, then φ_{V,c^0,w^0} is strongly $\mathbb{R}^n_+$-increasing.*

We conclude this section by showing that, in certain situations, a nonlinear sorting function of type φ_{V,c^0,w^0} could be preferred to a linear one. For convenience, given any $\lambda \in \mathbb{R}^n$ we denote by $\varphi_\lambda : \mathbb{R}^n \to \mathbb{R}$ the linear function defined by

$$\varphi_\lambda(x) := \langle \lambda, x \rangle \quad \text{for all } x \in \mathbb{R}^n. \tag{15}$$

Example 4.12. Let $\mathbb{R}^2$ be endowed with the standard ordering cone $K := \mathbb{R}^2_+$ and let $A := \{a^1, \ldots, a^{10}\} \subseteq \mathbb{R}^2$ be a finite set. In Figure 1 we illustrate an enumeration $A = \{a^{j_1}, \ldots, a^{j_{10}}\}$ satisfying (3) for a nonlinear sorting function $\varphi = \varphi_{V,c^0,w^0}$ with

$$V := \{(1,6),(6,1)\},\ c^0 := (1,1) \ \text{ and } \ w^0 := (0,0).$$

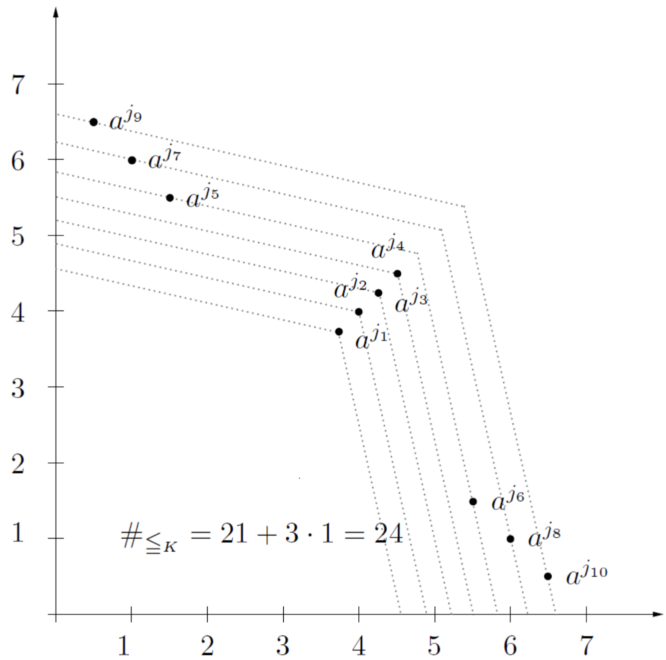

Figure 1. Level lines of the nonlinear sorting function φ_{V,c^0,w^0}.

Notice that φ_{V,c^0,w^0} is strongly $\mathbb{R}^2_+$-increasing by Proposition 4.11. Denote by the symbol "$\#_{\leqq_K}$" the number of pairwise comparisons of points with respect to the ordering relation $\leqq_K$. By using the nonlinear sorting function φ_{V,c^0,w^0}, we get $\#_{\leqq_K} = 21 + 3 \cdot 1 = 24$. In contrast, it can be easily seen that for a linear sorting function $\varphi = \varphi_\lambda$ with $\lambda \in \operatorname{int} \mathbb{R}^2_+$, we have

$$33 = 21 + 3 \cdot 4 \leq \#_{\leqq_K} \leq 21 + 3 \cdot 7 = 42.$$

In Figure 2 one can see that for the case $\lambda = (6,1)$ we have $\#_{\leqq_K} = 21 + 3 \cdot 4 = 33$. Hence, in this example a nonlinear sorting function could be preferred to a linear one.

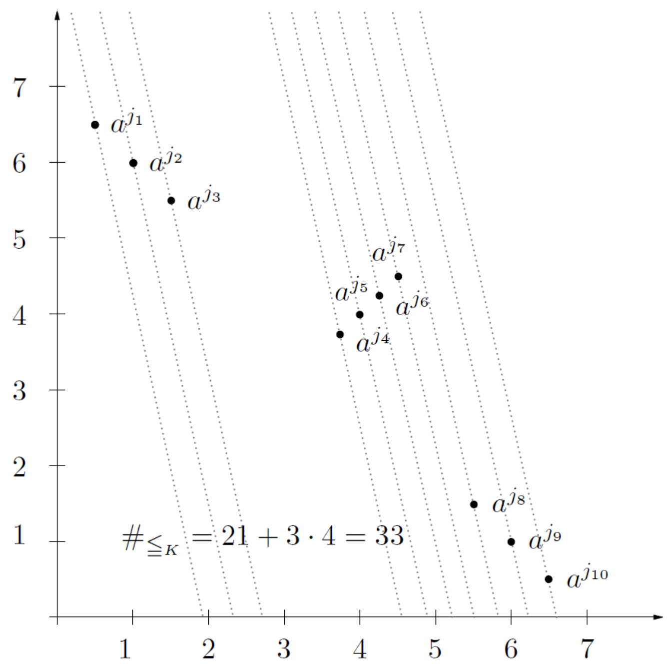

Figure 2. Level lines of the linear sorting function φ_λ for $\lambda = (6, 1)$.

5. Implementation of the new algorithms for polyhedral ordering cones and linear sorting functions

In this section we present implementable versions of our Algorithms 4 and 5 for computing the minimal elements of a finite set with respect to a polyhedral cone, by using linear sorting functions. As in Section 4.2, we consider a finite set

$$A := \{a^1, \ldots, a^p\} \subseteq E = \mathbb{R}^n,$$

where the points $a^1, \ldots, a^p$ are pairwise distinct. We are interested to compute the set $\mathrm{MIN}(A \mid K)$ of all minimal elements of A with respect to a polyhedral cone

$$K := U^+ \ \text{ with } \ U := \{u^1, \ldots, u^m\} \subseteq \mathbb{R}^n \setminus \{0\}, \tag{16}$$

where $u^1, \ldots, u^m$ are pairwise distinct, $0 \notin \operatorname{int} \operatorname{conv} U$ and $U^\perp = \{0\}$.

In what follows, for any $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$, we denote

$$\lambda_\alpha := \sum_{t \in I_m} \alpha_t u^t$$

and we introduce the linear scalar function $\eta_\alpha := \varphi_{\lambda_\alpha}$, defined according to (15) as

$$\eta_\alpha(x) := \varphi_{\lambda_\alpha}(x) = \langle \lambda_\alpha, x \rangle = \sum_{t \in I_m} \alpha_t \langle u^t, x \rangle = \sum_{t \in I_m} \alpha_t \varphi_{u^t}(x) \tag{17}$$

for all $x \in \mathbb{R}^n$. The following result shows that, by an appropriate choice of α, the corresponding function η_α can be used for the sorting procedures of our algorithms.

Proposition 5.1. *If $\alpha = (\alpha_1, \ldots, \alpha_m) \in \operatorname{int} \mathbb{R}^m_+$, then the function η_α (i.e., φ_{λ_α}) is strongly K-increasing.*

Proof. By Proposition 2.3 it suffices to prove that $\lambda_\alpha \in \operatorname{int} K^+$, i.e.,

$$\langle \lambda_\alpha, x \rangle > 0 \quad \text{for any} \quad x \in K \setminus \{0\} \tag{18}$$

according to Proposition 4.7 (4°), applied to K in the role of C.

To this aim, consider an arbitrary point $x \in K \setminus \{0\}$. Since

$$\alpha_t > 0 \quad \text{and} \quad \langle u^t, x \rangle \geq 0 \quad \text{for all} \quad t \in I_m, \tag{19}$$

we can easily deduce by (17) that $\langle \lambda_\alpha, x \rangle \geq 0$. Actually, this inequality is strict. Indeed, assume to the contrary that $\langle \lambda_\alpha, x \rangle = 0$. This means that $\sum_{t \in I_m} \alpha_t \langle u^t, x \rangle = 0$ by (17). In view of (19), it follows that $\langle u^t, x \rangle = 0$ for all $t \in I_m$. By assumption $U^\perp = \{0\}$, we get $x = 0$, a contradiction. Thus (18) holds. □

Now, we are ready to present the implementation of Algorithm 4 for computing the minimal elements of A with respect to K (namely Algorithm 6).

Algorithm 6: Adaptation of Algorithm 4 for a polyhedral ordering cone and a linear sorting function

Input: The cone $K = U^+$ where $U = \{u^1, \dots, u^m\}$, the set $A = \{a^1, \dots, a^p\}$, and some $\alpha = (\alpha_1, \dots, \alpha_m) \in \operatorname{int} \mathbb{R}^m_+$.

```
/* Phase 1                                                    */
for j ← 1 to p do
    η^j ← 0;
    for t ← 1 to m do
        φ_t^j ← ⟨u^t, a^j⟩;
        η^j ← α_t · φ_t^j + η^j;
    end
end
/* Phase 2 (Pre-sorting procedure)                            */
Compute an enumeration of the given points of the set A
such that A = {a^{j_1}, ..., a^{j_p}} and η^{j_1} ≤ η^{j_2} ≤ ... ≤ η^{j_p}.
/* Phase 3 (Reduction procedure)                              */
i ← 1;
ind_i ← j_1;
B ← {a^{ind_i}};
for k ← 2 to p do
    s ← 0;
    for l ← 1 to i do
        for t ← 1 to m do
            if φ_t^{ind_l} > φ_t^{j_k} then
                s ← s + 1;
                break;
            end
        end
        if s < l then
            break;
        end
    end
    if s = i then
        i ← i + 1;
        ind_i ← j_k;
        B ← B ∪ {a^{ind_i}};
    end
end
T ← B;
```

Output: The set T (representing the set of all minimal elements of A).

Remark 17. Algorithm 6 is an output-sensitive method, in view of Remark 12 (as being the implementation of Algorithm 4 for $E = \mathbb{R}^n$ and $K = U^+$). It computes the set $T = \mathrm{MIN}(A \mid K)$ with complexity

$$\mathcal{O}(|A| \cdot |U| \cdot n + |A| \cdot \log(|A|) + |A| \cdot |T| \cdot |U|). \tag{20}$$

In particular, when $K = \mathbb{R}^n_+$ we have $|U| = n$, hence (20) becomes

$$\mathcal{O}(|A| \cdot n^2 + |A| \cdot \log(|A|) + |A| \cdot |T| \cdot n).$$

Actually, when $K = \mathbb{R}^n_+$ one can also compute T by any method known in the literature for discrete vector optimization with respect to the standard ordering cone. For instance, the classical algorithm of Kung, Luccio and Preparata [17] computes T with complexity $\mathcal{O}(|A| \cdot \log(|A|))$ for $n \in \{2, 3\}$ and $\mathcal{O}(|A| \cdot \log^{n-2}(|A|))$ for $n \geq 4$.

Remark 18. Given a nontrivial pointed polyhedral cone $K = U^+ \subseteq \mathbb{R}^n$, there exists a $m \times n$ matrix W with $m = |U| \geq n$ such that

$$K = \{y \in \mathbb{R}^n \mid Wy \in \mathbb{R}^m_+\}.$$

Then, according to Engau and Wiecek [18, Th. 4.1] (see also Dempe, Eichfelder and Fliege [19]), we have

$$\mathrm{MIN}(A \mid K) = \{a \in A \mid Wa \in \mathrm{MIN}(\tilde{A} \mid \mathbb{R}^m_+)\},$$

where $\tilde{A} := \{Wa^1, \cdots, Wa^p\}$. Therefore, an alternative way for computing the set $\mathrm{MIN}(A \mid K)$ is to derive it from $\mathrm{MIN}(\tilde{A} \mid \mathbb{R}^m_+)$, which on its turn can be generated for instance by the algorithm of Kung, Luccio and Preparata [17] with complexity $\mathcal{O}(|\tilde{A}| \cdot \log(|\tilde{A}|))$ for $m \in \{2, 3\}$ and $\mathcal{O}(|\tilde{A}| \cdot \log^{m-2}(|\tilde{A}|))$ when $m \geq 4$. However, whenever $n \geq 3$ it may happen that $m \gg n$. For instance if $K \subseteq \mathbb{R}^3$ is a polyhedral ordering cone which gives a good approximation of a Bishop-Phelps ("ice cream" type) cone, then $m \gg n = 3$. In this case our Algorithm 6 could be a better approach.

The corresponding algorithmic implementation to Algorithm 5 for computing $\mathrm{MIN}(A \mid K)$ is formulated in Algorithm 7.

Algorithm 7: Adaptation of Algorithm 5 for a polyhedral ordering cone and a linear sorting function

Input: The cone $K = U^+$ where $U = \{u^1, \dots, u^m\}$, the set $A = \{a^1, \dots, a^p\}$, and some $\alpha = (\alpha_1, \dots, \alpha_m) \in \operatorname{int} \mathbb{R}^m_+$.

```
/* Phase 1                                                        */
for j ← 1 to p do
    for t ← 1 to m do
        φ_t^j ← ⟨u^t, a^j⟩;
    end
end
/* Phase 2 (Reduction procedure)                                  */
i ← 1;
ind_i ← 1;
B ← {a^{ind_i}};
for j ← 2 to p do
    s ← 0;
    for l ← 1 to i do
        for t ← 1 to m do
            if φ_t^{ind_l} > φ_t^j then
                s ← s + 1;
                break;
            end
        end
        if s < l then
            break;
        end
    end
    if s = i then
        i ← i + 1;
        ind_i ← j;
        B ← B ∪ {a^{ind_i}};
    end
end
/* Phase 3 (Intermediate sorting procedure & reduction procedure) */
Apply Algorithm 6 for the set B (in the role of A) as input to get the set T.
```

Output: The set T (representing the set of all minimal elements of A).

Remark 19. The computational complexity of Algorithm 7 is

$$\mathcal{O}(|A|^2 + 2 \cdot |A| \cdot |U| \cdot n + |A| \cdot \log(|A|) + |A| \cdot |T| \cdot |U|).$$

6. Application to a bi-objective optimization problem

Our new algorithms introduced in Sections 3 and 5 can be used for approximating the sets of minimal outcomes of certain continuous vector optimization problems, via a discretization approach proposed by Jahn [4], namely the "Multiobjective search algorithm with subdivision technique" (MOSAST).

In this section we will apply our algorithms to a particular continuous bi-objective test problem (known in the literature as being very difficult to solve). A detailed comparative analysis of our algorithms and other classical methods is provided, based on computational experiments in MATLAB.

6.1. *Continuous bi-objective optimization problems and the approximation of their solution sets*

Consider a continuous vector-valued function $f = (f_1, f_2) : S \to \mathbb{R}^2$, defined on a nonempty compact set $S \subseteq \mathbb{R}^2$, and let $f(S) := \{f(x) \mid x \in S\}$. Assume that $K \subseteq \mathbb{R}^2$ is a nontrivial pointed polyhedral cone. An element $x \in S$ is said to be an efficient solution of the vector optimization problem

$$\begin{cases} f(x) = (f_1(x), f_2(x)) \to \min \quad \text{w.r.t. } K \\ x \in S \end{cases} \tag{21}$$

if $f(x) \in \mathrm{MIN}(f(S) \mid K)$. In what follows we denote by

$$\mathrm{EFF}(S \mid f, K) := f^{-1}(\mathrm{MIN}(f(S) \mid K))$$

the set of all efficient solutions to problem (21).

Since the computation of the set $\mathrm{MIN}(f(S) \mid K)$ is in general a very difficult task, we are interested in finding a good enough approximation of $\mathrm{MIN}(f(S) \mid K)$. One possibility is to generate a finite set $\widetilde{S} \subseteq S$ and to compute the set $\mathrm{MIN}(f(\widetilde{S}) \mid K)$. Of course, in general there is no containment relation between the sets $\mathrm{MIN}(f(S) \mid K)$ and $\mathrm{MIN}(f(\widetilde{S}) \mid K)$, but we always have

$$\mathrm{MIN}(f(\widetilde{S}) \mid K) \subseteq \mathrm{MIN}(f(S) \mid K) + K,$$

by the domination property of the nonempty compact set $f(S)$ (see, e.g., Göpfert et al. [6, Prop. 3.2.20]). In what follows $\widetilde{S}$ will be generated by an iterative procedure, as a union of finite sets

$$\widetilde{S} := \widetilde{S}_0 \cup \widetilde{S}_1 \cup \ldots \cup \widetilde{S}_l.$$

Then, for each $i \in \{0, 1, \ldots, l\}$ we will compute the set

$$T_i := \mathrm{MIN}(f(\widetilde{S}_i) \mid K)$$

by our methods from the previous sections for the finite set $A := f(\widetilde{S}_i)$.

Denoting

$$T := T_0 \cup T_1 \cup \ldots \cup T_l,$$

we can deduce by Remark 4 and Lemma 2.7 that

$$\mathrm{MIN}(f(\widetilde{S}) \mid K) = \mathrm{MIN}\,(T \mid K)\,. \tag{22}$$

The right-hand side term of (22) will be computed by applying our methods for the set $A := T$. Next, we present the main steps of this iterative procedure:

Step 1. Compute a first approximation of $\mathrm{MIN}(f(S) \mid K)$:

- Consider a box (i.e., an axis-parallel rectangle) $B_0 \subseteq \mathbb{R}^2$ such that $S \subseteq B_0$. Generate a finite set of random points $\widetilde{B_0} \subseteq B_0$ and define $\widetilde{S}_0 := \widetilde{B}_0 \cap S$.
- Compute the sets $T_0 := \mathrm{MIN}(f(\widetilde{S}_0) \mid K)$ and $\mathrm{EFF}(\widetilde{S}_0 \mid f, K) := f^{-1}(T_0)$.

Step 2. Apply a subdivision technique similar to that introduced by Dellnitz et al. [20] (see also Jahn [4]) in order to improve the approximation of the set $\mathrm{MIN}(f(S) \mid K)$:

- Consider a system $\mathcal{B}$ of boxes in $\mathbb{R}^2$ that covers the set S, i.e., $S \subseteq \bigcup_{B \in \mathcal{B}} B$.
- Determine the subsystem of boxes

  $$\mathcal{B}_{act} := \{B \in \mathcal{B} \mid B \cap \mathrm{EFF}(\widetilde{S}_0 \mid f, K) \neq \emptyset\}$$

  and let $\{B_1, \ldots, B_l\} = \mathcal{B}_{act}$ be an enumeration of $\mathcal{B}_{act}$ with $l = |\mathcal{B}_{act}|$.
- For each $i \in I_l$ generate a finite set of random points $\widetilde{B}_i \subseteq B_i$ and let $\widetilde{S}_i := \widetilde{B}_i \cap S$.
- For every $i \in I_l$ compute the set $T_i := \mathrm{MIN}(f(\widetilde{S}_i) \mid K)$.

Step 3. Compute the set $\mathrm{MIN}\,(T \mid K)\,.$

Remark 20. The iterative procedure described above represents a modification of the MOSAST method. In contrast to the classical approach by Jahn [4] (see also Limmer et al. [21]), we use a fixed number of randomly generated points. In order to get a good approximation of the set $\mathrm{MIN}(f(S) \mid K)$, the cardinality of $\widetilde{S}_0$ should be high enough. In Step 2 we have to solve a family of l discrete vector optimization problems. Since l can be very high, it is convenient to consider sets $\widetilde{S}_i$, $i \in I_l$, with reasonable small cardinality (significantly smaller than $|\widetilde{S}_0|$).

In what follows we present the pseudo-code (Algorithm 8) of our iterative procedure.

Algorithm 8: SPECIAL INSTANCE OF MOSAST

Input: The nonempty bounded set $S \subseteq \mathbb{R}^2$, the objective function $f : S \to \mathbb{R}^2$, a box $B_0 := [P_1^{min}, P_1^{max}] \times [P_2^{min}, P_2^{max}] \subseteq \mathbb{R}^2$, such that $S \subseteq B_0$, and four positive integers (custom parameters) $\#_{step1}$, $\#_{step2}$, $\#_{int1}$, $\#_{int2}$.

/* Step 1 */

Generate a set of random points, $\widetilde{B_0} \subseteq B_0$ with $|\widetilde{B_0}| = \#_{step1}$;

$\widetilde{S}_0 \leftarrow \widetilde{B_0} \cap S$;

$T_0 \leftarrow \mathrm{MIN}(f(\widetilde{S}_0) \mid K)$;

$\mathrm{EFF}(\widetilde{S}_0 \mid f, K) \leftarrow f^{-1}(T_0)$;

/* Step 2 */

$T \leftarrow T_0$;

for $k \leftarrow 1$ **to** $\#_{int1} + 1$ **do**

    $P_1^k \leftarrow P_1^{min} + \frac{k-1}{\#_{int1}} \cdot (P_1^{max} - P_1^{min})$;

end

for $t \leftarrow 1$ **to** $\#_{int2} + 1$ **do**

    $P_2^t \leftarrow P_2^{min} + \frac{t-1}{\#_{int2}} \cdot (P_2^{max} - P_2^{min})$;

end

$l \leftarrow 0$;

for $k \leftarrow 1$ **to** $\#_{int1}$ **do**

    for $t \leftarrow 1$ **to** $\#_{int2}$ **do**

        $B \leftarrow [P_1^k, P_1^{k+1}] \times [P_2^t, P_2^{t+1}]$;

        if $|B \cap \mathrm{EFF}(\widetilde{S}_0 \mid f, K)| > 0$ **then**

            $l \leftarrow l + 1$;

            $B_l \leftarrow B$;

            Generate a set of random points, $\widetilde{B_l} \subseteq B_l$ with $|\widetilde{B_l}| = \#_{step2}$;

            $\widetilde{S}_l \leftarrow \widetilde{B_l} \cap S$;

            $T_l \leftarrow \mathrm{MIN}(f(\widetilde{S}_l) \mid K)$;

            $T \leftarrow T \cup T_l$;

        end

    end

end

/* Step 3 */

$\widetilde{T} \leftarrow \mathrm{MIN}(T \mid K)$;

$\mathrm{EFF}(\widetilde{S} \mid f, K) \leftarrow f^{-1}(\widetilde{T})$;

Output: The set $\widetilde{T}$ (representing the minimal elements of the set $f(\widetilde{S})$) and the set $\mathrm{EFF}(\widetilde{S} \mid f, K)$ (representing the corresponding set of efficient solutions).

6.2. *Comparative analysis of our algorithms for Jahn's test problem*

As test problem for our numerical experiments we will consider a particular vector optimization problem of type (21), known in the literature for being difficult to solve (see Jahn [4, Ex 2]). We mention that our comparative analysis is based on numerical results obtained by implementing the algorithms in MATLAB 2016a on a Core i5-7200U 2x 2.50GHz CPU, 16GB ram computer. Sorting procedures are based on the MATLAB function "sort".

Example 6.1 (Jahn's test problem)**.** Consider the feasible set $S \subseteq \mathbb{R}^2$ of all points $x = (x_1, x_2) \in \mathbb{R}^2$ that satisfy the following constraints

$$\begin{cases} -1.5 \leq x_1 \leq 1 \\ 0 \leq x_2 \leq 2.25 \\ x_1^2 - x_2 \leq 0 \\ x_1 + 2x_2 - 3 \leq 0. \end{cases}$$

Define the objective function $f : S \to \mathbb{R}^2$ for all $x = (x_1, x_2) \in S$ by

$$f(x) = (f_1(x), f_2(x)) := (-x_1, x_1 + x_2^2 - \cos(50x_1)).$$

The set S is illustrated in Figure 3 (color light grey) while the outcome set $f(S)$ is illustrated in Figure 4 (color light grey).

In the following two sections, 6.2.1 and 6.2.2, we will study this problem when $K = \mathbb{R}^2_+$ is the standard ordering cone, and when $K = U^+$ is an another polyhedral ordering cone, respectively.

We start our analysis by applying Algorithm 8 to Jahn's test problem, where $B_0 := [-1.5, 1] \times [0, 2.25]$ and $\#_{int} := \#_{int1} = \#_{int2} = 30$, while the sets $\mathrm{MIN}(f(\widetilde{S}_0) \mid K)$, $\mathrm{MIN}(f(\widetilde{S}_i) \mid K)$, $i \in I_l$, and $\mathrm{MIN}\,(T \mid K)$, are computed by means of Algorithm 3 with:

- $\varphi := \varphi_\lambda$, $\lambda = (1, 1)$, when $K = \mathbb{R}^2_+$ (in Section 6.2.1)
- $\varphi := \varphi_{\lambda_\alpha} = \eta_\alpha$, $\alpha = (1, 1)$, when $K = U^+$ (in Section 6.2.2).

Then, we use the sets $\widetilde{S}_0, \widetilde{S}_1, \ldots, \widetilde{S}_l$ as input data for the next Algorithm 9, where all the sets $\mathrm{MIN}(f(\widetilde{S}_0) \mid K)$, $\mathrm{MIN}(f(\widetilde{S}_i) \mid K)$, $i \in I_l$, and $\mathrm{MIN}\,(T \mid K)$ are computed by the same algorithm, corresponding to the so-called "Procedure $*$" indicated in Table 1 (within Section 6.2.1) and Table 3 (within Section 6.2.2).

Algorithm 9: PROCEDURE $*$

Input: The sets $\widetilde{S}_0, \widetilde{S}_1, \ldots, \widetilde{S}_l$.
$T \leftarrow \mathrm{MIN}(f(\widetilde{S}_0) \mid K)$;
for $t \leftarrow 1$ **to** l **do**
  $T \leftarrow T \cup \mathrm{MIN}(f(\widetilde{S}_t) \mid K)$;
end
$\widetilde{T} \leftarrow \mathrm{MIN}(T \mid K)$;
Output: The set $\widetilde{T}$ (representing an approximation of the set $\mathrm{MIN}(f(S) \mid K)$).

6.2.1. Jahn's test problem when K is the standard ordering cone

In this subsection we consider the standard ordering cone $K = \mathbb{R}^2_+$. We analyse twelve different procedures, as listed in Table 1. Among them, Procedures III-VIII use linear sorting functions of type $\varphi = \varphi_\lambda : \mathbb{R}^2 \to \mathbb{R}$, $\lambda \in \operatorname{int} \mathbb{R}^2_+$, given by (15), while the Procedures IX - XII use nonlinear sorting functions of type $\varphi = \varphi_{V,c^0,w^0} : \mathbb{R}^2 \to \mathbb{R}$, given by (14), where $V \subseteq \mathbb{R}^2$, $c^0 := (1,1)$ and $w^0 := (0,0)$. All considered sorting functions are strongly $\mathbb{R}^2_+$-increasing, in view of Propositions 2.3 and 4.11.

Proc.∗	Alg.	Pre-sorting	Intermediate sorting	Parameters
I	1	-	-	-
II	3	-	-	-
III	4	$\varphi := \varphi_\lambda$	-	$\lambda = (1,2)$
IV	5	-	$\varphi := \varphi_\lambda$	$\lambda = (1,2)$
V	4	$\varphi := \varphi_\lambda$	-	$\lambda = (1,1)$
VI	5	-	$\varphi := \varphi_\lambda$	$\lambda = (1,1)$
VII	4	$\varphi := \varphi_\lambda$	-	$\lambda = (2,1)$
VIII	5	-	$\varphi := \varphi_\lambda$	$\lambda = (2,1)$
IX	4	$\varphi := \varphi_{V,c^0,w^0}$	-	$V = \{(1,2),(2,1)\}$
X	5	-	$\varphi := \varphi_{V,c^0,w^0}$	$V = \{(1,2),(2,1)\}$
XI	4	$\varphi := \varphi_{V,c^0,w^0}$	-	$V = \{(1,3),(2.5,2.5),(3,1)\}$
XII	5	-	$\varphi := \varphi_{V,c^0,w^0}$	$V = \{(1,3),(2.5,2.5),(3,1)\}$

Table 1. Procedures applied for Jahn's test problem when $K = \mathbb{R}^2_+$ is the standard ordering cone.

For convenience we denote by $\widetilde{B} := \widetilde{B}_1 \cup \widetilde{B}_2 \cup \ldots \cup \widetilde{B}_l$ the set of all randomly generated points, hence $\widetilde{S} = \widetilde{B} \cap S$. Also, we adopt the notation $\#_{\leqq_K}$ from Example 4.12 for the number of pairwise comparisons with respect to the ordering relation $\leqq_K$.

In what follows we will analyse the computational results obtained by applying Algorithm 9 for Jahn's test problem. The running times (in seconds) and the number of pairwise comparisons needed to compute the set $\widetilde{T} = \mathrm{MIN}(f(\widetilde{S}) \mid \mathbb{R}^2_+)$ by Algorithm 9 are listed in Table 2.

	$\#_{step1}$	10^6	10^7	10^7
	$\#_{step2}$	10^4	10^4	10^5
	$\#_{int}$	30	30	30
	$\lvert\widetilde{B}\rvert$	1 260 000	10 220 000	12 200 000
	$\lvert\widetilde{S}\rvert$	627 047	4 758 085	5 896 073
	l	26	22	22
	$\lvert T\rvert$	15 786	14 832	49 596
	$\lvert\widetilde{T}\rvert$	1 964	2 808	6 986
I	Runtime	27.6	672.8	940.0
	$\#_{\leqq_K}$	684 510 944	16 952 327 907	23 032 520 468
II	Runtime	2.4	5.3	37.9
	$\#_{\leqq_K}$	50 301 957	107 473 149	843 505 826
III	Runtime	4.9	86.4	127.3
	$\#_{\leqq_K}$	129 438 579	2 440 717 924	4 123 833 895
IV	Runtime	2.8	7.8	37.7
	$\#_{\leqq_K}$	54 701 638	117 968 982	907 403 128
V	Runtime	2.8	45.6	87.6
	$\#_{\leqq_K}$	87 408 106	1 192 388 508	2 823 993 735
VI	Runtime	2.3	6.3	35.3
	$\#_{\leqq_K}$	52 968 556	113 309 910	889 035 465
VII	Runtime	2.4	28.7	72.4
	$\#_{\leqq_K}$	67 748 612	501 887 794	2 125 652 797
VIII	Runtime	2.3	6.7	35.9
	$\#_{\leqq_K}$	52 950 686	113 172 781	888 846 269
IX	Runtime	2.9	50.8	86.1
	$\#_{\leqq_K}$	77 731 403	856 876 870	2 492 171 225
X	Runtime	2.3	9.9	39.6
	$\#_{\leqq_K}$	53 035 032	113 716 565	889 756 299
XI	Runtime	2.9	40.7	86.3
	$\#_{\leqq_K}$	77 565 368	858 921 388	2 457 803 337
XII	Runtime	2.3	6.8	35.9
	$\#_{\leqq_K}$	52 995 315	113 590 919	889 759 216

Table 2. Computational results for Jahn's test problem with standard ordering cone $K = \mathbb{R}^2_+$.

Figure 3 illustrates the initial feasible set S (color lightgrey), the set $\widetilde{S}$ (color darkgrey), the system of boxes $\mathcal{B}_{act} = \{B_1, \ldots, B_l\}$ (color red) and the set $\mathrm{EFF}(\widetilde{S} \mid f, \mathbb{R}^2_+)$ (color black), generated by Algorithm 8 for Jahn's test problem ($K = \mathbb{R}^2_+$, $\#_{step1} = 10^5$, $\#_{step2} = 10^4$, $\#_{int} = 30$). The outcome sets $f(S)$ (color lightgrey) and $f(\widetilde{S})$ (color darkgrey) as well as the set of minimal elements $\widetilde{T} = \mathrm{MIN}(f(\widetilde{S}) \mid \mathbb{R}^2_+)$ (color black) are represented in Figure 4.

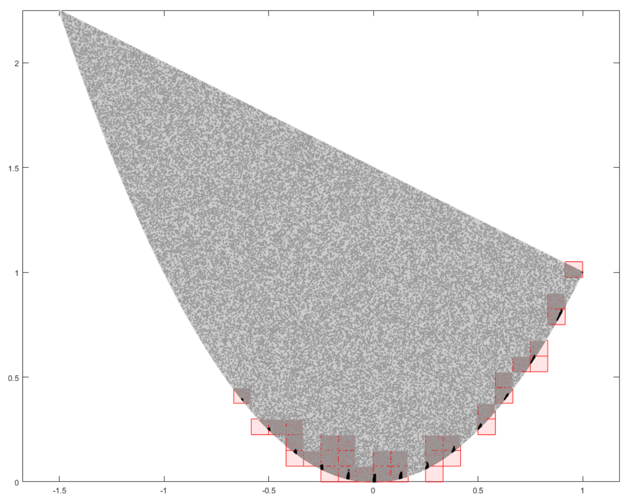

Figure 3. The set S (color lightgrey), the set $\widetilde{S}$ (color darkgrey), the system of boxes $\mathcal{B}_{act} = \{B_1, \ldots, B_{33}\}$ (color red) and the set $\mathrm{EFF}(\widetilde{S} \mid f, \mathbb{R}^2_+)$ (color black) for Jahn's test problem with $K = \mathbb{R}^2_+$.

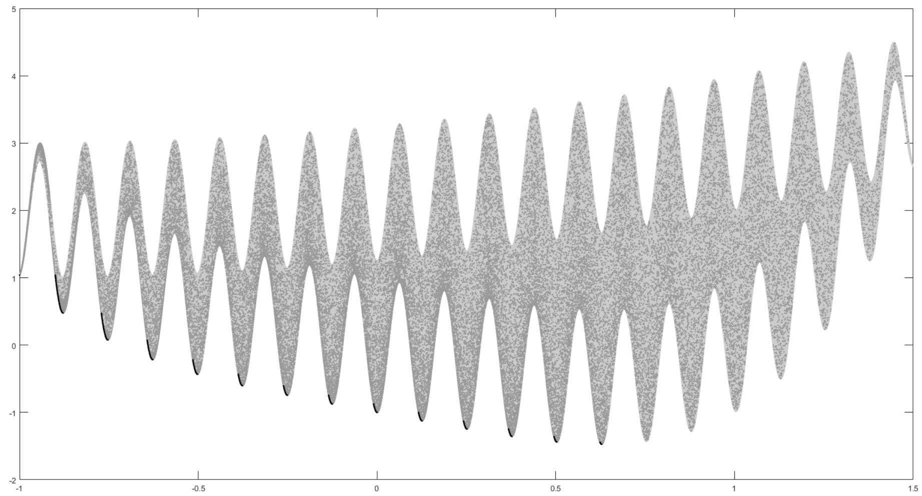

Figure 4. The set $f(S)$ (color lightgrey), the set $f(\widetilde{S})$ (color darkgrey) and the set of minimal elements $\widetilde{T} = \mathrm{MIN}(f(\widetilde{S}) \mid \mathbb{R}^2_+)$ (color black) for Jahn's test problem with $K = \mathbb{R}^2_+$.

By analyzing Table 2, one can see that the running times depend essentially on the corresponding sorting functions (listed in Table 1). More precisely, the results produced by Algorithm 5 with a strongly $\mathbb{R}^2_+$-increasing sorting function (Procedures IV, VI, VIII, X, XII) are comparable with those produced by the original Jahn-Graef-Younes method (Procedure II). In contrast, Procedures III, V, VII, IX and XI, based on Algorithm 4, seem to be slower than Procedure II for our test instances, since the number of pairwise comparisons with respect to the ordering relation $\leqq_K$ are higher. Naturally, the Jahn-Graef-Younes Algorithm 3 as well as our new Algorithms 4 and 5 are significantly faster than Algorithm 1 (the naive method). Notice that sometimes a procedure may be faster than another one (in running time), even if it requires a larger number of pairwise comparisons with respect to the ordering $\leqq_K$. This is due to the different costs for the evaluation of $\leqq_K$.

6.2.2. Jahn's test problem when K is a polyhedral ordering cone

In this subsection we consider a particular polyhedral cone $K = U^+ \subseteq \mathbb{R}^2$, where $U := \{(100,1),(-100,1)\}$. We analyze eight procedures for Jahn's test problem, as listed in Table 3. The last six of them use linear sorting functions as defined in Section 5. The running times (in seconds) and the number of pairwise comparisons based on (12) needed to compute $\widetilde{T} = \mathrm{MIN}(f(\widetilde{S}) \mid K)$ by Algorithm 9 are listed in Table 4.

Proc.$*$	Alg.	Pre-sorting	Intermediate sorting	Parameters
I	1	-	-	-
II	3	-	-	-
III	6	$\varphi := \varphi_{\lambda_\alpha} = \eta_\alpha$	-	$\alpha = (1,2)$
IV	7	-	$\varphi := \varphi_{\lambda_\alpha} = \eta_\alpha$	$\alpha = (1,2)$
V	6	$\varphi := \varphi_{\lambda_\alpha} = \eta_\alpha$	-	$\alpha = (1,1)$
VI	7	-	$\varphi := \varphi_{\lambda_\alpha} = \eta_\alpha$	$\alpha = (1,1)$
VII	6	$\varphi := \varphi_{\lambda_\alpha} = \eta_\alpha$	-	$\alpha = (2,1)$
VIII	7	-	$\varphi := \varphi_{\lambda_\alpha} = \eta_\alpha$	$\alpha = (2,1)$

Table 3. Procedures applied for Jahn's test problem when $K = U^+$ is a polyhedral ordering cone.

	$\#_{step1}$	10^5	10^6	10^6
	$\#_{step2}$	10^4	10^4	10^5
	$\#_{int}$	30	30	30
	$\lvert\widetilde{B}\rvert$	1 200 000	1 870 000	9 700 000
	$\lvert\widetilde{S}\rvert$	836 057	992 983	5 664 607
	l	110	87	87
	$\lvert T\rvert$	113 601	94 200	291 481
	$\lvert\widetilde{T}\rvert$	42 784	45 485	154 145
I	Runtime	346.1	698.5	5356.9
	$\#_{\leqq_K}$	8 830 661 499	16 371 742 342	101 963 388 244
II	Runtime	148.6	194.9	1762.2
	$\#_{\leqq_K}$	3 429 003 410	4 111 754 987	38 244 516 704
III	Runtime	93.1	160.0	1198.2
	$\#_{\leqq_K}$	2 902 570 705	4 794 331 614	35 225 370 700
IV	Runtime	122.9	178.6	1520.1
	$\#_{\leqq_K}$	3 485 894 921	4 243 658 305	38 897 471 354
V	Runtime	116.7	201.8	1451.4
	$\#_{\leqq_K}$	2 901 559 998	4 438 000 337	35 456 839 150
VI	Runtime	130.3	194.9	1638.5
	$\#_{\leqq_K}$	3 491 033 748	4 272 269 882	38 996 647 665
VII	Runtime	137.4	296.2	1697.6
	$\#_{\leqq_K}$	3 023 916 712	5 813 641 555	36 856 821 409
VIII	Runtime	138.4	196.9	1722.5
	$\#_{\leqq_K}$	3 497 988 473	4 278 940 881	38 992 693 160

Table 4. Computational results for Jahn's test problem with polyhedral ordering cone $K = U^+$.

Figure 5 illustrates the set S (color lightgrey), the set $\widetilde{S}$ (color darkgrey), the system of boxes $\mathcal{B}_{act} = \{B_1, \ldots, B_l\}$ (color red) and the set $\mathrm{EFF}(\widetilde{S} \mid f, K)$ (color black), generated by Algorithm 8 for Jahn's test problem ($K = U^+$, $\#_{step1} = 10^5$, $\#_{step2} = 10^4$, $\#_{int} = 30$). The corresponding outcome sets $f(S)$ (color lightgrey) and $f(\widetilde{S})$ (color darkgrey) as well as the set of minimal elements $\widetilde{T} = \mathrm{MIN}(f(\widetilde{S}) \mid K)$ (color black) are represented in Figure 6.

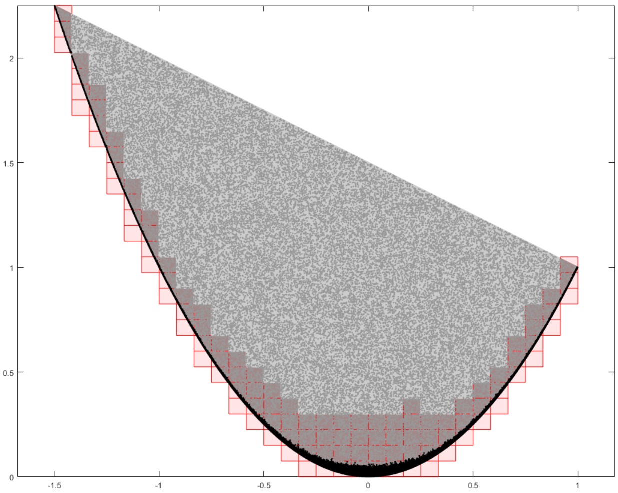

Figure 5. The set S (color lightgrey), the set $\widetilde{S}$ (color darkgrey), the system of boxes $\mathcal{B}_{act} = \{B_1, \ldots, B_{110}\}$ (color red) and the set $\mathrm{EFF}(\widetilde{S} \mid f, K)$ (color black) for Jahn's test problem with $K = U^+$.

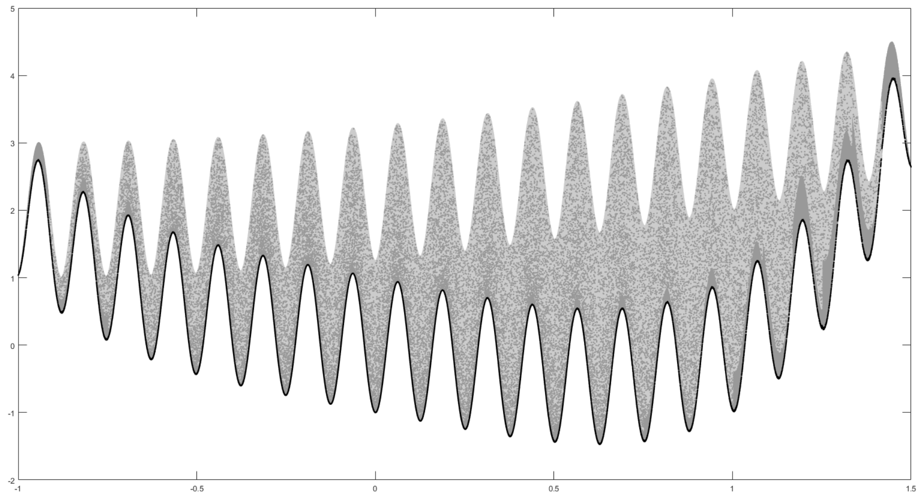

Figure 6. The set $f(S)$ (color lightgrey), the set $f(\widetilde{S})$ (color darkgrey) and the set of minimal elements $\widetilde{T} = \mathrm{MIN}(f(\widetilde{S}) \mid K)$ (color black) for Jahn's test problem with $K = U^+$.

Finally, observe that the computational results listed in Table 4 show that a presorting phase can improve the performance of the original Jahn-Graef-Younes method. In this regard Procedures III and V (based on Algorithm 6) seem to be the best choices. Also our Algorithm 7 performs quite well within Procedures IV and VI.

7. Conclusions

The problem of computing all minimal elements of a finite set A with respect to a pointed convex ordering cone K in a linear space E is important from both theoretical and computational points of view. It naturally occurs in discrete vector optimization problems, but also in continuous vector optimization, as an intermediate-step within different population-based algorithms (see, e.g., Hirsch, Shukla and Schmeck [22], Jahn [4], Shukla and Deb [23]) or other algorithms, conceived for specific problems (see, e.g., the Rectangular Decomposition Algorithm for solving planar location problems, developed by Alzorba et al. [24]).

Our main theoretical result (Theorem 3.1) reveals that the well-known Graef-Younes method, originally conceived to generate a reduced set $B \subseteq A$ whose minimal elements are the same as those of A, has a more special feature, namely to generate precisely the set of all minimal elements of A when applied to an appropriate enumeration of A with respect to a strongly K-increasing sorting function. Therefore, our new numerical methods (Algorithms 4 and 5) can be used in many applications as an alternative to other methods known in the literature (for instance, instead of the well-known Jahn-Graef-Younes method, used among others by Eichfelder [7,25] and Alzorba et al. [24]).

In forthcoming papers we intend to generalize our approach for vector optimization problems with variable ordering structures (see, e.g., Eichfelder [26,27] and Hirsch, Shukla and Schmeck [22]) as well as for set optimization problems (see, e.g., Jahn [28], Köbis, Kuroiwa and Tammer [29]).

Acknowledgements: Nicolae Popovici's research was supported by a grant of the Romanian Ministry of Research and Innovation, CNCS-UEFISCDI, project number PN-III-P4-ID-PCE-2016-0190, within PNCDI III. The authors are grateful to the anonymous referees for their valuable and constructive remarks.

References

[1] Jahn J. Vector optimization - theory, applications, and extensions. 2nd ed. Berlin, Heidelberg: Springer; 2011.
[2] Younes YM. Studies on discrete vector optimization [dissertation]. University of Demiatta; 1993.
[3] Jahn J, Rathje U. Graef-Younes method with backward iteration. In: Küfer KH, Rommelfanger H, Tammer C, Winkler K. Multicriteria decision making and fuzzy systems - theory, methods and applications. Aachen: Shaker Verlag; 2006. p. 75–81.
[4] Jahn J. Multiobjective search algorithm with subdivision technique. Comput Optim Appl. 2006;35(2):161–175.
[5] Luc DT. Theory of vector optimization. Berlin, Heidelberg: Springer; 1989.
[6] Göpfert A, Riahi H, Tammer C, Zălinescu C. Variational methods in partially ordered spaces. New York: Springer; 2003.
[7] Eichfelder G. Ordering structures in vector optimization and applications in medical engineering. Ilmenau University of Technology, Preprint-Series of the Institute of Mathematics; 2013.
[8] Gerth (Tammer) C, Weidner P. Nonconvex separation theorems and some applications in vector optimization. J Optim Theory Appl. 1990;67(2):297–320.
[9] Popovici N. Explicitly quasiconvex set-valued optimization. J Global Optim. 2007;38(1):103–118.
[10] Tanaka T, Kuroiwa D. The convexity of A and B assures $\mathrm{int}A + B = \mathrm{int}(A + B)$. Appl Math Lett. 1993;6(1):83–86.

[11] Urbański R. A generalization of the Minkowski-Rådström-Hörmander theorem. Bull Polish Acad Sci Math. 1976;24:709–715.
[12] Aliprantis CD, Tourky R. Cones and duality. American Mathematical Society; 2007.
[13] Borwein JM, Lewis AS. Convex analysis and nonlinear optimization. 2nd ed. New York: Springer; 2006.
[14] Rockafellar RT. Convex analysis. Princeton: Princeton University Press; 1970.
[15] Tammer C, Winkler K. A new scalarization approach and applications in multicriteria d.c. optimization. J Nonlinear Convex Anal. 2003;4(3):365–380.
[16] Schandl B, Klamroth, Wiecek MM. Introducing oblique norms into multiple criteria programming. J Global Optimization. 2002;23(1):81–97.
[17] Kung HT, Luccio F, Preparata FP. On finding the maxima of a set of vectors. J Assoc Comput Mach. 1975;22(4):469-476.
[18] Engau A, Wiecek MM. Cone characterizations of approximate solutions in real vector optimization. J Optim Theory Appl. 2007;134(3):499–513.
[19] Dempe S, Eichfelder G, Fliege J. On the effects of combining objectives in multi-objective optimization. Math Meth Oper Res. 2015;82(1):1-18.
[20] Dellnitz M, Schütze O, Hestermeyer T. Covering Pareto sets by multilevel subdivision techniques. J Optim Theory Appl. 2005;124(1):113–136.
[21] Limmer S, Fey D, Jahn J. GPU implementation of a multiobjective search algorithm. Positivity. 2012;16(3):397–404.
[22] Hirsch C, Shukla PK, Schmeck H. Variable preference modeling using multi-objective evolutionary algorithms. In: Takahashi RHC, Deb K, Wanner EF, Greco S, editors. Evolutionary multi-criterion optimization. EMO 2011. Lecture Notes in Computer Science, vol. 6576. Berlin, Heidelberg: Springer; 2011. p. 91–105
[23] Shukla PK, Deb K. On finding multiple Pareto-optimal solutions using classical and evolutionary generating methods. European J Oper Res. 2007;181(3):1630–1652.
[24] Alzorba S, Günther C, Popovici N, Tammer C. A new algorithm for solving planar multiobjective location problems involving the Manhattan norm. European J Oper Res. 2017;258(1):35–46.
[25] Eichfelder G. Multiobjective bilevel optimization. Math Program. 2010;123(2):419–449.
[26] Eichfelder G. Variable ordering structures in vector optimization. Berlin, Heidelberg: Springer; 2014.
[27] Eichfelder G. Numerical procedures in multiobjective optimization with variable ordering structures. J Optim Theory Appl. 2014;162(2):489–514.
[28] Jahn J. Vectorization in set optimization. J Optim Theory Appl. 2015;167(3):783–795.
[29] Köbis E, Kuroiwa D, Tammer C. Generalized set order relations and their numerical treatment. Applied Anal Optim. 2017;1(1): 45–66.