

Chambolle-Pock and Tseng's methods: relationship and extension to the bilevel optimization

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Abstract

In the first part of the paper we focus on two problems: (a) regularized least squares and (b) nonsmooth minimization over an affine subspace. For these problems we establish the connection between the primal-dual method of Chambolle-Pock and Tseng's proximal gradient method. For problem (a) it allows us to derive a nonergodic $O(1/k^2)$ convergence rate of the objective function for the primal-dual method. For problem (b) we show that the primal-dual method converges even when the duality does not hold or when the linear system is inconsistent. We also obtain new convergence rates which are more suitable in practice.

In the second part, using the established connection, we develop a novel method for the bilevel composite optimization problem. We prove its convergence without the assumption of strong convexity which is typical for many existing methods.

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1 Introduction

Let X, Y be two finite-dimensional real vector spaces equipped with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$. Consider the following saddle point problem

$$\min_{x \in X} \max_{y \in Y} g(x) + \langle Ax, y \rangle - p^*(y), \quad (1)$$

where

- $A: X \rightarrow Y$ is a bounded linear operator, with the operator norm $L = \|A^*A\|$;
- $g: X \rightarrow (-\infty, +\infty]$ and $p^*: Y \rightarrow (-\infty, +\infty]$ are proper lower semicontinuous (l.s.c.) convex functions.

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A popular simple first-order method for solving (1) is the primal-dual algorithm (PDA) due to Chambolle and Pock [6]:

$$\begin{aligned}\bar{x}^k &= x^k + \theta_k(x^k - x^{k-1}) \\ y^{k+1} &= \text{prox}_{\sigma_k p^*}(y^k + \sigma_k A \bar{x}^k) \\ x^{k+1} &= \text{prox}_{\tau_k g}(x^k - \tau_k A^* y^{k+1})\end{aligned}\tag{2}$$

For possible extensions and applications we refer the reader to [7–9, 11, 15, 20, 29]. The convergence of PDA holds under the following assumptions:

- (i) If $\sigma_k = \sigma$, $\tau_k = \tau$, $\theta_k = 1$, $\tau\sigma L < 1$, (2) corresponds to the basic primal-dual algorithm.
- (ii) If g is γ -strongly convex, $\tau_k = \frac{\tau_{k-1}}{\sqrt{1+\gamma\tau_{k-1}}}$, $\theta_k = \frac{\tau_k}{\tau_{k-1}} = \frac{\sigma_{k-1}}{\sigma_k}$, $\tau_k\sigma_k L < 1$, (2) corresponds to the accelerated primal-dual algorithm (primal is strongly convex).
- (iii) If f^* is γ -strongly convex, $\sigma_k = \frac{\sigma_{k-1}}{\sqrt{1+\gamma\sigma_{k-1}}}$, $\theta_k = \frac{\tau_k}{\tau_{k-1}} = \frac{\sigma_{k-1}}{\sigma_k}$, $\tau_k\sigma_k L < 1$, (2) corresponds to the accelerated primal-dual algorithm (dual is strongly convex).
- (iv) If in every iteration $\tau_k\sigma_k\|A^*y^{k+1} - Ay^k\|^2 \leq \|y^{k+1} - y^k\|^2$ and $\theta_k = \frac{\tau_k}{\tau_{k-1}} = \frac{\sigma_k}{\sigma_{k-1}}$, (2) corresponds to the primal-dual algorithm with linesearch [20].

Since we will often mix PDA with fixed steps and with variable, it is useful to remember that $\theta_k = \frac{\tau_k}{\tau_{k-1}}$. Thus, when we consider algorithm with fixed steps, $\theta_k = 1$ for all k .

Contribution. In the first part of this work we study two particular cases of (1): for $p^*(y) = \frac{1}{2}\|y + b\|^2$ and for $p^*(y) = \langle b, y \rangle$. With the former choice of p^* , (1) is equivalent to the regularized least squares (RLS) problem $\min_x g(x) + \frac{1}{2}\|Ax - b\|^2$ and with the latter to the minimization of g over an affine subspace $Ax = b$. Although we consider only very particular problems, they cover most of the applications of PDA in the literature. For both of these problems we show how PDA relates to the Tseng proximal gradient method [28]. This connection by itself is a quite surprising result, as Tseng’s method does not use any duality arguments. Apart from that, it also leads to several new consequences. For the first problem we (a) enlarge bounds for the step sizes, (b) establish a nonergodic $O(1/k^2)$ convergence rate of the accelerated PDA for the objective function. It is known experimentally that the primal-dual method (or its extensions) often converges at the similar rate as accelerated methods [3, 24]. Our result confirms this theoretically for one important class of problems. For the second problem we (a) prove convergence of PDA when duality does not hold or when the system $Ax = b$ is inconsistent, (b) establish new rates of convergence.

In the second part of this paper we switch to a much more general problem of a bilevel composite (or structural) optimization. This problem aims to solve a composite minimization problem over the solution set of another composite minimization problem (this problem is also known as the hierarchical optimization). To our knowledge, all existing methods in the literature either consider only special cases of such problems or converge under very restrictive assumptions. Using the insight obtain from the interpretation of the primal-dual method as Tseng’s scheme, we develop the modification of the latter method, that is able to solve the considered problem.

We do not provide much details in the introduction, since the paper covers quite different topics. Instead, in every new section we give some relevant comments and important references.

Paper outline. In section 2 we briefly recall the standard notation from convex analysis and establish several necessary lemmas. Section 3 considers the regularized least squares

problem. Our main result in this section is Theorem 2, in which we derive $O(1/k^2)$ convergence rate of PDA. In section 4 we study the problem of convex function minimization over the linear constraints. Here our main results are Theorem 4 and 5 in which we prove convergence of the generalized (and accelerated) PDA when the linear system is inconsistent and/or strong duality does not hold. In section 5 we consider the problem of a bilevel composite minimization. We state our algorithm and prove its convergence. Finally, section 6 collects some numerical experiments which confirm our findings.

2 Preliminaries

Let X be a finite-dimensional vector space \mathbb{R}^n equipped with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$. For a convex lower semi-continuous (l.s.c.) function $g: X \rightarrow (-\infty, +\infty]$ by g^* we denote the Legendre-Fenchel conjugate of g and by $\text{dom } g$ the domain of g , i.e., the set $\{x: g(x) < +\infty\}$. A^* stands for the adjoint of the linear operator A and δ_C for the indicator function of a closed convex set C . A function g is called γ -strongly convex function, if $g - \frac{\gamma}{2} \| \cdot \|^2$ is convex. The proximal operator prox_g for a proper l.s.c. convex function $g: X \rightarrow (-\infty, +\infty]$ is defined as $\text{prox}_g(z) = \text{argmin}_x \{g(x) + \frac{1}{2} \|x - z\|^2\}$. The following characteristic property (prox-inequality) will be often used:

$$\bar{x} = \text{prox}_g z \quad \Leftrightarrow \quad \langle \bar{x} - z, x - \bar{x} \rangle \geq g(\bar{x}) - g(x) \quad \forall x \in X. \quad (3)$$

When g is γ -strongly convex, the above inequality can be strengthened:

$$\bar{x} = \text{prox}_g z \quad \Leftrightarrow \quad \langle \bar{x} - z, x - \bar{x} \rangle \geq g(\bar{x}) - g(x) + \frac{\gamma}{2} \|\bar{x} - x\|^2 \quad \forall x \in X. \quad (4)$$

For the quadratic function $f(x) = \frac{1}{2} \|Ax - b\|^2$ it holds:

$$\alpha f(x) + (1 - \alpha)f(y) = f(\alpha x + (1 - \alpha)y) + \frac{\alpha(1 - \alpha)}{2} \|A(x - y)\|^2 \quad \forall x, y \in X \quad \forall \alpha \in \mathbb{R}. \quad (5)$$

Another useful identity obviously holds:

$$2\langle \bar{x} - z, x - \bar{x} \rangle = \|x - z\|^2 - \|\bar{x} - z\|^2 - \|\bar{x} - x\|^2.$$

We conclude our preliminary section by two important lemmas.

Lemma 1. Suppose that sequences $(x^k) \subset \mathbb{R}^n$, $(b_k) \subset \mathbb{R}$ and a set $D \subseteq \mathbb{R}^n$ satisfy:

- (i) All limit points of (x^k) belong to D ;
- (ii) for all $x \in D$ the sequence $(\|x^k - x\|^2 + b_k)$ is nonincreasing and bounded from below.

Then the sequence (x^k) converges to some point in D .

Proof. Suppose, on the contrary, that there exist two different subsequences (x^{k_i}) and (x^{k_j}) such that $x^{k_i} \rightarrow \tilde{x}_1$, $x^{k_j} \rightarrow \tilde{x}_2$ and $\tilde{x}_1 \neq \tilde{x}_2$. For simplicity, let $a_k(x) := \|x^k - x\|^2 + b_k$. By (ii), the sequence $(a_k(x))$ is convergent for any $x \in D$. Setting $x := \tilde{x}_1$, we obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} a_k(\tilde{x}_1) &= \lim_{i \rightarrow \infty} a_{k_i}(\tilde{x}_1) = \lim_{i \rightarrow \infty} (\|x^{k_i} - \tilde{x}_1\|^2 + b_{k_i}) = \lim_{i \rightarrow \infty} b_{k_i} \\ &= \lim_{j \rightarrow \infty} a_{k_j}(\tilde{x}_1) = \lim_{j \rightarrow \infty} (\|x^{k_j} - \tilde{x}_1\|^2 + b_{k_j}) = \|\tilde{x}_2 - \tilde{x}_1\|^2 + \lim_{j \rightarrow \infty} b_{k_j}, \end{aligned}$$

from which $\lim_{i \rightarrow \infty} b_{k_i} = \|\tilde{x}_2 - \tilde{x}_1\|^2 + \lim_{j \rightarrow \infty} b_{k_j}$ follows. Evidently, setting $x = \tilde{x}_2$, we analogously derive

$$\lim_{j \rightarrow \infty} b_{k_j} = \|\tilde{x}_1 - \tilde{x}_2\|^2 + \lim_{i \rightarrow \infty} b_{k_i},$$

from which we conclude that $\tilde{x}_1 = \tilde{x}_2$. Therefore, the whole sequence (x^k) converges to some point in D . \square

Lemma 2. For a convex smooth $h: \mathbb{R}^n \rightarrow \mathbb{R}$ with L_h -Lipschitz-continuous gradient and any $u, v, w \in \mathbb{R}^n$ it holds

$$\langle \nabla h(u), v - w \rangle \leq h(v) - h(w) + \frac{L_h}{2} \|w - u\|^2.$$

Proof. By the descent lemma [24] for h ,

$$h(w) - h(u) - \langle \nabla h(u), w - u \rangle \leq \frac{L_h}{2} \|w - u\|^2,$$

By convexity of h ,

$$\langle \nabla h(u), v - u \rangle \leq h(v) - h(u).$$

Adding up the above two inequalities, we obtain the desired. \square

3 Regularized least squares

3.1 Problem description

We are concerned with the regularized least squares problem

$$\min_x \frac{1}{2} \|Ax - b\|^2 + g(x) =: F(x), \quad (6)$$

where

- $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$;
- $g: \mathbb{R}^n \rightarrow (-\infty, +\infty]$ is a proper l.s.c. convex and possibly non-smooth function.

Problem (6) is overwhelmingly important in applications, since it serves as a model for solving linear inverse problems $Ax = b$. The regularizer $g(x)$ describes some constraints which we want to impose due to the fact that usually that system is ill-conditioned and the observed data A, b is noisy, and hence the obtained solution might be meaningless. For $g(x) = \frac{\lambda}{2} \|x\|^2$ problem (6) is known as the Tikhonov regularization; for $g(x) = \lambda \|x\|_1$ as the lasso regression (or basis pursuit denoising); for $g(x) = \frac{\lambda_1}{2} \|x\|^2 + \lambda_2 \|x\|_1$ as the elastic net; for $g(x) = \delta_C(x)$ as the constrained least squares problem; for $g(x) = \|x\|_*$, when x is a $\sqrt{n} \times \sqrt{n}$ matrix, as the nuclear minimization, etc.

It is also important to remark that another common problem in applications

$$\min_y g^*(A^*y) + \frac{1}{2} \|y - b\|^2 \quad (7)$$

is the dual problem of (6). In particular, $g^*(A^*y)$ might describe some loss function as in the linear support vector machine or logistic regression. In this case the common regularizer is $\|y\|^2$ which corresponds to (7) with $b = 0$. Alternatively, $g^*(A^*y)$ may describe some regularizer and $\|y - b\|^2$ is a fidelity term. The famous total variation denoising problem falls into this latter category.

The number of methods for solving (6) is enormous. Some of them are developed specifically for a particular choice of g , some can be applied for any (convex) g . Problem (6) often serves as a standard benchmark for testing general convex optimization algorithms.

3.2 Primal-dual approach

The primal-dual form of (6) is:

$$\min_x \max_y g(x) + \langle Ax, y \rangle - \frac{1}{2} \|y + b\|^2.$$

The PDA generates sequences (x^k) and (y^k) as:

$$\begin{aligned} y^{k+1} &= \text{prox}_{\sigma_k \frac{1}{2} \|\cdot + b\|^2}(y^k + \sigma_k A \bar{x}^k) = \frac{y^k + \sigma_k(A \bar{x}^k - b)}{1 + \sigma_k} \\ x^{k+1} &= \text{prox}_{\tau_k g}(x^k - \tau_k A^* y^{k+1}) \end{aligned} \quad (8)$$

Iterating the first equation in (8), we can derive

$$\begin{aligned} y^{k+1} &= \frac{1}{1 + \sigma_k}(y^k + \sigma_k(A \bar{x}^k - b)) = \frac{\sigma_k}{1 + \sigma_k}(A \bar{x}^k - b) + \frac{1}{1 + \sigma_k}y^k \\ &= \frac{\sigma_k}{1 + \sigma_k}(A \bar{x}^k - b) + \frac{1}{1 + \sigma_k}(\frac{\sigma_{k-1}}{1 + \sigma_{k-1}}(A \bar{x}^{k-1} - b) + \frac{1}{1 + \sigma_{k-1}}y^{k-1}) \\ &= A[\frac{\sigma_k}{1 + \sigma_k}\bar{x}^k + \frac{\sigma_{k-1}}{(1 + \sigma_k)(1 + \sigma_{k-1})}\bar{x}^{k-1} + \dots + \frac{\sigma_1}{(1 + \sigma_k)(1 + \sigma_{k-1})\dots(1 + \sigma_1)}\bar{x}^1] \\ &\quad - b[\frac{\sigma_k}{1 + \sigma_k} + \dots + \frac{\sigma_1}{(1 + \sigma_k)(1 + \sigma_{k-1})\dots(1 + \sigma_1)}] + \frac{1}{(1 + \sigma_k)\dots(1 + \sigma_1)}y^1. \end{aligned}$$

Suppose for simplicity that PDA starts from (x^0, y^0) with $\bar{x}^0 = x^0$ and $y^0 = Ax^0 - b$. In this case $y^1 = y^0 = A\bar{x}^0 - b$ and the above formula simplifies to

$$y^{k+1} = A[\frac{\sigma_k}{1 + \sigma_k}\bar{x}^k + \dots + \frac{\sigma_1}{(1 + \sigma_k)\dots(1 + \sigma_1)}\bar{x}^1 + \frac{1}{(1 + \sigma_k)\dots(1 + \sigma_1)}\bar{x}^0] - b$$

Denote

$$z^k = \frac{\sigma_k}{1 + \sigma_k}\bar{x}^k + \dots + \frac{\sigma_1}{(1 + \sigma_k)\dots(1 + \sigma_1)}\bar{x}^1 + \frac{1}{(1 + \sigma_k)\dots(1 + \sigma_1)}\bar{x}^0$$

Then $y^{k+1} = Az^k - b$ and the following relation holds

$$\begin{aligned} z^k &= \frac{\sigma_k}{1 + \sigma_k}\bar{x}^k + \frac{1}{1 + \sigma_k}z^{k-1} \\ x^{k+1} &= \text{prox}_{\tau_k g}(x^k - \tau_k A^*(Az^k - b)) \end{aligned} \quad (9)$$

with $z^0 = \bar{x}^0 = x^0$. This scheme resembles the proximal gradient method applied to (6):

$$x^{k+1} = \text{prox}_{\tau g}(x^k - \tau A^*(Ax^k - b)), \quad (10)$$

where $\tau \in (0, \frac{2}{L})$. With the fixed steps $\sigma_k \equiv \sigma$, $\tau_k \equiv \tau$, (9) becomes

$$\begin{aligned} z^k &= \frac{\sigma}{1 + \sigma}\bar{x}^k + \frac{1}{1 + \sigma}z^{k-1} \\ x^{k+1} &= \text{prox}_{\tau g}(x^k - \tau A^*(Az^k - b)). \end{aligned} \quad (11)$$

It is not difficult to prove by induction that

- (i) if $\sigma \leq 1$ then $z^k \in \text{conv}\{x^k, \dots, x^0\}$;

- (ii) if $\sigma = 1$ then $z^k = x^k$;
- (iii) if $\sigma \rightarrow \infty$ then $z^k \rightarrow \bar{x}^k$.

We do not prove this now, as later we will give a simpler explanation why this is true.

The first implication of the above is that PDA with $\sigma = 1$ coincides with the proximal gradient method (10), however the analysis of the primal-dual method guarantees convergence only for $\tau < \frac{1}{L}$, in contrast to $\tau < \frac{2}{L}$ for (10).

Second, it is interesting to see that PDA with $\sigma < 1$ allows us to make larger steps than proximal gradient method does. For example, for $\sigma = \frac{1}{\sqrt{L}}$ we have to ensure that $\tau < \frac{1}{\sqrt{L}}$, in the same time for (10) we need $\tau < \frac{2}{L}$. Of course, instead of evaluating the gradient at a current point we use some convex combination of all previous iterations. This gives us some inertia for the trajectory of iterates. To our knowledge, in all successful applications of the primal-dual algorithm to problem (6), PDA uses $\sigma < 1$, thus the scheme (11) uses larger steps than (10). We believe that this is the major success factor of PDA. The practical problem is of course to know in advance which $\sigma \in (0, 1)$ will provide the fastest convergence.

Third, for a very large σ , z^k almost coincides with \bar{x}^k . In this case PDA is almost identical to the proximal reflected gradient method [18, 19]:

$$x^{k+1} = \text{prox}_{\tau g}(x^k - \tau A^*(A\bar{x}^k - b)). \quad (12)$$

It is interesting to compare convergence guarantees for both methods. PDA requires τ to be taken very small, since $\tau < \frac{1}{\sigma L^2} \approx 0$. In contrast, (12) requires $\tau < \frac{\sqrt{2}-1}{L^2}$. This tells us that for problem (6) convergence guarantees of PDA might be improvable.

3.3 Connection with Tseng's method

Let (s^k) be a sequence defined recursively as $s^{k+1} = \frac{1}{1+\sigma}(\sigma x^{k+1} + s^k)$, $s^0 = x^0$. It is not difficult to prove by induction that $z^k = \sigma x^k + (1-\sigma)s^k$. In fact, it holds for $k=0$ and let it hold for $k-1$. This means that

$$z^{k-1} = \sigma x^{k-1} + (1-\sigma)s^{k-1} = \sigma x^{k-1} + (1-\sigma)((1+\sigma)s^k - \sigma x^k).$$

Then we have

$$z^k = \frac{\sigma(2x^k - x^{k-1}) + z^{k-1}}{1+\sigma} = \sigma x^k + (1-\sigma)s^k.$$

Hence, having in mind that $f(x) = \frac{1}{2}\|Ax - b\|^2$, we can rewrite method (11) as

$$\begin{aligned} z^k &= \sigma x^k + (1-\sigma)s^k \\ x^{k+1} &= \text{prox}_{\tau g}(x^k - \tau \nabla f(z^k)) \\ s^{k+1} &= \frac{1}{1+\sigma}(\sigma x^{k+1} + s^k). \end{aligned} \quad (13)$$

The above scheme has some striking similarities with the well-known Tseng's proximal gradient method [28]. In that paper Tseng based on [1, 3, 17, 23, 24] proposed a series of different accelerated algorithms for the problem $\min_x g(x) + h(x)$, one of which is given below (we slightly changed notation to make it closer to our settings):

$$\begin{aligned} z^k &= \theta_k x^k + (1-\theta_k)s^k \\ x^{k+1} &= \text{prox}_{\frac{\lambda}{\theta_k} g}(x^k - \frac{\lambda}{\theta_k} \nabla h(z^k)) \\ s^{k+1} &= \theta_k x^{k+1} + (1-\theta_k)s^k. \end{aligned} \quad (14)$$

Convergence of (14) was proved under the assumption that ∇h is $\frac{1}{\lambda}$ -Lipschitz continuous and $\theta_k \in (0, 1]$ satisfies $\frac{1-\theta_k}{\theta_k^2} \leq \frac{1}{\theta_{k-1}^2}$.

Consider the case with fixed $\theta_k \equiv \theta$, which evidently fulfills the assumption. If we denote $\theta = \sigma$, $\tau = \frac{\lambda}{\theta_k} = \frac{\lambda}{\sigma}$, the schemes (13) and (14) with $h(x) = \frac{1}{2}\|Ax - b\|^2$ almost coincide: only updates for s^{k+1} are different. However, for small values σ , which are mostly used in practice, this is not essential.

In the next subsection we give a new proof of convergence for (13). Unfortunately, we cannot apply Tseng's argument from [28], as we can not use the relation $s^{k+1} - z^k = \theta(x^{k+1} - x^k)$, that holds for Tseng's method. Instead, we explicitly use the structure of our problem, namely, that f is quadratic.

3.4 Proof of convergence

Lemma 3. Let $f(x) = \frac{1}{2}\|Ax - b\|^2$. Then for any u, v, w it holds

$$\langle \nabla f(u), v - w \rangle = f(v) - f(w) + \frac{1}{2}\|A(w - u)\|^2 - \frac{1}{2}\|A(u - v)\|^2.$$

Proof. As we work with a quadratic function f , we can use a stronger version of the descent lemma, which in our case will be just an identity:

$$\begin{aligned} f(w) - f(u) - \langle \nabla f(u), w - u \rangle &= \frac{1}{2}\|A(w - u)\|^2, \\ f(v) - f(u) - \langle \nabla f(u), v - u \rangle &= \frac{1}{2}\|A(u - v)\|^2. \end{aligned}$$

Subtraction one from another gives the desired. \square

Assume that the solution set S of (6) is nonempty and let $F_* = \min_x F(x)$ and $\lambda = \tau\sigma$.

Theorem 1. Let (x^k) be generated by (13). Suppose that either (a) $\sigma \leq 1$ and $\lambda(1 - \frac{\sigma}{2})L < 1$ or (b) $\sigma > 1$ and $\tau\sigma \leq \frac{1}{L}$. Then (x^k) converges to a solution of (6).

In general, when $\sigma\tau L < 1$, the direct proof of this theorem would be much simpler (alternatively one can apply the standard arguments of PDA). However, case (a) is more general, because it allows us to take a slightly larger stepsize than the condition $\tau\sigma L < 1$ does. In particular, when $\sigma = 1$, we can choose the same $\tau < \frac{2}{L}$ as in the proximal gradient method. We could have also increased the bounds similarly in case (b), but as this is not important in applications, we decided to avoid that. However, the main reason to prove this theorem is that some parts of it will be reused in the subsequent statements. Notice, that these parts are only the few equations until (17), so one can easily skip everything below (17) without loss of further understanding.

Proof. By the prox-inequality (3),

$$\langle x^{k+1} - x^k, x - x^{k+1} \rangle + \tau \langle \nabla f(z^k), x - x^{k+1} \rangle \geq \tau(g(x^{k+1}) - g(x)). \quad (15)$$

From Lemma 3 it follows

$$\begin{aligned} \langle \nabla f(x^k), x - x^{k+1} \rangle &= f(x) - f(x^{k+1}) + \frac{1}{2}\|A(x^{k+1} - x^k)\|^2 - \frac{1}{2}\|A(x^k - x)\|^2, \\ \langle \nabla f(s^k), x - x^{k+1} \rangle &= f(x) - f(x^{k+1}) + \frac{1}{2}\|A(x^{k+1} - s^k)\|^2 - \frac{1}{2}\|A(s^k - x)\|^2. \end{aligned}$$

Therefore, using that $\nabla f(z^k) = \sigma \nabla f(x^k) + (1 - \sigma) \nabla f(s^k)$, we have

$$\begin{aligned} & \|x^{k+1} - x\|^2 + \|x^{k+1} - x^k\|^2 - \lambda \|A(x^{k+1} - x^k)\|^2 + 2\tau(F(x^{k+1}) - F(x)) \\ & \quad + \tau(1 - \sigma) \|A(s^k - x)\|^2 + \lambda \|A(x^k - x)\|^2 \\ & \leq \|x^k - x\|^2 + \tau(1 - \sigma) \|A(x^{k+1} - s^k)\|^2. \end{aligned} \quad (16)$$

Case (a): $\sigma \leq 1$. As $s^{k+1} = \frac{\sigma x^{k+1} + s^k}{1 + \sigma}$, and $F = f + g$ with a quadratic f , we observe

$$F(x^{k+1}) \geq \frac{1 + \sigma}{\sigma} F(s^{k+1}) - \frac{1}{\sigma} F(s^k) + \frac{1}{2(1 + \sigma)} \|A(x^{k+1} - s^k)\|^2. \quad (17)$$

Using (17) to estimate the term $\tau(1 - \sigma) \|A(x^{k+1} - s^k)\|^2$ in (16), we deduce

$$\begin{aligned} & \|x^{k+1} - x\|^2 + \|x^{k+1} - x^k\|^2 - \lambda \|A(x^{k+1} - x^k)\|^2 + \tau(1 - \sigma) \|A(s^k - x)\|^2 + \\ & 2\tau \frac{(1 - \sigma^2)(1 + \sigma)}{\sigma} (F(s^{k+1}) - F(x)) + 2\tau\sigma^2 (F(x^{k+1}) - F(x)) + \lambda \|A(x^k - x)\|^2 \\ & \leq \|x^k - x\|^2 + 2\tau \frac{1 - \sigma^2}{\sigma} (F(s^k) - F(x)). \end{aligned} \quad (18)$$

Choose $x = \bar{x} \in S$. From $F(x^{k+1}) - F(\bar{x}) \geq \frac{1}{2} \|A(x^{k+1} - \bar{x})\|^2$ it follows

$$2\tau\sigma^2 (F(x^{k+1}) - F(\bar{x})) + \lambda \|A(x^k - \bar{x})\|^2 \geq \frac{\lambda\sigma}{2} \|A(x^{k+1} - x^k)\|^2.$$

Applying this inequality in (18) and using that $\|A(x^{k+1} - x^k)\| \leq L \|x^{k+1} - x^k\|$, we derive

$$\begin{aligned} & \|x^{k+1} - \bar{x}\|^2 + (1 - \lambda(1 - \frac{\sigma}{2})L) \|x^{k+1} - x^k\|^2 + \tau(1 - \sigma) \|A(s^k - \bar{x})\|^2 \\ & + 2\tau \frac{(1 - \sigma^2)(1 + \sigma)}{\sigma} (F(s^{k+1}) - F_*) \leq \|x^k - \bar{x}\|^2 + 2\tau \frac{1 - \sigma^2}{\sigma} (F(s^k) - F_*). \end{aligned} \quad (19)$$

From the above we conclude that (x^k) is bounded, $x^{k+1} - x^k \rightarrow 0$ and $\|A(s^k - \bar{x})\| \rightarrow 0$. As $Ax^k = \frac{1+\sigma}{\sigma} As^{k+1} - \frac{1}{\sigma} As^k$, we have $Ax^k \rightarrow A\bar{x}$ and hence, $Az^k \rightarrow A\bar{x}$ as well. Let us show that all limit points of (x^k) belong to S . Let $x^{k_i} \rightarrow \tilde{x}$. Then by above arguments we know that $A\tilde{x} = A\bar{x}$, and hence $\nabla f(z^k) \rightarrow \nabla f(\tilde{x})$. Passing to the limit in (15) and using that $x^{k_i+1} - x^{k_i} \rightarrow 0$, we obtain $\langle \nabla f(\tilde{x}), x - \tilde{x} \rangle \geq g(\tilde{x}) - g(x)$. Therefore $\tilde{x} \in S$. Then Lemma 1 with $b_k = 2\tau \frac{1-\sigma^2}{\sigma} (F(s^k) - F_*)$ applied to (19) yields that the whole sequence (x^k) converges to some element in S .

Case (b): We omit the direct proof for this case, since it follows from the standard PDA analysis. However, we note that the direct proof is, in some sense, similar to the above. It requires the use of the following identity:

$$\|A(x^k - x)\|^2 = \frac{1 + \sigma}{\sigma} \|A(s^k - x)\|^2 - \frac{1}{\sigma} \|A(s^{k-1} - x)\|^2 + (1 + \sigma) \|A(x^k - s^k)\|^2.$$

□

3.5 Acceleration

Observe that the objective $\frac{1}{2} \|y + b\|^2$ in (8) is 1-strongly convex. Hence, one can apply the accelerated primal-dual algorithm (APDA). By this, we arrive at scheme (9), where

$$\sigma_{k+1} = \frac{\sigma_k}{\sqrt{1 + \sigma_k}}, \quad \tau_k \sigma_k = \lambda. \quad (20)$$

With such choice of steps σ_k, τ_k it is known that the primal-dual method applied to (6) gives us $O(1/k)$ convergence rate for the sequence $\|y^k - y^*\|$. In our settings it means $O(1/k)$ convergence rate for the $\|A(z^k - x^*)\|$, that is clearly not very informative unless A^*A is positive definite. On the other hand, one can derive [7] an ergodic $O(1/k^2)$ rate for the primal-dual gap, that for problem (6) bounds $F(\cdot) - F_*$ from above. In this section we show a nonergodic $O(1/k^2)$ rate of PDA for the objective F , that is the same rate as accelerated gradient methods [3, 24, 28] provide.

If we define s^k implicitly in a similar way, as in Section 3.3: $z^k = \sigma_k x^k + (1 - \sigma_k)s^k$, then it is not difficult to check by the same logic that s^k must satisfy:

$$s^{k+1} = \frac{\sigma_k - \sigma_{k+1}^2}{1 - \sigma_{k+1}^2} x^{k+1} + \frac{1 - \sigma_k}{1 - \sigma_{k+1}^2} s^k$$

with $s^0 = x^0$. Note that in case $\sigma_k = \sigma_{k+1}$ the above relation corresponds to the third line in (13). With the steps σ_k defined in (20), the formula for s^{k+1} boils down to

$$s^{k+1} = \frac{\sigma_k}{1 + \sigma_k - \sigma_k^2} x^{k+1} + \frac{1 - \sigma_k^2}{1 + \sigma_k - \sigma_k^2} s^k.$$

Finally, we obtain the following scheme:

$$\begin{aligned} z^k &= \sigma_k x^k + (1 - \sigma_k)s^k \\ x^{k+1} &= \text{prox}_{\tau_k g}(x^k - \tau_k \nabla f(z^k)) \\ s^{k+1} &= \frac{\sigma_k}{1 + \sigma_k - \sigma_k^2} x^{k+1} + \frac{1 - \sigma_k^2}{1 + \sigma_k - \sigma_k^2} s^k, \end{aligned} \tag{21}$$

where $s^0 = x^0$, stepsizes τ_k, σ_k satisfy recursion (20), and $f(x) = \frac{1}{2}\|Ax - b\|^2$.

Theorem 2. *Let $(x^k), (s^k), (z^k)$ be generated by (21), $\sigma_0 \leq 1$, and $\lambda L < 1$. Then all three sequences are bounded, the limit points of $(s^k), (z^k)$ are solutions of (6) and $F(s^k) - F_* = O(1/k^2)$.*

By this, we show that APDA, applied to (6), has the same rate of convergence as Nesterov-type methods [3, 24].

Proof. Clearly, in these settings, instead of (16), we have

$$\begin{aligned} \|x^{k+1} - x\|^2 + (1 - \lambda L^2)\|x^{k+1} - x^k\|^2 + 2\tau_k(F(x^{k+1}) - F(x)) \\ \leq \|x^k - x\|^2 + \tau_k(1 - \sigma_k)\|A(x^{k+1} - s^k)\|^2. \end{aligned} \tag{22}$$

Next, instead of (17), we have to use

$$F(x^{k+1}) \geq \frac{1 + \sigma_k - \sigma_k^2}{\sigma_k} F(s^{k+1}) - \frac{1 - \sigma_k^2}{\sigma_k} F(s^k) + \frac{1 - \sigma_k^2}{2(1 + \sigma_k - \sigma_k^2)} \|A(x^{k+1} - s^k)\|^2.$$

Applying the latter inequality to (22) and using that $\lambda \leq \frac{1}{L}$, we deduce

$$\begin{aligned} \|x^{k+1} - x\|^2 + 2\tau_k \frac{1 + \sigma_k - \sigma_k^2}{\sigma_k} (F(s^{k+1}) - F(x)) + \tau_k \left(\frac{1 - \sigma_k^2}{1 + \sigma_k - \sigma_k^2} - (1 - \sigma_k) \right) \|A(x^{k+1} - s^k)\|^2 \\ \leq \|x^k - x\|^2 + 2\tau_k \frac{1 - \sigma_k^2}{\sigma_k} (F(s^k) - F(x)). \end{aligned}$$

Notice that $\frac{1-\sigma_k^2}{1+\sigma_k-\sigma_k^2} \geq 1 - \sigma_k$ for $\sigma_k \in (0, 1]$ and $\frac{1+\sigma_k-\sigma_k^2}{\sigma_k^2} = \frac{1-\sigma_{k+1}^2}{\sigma_{k+1}^2}$. This gives us

$$\|x^{k+1} - x\|^2 + 2\lambda \frac{1-\sigma_{k+1}^2}{\sigma_{k+1}^2} (F(s^{k+1}) - F(x)) \leq \|x^k - x\|^2 + 2\lambda \frac{1-\sigma_k^2}{\sigma_k^2} (F(s^k) - F(x)). \quad (23)$$

Setting $x = x^* \in S$ and iterating the above inequality, we derive

$$\|x^{k+1} - x^*\|^2 + 2\lambda \frac{1-\sigma_{k+1}^2}{\sigma_{k+1}^2} (F(s^{k+1}) - F_*) \leq \|x^0 - x^*\|^2 + 2\lambda \frac{1-\sigma_0^2}{\sigma_0^2} (F(s^0) - F_*).$$

The last equation ensures that (x^k) is bounded. It is also not difficult to show by induction that $\sigma_k \leq \frac{\sqrt{6}}{k}$ for $k \geq 3$. This implies $\frac{1-\sigma_k^2}{\sigma_k^2} \geq \frac{k^2}{6} - 1$ and hence, (23) yields $O(1/k^2)$ convergence rate for $F(s^k) - F_*$. In particular, if σ_0 was chosen as 1, we would obtain

$$F(s^k) - F_* \leq \frac{3L\|x^0 - x^*\|^2}{k^2 - 6}, \quad \forall k \geq 3. \quad (24)$$

Since (x^k) is bounded, it follows from (21) that (s^k) and (z^k) are bounded as well. From (24) we have that all limit points of (s^k) and, hence (z^k) , are solutions of (6). \square

Unfortunately, it is not clear how to prove convergence of the whole sequences $(s^k), (z^k)$. This is a common drawback of the accelerated methods with a notable exception of [5]. Notice also that we are only able to prove boundedness of (x^k) . This is what can be proved by APDA as well.

4 Minimization over an affine subspace

4.1 Problem description

Consider the following minimization problem

$$\begin{aligned} \min \quad & g(x) \\ \text{s.t.} \quad & Ax = b, \end{aligned} \quad (25)$$

where

- $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$;
- $g: \mathbb{R}^n \rightarrow (-\infty, +\infty]$ is a proper l.s.c. convex and possibly non-smooth function.

As we allow g to be infinite-valued, problem (25) is very general. In particular, it includes conic optimization, which in turn includes linear and semidefinite optimization. (25) often arises in inverse problems, distributed optimization, machine learning. Notice that every composite optimization problem $g_1(x) + g_2(Kx)$ can be also written in the form (25). We restrict our attention to the large-scale problems where computing the projection onto the subspace $\{x: Ax = b\}$ is expensive or even practically impossible.

In this section we show that the primal-dual method applied to (25) can be seen as a modified Tseng's method, whose stepsizes tend to infinity. This interpretation provides several new results. When duality holds for problem (25), the only known result regarding the convergence rate is an ergodic $O(1/k)$ rate for the primal-dual gap. Unfortunately, for general problems like (25) the primal-dual gap is difficult to compute and hence, in practice

we can not use it as a stopping criteria. Instead, we obtain new rates in terms of the objective function and the feasibility gap which are both easy to compute.

It is known that for problem (25) strong duality holds whenever $b \in \text{ri}(A \text{ dom } g)$, where $\text{ri}(C)$ stands for the relative interior of C . This condition is often difficult or even impossible to check. All proofs of the standard methods (PDA, ADMM, variational inequality methods) that can be used to solve (25) require duality arguments.

Another approach to solve (25) is the penalty method. It consists in solving a sequence of unconstrained optimization problems

$$\min_x g(x) + \frac{\sigma k}{2} \|Ax - b\|^2 =: F_k(x). \quad (26)$$

We use parameter σ here only for the ease of presentation in the future. Intuitively it is clear that with larger k , solutions of (26) become closer to a solution of our constrained problem (25). For more rigorous treatment on this subject, see [13, 25]. Since penalty methods are entirely primal, they do not use duality arguments. However, one still needs them in order to obtain some convergence rates [13]. As an exception, there is a recent paper [22] that studies the conic optimization problem without assuming that there exists a Lagrange multiplier. The authors applied the accelerated gradient method for the penalized objective (although different from (26)) and derived $O(1/k^{\frac{2}{3}})$ estimation for the feasibility gap.

Let g_* be the optimal value of the objective in (25) and $\hat{x}^k \in \text{argmin}_x F_k(x)$. Generally speaking, we cannot solve just one problem (26) for k large enough. First, because we do not know which k is large enough in order to approximate the true solution x^* by \hat{x}^k . And second, because solving (26) in practice becomes difficult for large k . As we show later, the primal-dual method provides a nice alternative. It is not necessary to directly solve any ill-condition unconstrained problem and the ergodic sequence (s^k) satisfies

$$F_k(\hat{x}^k) - g_* \leq F_k(s^k) - g_* \leq \frac{C}{k},$$

where $C > 0$ is some constant. Notice that in order to achieve this, we do not need to invoke the duality arguments and, consequently, require strong duality to hold. Moreover, when the solution set of (25) is bounded, we get $o(1/k)$ estimation for the feasibility gap.

4.2 Primal-dual approach

The primal-dual problem corresponding to (25) is:

$$\min_x \max_y g(x) + \langle Ax, y \rangle - \langle b, y \rangle.$$

PDA generates sequence (x^k) and (y^k) as:

$$\begin{aligned} y^{k+1} &= \text{prox}_{\sigma_k \langle b, \cdot \rangle} (y^k + \sigma_k A \bar{x}^k) = y^k + \sigma_k (A \bar{x}^k - b) \\ x^{k+1} &= \text{prox}_{\tau_k g} (x^k - \tau_k A^* y^{k+1}) \end{aligned}$$

One can derive

$$\begin{aligned} y^{k+1} &= y^k + \sigma_k (A \bar{x}^k - b) = y^{k-1} + A(\sigma_k \bar{x}^k + \sigma_{k-1} \bar{x}^{k-1}) - (\sigma_k + \sigma_{k-1})b = \dots \\ &= y^0 + A(\sigma_k \bar{x}^k + \dots + \sigma_0 \bar{x}^0) - (\sigma_k + \dots + \sigma_0)b. \end{aligned}$$

First, we concentrate on the PDA with fixed steps $\sigma_k = \sigma$, $\tau_k = \tau$. We also assume that PDA starts from (x^0, y^0) with $\bar{x}^0 = x^0$ and $y^0 = 0$. Then the above formula simplifies to

$$y^{k+1} = \sigma A(\bar{x}^k + \dots + \bar{x}^0) - \sigma(k+1)b = \sigma A(2x^k + x^{k-1} + \dots + x^1) - \sigma(k+1)b.$$

Let us introduce a new sequence (z^k) with $z^0 = x^0$ and $z^k = \frac{1}{k+1}(2x^k + x^{k-1} + \dots + x^1)$. Then $y^{k+1} = (k+1)\sigma(Az^k - b)$ and the primal-dual scheme can be written as

$$\begin{aligned} z^k &= \frac{k}{k+1}z^{k-1} + \frac{1}{k+1}\bar{x}^k \\ x^{k+1} &= \text{prox}_{\tau g}(x^k - (k+1)\tau\sigma A^*(Az^k - b)) \end{aligned} \quad (27)$$

with $z^0 = x^0 = \bar{x}^0$. The scheme (27) is somehow surprising. It uses the same idea of penalization of the constraints, however the penalty parameter is not fixed but increases in every iteration. At the same time, this method, in contrast to penalty methods, is direct, that is we need to run it only once.

4.3 Connection with Tseng's method

Define a new sequence (s^k) with $s^0 = x^0$ and $s^k = \frac{x^1 + \dots + x^k}{k}$. Then we can rewrite scheme (27) as

$$\begin{aligned} z^k &= \frac{x^k + ks^k}{k+1} \\ x^{k+1} &= \text{prox}_{\tau g}(x^k - (k+1)\lambda\nabla f(z^k)) \\ s^{k+1} &= \frac{x^{k+1} + ks^k}{k+1}, \end{aligned} \quad (28)$$

where as usually $f(x) = \frac{1}{2}\|Ax - b\|^2$ and $\lambda = \tau\sigma$. Again this scheme is reminiscent of the Tseng method. Notice, however, that the stepsize in k -th iteration is $\lambda(k+1)$.

Theorem 3. *Assume that the solution set S of (25) is nonempty and $\lambda L < 1$. Then for sequences (x^k) and (s^k) , generated by (28), it holds*

- (i) $F_k(\hat{x}^k) - g_* \leq F_k(s^k) - g_* = O(1/k)$.
- (ii) *If there exists a Lagrange multiplier for problem (25), then (x^k) and (s^k) converge to a solution of (25) and $f(z^k) = O(1/k^2)$.*
- (iii) *If S is bounded, then all limit points of (s^k) belong to S , $f(s^k) = o(1/k)$.*

Therefore in the most general case one can consider s^k as an approximated minimizer of the problem $\min_x F_k(x)$. Under some additional assumptions it is possible to prove convergence of the iterates. Finally, even when a dual solution y^* does not exist, but the solution set is bounded, we still can say something about convergence of the iterates (s^k) and feasibility gap $f(s^k)$.

Proof. By the proximal property (3),

$$\frac{1}{\tau}\langle x^{k+1} - x^k, x - x^{k+1} \rangle + \sigma\langle \nabla f(x^k), x - x^{k+1} \rangle + \sigma k\langle \nabla f(s^k), x - x^{k+1} \rangle \geq g(x^{k+1}) - g(x).$$

From Lemma 3 it follows

$$\begin{aligned} \langle \nabla f(x^k), x - x^{k+1} \rangle &= f(x) - f(x^{k+1}) + \frac{1}{2}\|A(x^{k+1} - x^k)\|^2 - \frac{1}{2}\|A(x^k - x)\|^2, \\ \langle \nabla f(s^k), x - x^{k+1} \rangle &= f(x) - f(x^{k+1}) + \frac{1}{2}\|A(x^{k+1} - s^k)\|^2 - \frac{1}{2}\|A(s^k - x)\|^2. \end{aligned} \quad (29)$$

Using the above identities, we deduce

$$\begin{aligned} \frac{1}{2\tau}\|x^{k+1} - x\|^2 + \frac{1}{2\tau}\|x^{k+1} - x^k\| - \frac{\sigma}{2}\|A(x^{k+1} - x^k)\|^2 + \sigma(k+1)(f(x^{k+1}) - f(x)) \\ + \frac{\sigma}{2}\|A(x^k - x)\|^2 + \frac{\sigma k}{2}\|A(s^k - x)\|^2 + g(x^{k+1}) - g(x) \\ \leq \frac{1}{2\tau}\|x^k - x\|^2 + \frac{\sigma k}{2}\|A(x^{k+1} - s^k)\|^2, \end{aligned} \quad (30)$$

Convexity of $F_{k+1}(x) = g(x) + \sigma(k+1)f(x)$ and the property (5) for f yield

$$F_{k+1}(x^{k+1}) + kF_{k+1}(s^k) \geq (k+1)F_{k+1}(s^{k+1}) + \frac{k}{2(k+1)}\|A(x^{k+1} - s^k)\|^2. \quad (31)$$

Combining this with $\lambda = \tau\sigma \leq \frac{1}{L^2}$ and with (30), we derive

$$\begin{aligned} \frac{1}{2\tau}\|x^{k+1} - x\|^2 + \sigma(k+1)(F_{k+1}(s^{k+1}) - F_{k+1}(x)) + \frac{\sigma k}{2}\|A(s^k - x)\|^2 \\ \leq \frac{1}{2\tau}\|x^k - x\|^2 + \sigma k F_{k+1}(s^k). \end{aligned}$$

If we choose $x = \bar{x}$ to be feasible, i.e., $A\bar{x} = b$, then $f(\bar{x}) = 0$, $F_{k+1}(\bar{x}) = g_*$ and $\frac{1}{2}\|A(s^k - \bar{x})\|^2 = f(s^k)$. Therefore,

$$\frac{1}{2\tau}\|x^{k+1} - \bar{x}\|^2 + (k+1)(F_{k+1}(s^{k+1}) - g_*) \leq \frac{1}{2\tau}\|x^k - \bar{x}\|^2 + k(F_k(s^k) - g_*). \quad (32)$$

Iterating the above, we obtain

$$\frac{1}{2\tau}\|x^{k+1} - \bar{x}\|^2 + (k+1)(F_{k+1}(s^{k+1}) - g_*) \leq \frac{1}{2\tau}\|x^1 - \bar{x}\|^2 + (F_1(s^1) - g_*) =: C. \quad (33)$$

It follows that $F_k(s^k) - g_* \leq \frac{C}{k}$. Let $\hat{x}^k \in \operatorname{argmin}_x F_k(x)$ for any $k > 0$. Thus,

$$F_k(\hat{x}^k) - g_* \leq F_k(s^k) - g_* \leq \frac{C}{k}. \quad (34)$$

(ii) Assume there exists a Lagrange multiplier. In this case the analysis of the primal-dual method [6] says that (x^k, y^k) converges to a solution (x^*, y^*) . In our notations it means that $y^{k+1} = (k+1)\sigma(Az^k - b)$ converges to some y^* . Hence, $f(z^k) = O(1/k^2)$.

(iii) First, we show that the sequence $(s^i)_{i \in \mathcal{I}}$ with $\mathcal{I} = \{i: g(s^i) < g_*\}$ is bounded. To this end, we use arguments due to Solodov [27]:

By assumption the set $S = \{x: g(x) \leq g_*, f(x) \leq 0\}$ is nonempty and bounded. Consider the convex function $\varphi(x) = \max\{g(x) - g_*, f(x)\}$. Notice that S coincides with the level set $\mathcal{L}(0)$ of φ :

$$S = \mathcal{L}(0) = \{x: \varphi(x) \leq 0\}.$$

Since $\mathcal{L}(0)$ is bounded, it is known then $\mathcal{L}(c) = \{x: \varphi(x) \leq c\}$ is bounded for any $c \in \mathbb{R}$. Fix any $c \geq 0$ such that $f(s^k) \leq c$ for all k . Since $g(s^i) - g_* < 0 \leq c$ for $i \in \mathcal{I}$, we have that $s^i \in \mathcal{L}(c)$, which is a bounded set. Hence, $(s^i)_{i \in \mathcal{I}}$ is bounded.

Let $M > 0$ be any constant that bounds from above $(\|s^i\|)_{i \in \mathcal{I}}$ and $\sqrt{C} + \|\bar{x}\|$, where $C > 0$ is a constant from (33). For every index k we have two alternatives: either $g(s^k) < g_*$ or $g(s^k) \geq g_*$. If the latter holds, then $\|x^k\| \leq \sqrt{C} + \|\bar{x}\| \leq M$. If the former holds, then by the above arguments we know that $\|s^k\| \leq M$. We now want to prove by induction that the sequence (s^k) is bounded. Assume that for the index k , $\|s^k\| \leq M$. If for the index $k+1$,

$g(s^{k+1}) < g_*$ then we are done: $k+1 \in \mathcal{I}$ and hence, $\|s^{k+1}\| \leq M$. If $g(s^{k+1}) \geq g_*$, then $\|x^{k+1}\| \leq M$. Observe that

$$\|s^{k+1}\| = \frac{\|ks^k + x^{k+1}\|}{k+1} \leq \frac{k}{k+1}M + \frac{1}{k+1}M = M,$$

which completes the proof that (s^k) is bounded. From (34) it follows that all limit points of (s^k) belong to S .

Notice that $\liminf_{k \rightarrow \infty} g(s^k) \geq g_*$. If we assume that there exists a subsequence (s^{k_j}) such that $\lim_{j \rightarrow \infty} g(s^{k_j}) > g_*$, then taking the limit in

$$g(s^{k_j}) + \sigma k_j f(s^{k_j}) \leq g_* + \frac{C}{k_j}$$

yields a contradiction. Hence, $g(s^k) \rightarrow g_*$. Using this in (34), we conclude that

$$f(s^k) \leq \frac{C}{k^2} + \frac{g_* - g(s^k)}{k} = o(1/k).$$

□

Remark 1.

- (a) Let $g(x) = \delta_C(x)$ for some closed convex set C . Then, first, in this case the performance of primal-dual method does not depend on the ratio σ/τ . And second, for $f(s^k)$, we obtain the same $O(1/k^2)$ rate of convergence which accelerated gradient methods provide for problem $\min_{x \in C} \frac{1}{2} \|Ax - b\|^2$.
- (b) Item (i) says that if by some analysis (which might be entirely independent from our settings) we know the convergence rate for $(F(\hat{x}^k) - g_*)$, then we can easily derive the rate for the $(F(s^k) - g_*)$.
- (c) If $m \gg n$ it might be cheaper to compute once $A^*A \in \mathbb{R}^{n \times n}$ and use it in all iterations of (28).

4.4 Minimization over least squares solutions

Consider a generalization of the problem (25):

$$\begin{aligned} \min \quad & g(x) + h(x) =: \phi(x) \\ \text{s.t.} \quad & x \in \operatorname{argmin}_y \|Ay - b\|^2, \end{aligned} \tag{35}$$

where in addition to the assumptions given in Section 4.1, we require that

- $h: \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex smooth functions with L_h -Lipschitz-continuous gradient ∇h .

First, for the discussion below consider the case $h \equiv 0$. From the theoretical point of view, (35) and (25) are of the same type, as the constraint in (35) is equivalent to a new linear system $A^*Ax = A^*b$. Hence, one can apply PDA using the latter equation instead of $Ax = b$. However, practically, it is not desirable. First, because now we have to deal with $A^*A \in \mathbb{R}^{n \times n}$ that for many applications where typically $m \ll n$ is much larger than just $A \in \mathbb{R}^{m \times n}$. Second, if the matrix A is sparse, A^*A will probably have a worse sparsity which will lead to a more expensive iteration. Third, it is known that the conditional number of A^*A is a square of the conditional number of A . Hence, when A is ill-conditioned, working with

A^*A will be much harder than with A . As an illustration, in Section 6 we present a simple numerical experiment to support our claim. Finally, we may not know in advance whether the system $Ax = b$ is consistent. Consequently, while applying PDA to the inconsistent system $Ax = b$, we would like to have something meaningful. In this subsection we show that under the similar conditions as in Theorem 3, the scheme (28), and hence the primal-dual algorithm, converges to a solution of the problem (35).

Now consider a more general problem with an additional smooth term h in (35). When the system $Ax = b$ is consistent and strong duality holds, it is known that the generalized primal-dual method [9, 29] converges to a solution of (35).

As it was before, let $f(x) = \frac{1}{2}\|Ax - b\|^2$. Similar to (28), consider the scheme

$$\begin{aligned} z^k &= \frac{x^k + ks^k}{k+1} \\ x^{k+1} &= \text{prox}_{\tau g}(x^k - \tau(\nabla h(x^k) + (k+1)\sigma\nabla f(z^k))) \\ s^{k+1} &= \frac{x^{k+1} + ks^k}{k+1}. \end{aligned} \tag{36}$$

Of course, it is clear that when $h \equiv 0$, both schemes (36) and (28) coincide. It should also be clear that the sequence (x^k) is the same sequence that the generalized primal-dual method with arbitrary x^0 and $y^0 = 0$ generates.

Let $f_* = \min_x f(x)$, ϕ_* be an optimal value of (35), $F_k(x) := \phi(x) + \sigma k(f(x) - f_*)$, and \hat{x}^k be any minimizer of F_k .

Theorem 4. *Assume that the solution set S of (35) is nonempty and $\tau(\sigma L + L_h) < 1$. Then for sequences (x^k) , (z^k) , and (s^k) , generated by (36), it holds*

- (i) $F_k(\hat{x}^k) - \phi_* \leq F_k(s^k) - \phi_* = O(1/k)$.
- (ii) *If there exists a Lagrange multiplier u^* for problem (35), then (x^k) , (z^k) , and (s^k) converge to a solution of (35) and $f(x^k) - f_* = o(1/k)$, $f(z^k) - f_* = O(1/k^2)$, $f(s^k) - f_* = O(1/k^2)$, $|F_k(s^k) - \phi_*| = O(1/k)$.*
- (iii) *If S is bounded, then all limit points of (s^k) belong to S , $f(s^k) - f_* = o(1/k)$.*

The general line of proof is the same as in Theorem 3. However, now in (ii) we can not use the result from standard analysis of PDA. This will be the most tricky part of our proof.

Proof. Before proceeding, notice that for any $\bar{x} \in S$ and $x \in \mathbb{R}^n$, we have

$$f(x) - f_* = f(x) - f(\bar{x}) = \frac{1}{2}\|A(x - \bar{x})\|^2. \tag{37}$$

Using (30) and taking into account the additional term $\nabla h(x^k)$ in (36), we obtain

$$\begin{aligned} &\frac{1}{2\tau}\|x^{k+1} - x\|^2 + \frac{1}{2\tau}\|x^{k+1} - x^k\|^2 - \frac{\sigma}{2}\|A(x^{k+1} - x^k)\|^2 + \sigma(k+1)(f(x^{k+1}) - f(x)) \\ &\quad + \frac{\sigma}{2}\|A(x^k - x)\|^2 + \frac{\sigma k}{2}\|A(s^k - x)\|^2 + g(x^{k+1}) - g(x) + \langle \nabla h(x^k), x^{k+1} - x \rangle \\ &\leq \frac{1}{2\tau}\|x^k - x\|^2 + \frac{\sigma k}{2}\|A(x^{k+1} - s^k)\|^2. \end{aligned}$$

Take $x = \bar{x} \in S$ and apply Lemma 2 and identity (37). This yields

$$\begin{aligned} & \frac{1}{2\tau} \|x^{k+1} - \bar{x}\|^2 + \frac{1}{2\tau} \|x^{k+1} - x^k\|^2 - \frac{\sigma}{2} \|A(x^{k+1} - x^k)\|^2 - \frac{L_h}{2} \|x^{k+1} - x^k\|^2 \\ & + \sigma(k+1)(f(x^{k+1}) - f_*) + \sigma(f(x^k) - f_*) + \sigma k(f(s^k) - f_*) + \phi(x^{k+1}) - \phi_* \\ & \leq \frac{1}{2\tau} \|x^k - \bar{x}\|^2 + \frac{\sigma k}{2} \|A(x^{k+1} - s^k)\|^2. \end{aligned}$$

Let $\beta = \frac{1-\tau\sigma L-\tau L_h}{2\tau}$. Using inequality (31), which is also true for our $F_{k+1}(x) = \phi(x^{k+1}) + \sigma(k+1)(f(x^{k+1}) - f_*)$, we obtain

$$\begin{aligned} & \frac{1}{2\tau} \|x^{k+1} - \bar{x}\|^2 + \beta \|x^{k+1} - x^k\|^2 + \sigma(f(x^k) - f_*) + (k+1)(F_{k+1}(s^{k+1}) - \phi_*) \\ & \leq \frac{1}{2\tau} \|x^k - \bar{x}\|^2 + k(F_k(s^k) - \phi_*). \quad (38) \end{aligned}$$

Iterating the above equation, we derive

$$\begin{aligned} & \frac{1}{2\tau} \|x^k - \bar{x}\|^2 + \beta \sum_{i=1}^{k-1} \|x^{i+1} - x^i\|^2 + \sigma \sum_{i=1}^{k-1} (f(x^i) - f_*) + k(F_k(s^k) - \phi_*) \\ & \leq \frac{1}{2\tau} \|x^1 - \bar{x}\|^2 + F_1(s^1) - \phi_* =: C. \quad (39) \end{aligned}$$

From that we see that

$$\phi(s^k) - \phi_* \leq \frac{C}{k} \quad \text{and} \quad F_k(\hat{x}^k) - \phi_* \leq F_k(s^k) - \phi_* \leq \frac{C}{k}, \quad (40)$$

which proves (i).

(ii). There exists a Lagrange multiplier $u^* \in \mathbb{R}^n$. This gives us an important estimation:

$$\phi(s^k) - \phi_* = \phi(s^k) - \phi(\bar{x}) \geq \langle A^* A u^*, s^k - \bar{x} \rangle \geq -\|A u^*\| \cdot \|A(s^k - \bar{x})\| = -\delta \cdot \sqrt{f(s^k) - f_*}, \quad (41)$$

where for simplicity $\delta := \|A^* u^*\|$. Our goal is to show that $k(F_k(s_k) - \phi_*)$ is bounded from below. We have

$$\begin{aligned} k(F_k(s_k) - \phi_*) &= k(\phi(s^k) - \phi_*) + \sigma k^2(f(s^k) - f_*) \\ &\geq \frac{\sigma k^2}{2}(f(s^k) - f_*) - \delta k \sqrt{f(s^k) - f_*} + \frac{\sigma k^2}{2}(f(s^k) - f_*). \quad (42) \end{aligned}$$

An easy algebra shows that the function $t \mapsto \frac{\sigma k^2}{2} t^2 - \delta k t$ attains its smallest value $-\frac{\delta^2}{2\sigma}$ at $t = \frac{\delta}{\sigma k}$. Hence, we can deduce in (42) that

$$k(F_k(s_k) - \phi_*) \geq -\frac{\delta^2}{2\sigma} + \frac{\sigma k^2}{2}(f(s^k) - f_*). \quad (43)$$

This inequality and (39) lead to

$$\frac{1}{2\tau} \|x^k - \bar{x}\|^2 + \beta \sum_{i=1}^{k-1} \|x^{i+1} - x^i\|^2 + \sigma \sum_{i=1}^{k-1} (f(x^i) - f_*) + \frac{\sigma k^2}{2}(f(s^k) - f_*) \leq C + \frac{\delta^2}{2\sigma}.$$

This shows that (x^k) is bounded, $\|x^k - x^{k-1}\| \rightarrow 0$, $f(x^k) - f_* = o(1/k)$, and $f(s^k) - f_* = O(1/k^2)$. Using the latter estimation in (41) and combining it with (40), we obtain that

$|\phi(s^k) - \phi_*| = O(1/k)$. The rate $f(z^k) - f_* = O(1/k^2)$ follows from the first equation in (36) and the rate $|F_k(s^k) - \phi_*| = O(1/k)$ from (43) and (40).

It only remains to prove that (x^k) is convergent. First we show that all limit points of (x^k) belong to S . Let (x^{k_i}) be any subsequence that converges to \tilde{x} . By the above, we know that \tilde{x} is feasible, that is $f(\tilde{x}) = f_*$. By prox-inequality, we have

$$\langle x^{k_i} - x^{k_i-1}, \bar{x} - x^{k_i} \rangle + \tau \langle \nabla h(x^{k_i}), \bar{x} - x^{k_i} \rangle + k_i \sigma \langle A^*(Az^{k_i} - b), \bar{x} - x^{k_i} \rangle \geq \tau(g(x^{k_i}) - g(\bar{x})).$$

and consequently by convexity of h ,

$$\langle x^{k_i} - x^{k_i-1}, \bar{x} - x^{k_i} \rangle + k_i \sigma \langle A^*(Az^{k_i} - b), \bar{x} - x^{k_i} \rangle \geq \tau(\phi(x^{k_i}) - \phi_*). \quad (44)$$

If we want to tend $k_i \rightarrow \infty$, we need to know how to estimate the second item in the left-hand side of (44). Using that $A^*b = A^*A\bar{x}$, we derive

$$\begin{aligned} k_i \langle A^*(Az^{k_i} - b), \bar{x} - x^{k_i} \rangle &= k_i \langle A(z^{k_i} - \bar{x}), A(\bar{x} - x^{k_i}) \rangle \leq k_i \|A(z^{k_i} - \bar{x})\| \cdot \|A(\bar{x} - x^{k_i})\| \\ &= k_i \sqrt{f(z^{k_i}) - f_*} \sqrt{f(x^{k_i}) - f_*} \rightarrow 0, \end{aligned}$$

due to the obtained asymptotics for $f(z^k)$ and $f(x^k)$. Hence, passing to the limit in (44) and using that $x^k - x^{k-1} \rightarrow 0$, we deduce $0 \geq \tau(\phi(\tilde{x}) - \phi_*)$. This means that $\tilde{x} \in S$ and therefore all limit points of (x^k) belong to S . From (38) it follows that

$$\frac{1}{2\tau} \|x^{k+1} - \bar{x}\|^2 + (k+1)(F_{k+1}(s^{k+1}) - \phi_*) \leq \frac{1}{2\tau} \|x^k - \bar{x}\|^2 + k(F_k(s^k) - \phi_*).$$

As $(F_k(s^k) - \phi_*)$ is bounded from below, we can apply Lemma 1 and conclude that the sequence (x^k) converges to some element in S .

(iii). In this case the proof is identical (with ϕ instead of g) to the case (iii) in Theorem 3. \square

Remark 2.

- (a) Notice that when $g = \delta_C$ and $h \equiv 0$, $x^k \in C$ and, consequently, $\phi(x^k) = \phi(s^k) = \phi_* = 0$. Hence, in this case the statement (ii) of Theorem 4 will be valid without any duality assumptions.

4.5 g is strongly convex

When g is γ -strongly convex, we can obtain even better estimations. Although our results presented below will be valid for the general case as in (35), for the clarity of presentation we consider the case when $h \equiv 0$. Hence, now our problem reads as:

$$\begin{aligned} \min \quad & g(x) \\ \text{s.t.} \quad & x \in \operatorname{argmin}_y \|Ay - b\|^2, \end{aligned} \quad (45)$$

where g is 1-strongly convex function that we assume without loss of generality.

First, let us consider the case when the linear system $Ax = b$ is consistent and there exists a Lagrange multiplier. In this case one can apply the accelerated primal-dual method (2) with stepsizes given by

$$\tau_{k+1} = \frac{\tau_k}{\sqrt{1 + \tau_k}}, \quad \tau_k \sigma_k = \lambda.$$

Similarly to section 4.2, one may introduce z^k , defined by

$$z^k = \frac{\sigma_k \bar{x}^k + \cdots + \sigma_0 \bar{x}^0}{\sigma_k + \cdots + \sigma_0}$$

and notice that $y^{k+1} = (\sigma_k + \cdots + \sigma_0)(Az^k - b)$ and $x^{k+1} = \text{prox}_{\tau_k g}(x^k - \tau_k A^* y^{k+1})$. Let $\Sigma_k = \sigma_k + \cdots + \sigma_0$. Define sequence (s^k) as $s^0 = x^0$ and $s^k = \frac{\sigma_{k-1}x^k + \cdots + \sigma_0 x^1}{\Sigma_{k-1}}$. Then the primal-dual scheme might be written in the Tseng form:

$$\begin{aligned} z^k &= (\sigma_k x^k + \Sigma_{k-1} s^k) / \Sigma_k \\ x^{k+1} &= \text{prox}_{\tau_k g}(x^k - \tau_k \Sigma_k \nabla f(z^k)) \\ s^{k+1} &= (\sigma_k x^{k+1} + \Sigma_{k-1} s^k) / \Sigma_k, \end{aligned} \tag{46}$$

where as usually $f(x) = \frac{1}{2} \|Ax - b\|^2$.

Let $f_* = \min_x f(x)$, g_* be an optimal value of (45), $F_k(x) = g(x) + \Sigma_k(f(x) - f_*)$ be the penalty function, and \hat{x}^k be the unique minimizer of F_k .

Theorem 5. *Let (x^k) , (s^k) be generated by (46), $\lambda L \leq 1$, and the solution set $S = \{\bar{x}\}$. Then it holds*

- (i) (s^k) converges to \bar{x} , $F_k(\hat{x}^k) - g_* \leq F_k(s^k) - g_* = O(1/k^2)$, $f(s^k) = o(1/k^2)$.
- (ii) *If there exists a Lagrange multiplier u^* for problem (45), then (x^k) also converges to \bar{x} at the rate $\|x^k - \bar{x}\| = O(1/k)$ and $f(x^k) - f_* = o(1/k^3)$, $f(z^k) - f_* = O(1/k^4)$, $f(s^k) - f_* = O(1/k^4)$, $|F_k(s^k) - \phi_*| = O(1/k^2)$.*

Proof. By the proximal property (4),

$$\begin{aligned} \langle x^{k+1} - x^k, \bar{x} - x^{k+1} \rangle + \tau_k \sigma_k \langle \nabla f(x^k), \bar{x} - x^{k+1} \rangle + \tau_k \Sigma_{k-1} \langle \nabla f(s^k), \bar{x} - x^{k+1} \rangle \\ \geq \tau_k (g(x^{k+1}) - g(\bar{x}) + \|x^{k+1} - \bar{x}\|^2). \end{aligned}$$

Using identities (29), we obtain

$$\begin{aligned} \frac{1 + \tau_k}{2} \|x^{k+1} - \bar{x}\|^2 + \frac{1}{2} \|x^{k+1} - x^k\|^2 - \frac{\tau_k \sigma_k}{2} \|A(x^{k+1} - x^k)\|^2 + \tau_k \Sigma_k (f(x^{k+1}) - f(\bar{x})) \\ + \frac{\tau_k \sigma_k}{2} \|A(x^k - \bar{x})\|^2 + \frac{\tau_k \Sigma_{k-1}}{2} \|A(s^k - \bar{x})\|^2 + \tau_k (g(x^{k+1}) - g(\bar{x})) \\ \leq \frac{1}{2} \|x^k - \bar{x}\|^2 + \frac{\tau_k \Sigma_{k-1}}{2} \|A(x^{k+1} - s^k)\|^2. \end{aligned}$$

From (37) and from $\tau_k \sigma_k = \lambda \leq \frac{1}{L}$ it follows

$$\begin{aligned} \frac{1 + \tau_k}{2} \|x^{k+1} - \bar{x}\|^2 + \tau_k \Sigma_k (f(x^{k+1}) - f_*) \\ + \lambda (f(x^k) - f_*) + \tau_k \Sigma_{k-1} (f(s^k) - f_*) + \tau_k (g(x^{k+1}) - g_*) \\ \leq \frac{1}{2} \|x^k - \bar{x}\|^2 + \frac{\tau_k \Sigma_{k-1}}{2} \|A(x^{k+1} - s^k)\|^2. \end{aligned} \tag{47}$$

Using that

$$\begin{aligned} f(x^{k+1}) &= \frac{\Sigma_k}{\sigma_k} f(s^{k+1}) - \frac{\Sigma_{k-1}}{\sigma_k} f(s^k) + \frac{\Sigma_{k-1}}{2\Sigma_k} \|A(x^{k+1} - s^k)\|^2, \\ g(x^{k+1}) &\geq \frac{\Sigma_k}{\sigma_k} g(s^{k+1}) - \frac{\Sigma_{k-1}}{\sigma_k} g(s^k), \end{aligned}$$

we derive

$$\begin{aligned} \frac{1+\tau_k}{2}\|x^{k+1}-\bar{x}\|^2 + \frac{\tau_k}{\sigma_k}\Sigma_k(F_{k+1}(s^{k+1})-g_*) + \lambda(f(x^k)-f_*) \\ \leq \frac{1}{2}\|x^k-\bar{x}\|^2 + \frac{\tau_k}{\sigma_k}\Sigma_{k-1}(F_k(s^k)-g_*). \end{aligned} \quad (48)$$

By the definition of (τ_k) , we have $\frac{\tau_k\sigma_{k+1}}{\sigma_k\tau_{k+1}} = \frac{\tau_k^2}{\tau_{k+1}^2} = (1+\tau_k)$. Hence, from (48) it follows that

$$\begin{aligned} \frac{1}{\tau_{k+1}^2}(\|x^{k+1}-\bar{x}\|^2 + \frac{\tau_{k+1}}{\sigma_{k+1}}\Sigma_k(F_{k+1}(s^{k+1})-g_*)) + \frac{\lambda}{\tau_k^2}(f(x^k)-f_*) \\ \leq \frac{1}{\tau_k^2}(\|x^k-\bar{x}\|^2 + \frac{\tau_k}{\sigma_k}\Sigma_{k-1}(F_k(s^k)-g_*)). \end{aligned}$$

Iterating the above, we obtain

$$\begin{aligned} \frac{1}{\tau_{k+1}^2}\|x^{k+1}-\bar{x}\|^2 + \frac{\Sigma_k}{\lambda}(F_{k+1}(s^{k+1})-g_*) + \sum_{i=1}^k \frac{\lambda}{\tau_i^2}(f(x^i)-f_*) \\ \leq \frac{1}{\tau_1^2}(\|x^1-\bar{x}\|^2 + \frac{\tau_1}{\sigma_1}\Sigma_0(F_1(s^1)-g_*)) =: C. \end{aligned} \quad (49)$$

For simplicity, assume that $\tau_0 = 1$. Then it is not difficult to prove by induction that $\frac{2}{k+2} \leq \tau_k \leq \frac{3}{k+2}$. In the general case, all results will be the same up to some constants, as it is known [6] that $\tau_k \sim 1/k$. In our case we have $\frac{k+2}{2}\lambda \geq \sigma_k \geq \frac{k+2}{3}\lambda$, hence

$$\frac{(k+4)(k+1)}{4}\lambda \geq \Sigma_k \geq \frac{(k+4)(k+1)}{6}\lambda.$$

First, let us estimate the term $F_k(s^k)-g_*$. From (49) it follows that $\Sigma_k(F_{k+1}(s^{k+1})-g_*) \leq \lambda C$, and thus

$$F_{k+1}(s^{k+1})-g_* \leq \frac{\lambda C}{\Sigma_k} = O(1/k^2). \quad (50)$$

As $S = \{\bar{x}\}$, by the same arguments as in part (iii) of Theorem 3, we have that (s^k) is bounded. From (50) one can derive that $s^k \rightarrow \bar{x}$ and $g(s^k) \rightarrow g_*$. Equation (50) also yields $f(s^{k+1}) \leq \frac{2\lambda C}{\Sigma_k^2} + \frac{2(g_*-g(s^k))}{\Sigma_k} = o(1/k^2)$, where we have used that $g(s^k) \rightarrow g_*$ and $\Sigma_k \sim O(1/k^2)$.

Case (ii). There exists a Lagrange multiplier $u^* \in \mathbb{R}^n$. Similarly to (41), we have

$$g(s^k)-g_* = g(s^k)-g(\bar{x}) \geq \langle A^*Au^*, s^k-\bar{x} \rangle \geq -\|Au^*\| \cdot \|A(s^k-\bar{x})\| = -\delta \cdot \sqrt{f(s^k)-f_*},$$

where $\delta := \|A^*u^*\|$. Next, we have

$$\begin{aligned} \Sigma_{k-1}(F_k(s_k)-g_*) &= \Sigma_{k-1}(g(s^k)-g_*) + \Sigma_{k-1}^2(f(s^k)-f_*) \\ &\geq \frac{\Sigma_{k-1}^2}{2}(f(s^k)-f_*) - \delta\Sigma_{k-1}\sqrt{f(s^k)-f_*} + \frac{\Sigma_{k-1}^2}{2}(f(s^k)-f_*). \end{aligned} \quad (51)$$

An easy algebra shows that the function $t \mapsto \frac{\Sigma_{k-1}^2}{2}t^2 - \delta\Sigma_{k-1}t$ attains its smallest value $-\frac{\delta^2}{2}$ at $t = \frac{\delta}{\Sigma_{k-1}}$. Hence, we can deduce in (51) that

$$\Sigma_{k-1}(F_k(s_k)-g_*) \geq -\frac{\delta^2}{2} + \frac{\Sigma_{k-1}^2}{2}(f(s^k)-f_*). \quad (52)$$

Combining (52) with (50), we derive $|F_k(s^k) - g_*| = O(1/k^2)$. Using (52) in (49), we obtain

$$\|x^{k+1} - \bar{x}\|^2 \leq \tau_{k+1}^2(C + \frac{\delta^2}{2}), \quad \frac{\sum_k}{2}(f(s^{k+1}) - f_*) \leq \lambda C, \quad \sum_{i=1}^k \frac{\lambda}{\tau_i^2}(f(x^k) - f_*) \leq C.$$

The above three inequalities yield respectively $\|x^k - \bar{x}\| = O(1/k)$, $f(s^k) - f_* = O(1/k^4)$ and $f(x^k) - f_* = o(1/k^3)$. The latter two asymptotics give $f(z^k) - f_* = O(1/k^4)$. \square

Remark 3. Consider the following feasibility problem: find x in the intersection of a closed convex set C and an affine subspace $\mathcal{L} := \{w \in \mathbb{R}^n : Aw = b\}$. This can be written as:

$$\begin{aligned} \min \quad & \frac{1}{2}\|x\|^2 + \delta_C(x) \\ \text{s.t.} \quad & Ax = b \end{aligned} \tag{53}$$

For problem (53) we can apply APDA in the form (46) with strongly convex $g(x) = \frac{1}{2}\|x\|^2 + \delta_C(x)$. Evidently, all iterates x^k and s^k belong to C . Under the assumption that strong duality holds, it follows from Theorem 5 that $f(s^k) - f_* = O(1/k^4)$. This means $\|As^k - b\|^2 = O(1/k^4)$. By Hoffmann Lemma we can deduce that $\text{dist}(s^k, \mathcal{L}) = O(1/k^2)$; and since $s^k \in C$, that $\text{dist}(s^k, C \cap \mathcal{L}) = O(1/k^2)$.

On the other hand, if we formulate the given problem as $\min_{x \in C} \|Ax - b\|^2$ and apply the projected gradient or accelerated projected gradient methods (or primal-dual method in the form (21)), we obtain $f(x^k) - f_* = O(1/k)$ or $f(x^k) - f_* = O(1/k^2)$ respectively. Combining this with Hoffmann Lemma, we derive $\text{dist}(x^k, C \cap \mathcal{L}) = O(1/\sqrt{k})$ or $\text{dist}(x^k, C \cap \mathcal{L}) = O(1/k)$. Notice that our discussion above does not assume any special properties of C , like, for example, polyhedrality of C . If the latter holds then it is known that projected gradient method converges at least linearly [16].

5 Bilevel composite minimization

Bilevel convex optimization aims to minimize some convex function (outer problem) over the solution set of another convex function (inner problem). This field is rich in applications; however, the implicit constraints imposed by the inner problem make such problems difficult for developing algorithms or even establishing the optimality conditions; for more in-depth reference see [10]. In this section we focus on a convex problem where both objectives in the inner and outer problems are instances of composite optimization problems [24]. Below we list two most frequent sources, where such problems arise.

Assume we want to solve a composite optimization problem whose objective is not strictly convex, in this case it is likely that the problem will have multiple solutions (infinitely many). Naturally, one can add some a priori knowledge about the desired solution: smoothness, sparsity, closeness to some given point, etc. Such an approach gives rise to a bilevel composite optimization.

Another source emerges when we want to minimize a composite function over some difficult constraints. In this case, the inner problem can describe those constraints. For example, for a family of closed convex sets C_i we can use $\bigcap_i C_i = \text{argmin}_y \{\sum_i \text{dist}(y, C_i)^2\}$. This recasting might be even preferable when the intersection of C_i is empty.

5.1 Problem description

Consider the problem of a bilevel composite minimization

$$\begin{aligned} \min_x \quad & \phi_1(x) := g_1(x) + h_1(x) \\ \text{s.t.} \quad & x \in \operatorname{argmin}_y \phi_2(y) := g_2(y) + h_2(y), \end{aligned} \quad (54)$$

where

- $g_1: \mathbb{R}^n \rightarrow (-\infty, +\infty]$, $g_2: \mathbb{R}^n \rightarrow (-\infty, +\infty]$ are proper l.s.c. convex functions;
- $h_1: \mathbb{R}^n \rightarrow \mathbb{R}$, $h_2: \mathbb{R}^n \rightarrow \mathbb{R}$ are convex smooth functions with L_{h_1} - and L_{h_2} - Lipschitz-continuous gradients ∇h_1 , ∇h_2 respectively;

Denote by ϕ_1^* , ϕ_2^* , S_1 , S_2 the optimal values and the solution sets of the outer and inner problems respectively.

In [4] for problem (54) Cabot proposed the proximal point algorithm with slowly vanishing steps:

$$x^{k+1} = \operatorname{prox}_{\alpha_k \phi_1 + \phi_2}(x^k), \quad (55)$$

where $\sum_k \alpha_k = +\infty$. This method is implicit, since even for simple practical problems the evaluation of the prox-operator in (55) is not trivial. For the particular case of (54) with $g_1 \equiv 0$ and $g_2 = \delta_C(x)$ for some closed convex set C , Solodov proposed an explicit method [27]:

$$x^{k+1} = P_C(x^k - \lambda_k(\alpha_k \nabla h_1(x) + \nabla h_2(x))),$$

where (α_k) is the sequence as defined above and λ_k is a stepsize defined by some simple backtracking procedure.

There are two further simple schemes for the case h_1 is strongly convex and $g_1 \equiv 0$: the viscosity approximation method [21, 30] and the hybrid steepest descent method [31]. The analysis of these two schemes is based on the fixed point theory which involves nonexpansive and contractive operators. Hence, it is not so easy to introduce some backtracking procedure for them (in case we do not know the Lipschitz constant) or to derive convergence rates (see the recent paper [26] for the viscosity approximation algorithm). On the other hand, they are more general because the inner problem is not required to be an optimization problem.

The characteristic feature of all four aforementioned methods is that they use vanishing stepsizes with respect to the outer objective. In contrast, our method uses a fixed stepsize with respect to the outer objective and gradually increasing stepsizes for the inner problem. It does not require strong convexity assumption nor evaluating any difficult prox-operators.

5.2 New method

Here is our proposed extension of the Tseng scheme:

$$\begin{aligned} z^k &= \alpha_k x^k + (1 - \alpha_k) s^k \\ x^{k+1} &= \operatorname{prox}_{\sigma g_1 + \frac{\tau}{\alpha_k} g_2}(x^k - (\sigma \nabla h_1(x^k) + \frac{\tau}{\alpha_k} \nabla h_2(z^k))) \\ s^{k+1} &= \alpha_k x^{k+1} + (1 - \alpha_k) s^k \end{aligned} \quad (56)$$

with $s^1 = x^1 \in \mathbb{R}^n$. We prove the following result:

Theorem 6. *Let $(s^k), (x^k)$ be generated by method (56). Suppose that ϕ_1 is bounded from below, $S_1 \neq \emptyset$, $\sigma L_{h_1} + \tau L_{h_2} \leq 1$ and the sequence $(\alpha_k) \subset (0, 1]$ satisfies $\frac{1 - \alpha_k}{\alpha_k^2} \leq \frac{1}{\alpha_{k-1}^2}$. Then*

(i) $\phi_2(s^k) - \phi_2^* = O(1/k)$;

(ii) If S_2 is a bounded set, then $\phi_2(s^k) - \phi_2^* = o(1/k)$, $\phi_1(s^k) \rightarrow \phi_1^*$, the sequence (s^k) is bounded and all its limit points belong to S_1 .

For simplicity, for the proof given below, we consider the case with $\alpha_k = \frac{2}{k+2}$, which fulfills the assumption of the theorem.

Proof. By the prox-inequality, we have

$$(x^{k+1} - x^k + (\sigma h_1(x^k) + \frac{\tau}{\alpha_k} \nabla h_2(z^k)), x - x^{k+1}) \geq \sigma[g_1(x^{k+1}) - g_1(x)] + \frac{\tau}{\alpha_k}[g_2(x^{k+1}) - g_2(x)]. \quad (57)$$

By the descent lemma for h_2 ,

$$h_2(s^{k+1}) - h_2(z^k) - \langle \nabla h_2(z^k), s^{k+1} - z^k \rangle \leq \frac{L_{h_2}}{2} \|s^{k+1} - z^k\|^2,$$

which we can rewrite as

$$\frac{1}{\alpha_k^2} (h_2(s^{k+1}) - h_2(z^k)) - \frac{1}{\alpha_k} \langle \nabla h_2(z^k), x^{k+1} - x^k \rangle \leq \frac{L_{h_2}}{2} \|x^{k+1} - x^k\|^2, \quad (58)$$

using that $s^{k+1} - z^k = \alpha_k(x^{k+1} - x^k)$. By convexity of h_2 ,

$$\frac{1 - \alpha_k}{\alpha_k^2} (h_2(s^k) - h_2(z^k)) \geq \frac{1 - \alpha_k}{\alpha_k^2} \langle \nabla h_2(z^k), s^k - z^k \rangle = \frac{1}{\alpha_k} \langle \nabla h_2(z^k), z^k - x^k \rangle, \quad (59)$$

and

$$\frac{1}{\alpha_k} (h_2(x) - h_2(z^k)) \geq \frac{1}{\alpha_k} \langle \nabla h_2(z^k), x - z^k \rangle. \quad (60)$$

Summing up (58), (59), and (60), we obtain

$$\frac{1}{\alpha_k} h_2(x) + \frac{1 - \alpha_k}{\alpha_k^2} h_2(s^k) - \frac{1}{\alpha_k^2} h_2(s^{k+1}) + \frac{L_{h_2}}{2} \|x^{k+1} - x^k\|^2 \geq \frac{1}{\alpha_k} \langle \nabla h_2(z^k), x - x^{k+1} \rangle. \quad (61)$$

Applying Lemma 2 to h_1 , we derive

$$h_1(x) - h_1(x^{k+1}) + \frac{L_{h_1}}{2} \|x^{k+1} - x^k\|^2 \geq \langle \nabla h_1(x^k), x - x^{k+1} \rangle. \quad (62)$$

As g_2 is convex,

$$g_2(x^{k+1}) \geq \frac{1}{\alpha_k} g_2(s^{k+1}) - \frac{1 - \alpha_k}{\alpha_k} g_2(s^k). \quad (63)$$

Now summation of (57), (61), multiplied by τ , (62), multiplied by σ , and (63), multiplied by $\frac{\tau}{\alpha_k}$, yields

$$\begin{aligned} \frac{1}{2} \|x^{k+1} - x\|^2 + \frac{1 - \sigma L_{h_1} - \tau L_{h_2}}{2} \|x^{k+1} - x^k\|^2 + \sigma(\phi_1(x^{k+1}) - \phi_1(x)) + \frac{\tau}{\alpha_k} (\phi_2(s^{k+1}) - \phi_2(x)) \\ \leq \frac{1}{2} \|x^k - x\|^2 + \frac{\tau(1 - \alpha_k)}{\alpha_k^2} (\phi_2(s^k) - \phi_2(x)). \end{aligned} \quad (64)$$

Take $x = x^* \in S$. Since $1 \geq \sigma L_{h_1} + \tau L_{h_2}$ and $\frac{1-\alpha_k}{\alpha_k^2} \leq \frac{1}{\alpha_{k-1}^2}$ for $k \geq 1$, from (64) it follows

$$\begin{aligned} \frac{1}{2} \|x^{k+1} - x^*\|^2 + \sigma(\phi_1(x^{k+1}) - \phi_1^*) + \frac{\tau}{\alpha_k^2}(\phi_2(s^{k+1}) - \phi_2^*) \\ \leq \frac{1}{2} \|x^k - x^*\|^2 + \frac{\tau}{\alpha_{k-1}^2}(\phi_2(s^k) - \phi_2^*). \end{aligned}$$

Also, as $\alpha_k \phi_1(x^{k+1}) \geq \phi_1(s^{k+1}) - (1 - \alpha_k)\phi_1(s^k)$, we have

$$\begin{aligned} \frac{1}{2} \|x^{k+1} - x^*\|^2 + \frac{\sigma}{\alpha_k}(\phi_1(s^{k+1}) - \phi_1^*) + \frac{\tau}{\alpha_k^2}(\phi_2(s^{k+1}) - \phi_2^*) \\ \leq \frac{1}{2} \|x^k - x^*\|^2 + \frac{\sigma(1 - \alpha_k)}{\alpha_k}(\phi_1(s^k) - \phi_1^*) + \frac{\tau}{\alpha_{k-1}^2}(\phi_2(s^k) - \phi_2^*). \quad (65) \end{aligned}$$

Let $\alpha_k = \frac{2}{k+2}$. From (65) we have

$$\begin{aligned} \|x^{k+1} - x^*\|^2 + \sigma(k+2)(\phi_1(s^{k+1}) - \phi_1^*) + \frac{\tau(k+2)^2}{2}(\phi_2(s^{k+1}) - \phi_2^*) \\ \leq \|x^k - x^*\|^2 + \sigma k(\phi_1(s^k) - \phi_1^*) + \frac{\tau(k+1)^2}{2}(\phi_2(s^k) - \phi_2^*). \end{aligned}$$

Let $a_k = \|x^k - x^*\|^2$, $b_k = \sigma(\phi_1(s^k) - \phi_1^*)$, $c_k = \frac{\tau}{2}(\phi_2(s^k) - \phi_2^*)$. Then

$$a_{k+1} + (k+2)b_{k+1} + (k+2)^2 c_{k+1} \leq a_k + (k+1)b_k + (k+1)^2 c_k - b_k. \quad (66)$$

Unfortunately, because of the term $-b_k$, we can not iterate (66) as it was before in (32)–(33). Instead, we derive

$$a_{k+1} + (k+1)b_{k+1} + \sum_{i=1}^{k+1} b_i + (k+2)^2 c_{k+1} \leq a_1 + 2b_1 + 4c_1. \quad (67)$$

As b_k is bounded from below, say $b_k \geq -l$ with $l \geq 0$, we deduce

$$c_{k+1} \leq \frac{(k+2)^2}{(k+1)^2} c_{k+1} \leq \frac{a_1 + 2b_1 + 4c_1 + 2(k+1)l}{(k+1)^2} \leq \frac{a_1 + 2b_1 + 4c_1}{(k+1)^2} + \frac{2l}{k+1} \quad (68)$$

Thus, there exists some $C > 0$ such that $\phi_2(s^k) - \phi_2^* \leq \frac{C}{k}$. This proves (i).

By our assumption in (ii), the solution set S_2 is bounded. As it coincides with the level set $\mathcal{L}(\phi_2^*)$ of ϕ_2 , it follows that all level sets $\mathcal{L}(c)$ are bounded for all $c \in \mathbb{R}$. Hence, the sequence (s^k) is bounded. Since ϕ_2 is l.s.c., all limit points of (s^k) belong to S_2 . Evidently, this means that $\liminf_{k \rightarrow \infty} \phi_1(s^k) \geq \phi_1^*$, or $\liminf_{k \rightarrow \infty} b_k \geq 0$. Assume that there exist a subsequence (s^{k_i}) such that $\lim_{i \rightarrow \infty} \phi_1(s^{k_i}) > \phi_1^*$, that is $\lim_{i \rightarrow \infty} b_{k_i} > 0$. Taking the \liminf in both sides of (67) indexed by k_i , instead of $k+1$, we conclude

$$\lim_{i \rightarrow \infty} b_{k_i} + \liminf_{i \rightarrow \infty} \frac{b_1 + \dots + b_{k_i}}{k_i} \leq 0.$$

Notice that $\liminf_{i \rightarrow \infty} \frac{b_1 + \dots + b_{k_i}}{k_i} \geq \liminf_{k \rightarrow \infty} \frac{b_1 + \dots + b_k}{k} \geq \liminf_{k \rightarrow \infty} b_k \geq 0$. From this it follows that $\lim_{i \rightarrow \infty} b_{k_i} \leq 0$ which provides us a contradiction. Hence, $\lim_{k \rightarrow \infty} \phi_1(s^k) = \phi_1^*$ and consequently, $\lim_{k \rightarrow \infty} \frac{\phi_1(s^1) + \dots + \phi_1(s^k)}{k} = \phi_1^*$. As the sequence (s^k) is bounded, the obtained result means that all limit points of (s^k) belong to S_1 .

Now we can improve our estimation for c_{k+1} . In fact, from (68) we have

$$c_{k+1} \leq \frac{a_1 + 2b_1 + 4c_1}{(k+1)^2} + \frac{\phi_1^* - \phi_1(s^k)}{k+1} + \frac{\phi_1^* - \frac{\phi_1(s^1) + \dots + \phi_1(s^{k+1})}{k+1}}{k+1}.$$

This yields the $o(1/k)$ convergence rate for $\phi_2(s^k) - \phi_2^*$. \square

Although our convergence result does not seem to be very strong: in particular we have not proved convergence of the whole sequences (s^k) or (x^k) , we consider a very general problem without assumption of strong convexity and any regularity assumptions. We believe that further investigation should be done in this direction.

Remark 4.

- (a) Notice that the condition on (α_k) is the same as in Tseng's method or other accelerated gradient methods.
- (b) Although problem (54) includes (35) as a particular example, the results in Theorem 4 are stronger. First, it works even when the solution set S_2 is unbounded (which for (35) is always the case unless for the trivial case when it is a singleton). Second, for (35) we obtained better convergence results.
- (c) Of course the proposed method will be most useful if we know an explicit formula for $\text{prox}_{\lambda_1 g_1 + \lambda_2 g_2}$ or know an efficient way how to compute it.
- (d) Instead of the assumption on S_2 to be bounded, we can check whether (s^k) is bounded directly during the numerical experiment. If this is the case, the statement (ii) of Th. 6 is also valid.
- (e) If both h_1 and h_2 are present in problem (54), it is important to understand what is a good way to choose σ and τ in condition $\sigma L_{h_1} + \tau L_{h_2} \leq 1$. This relates to the known problem of choosing the ratio between steps in PDA.
- (f) During the presentation in Theorems 3, 4, 5, 6 we consider the case when all Lipschitz constants were known. As they appeared when we apply the descent lemma, one may easily extend our analysis for the case when it is used the standard Goldschtein-Armijo backtracking rule. What is not clear is how to combine our analysis with linesearch for PDA proposed in [20].

6 Numerical experiments

This section collects several numerical tests to illustrate the performance of the proposed methods. Computations¹ were performed using Python 3.5 running 64-bit Debian GNU/Linux.

Elastic net minimization over an inconsistent linear system. Consider the following problem

$$\begin{aligned} \min_x \quad & \|x\|_1 + \frac{\gamma}{2} \|x\|^2 \\ \text{s.t.} \quad & x \in \text{argmin}_y \|Ay - b\|^2, \end{aligned} \tag{69}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $\gamma > 0$. For generating the random test problem, we apply the following strategy:

¹Codes can be found on https://gitlab.gwdg.de/malitskyi/pd_and_tseng_methods

1. Set $A = Q_1 Q_2$ for $Q_1 \in \mathbb{R}^{m \times r}$ and $Q_2 \in \mathbb{R}^{r \times n}$, whose entries drawn from the normal distribution and $r \in \mathbb{N}$.
2. Choose $w \in \mathbb{R}^n$ with 20 nonzero elements drawn from the uniform distribution in $(-1, 1)$.
3. Generate randomly a vector $\nu \in \mathbb{R}^m$ with elements drawn from the normal distribution.
4. Set $b = Aw + \nu$.

Note that by construction, $\text{rank } A \leq r$. If $r < m$, the linear system $Ax = b$ is inconsistent with a high probability. We apply two primal-dual methods with the objective from (69) and the constraints $Ax = b$ (PDA-1) and $A^*Ax = A^*b$ (PDA-2) respectively. As the objective is strongly convex, the solution x^* of (69) is unique. For both methods we compare $\|x^k - x^*\|$ v.s. the number of iterations. In order to guarantee the good choice of parameters τ and σ for primal-dual methods we run several instances of each method with steps $\sigma = \frac{1}{2^i \sqrt{L}}$, $\tau = \frac{2^i}{\sqrt{L}}$ for PDA-1 and $\sigma = \frac{1}{2^i L}$, $\tau = \frac{2^i}{L}$ for PDA-2, where $i = -5, -4, \dots, 10$, and choose the best instances. A good approximation of the true solution x^* is found as the result of the best method after 10^5 iterations. For our four test problems the parameters are chosen as follows: $\gamma = 0.1$, $r = 50$ and $(m, n) \in \{(70, 100), (100, 200), (100, 500), (300, 500)\}$. The starting primal point for both methods is $x^0 = (0, \dots, 0) \in \mathbb{R}^n$, and the dual ones are $y^0 = (0, \dots, 0) \in \mathbb{R}^m$ for PDA-1 and $y^0 = (0, \dots, 0) \in \mathbb{R}^n$ for PDA-2. The results are presented in Fig. 1.

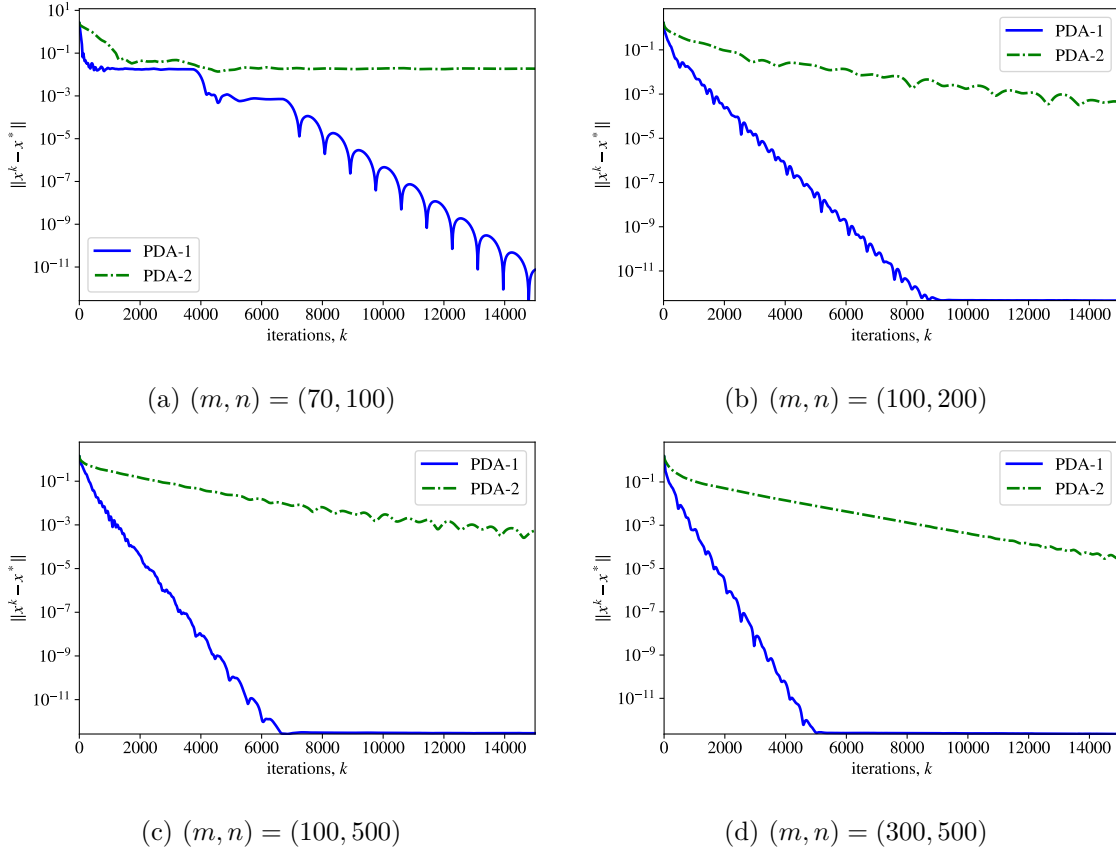


Figure 1: Convergence plots for problem (69)

Bilevel composite optimization. In this experiment we are seeking the closest solution of the nonnegative least squares problem to a given element u :

$$\begin{aligned} \min_x \quad & \frac{1}{2} \|x - u\|^2 \\ \text{s.t.} \quad & x \in \operatorname{argmin}_y \frac{1}{2} \|Ay - b\|^2 + \delta_{\mathbb{R}_+^n}(y). \end{aligned} \tag{70}$$

Let $g_2(x) = \delta_{\mathbb{R}_+^n}(x)$, $h_2(x) = \frac{1}{2} \|Ax - b\|^2$ and $g_1(x) = \frac{1}{2} \|x - u\|^2$.

Note that both the viscosity approximation method and the hybrid steepest descent method for this problem will be the same and in turn will coincide with the classical Halpern scheme [14]: $x^{k+1} = \alpha_k u + (1 - \alpha_k) \operatorname{prox}_{\tau g_2}(x^k - \tau \nabla h_2(x^k))$, where (α_k) is a nonnegative sequence that satisfies some conditions. The standard choice is $\alpha_k = \frac{1}{k}$ which fulfills the assumption in [30]. Therefore, we compare only our proposed method (56) with different steps σ and the Halpern scheme. Notice that the cost of each iteration in both methods is approximately the same. Although the solution set of the inner problem may be unbounded, in our experiments the sequences (s^k) and (x^k) remained bounded (see Remark 4 (d)). For simplicity, we use two test problems from the previous example, but now the elements of $w \in \mathbb{R}^n$ are generated from the uniform distribution in $(0, 10)$. The vector u is generated by the same way. We run two instances of the proposed method for $\sigma = 10^{-3}$ and $\sigma = 10^{-4}$. As the outer problem is strongly convex, the solution is unique and its approximation x^* is found as the best result among all methods after $2 \cdot 10^6$ iterations. The minimal value ϕ_2^* is found as the smallest value among all function values obtained by applying FISTA for the inner problem after 10^5 iterations. For the proposed method we present both residuals $\|x^k - x^*\|$, $\phi_2(x^k) - \phi_2^*$ (solid line) and $\|s^k - x^*\|$, $\phi_2(s^k) - \phi_2^*$ (dashed line). The starting point for all methods is $x^0 = (0, \dots, 0) \in \mathbb{R}^n$, the stepsize is $\tau = \frac{1}{\|A\|}$. All methods converge quite fast in the beginning and much slower after that, for the feasibility gap the convergence in the beginning is even more drastic. The results are presented in Fig. 2.

7 Conclusion and further research

Our main result in this paper is the obtained relationship between the primal-dual algorithm of Chambolle-Pock and the proximal gradient method of Tseng. This allows us to deduce several new results. First, we have showed that for the regularized least squares problem the primal-dual method converges at $O(1/k^2)$ rate for the objective function. Second, for the problem of minimization over a linear system we have proved the PDA converges even when the system is inconsistent and/or the duality does not hold. Third, we have proposed an extension of Tseng's method to solve a bilevel composite optimization problem.

There are many things that remain to be done; some of them we list below:

1. Paper [12] proposed a coordinate version of Tseng's method applied to a composite minimization problem. It is interesting to study if it is possible to do the same for the schemes (28) or (36), as they both are also extensions of Tseng's method.
2. Paper [20] proposed linesearch for PDA, which for problems (6) and (25) does not use additional expensive operations like matrix-vector multiplications or evaluation of the prox-operators. However, its analysis is also based on the duality argument. It is interesting to see whether the proposed approach can be extended to that non-standard linesearch.
3. Investigation of regularity condition for problem (56) which will guarantee better convergence properties.

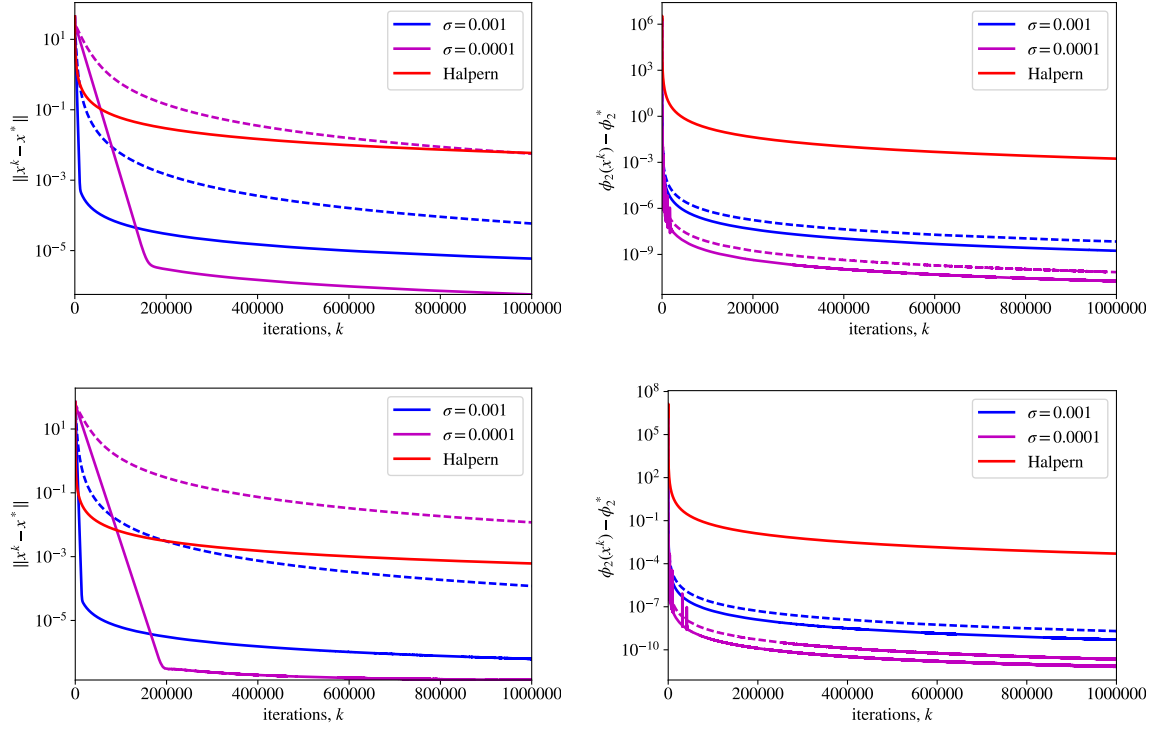


Figure 2: Convergence plots for problem (70). Top plots show results for problem with $(m, n) = (70, 100)$, bottom for $(m, n) = (100, 200)$. Left plots depict residuals $\|x^k - x^*\|$ (solid line) and $\|s^k - x^*\|$ (dashed line). Right plots depict respectively $\phi_2(x^k) - \phi_2^*$ (solid line) and $\phi_2(s^k) - \phi_2^*$ (dashed line).

4. It is worth studying whether similar results can be obtained for the methods with Bregman distance and/or without applying the descent lemma; see [2, 7].
5. For problems (25), (35) it is interesting to derive convergence rates for PDA that explicitly depend on the properties of the matrix A .

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