Bi-objective autonomous vehicle repositioning problem with travel time uncertainty

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Abstract

We study the problem of repositioning autonomous vehicles in a shared mobility system in order to simultaneously minimize the unsatisfied demand and the total operating cost. We first present a mixed integer linear programming formulation for the deterministic version of the problem, and based on that we develop an extended formulation that is easier to work with in the non-deterministic setting that we aim to explore. We then show how the travel time uncertainty can be incorporated into the extended deterministic formulation using chance-constraint programming. Finally, two new reformulations for the proposed chance-constraint program are developed. We show a critical result that the size of one of the reformulations (in terms of the number of variables and constraints) does not depend on the number of scenarios, and so it outperforms the other reformulation. Both reformulations are bi-objective mixed integer linear programs with finite number of nondominated points and so they can be solved directly by algorithms such as the balanced box method (Boland et al., 2015). A computational study demonstrates the efficacy of the proposed reformulations.

Keywords: autonomous vehicle repositioning problem, bi-objective integer linear programming, chanced-constraint programming

1. Introduction

In transportation science, carsharing is known to have many benefits (Jorge and de Almeida Correia, 2013). Martin et al. (2010) report that carsharing systems can substantially reduce the total number of vehicles held by household members. Shaheen and Cohen (2007) report that a single carsharing vehicle can reduce the need for 6 to 23 cars in North America. Shaheen et al. (2006) argue that carsharing can also reduce congestion, deliver cost savings through economies of scale, reduce emissions through deployment of clean technology and fuels, facilitate more efficient land use (for example by reducing the number of parking spaces needed), and increase mobility options and connectivity among other transportation modes. Overall, the prospects of carsharing systems look promising because more than a million people use carsharing around the world (Shaheen and Cohen, 2013) and the benefits of carsharing cannot be overstated.

In recent years, autonomous (or driverless) vehicles have garnered increasing interest from car manufacturers and the community of transportation researchers and practitioners (Fagnant et al.,

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2015; Fagnant and Kockelman, 2015; Kek et al., 2009). For example, Google has driven more than two million miles on public streets (https://waymo.com/ontheroad/) under autonomous mode since 2009. In October 2016, Uber's Otto used a self driving truck to make a delivery for Budweiser by driving 125 miles without human input (http://ot.to/). In addition to this, many automobile manufactures like General Motor, Mercedes-Benz, Audi, Nissan, BMW and Renault expect to sell driverless vehicles by 2020 (Liang et al., 2016).

In light of the above, the focus of this study is on the idea of combining carsharing with the technology of autonomous vehicles. It is worth mentioning that the business model for carsharing can take variety of forms including but not limited to the car rental companies and the traditional taxi services. The business model considered in this study is a free floating taxi service containing a fleet of autonomous vehicles. More specifically, we assume that there is a finite set of zones/stations, a finite set of time periods and a finite set of inter-zone demand, and also each autonomous vehicle can freely move from one zone to another (with or without a passenger). Two typical optimization problems arising in almost all carsharing systems are fleet sizing and fleet repositioning (or relocation). The second problem concerns with optimally repositioning vehicles between stations to serve the demand in future time periods (de Almeida Correia and Santos, 2014), and this is precisely what we explore in this study.

We note that there are several recent studies about operations of autonomous vehicles, but the relevant literature is not rich yet and so further research is required. For example, Chebbi and Chaouachi (2016) study the problem of repositioning a fleet of autonomous vehicles but in the framework of personal rapid transit system, i.e., autonomous vehicles should carry passengers between certain guideways. The goal of their study is to minimize the wasted capacity of the system. Lees-Miller (2016) studies a similar problem but with the goal of minimizing the total waiting time. In another research, Lam et al. (2016) explore the problem of scheduling and admission control for a taxi fleet of autonomous vehicles in an urban setting. They propose a mixed integer linear program for the former and a bi-level optimization model for the latter problem. In another recent study, Liang et al. (2016) propose two modeling schemes for optimizing automated taxi systems. In the first scheme, they assume that trip requests can be accepted or rejected according to the objective of profit maximization. In the second scheme, they assume that all travel requests must be satisfied. Interested readers may refer to Lam et al. (2016) for further details about studies on operations of autonomous vehicles, and to Jorge and de Almeida Correia (2013) and de Almeida Correia and Santos (2014) for further details about studies on operations of carsharing.

There are five distinguishing features between our research and the relevant existing studies in the literature. (1) Our proposed approach minimizes the total cost and the total unsatisfied demand at the same time. (2) Our proposed approach does not allow ride sharing or parallel demand fulfillment. In other words, it does not assign more than one passenger to an autonomous vehicle for a single journey. (3) Our proposed approach considers the uncertainty in the travel time since the underlying assumption is that a demand from zone i to zone j at time period t will be missed if there is no car available at zone i at time period t. (4) Our proposed approach does not consider uncertainty in the travel cost from zone i to zone j. We note that since our study is about driverless vehicles, it is not unreasonable to assume that the travel cost between two zones depends mainly on the travel distance and not the time taken to travel between the zones. This is highlighted by the fact that the fuel consumption of an idling engine is estimated to be about 0.6 litres/hr per litre of engine displacement (`http://www.ecomobile.gouv.qc.ca/en/ecomobilite/tips/idling_engine.php`), which is not too high. Furthermore, we can even assume that autonomous vehicles are electric and also smart enough to turn off their engines whenever appropriate to save energy. (5) Finally, our proposed approach does not consider uncertainty in demand from zone i to zone j at time period t. This can also be justified because, in the business model that we consider, customers are expected to contact the company to book cars in advance.

The main contributions of our research are as follows:

- We develop a bi-objective mixed integer linear programming formulation for the deterministic version of the problem, and based on that we develop an extended formulation that is easier to work with in the non-deterministic setting that we will explore. Note that the idea of employing multi-objective optimization techniques is not new in transportation science in general (see for instance Jozefowiez et al. (2008)). However, in the scope of autonomous vehicles, we were only able to find one relevant paper, i.e., Chebbi and Chaouachi (2016), in which the main focus of the authors is on heuristic approaches whereas our focus, in this paper, is on exact solution approaches.

- We use chance-constraint programming to incorporate the travel time uncertainty in the proposed extended deterministic formulation. It is worth mentioning that using stochastic and robust optimization techniques have been studied for a long time in transportation science (see for instance Laporte et al. (1992) and Gounaris et al. (2013)). However, there are not many (if any) studies in the scope of autonomous vehicles.

- We present two new mixed integer linear programming reformulations for the proposed chance-constraint program. We prove a critical result that the size of one of the reformulations in terms of the number of variables and constraints is independent of the number of scenarios. Consequently, we show numerically that this reformulation outperforms the other one significantly as the number of scenarios increases.

The remainder of the paper is organized as follows. In Section 2, we introduce notation and some fundamental concepts of bi-objective mixed integer linear programming. In Section 3, the deterministic version of the problem will be formulated using bi-objective mixed integer linear programming. In Section 4, the non-deterministic version of the problem is formulated using chance-constraint programming. In Section 5, we conduct a computational study. Finally, in Section 6, we give some concluding remarks.

2. Preliminaries

A *Bi-Objective Mixed Integer Linear Program* (BOMILP) can be stated as follows:

$$\min_{(\boldsymbol{x}_1,\boldsymbol{x}_2)\in\mathcal{X}} \{z_1(\boldsymbol{x}_1,\boldsymbol{x}_2), z_2(\boldsymbol{x}_1,\boldsymbol{x}_2)\}, \tag{1}$$

where $\mathcal{X} := \big\{(\boldsymbol{x}_1,\boldsymbol{x}_2) \in \mathbb{Z}^{n_1}_{\geq} \times \mathbb{R}^{n_2}_{\geq} : A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b}\big\}$ represents the *feasible set in the decision space*, $\mathbb{Z}^{n_1}_{\geq} := \{\boldsymbol{s} \in \mathbb{Z}^{n_1} : \boldsymbol{s} \geq \boldsymbol{0}\}$, $\mathbb{R}^{n_2}_{\geq} := \{\boldsymbol{s} \in \mathbb{R}^{n_2} : \boldsymbol{s} \geq \boldsymbol{0}\}$, $A_1 \in \mathbb{R}^{m\times n_1}$, $A_2 \in \mathbb{R}^{m\times n_2}$, and $\boldsymbol{b} \in \mathbb{R}^m$. It is assumed that $z_i(\boldsymbol{x}_1,\boldsymbol{x}_2) = \boldsymbol{c}_{i,1}^{\intercal}\boldsymbol{x}_1 + \boldsymbol{c}_{i,2}^{\intercal}\boldsymbol{x}_2$ where $\boldsymbol{c}_{i,1} \in \mathbb{R}^{n_1}$ and $\boldsymbol{c}_{i,2} \in \mathbb{R}^{n_2}$ for $i = 1, 2$ represents a linear objective function. The image $\mathcal{Y}$ of $\mathcal{X}$ under vector-valued function $\boldsymbol{z} := (z_1, z_2)^{\intercal}$ represents the *feasible set in the objective/criterion space*, that is $\mathcal{Y} := \{\boldsymbol{o} \in \mathbb{R}^2 : \boldsymbol{o} = \boldsymbol{z}(\boldsymbol{x}_1,\boldsymbol{x}_2)$ for all $(\boldsymbol{x}_1,\boldsymbol{x}_2) \in \mathcal{X}\}$.

Definition 1. *A feasible solution $(\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathcal{X}$ is called efficient or Pareto optimal, if there is no other $(\boldsymbol{x}'_1, \boldsymbol{x}'_2) \in \mathcal{X}$ such that $z_1(\boldsymbol{x}'_1, \boldsymbol{x}'_2) \leq z_1(\boldsymbol{x}_1, \boldsymbol{x}_2)$ and $z_2(\boldsymbol{x}'_1, \boldsymbol{x}'_2) < z_2(\boldsymbol{x}_1, \boldsymbol{x}_2)$ or $z_1(\boldsymbol{x}'_1, \boldsymbol{x}'_2) < z_1(\boldsymbol{x}_1, \boldsymbol{x}_2)$ and $z_2(\boldsymbol{x}'_1, \boldsymbol{x}'_2) \leq z_2(\boldsymbol{x}_1, \boldsymbol{x}_2)$. If $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is efficient, then $\boldsymbol{z}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is called a nondominated point. The set of all efficient solutions is denoted by $\mathcal{X}_E$. The set of all nondominated points $\boldsymbol{z}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ for $(\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathcal{X}_E$ is denoted by $\mathcal{Y}_N$ and referred to as the nondominated frontier.*

Definition 2. *If there exists a vector $(\lambda_1, \lambda_2)^\intercal \in \mathbb{R}^2_> := \{\boldsymbol{s} \in \mathbb{R}^2 : \boldsymbol{s} > \boldsymbol{0}\}$ such that $(\boldsymbol{x}^*_1, \boldsymbol{x}^*_2) \in \arg\min_{(\boldsymbol{x}_1,\boldsymbol{x}_2)\in\mathcal{X}} \lambda_1 z_1(\boldsymbol{x}_1, \boldsymbol{x}_2) + \lambda_2 z_2(\boldsymbol{x}_1, \boldsymbol{x}_2)$, then $(\boldsymbol{x}^*_1, \boldsymbol{x}^*_2)$ is called a supported efficient solution and $\boldsymbol{z}(\boldsymbol{x}^*_1, \boldsymbol{x}^*_2)$ is called a supported nondominated point.*

Definition 3. *Let $\mathcal{Y}^e$ be the set of extreme points of the convex hull of $\mathcal{Y}$, that is the smallest convex set containing the set $\mathcal{Y}$. A point $\boldsymbol{z}(\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathcal{Y}$ is called an extreme supported nondominated point, if $\boldsymbol{z}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is a supported nondominated point and $\boldsymbol{z}(\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathcal{Y}^e$.*

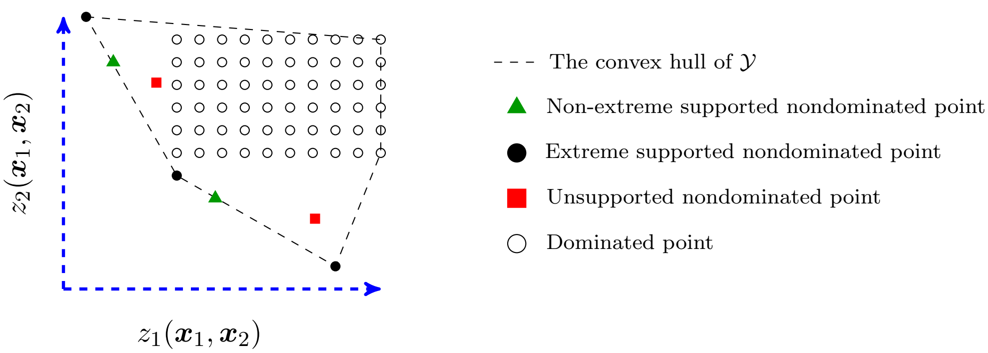
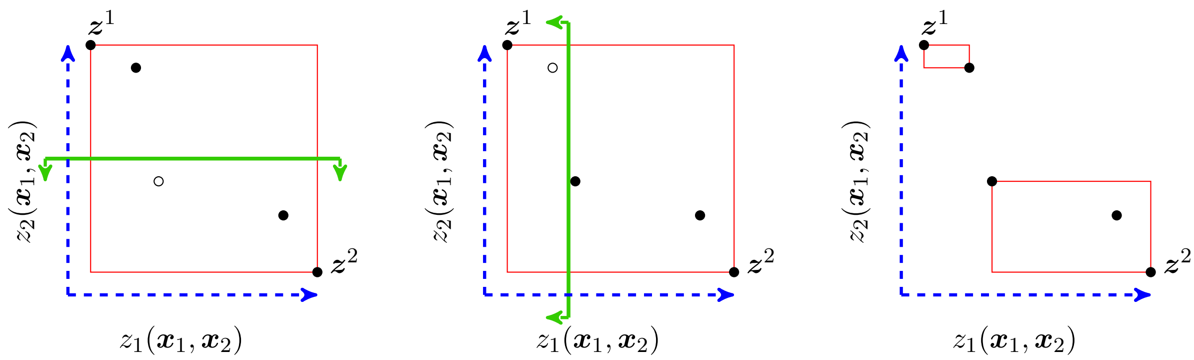


Figure 1: An illustration of different types of (feasible) points in the criterion space.

In summary, based on Definition 1, the elements of $\mathcal{Y}$ can be divided into two groups including dominated and nondominated points. Furthermore, based on Definitions 2 and 3, the nondominated points can be divided into unsupported nondominated points, non-extreme supported nondominated points and extreme supported nondominated points. Overall, bi-objective optimization problems are concerned with finding all elements of $\mathcal{Y}_N$, that is all nondominated points, including supported and unsupported nondominated points. An illustration of the set $\mathcal{Y}$ and its corresponding categories are shown in Figure 1.

In this study, the BOMILPs that we develop have two features. (1) The second objective function can only take integer values for any feasible solution, i.e., $z_2(\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathbb{Z}$ for all $(\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathcal{X}$; (2) The feasible set in the decision space, i.e., $\mathcal{X}$, is bounded. These two features guarantee that the BOMILPs that we develop have finite number of nondominated points (Stidsen et al., 2014). Recently, several studies have been conducted on exact solution approaches for BOMILPs with finite number of nondominated points, see for instance Boland et al. (2015), Dächert et al. (2012), and Stidsen et al. (2014). The balanced box method (BBM) is one of these algorithms (Boland et al., 2015). Since it is shown that BBM is promising for large size problems, we employ this algorithm in this study (Boland et al., 2015). Next, we provide a high-level description of this algorithm.

BBM maintains a priority queue of rectangles in the criterion space containing all not-yet found nondominated points. At the beginning, the priority queue is empty. So, the algorithm first finds



(a) Step 1 (b) Step 2 (c) Step 3

Figure 2: Progression of BBM in terms of the discovery of nondominated points.

the endpoints of the nondominated frontier denoted by $\boldsymbol{z}^T$ and $\boldsymbol{z}^B$. These two points result in defining the first rectangle, i.e., $R(\boldsymbol{z}^T, \boldsymbol{z}^B)$, containing all not yet found nondominated points. Next, we explain the workings of the algorithm in each iteration.

In each iteration, the algorithm pops out a rectangle, denoted by $R(\boldsymbol{z}^1, \boldsymbol{z}^2)$, from the priority queue. The algorithm then splits the rectangle horizontally into two equal parts. It first explores the bottom rectangle for a nondominated point with minimum value for the first objective function, which is illustrated with an unfilled circle in Figure 2a. Based on the position of the new point, it next splits the rectangle vertically. It then explores the left rectangle for a nondominated point with minimum value for the second objective function, which is illustrated with an unfilled circle in Figure 2b. It can be shown that by finding these two (new) nondominated points the rectangle can be split into (at most) two independent rectangles containing all not yet found nondominated points as shown in Figure 2c. So, a similar procedure can be repeated in each of them in the next iterations.

3. Deterministic formulations

In this section, we introduce two deterministic BOMILPs. In order to do so, we first present some basic assumptions and notation about the problem.

We divide the area under consideration into a finite number of zones/stations. We denote the index set of zones by $\mathcal{N} := \{1, \ldots, N\}$. The number of zones depends on the level of granularity desired. For the sake of simplicity, we assume that the starting and ending points of trips are the center of zones. We also divide the planning horizon under consideration into a finite number of time periods of equal length. We denote the index set of time periods by $\mathcal{T} := \{1, \ldots, T\}$. The number of time periods can vary depending on the level of detail desired. We assume that there is no car pooling option. In other words, each demand should be satisfied by a separate autonomous vehicle. We denote the cost of moving from zone $i \in \mathcal{N}$ to zone $j \in \mathcal{N}$ by $c_{ij} \in \mathbb{Z}_{\geq}$. As mentioned in Introduction, we assume that the cost does not depend on time and it is simply a function of distance between zone $i \in \mathcal{N}$ and zone $j \in \mathcal{N}$.

The list of mathematical notation used in the first formulation is given in Table 1. Using this

Table 1: Mathematical notation for the basic formulation.

$\mathcal{N}$	The index set of zones
$\mathcal{T}$	The index set of time periods in the planning horizon
d_{ijt}	Demand from zone $i \in \mathcal{N}$ to zone $j \in \mathcal{N}$ at time period $t \in \mathcal{T}$
$a_{jt'it}$	A binary parameter such that $a_{jt'it} = 1$ if a vehicle is expected to arrive to zone $i \in \mathcal{N}$ at time period $t \in \mathcal{T}$ by leaving zone $j \in \mathcal{N}$ at time period $t' \in \mathcal{T}$, and $a_{jt'it} = 0$ otherwise
c_{ij}	The cost of traversing from zone $i \in \mathcal{N}$ to zone $j \in \mathcal{N}$ for an autonomous vehicle
m_i	The number of autonomous vehicles available in zone $i \in \mathcal{M}$ at the beginning of time period one
y_{it}	A non-negative integer variable denoting the number of cars available in zone $i \in \mathcal{N}$ at the beginning of time period $t \in \mathcal{T}$
x_{ijt}	A non-negative integer variable denoting the number of cars moved from zone $i \in \mathcal{N}$ to zone $j \in \mathcal{N}$ at time period $t \in \mathcal{T}$
z_{ijt}	A non-negative continuous variable for capturing the value of $\max\{d_{ijt} - x_{ijt}, 0\}$, which is basically the number of unsatisfied demands from zone $i \in \mathcal{N}$ to zone $j \in \mathcal{N}$ at time period $t \in \mathcal{T}$

table, the first formulation of the problem can be stated follows:

$$\text{P1} \qquad \min \sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}}\sum_{t\in\mathcal{T}} z_{ijt} \tag{2}$$

$$\min \sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}}\sum_{t\in\mathcal{T}} c_{ij}x_{ijt} \tag{3}$$

s.t.

$$y_{i,1} = m_i \qquad \forall i \in \mathcal{N}, \tag{4}$$

$$y_{it} = \textsc{Recursive}(i,t) \qquad \forall i \in \mathcal{N}, \forall t \in \mathcal{T}\backslash\{1\}, \tag{5}$$

$$\sum_{j\in\mathcal{N}} x_{ijt} \leq y_{it} \qquad \forall i \in \mathcal{N}, \forall t \in \mathcal{T}, \tag{6}$$

$$d_{ijt} - x_{ijt} \leq z_{ijt} \qquad \forall i,j \in \mathcal{N}, \forall t \in \mathcal{T}, \tag{7}$$

$$x_{ijt}, y_{it} \in \mathbb{Z}_+ \qquad \forall i,j \in \mathcal{N}, \forall t \in \mathcal{T}, \tag{8}$$

$$z_{ijt} \in \mathbb{R}_+ \qquad \forall i,j \in \mathcal{N}, \forall t \in \mathcal{T}, \tag{9}$$

where

$$\textsc{Recursive}(i,t) = y_{i,t-1} + \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} a_{jt'it}.x_{jit'} - \sum_{j\in\mathcal{N}} x_{ij,t-1}.$$

The objective function (2) minimizes the unsatisfied demand. The second objective function (3) minimizes the total cost of operation over the entire time horizon. Constraints (4) and (5) are the inventory update constraints. The number of cars at zone i and time period t depends on the number of cars at zone i and time period $t-1$, the number of cars that arrived to zone i at the beginning of time period t, and the number of cars that left zone i at time period $t-1$. Constraint (6) ensures that the number of cars available at zone $i \in \mathcal{N}$ and time period $t \in \mathcal{T}$ serves as an upper bound for the total number of vehicles that can move to all other zones from zone i in time period t. Finally, Constraint (7) is introduced for computing the value of z_{ijt} for all $i, j \in \mathcal{N}$ and $t \in \mathcal{T}$. Note that since we are minimizing the first objective function, z_{ijt} naturally takes integer values in an efficient solution for each $i, j \in \mathcal{N}$ and $t \in \mathcal{T}$. It is also worth mentioning that the feasible set of this problem in the decision space is basically bounded since we can easily bound the decision variables of the problem, i.e., $0 \leq z_{ijt} \leq d_{ijt}$, $0 \leq y_{ijt} \leq \sum_{i \in \mathcal{M}} m_i$, and $0 \leq x_{ijt} \leq \sum_{i \in \mathcal{M}} m_i$, when we are defining them. Consequently, based on our discussion in Section 2, this implies that the nondominated points of this problem must be finite since for any feasible solution, the value of the second objective function is always integer.

Since the ultimate goal of this study is to solve a non-deterministic version of the problem, we next present an extended formulation of the problem. We note that working with the extended formulation in the non-deterministic setting is easier, and all the theoretical results that we will provide in Section 4 are valid because of employing this formulation.

Based on our assumptions for the basic formulation, we know that the total number of vehicles in the entire network is $\sum\limits_{i \in \mathcal{N}} m_i$. So, in the new formulation, we distinguish the vehicles in the entire network by giving each one a designated label. This helps us to know where each vehicle is located at any time. Table 2 details the mathematical notation used in the extended formulation which cannot be found in Table 1.

Table 2: Mathematical notation for the extended formulation.

$\mathcal{M}$	The index set of vehicles in the entire network, i.e., $\mathcal{M} = \{1, \ldots, \sum\limits_{i \in \mathcal{N}} m_i\}$
$\mathcal{M}_i^1$	The index set of vehicles that are located at zone i at the beginning of time period one. Without loss of generality, we assume that $\mathcal{M}_i^1 = \{\sum\limits_{j=1}^{i-1} m_j + r : r = 1, \ldots, m_i\}$
$\bar{y}_{it}^k$	A binary variable that takes the value of one if vehicle k is at zone i at the beginning of time period t, and zero otherwise.
$\bar{x}_{ijt}^k$	A binary variable that takes the value of one if vehicle k has moved from zone i to zone j at time period t, and zero otherwise.

Using the notation defined in this Tables 1 and 2, the problem can be stated as follows:

$$\text{P2} \quad \min \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} \sum_{t \in \mathcal{T}} z_{ijt} \tag{10}$$

$$\min \sum_{k \in \mathcal{M}} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} \sum_{t \in \mathcal{T}} c_{ij} \bar{x}_{ijt}^k \tag{11}$$

s.t.

$$\bar{y}_{i,1}^k = 1 \qquad \forall i \in \mathcal{N}, \forall k \in \mathcal{M}_i^1, \tag{12}$$

$$\bar{y}_{i,1}^k = 0 \qquad \forall i \in \mathcal{N}, \forall k \in \mathcal{M} \backslash \mathcal{M}_i^1, \tag{13}$$

$$\bar{y}_{it}^k = \textsc{Recursive}(k, i, t) \qquad \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \backslash \{1\}, \forall k \in \mathcal{M}, \tag{14}$$

$$\sum_{j \in \mathcal{N}} \bar{x}_{ijt}^k \leq \bar{y}_{it}^k \qquad \forall i \in \mathcal{N}, \forall t \in \mathcal{T}, \forall k \in \mathcal{M}, \tag{15}$$

$$d_{ijt} - \sum_{k \in \mathcal{M}} \bar{x}^k_{ijt} \leq z_{ijt} \qquad \forall i, j \in \mathcal{N}, \forall t \in \mathcal{T}, \tag{16}$$

$$\bar{x}^k_{ijt}, \bar{y}^k_{it} \in \{0,1\} \qquad \forall i, j \in \mathcal{N}, \forall t \in \mathcal{T}, \forall k \in \mathcal{M}, \tag{17}$$

$$z_{ijt} \in \mathbb{R}_+ \qquad \forall i, j \in \mathcal{N}, \forall t \in \mathcal{T}, \tag{18}$$

where

$$\textsc{Recursive}(k, i, t) = \bar{y}^k_{i,t-1} + \sum_{j \in \mathcal{N}} \sum_{t'=1}^{t} a_{jt'it}.\bar{x}^k_{jit'} - \sum_{j \in \mathcal{N}} \bar{x}^k_{ij,t-1}.$$

Similar to our discussion for P1, in P2 the objective function (10) minimizes the unsatisfied demand. Also, the second objective function (10) minimizes the total cost of operation over the entire time horizon. Constraints (12), (13) and (14) are the inventory update constraints. Constraint (15) ensures that the number of available cars at zone $i \in \mathcal{N}$ and time period $t \in \mathcal{T}$ serves as an upper bound for the total number of vehicles that can move to all other zones from zone i in time period t. Finally, Constraint (16) is introduced for computing the value of z_{ijt} for all $i, j \in \mathcal{N}$ and $t \in \mathcal{T}$.

The following valid inequalities ensure that each vehicle is at most in one zone at any time period:

$$\sum_{i \in \mathcal{N}} \bar{y}^k_{it} \leq 1 \qquad \forall t \in \mathcal{T}, \forall k \in \mathcal{M}. \tag{19}$$

Although P2 will be valid even without this set of valid inequalities, we assume that they are part of P2 in the rest of this paper because they are helpful under nondeterministic setting (see the proof of Lemma 6).

4. Non-deterministic formulations

For a carsharing company (with autonomous vehicles), time is an important factor for ensuring that the company remains competitive in an urban setting with traffic and congestion. Hence, the uncertainty in the travel time should be considered to alleviate its possible impact on demand fulfillment and relocation operations. So, we assume that $a_{jt'it}$ for each $i, j \in \mathcal{N}$ and $t, t' \in \mathcal{T}$ is uncertain. Consequently, Constraint (14) is uncertain for each $i \in \mathcal{N}, \forall t \in \mathcal{T} \backslash \{1\}, \forall k \in \mathcal{M}$ in P2.

We denote the index set of all scenarios by $\mathcal{S} := \{1, \ldots, S\}$. We also denote the scenario with index $s \in \mathcal{S}$ by $\mathcal{A}^s$, which is a four dimensional matrix that its entries are denoted by $a^s_{jt'it} \in \{0,1\}$ where $i, j \in \mathcal{N}$ and $t, t' \in \mathcal{T}$. Consequently, $\mathcal{S}$ is definitely a finite set since its cardinality, i.e., S, cannot be larger than $2^{N^2T^2}$. We assume that we have the full probabilistic knowledge of possible scenarios and to deal with the uncertainty, we apply the chance-constraint programming (Charnes and Cooper, 1959; Klein Haneveld, 1986). The following proposition is helpful.

Proposition 4. *P2 will be still a valid formulation if Constraint* (14) *is replaced by*

$$\bar{y}^k_{i,t} \leq \bar{y}^k_{i,t-1} + \sum_{j \in \mathcal{N}} \sum_{t'=1}^{t} a_{jt'it}.\bar{x}^k_{jit'} - \sum_{j \in \mathcal{N}} \bar{x}^k_{ij,t-1} \qquad \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \backslash \{1\}, \forall k \in \mathcal{M}.$$

Proof. Suppose that we use the new set of constraints, and there exists a solution such that for a given $i \in \mathcal{N}$, $t \in \mathcal{T} \backslash \{1\}$ and $\forall k \in \mathcal{M}$, we have that

$$\bar{y}^k_{i,t} < \bar{y}^k_{i,t-1} + \sum_{j\in\mathcal{N}} \sum_{t'=1}^{t} a_{jt'it}.\bar{x}^k_{jit'} - \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1}.$$

This immediately implies that $\bar{y}^k_{i,t} = 0$ and $\bar{y}^k_{i,t-1} + \sum_{j\in\mathcal{N}} \sum_{t'=1}^{t} a_{jt'it}.\bar{x}^k_{jit'} - \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1} = 1$. This means that vehicle k is available at zone i at time period t, but we simply ignored it. Therefore, based on Constraint (15), vehicle k will remain at zone i (and cannot move at all from this zone) at time period t and all subsequent iterations. However, this solution is obviously equivalent to just simply setting $\bar{y}^k_{i,t} = 1$ and forcing vehicle k to remain in zone i at time period t and all subsequent iterations. In other words, we have not generated a new solution using the new set of constraints. Basically, a different representation of a solution of P2 is generated. □

So, in the rest of this paper, we assume that Constraint (14) is replaced by the one introduced in Proposition 4. Note that this proposition is important since it implies that downward violation of Constraints (14) is not important. In other words, only upward infeasibility of these constraints is undesirable. So, we use this observation to develop our non-deterministic formulation. For notational convenience, given a linear constraint of the form $\boldsymbol{a}_i^\intercal \boldsymbol{x} - b_i \leq 0$, we define $I(\boldsymbol{a}_i^\intercal \boldsymbol{x} - b_i) \in \{0, 1\}$ such that $I(\boldsymbol{a}_i^\intercal \boldsymbol{x} - b_i) = 1$ if $\boldsymbol{a}_i^\intercal \boldsymbol{x} - b_i > 0$. Using this notation and P2, our proposed non-deterministic formulation is as follows,

$$\begin{aligned}
\text{P3} \qquad & \min \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}} \sum_{t\in\mathcal{T}} z_{ijt} \\
& \min \sum_{k\in\mathcal{M}} \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}} \sum_{t\in\mathcal{T}} c_{ij} \bar{x}^k_{ijt} \\
\text{s.t.} \quad & \\
& (12), (13), (15), (16), (17), (18), (19), \\
& \sum_{s\in\mathcal{S}} p_s.I(\bar{y}^k_{i,t} - \bar{y}^k_{i,t-1} - \sum_{j\in\mathcal{N}} \sum_{t'=1}^{t} a^s_{jt'it}.\bar{x}^k_{jit'} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1}) \leq \alpha^k_{it} \quad \forall i \in \mathcal{N}, \forall t \in \mathcal{T}\backslash\{1\}, \forall k \in \mathcal{M},
\end{aligned}$$

where $p_s \in (0, 1]$ is the probability of scenario s and $\alpha^k_{it} \in [0, 1]$ is the maximum probability allowed for violating the uncertain Constraint (14) for each $i \in \mathcal{N}$, $t \in \mathcal{T}\backslash\{1\}$ and $\forall k \in \mathcal{M}$. Note that we assume that all p_s and α^k_{it} are known (given as parameters by decision makers). Note too that $\sum_{s\in\mathcal{S}} p_s = 1$.

4.1. A small example

We later explain how P3 can be transformed to a BOMILP (to be solved by BBM). However, before that, we compare the deterministic formulation, i.e., P2, and the non-deterministic formulation, i.e., P3, on a small example with $N = 3$ and $T = 3$. We randomly generated three scenarios for this example (based on a method that we will explain in Section 5). P3 considers all scenarios at the same time but since P2 is a deterministic formulation, it only considers the first scenario. Figure 3 shows c_{ij} for each $i \in \mathcal{N}$ and $j \in \mathcal{N}\backslash\{i\}$ where nodes represent the zones. Also, d_{ijt} for

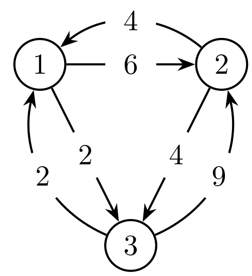
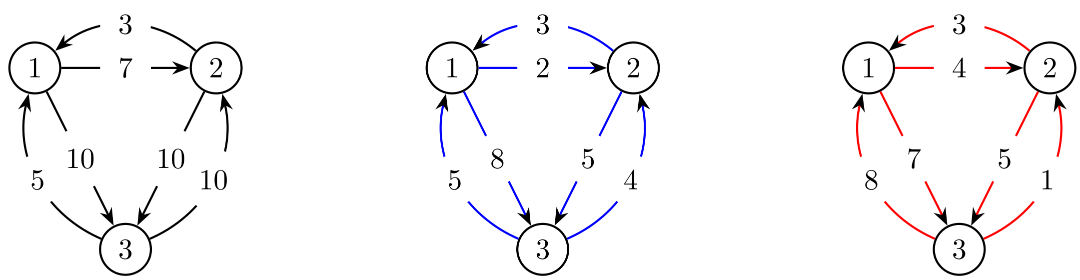


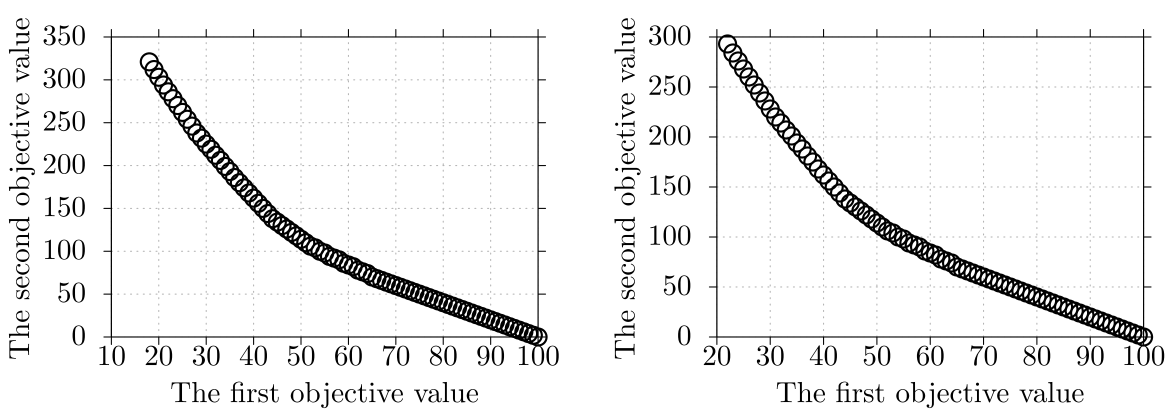
Figure 3: The cost of moving from zone $i \in \mathcal{N}$ to zone $j \in \mathcal{N}\backslash\{i\}$.



(a) $t = 1$ (b) $t = 2$ (c) $t = 3$

Figure 4: The demand from zone $i \in \mathcal{N}$ to zone $j \in \mathcal{N}\backslash\{i\}$ at each time period.

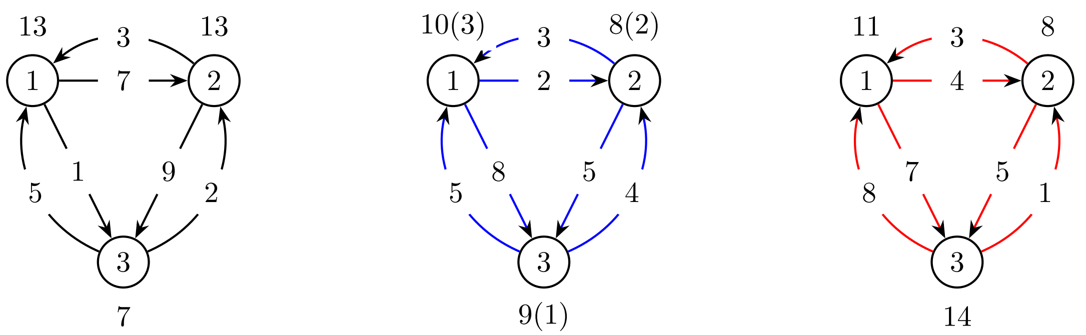
each $i \in \mathcal{N}$ and $j \in \mathcal{N}\backslash\{i\}$ and $t \in \mathcal{T}$ can be found in Figure 4. Note that we assume that there is no demand with the same starting and ending points. The nondominated frontier obtained by P2 and P3 are illustrated in Figure 5. Not surprisingly, we observe from the top endpoints of the nondominated frontiers that P2 can reach to a better, i.e., smaller, value for the first objective function. We next compare the top endpoints of the nondominated frontiers in terms of solution.



(a) Deterministic, i.e., P2 (b) Non-deterministic, i.e., P3

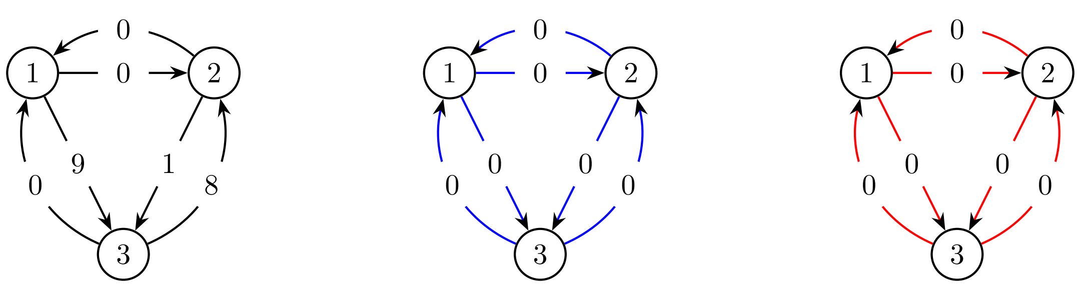
Figure 5: The nondominated frontier of the small example.

Figures 6 and 7 illustrate an efficient solution corresponding to the top endpoint of the nondominated frontier produced by solving P2. Similarly, Figures 8 and 9 illustrate an efficient solution corresponding to the top endpoint of the nondominated frontier produced by solving P3. Specifically, in Figures 6 and 8, the number of vehicles available at each zone at the beginning of each time period and the number of vehicles that will move from zone $i \in \mathcal{N}$ to zone $j \in \mathcal{N}\backslash\{i\}$ at each time period is shown. We note that some vehicles may be on their way to reach a zone since they may have moved to reach a zone in previous time periods. We note too that some vehicles may have reached to a particular zone but since we have applied Proposition 4, their corresponding inventory variable,, i.e., y_{it}^k, may take the value of zero and so they cannot move again. Therefore, we report the number of vehicles satisfying these two remarks in parentheses wherever appropriate in Figures 6 and 8. In Figures 7 and 9, the unsatisfied demand from zone $i \in \mathcal{N}$ to zone $j \in \mathcal{N}\backslash\{i\}$ at each time period is shown.



(a) $t = 1$ (b) $t = 2$ (c) $t = 3$

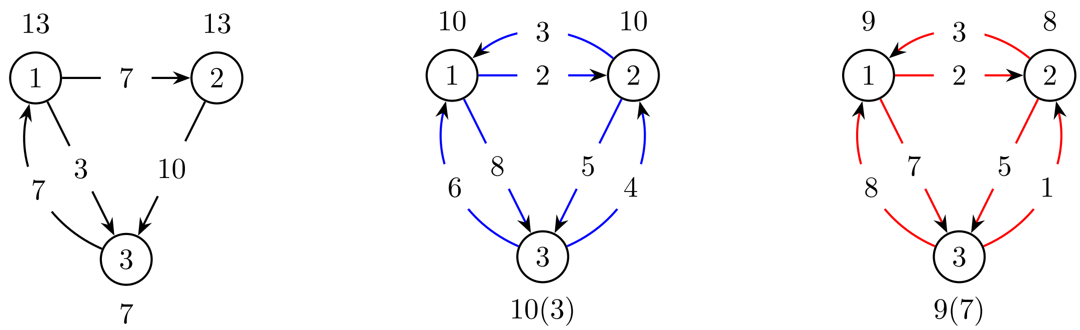
Figure 6: The number of vehicles available at each zone at the beginning of each time period and the number of vehicles that will move from zone $i \in \mathcal{N}$ to zone $j \in \mathcal{N}\backslash\{i\}$ at each time period for a solution corresponding to the top endpoint of the nondominated frontier produced by solving P2. Note that the number of vehicles that are either on their ways to reach a zone (but did not arrive yet) or they have previously arrived at that zone but they cannot technically move again (because of applying Proposition 4) is given in parentheses.



(a) $t = 1$ (b) $t = 2$ (c) $t = 3$

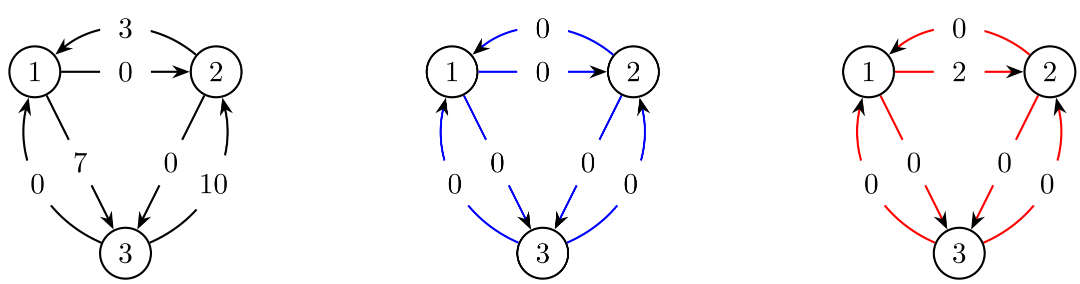
Figure 7: The unsatisfied demand from zone $i \in \mathcal{N}$ to zone $j \in \mathcal{N}\backslash\{i\}$ at each time period for a solution corresponding to the top endpoint of the nondominated frontier produced by solving P2.

For example, we observe from Figure 6a that there are 13 vehicles available at the beginning of time period $t = 1$ in zone 1. However, at the same time period, i.e., $t = 1$, we send seven vehicles



(a) $t = 1$ (b) $t = 2$ (c) $t = 3$

Figure 8: The number of vehicles available at each zone at the beginning of each time period and the number of vehicles that will move from zone $i \in \mathcal{N}$ to zone $j \in \mathcal{N} \backslash \{i\}$ at each time period for a solution corresponding to the top endpoint of the nondominated frontier produced by solving P3. Note that the number of vehicles that are either on their way to reach a zone (but have not arrived yet) or they have previously arrived at that zone but they cannot technically move again (because of applying Proposition 4) is given in parentheses.



(a) $t = 1$ (b) $t = 2$ (c) $t = 3$

Figure 9: The unsatisfied demand from zone $i \in \mathcal{N}$ to zone $j \in \mathcal{N} \backslash \{i\}$ at each time period for a solution corresponding to the top endpoint of the nondominated frontier produced by solving P3.

from zone 1 to zone 2, one vehicle from zone 1 to zone 3, three vehicles from zone 2 to zone 1, and five vehicles from zone 3 to zone 1. Consequently, we see that in Figure 6b, ten vehicles are available at zone 1 at $t = 2$, and the number of vehicles that are either on their ways to reach zone 1 (but have not arrived yet) or they have arrived at zone 1 but they cannot technically move again is three. For another example, consider Figure 4a. We observe from this figure that at $t = 1$, there are five customers that want to move from zone 3 to zone 1. Figure 8a shows that at zone 3 there are seven vehicles available at the beginning of time period $t = 1$. So, we send all of them to zone 1 at $t = 1$ in which only five of them are required for satisfying the existing demand and the others are empty, i.e., they are sent for repositioning purposes. So, we see that in Figure 9a, the unsatisfied demand from zone 3 to zone 1 is zero.

4.2. A basic reformulation of P3

By generating all possible scenarios explicitly, P3 can be written as a BOMILP as follows:

$$\text{P4} \quad \min \sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}}\sum_{t\in\mathcal{T}} z_{ijt}$$

$$\min \sum_{k\in\mathcal{M}}\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}}\sum_{t\in\mathcal{T}} c_{ij}\bar{x}^k_{ijt}$$

s.t.

$$(12),(13),(15),(16),(17),(18),(19),$$

$$\bar{y}^k_{it} - \bar{y}^k_{i,t-1} - \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} a^s_{jt'it}.\bar{x}^k_{jit'} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1} \leq M.b^k_s \quad \forall i\in\mathcal{N},\forall t\in\mathcal{T}\backslash\{1\},\forall k\in\mathcal{M}, s\in\mathcal{S},$$

$$\sum_{s\in\mathcal{S}} p_s.b^k_s \leq \alpha^k_{it} \quad \forall i\in\mathcal{N},\forall t\in\mathcal{T}\backslash\{1\},\forall k\in\mathcal{M},$$

$$b^k_s \in \{0,1\} \quad \forall k\in\mathcal{M}, s\in\mathcal{S},$$

where $M > 0$ is a sufficiently large value. This formulation simply removes the function $I(.)$ by introducing new binary variables, i.e., b^k_s for each $k \in \mathcal{M}$ and $s \in \mathcal{S}$. It is clear that if

$$\bar{y}^k_{it} - \bar{y}^k_{i,t-1} - \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} a^s_{jt'it}.\bar{x}^k_{jit'} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1} > 0$$

then $b^k_s = 1$. The main drawback of this method is its size since it can be exponentially large. It is worth mentioning that there are some approaches to approximately solve a single-objective version of this optimization problem using sample-based techniques (see for instance Campi and Garatti (2011); Pagnoncelli et al. (2009); Luedtke et al. (2010)). However, the focus of this study is on solving this problem, i.e., the bi-objective version, exactly.

We next prove that in P4, we can set $M = 1$ and relax the integrality condition of b^k_s for each $k \in \mathcal{M}$ and $s \in \mathcal{S}$.

Lemma 5. *For any solution satisfying* (12), (13), (15), (16), (17), (18), *and* (19) *we have* $-\bar{y}^k_{i,t-1} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1} \in \{-1, 0\}$ *for each* $i \in \mathcal{N}$, $t \in \mathcal{T}\backslash\{1\}$, *and* $k \in \mathcal{M}$.

Proof. We know that $\bar{y}^k_{i,t-1}\{0,1\}$ and $\bar{x}^k_{ij,t-1} \in \{0,1\}$ for each $j \in \mathcal{N}$. Therefore, based on Constraint (15), if $\sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1} = 1$ then $\bar{y}^k_{i,t-1} = 1$ and if $\bar{y}^k_{i,t-1} = 0$ then $\sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1} = 0$. So, the result follows.

Lemma 6. *For any solution satisfying* (12), (13), (15), (16), (17), (18), *and* (19) *we have* $\bar{y}^k_{it} - \sum_{j\in\mathcal{N}} \sum_{t'=1}^{t} a^s_{jt'it}.\bar{x}^k_{jit'} \in \{-1,0,1\}$ *for each* $i \in \mathcal{N}$, $t \in \mathcal{T}\backslash\{1\}$, $k \in \mathcal{M}$ *and* $s \in \mathcal{S}$.

Proof. We know that $\bar{y}^k_{it}$ and $\bar{x}^k_{jit'} \in \{0,1\}$ for each $j \in \mathcal{N}$ and $t' \in \{1,\ldots,t\}$. Therefore, based on Constraints (15) and (19), we have that $\sum_{j\in\mathcal{N}} \sum_{t'=1}^{t} a^s_{jt'it}.\bar{x}^k_{jit'} \in \{0,1\}$ since all $a^s_{jt'it} \in \{0,1\}$. So, the result follows.

Proposition 7. *For any solution satisfying* (12), (13), (15), (16), (17), (18), *and* (19) *we have*

$$\bar{y}^k_{it} - \bar{y}^k_{i,t-1} - \sum_{j\in\mathcal{N}} \sum_{t'=1}^{t} a^s_{jt'it}.\bar{x}^k_{jit'} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1} \in \{-2,-1,0,1\}, \tag{20}$$

for each $i \in \mathcal{N}$, $t \in \mathcal{T}\backslash\{1\}$, $k \in \mathcal{M}$, *and* $s \in \mathcal{S}$.

Proof. The result follows immediately from Lemmas 5 and 6.

Corollary 8. *P4 is equivalent to the following problem:*

$$\begin{aligned}
P5 \qquad & \min \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}} \sum_{t\in\mathcal{T}} z_{ijt} \\
& \min \sum_{k\in\mathcal{M}} \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}} \sum_{t\in\mathcal{T}} c_{ij}\bar{x}^k_{ijt} \\
s.t.\quad & \\
& (12),(13),(15),(16),(17),(18),(19), \\
& \bar{y}^k_{it} - \bar{y}^k_{i,t-1} - \sum_{j\in\mathcal{N}} \sum_{t'=1}^{t} a^s_{jt'it}.\bar{x}^k_{jit'} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1} \le b^k_s \quad && \forall i \in \mathcal{N}, \forall t \in \mathcal{T}\backslash\{1\}, \forall k \in \mathcal{M}, s \in \mathcal{S}, \\
& \sum_{s\in\mathcal{S}} p_s.b^k_s \le \alpha^k_{it} && \forall i \in \mathcal{N}, \forall t \in \mathcal{T}\backslash\{1\}, \forall k \in \mathcal{M}, \\
& 0 \le b^k_s \le 1 && \forall i \in \mathcal{N}, \forall t \in \mathcal{T}\backslash\{1\}, \forall k \in \mathcal{M}, s \in \mathcal{S}.
\end{aligned}$$

4.3. An advanced reformulation of P3

We now present a novel reformulation of P3, denoted by P6. It is wroth mentioning that the new reformulation is interesting since its size (in terms of the number of variables and constraints) is precisely equal to P2. In other words, the number of scenarios does not affect the size of this reformulation at all.

$$
\text{P6} \quad \min \sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}}\sum_{t\in\mathcal{T}} z_{ijt}
$$

$$
\min \sum_{k\in\mathcal{M}}\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}}\sum_{t\in\mathcal{T}} c_{ij}\bar{x}^k_{ijt}
$$

s.t.

$$
(12), (13), (15), (16), (17), (18), (19),
$$

$$
\bar{y}^k_{it} - \bar{y}^k_{i,t-1} - \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} \bar{a}_{jt'it}.\bar{x}^k_{jit'} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1} \leq \alpha^k_{it} \qquad \forall i\in\mathcal{N}, \forall t\in\mathcal{T}\backslash\{1\}, \forall k\in\mathcal{M},
$$

where $\bar{a}_{jt'it} := \sum_{s\in\mathcal{S}} p_s.a^s_{jt'it}$.

Proposition 9. *For any solution satisfying* (12), (13), (15), (16), (17), (18), *and* (19) *if*

$$
\sum_{s\in\mathcal{S}} p_s.I(\bar{y}^k_{i,t} - \bar{y}^k_{i,t-1} - \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} a^s_{jt'it}.\bar{x}^k_{jit'} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1}) \leq \alpha^k_{it}
$$

then

$$
\bar{y}^k_{it} - \bar{y}^k_{i,t-1} - \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} \bar{a}_{jt'it}.\bar{x}^k_{jit'} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1} \leq \alpha^k_{it}
$$

for each $i\in\mathcal{N}$, $t\in\mathcal{T}\backslash\{1\}$ and $k\in\mathcal{M}$.

Proof. By Proposition 7 and the assumption, we shall have

$$
\sum_{s\in\mathcal{S}} p_s.(\bar{y}^k_{it} - \bar{y}^k_{i,t-1} - \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} a^s_{jt'it}.\bar{x}^k_{jit'} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1}) \leq
$$

$$
\sum_{s\in\mathcal{S}} p_s.I(\bar{y}^k_{i,t} - \bar{y}^k_{i,t-1} - \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} a^s_{jt'it}.\bar{x}^k_{jit'} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1}) \leq \alpha^k_{it}.
$$

So, since $\sum_{s\in\mathcal{S}} p_s = 1$, we have

$$
\bar{y}^k_{it} - \bar{y}^k_{i,t-1} - \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} \bar{a}_{jt'it}.\bar{x}^k_{jit'} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1} =
$$

$$
\sum_{s\in\mathcal{S}} p_s.(\bar{y}^k_{it} - \bar{y}^k_{i,t-1} - \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} a^s_{jt'it}.\bar{x}^k_{jit'} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1}) \leq \alpha^k_{it}.
$$

□

Proposition 10. *For any solution satisfying* (12), (13), (15), (16), (17), (18), *and* (19) *if*

$$\sum_{s\in\mathcal{S}} p_s.I(\bar{y}^k_{i,t} - \bar{y}^k_{i,t-1} - \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} a^s_{jt'it}.\bar{x}^k_{jit'} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1}) > \alpha^k_{it}$$

then

$$\bar{y}^k_{it} - \bar{y}^k_{i,t-1} - \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} \bar{a}_{jt'it}.\bar{x}^k_{jit'} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1} > \alpha^k_{it}$$

for each $i \in \mathcal{N}$, $t \in \mathcal{T}\backslash\{1\}$ *and* $k \in \mathcal{M}$.

Proof. By Lemma 5, we know that $-\bar{y}^k_{i,t-1} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1} \in \{-1, 0\}$. Now, if $-\bar{y}^k_{i,t-1} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1} = -1$ then by Lemma 6, we shall have

$$\bar{y}^k_{i,t} - \bar{y}^k_{i,t-1} - \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} a^s_{jt'it}.\bar{x}^k_{jit'} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1} \leq 0$$

for each $s \in \mathcal{S}$. Therefore, we must have

$$\sum_{s\in\mathcal{S}} p_s.I(\bar{y}^k_{i,t} - \bar{y}^k_{i,t-1} - \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} a^s_{jt'it}.\bar{x}^k_{jit'} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1}) \leq \alpha^k_{it}.$$

However, this contradicts the assumption. Hence, we must have $-\bar{y}^k_{i,t-1} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1} = 0$ and that

$$\sum_{s\in\mathcal{S}} p_s.I(\bar{y}^k_{i,t} - \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} a^s_{jt'it}.\bar{x}^k_{jit'}) > \alpha^k_{it}.$$

Obviously, the latter implies that we must have $\bar{y}^k_{i,t} = 1$, and so by Lemma 6, we must have $\bar{y}^k_{i,t} - \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} a^s_{jt'it}.\bar{x}^k_{jit'} \in \{0, 1\}$ for each $s \in \mathcal{S}$. Consequently,

$$\sum_{s\in\mathcal{S}} p_s.(\bar{y}^k_{i,t} - \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} a^s_{jt'it}.\bar{x}^k_{jit'}) = \sum_{s\in\mathcal{S}} p_s.I(\bar{y}^k_{i,t} - \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} a^s_{jt'it}.\bar{x}^k_{jit'}) > \alpha^k_{it},$$

So, since $-\bar{y}^k_{i,t-1} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1} = 0$, we have

$$\sum_{s\in\mathcal{S}} p_s.(\bar{y}^k_{i,t} - \bar{y}^k_{i,t-1} - \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} a^s_{jt'it}.\bar{x}^k_{jit'} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1}) = \sum_{s\in\mathcal{S}} p_s.(\bar{y}^k_{i,t} - \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} a^s_{jt'it}.\bar{x}^k_{jit'}) > \alpha^k_{it}.$$

Finally, since $\sum_{s\in\mathcal{S}} p_s = 1$, we have

$$\bar{y}^k_{it} - \bar{y}^k_{i,t-1} - \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} \bar{a}_{jt'it}.\bar{x}^k_{jit'} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1} = \sum_{s\in\mathcal{S}} p_s.(\bar{y}^k_{i,t} - \bar{y}^k_{i,t-1} - \sum_{j\in\mathcal{N}}\sum_{t'=1}^{t} a^s_{jt'it}.\bar{x}^k_{jit'} + \sum_{j\in\mathcal{N}} \bar{x}^k_{ij,t-1}) > \alpha^k_{it}.$$

□

Theorem 11. *P6 is equivalent to P3.*

Proof. The result follows immediately from Propositions 9 and 10. □

5. Computational results

To evaluate the performance of the proposed BOMILPs for the non-deterministic version of the problem, i.e., P5 and P6, a computational study is conducted. We use Julia to implement all formulations and BBM. In this computational study, BBM uses GUROBI 7.0 as the single-objective integer programming solver. All computational experiments are carried out on a Dell PowerEdge R630 with two Intel Xeon E5-2650 2.2 GHz 12-Core Processors (30MB), 128GB RAM, and the RedHat Enterprise Linux 6.8 operating system, and using eight threads.

We generate 20 different instances to solve each one under three different settings including: 10, 15, and 100 scenarios denoted by S10, S15 and S100, respectively. We construct these settings such that all the scenarios of S10 exist in S15 (but with different probabilities), and all the scenarios of S15 exist in S100 (but with different probabilities).

To generate each instance, the values of N and T are randomly drawn from the discrete uniform distribution on interval $[5, 8]$. We set $\alpha_{it}^k = 0.1$ for each $i \in \mathcal{N}$, $t \in \mathcal{T}\backslash\{1\}$ and $k \in \mathcal{M}$. The value of d_{ijt} is randomly drawn from the discrete uniform distribution on interval $[1, 10]$ for each $i \in \mathcal{N}$, $j \in \mathcal{N}\backslash\{i\}$, and $t \in \mathcal{T}$. Also, we set $d_{iit} = 0$ for $i \in \mathcal{N}$ and $t \in \mathcal{T}$. Moreover, c_{ij} is randomly drawn from the discrete uniform distribution on interval $[1, 10]$ for each $i \in \mathcal{N}$ and $t \in \mathcal{T}$. Furthermore, the value of m_i is randomly drawn from the discrete uniform distribution on interval $[1, 5(N-1)]$. Finally, $p_1, \dots, p_S$ are randomly drawn from the uniform distribution on interval $(0, 1]$ but then we normalize these values to assure that $\sum_{s\in\mathcal{S}} p_s = 1$. Next, we explain how each scenario is generated.

To create the scenarios, the following three observations are taken into account:

- By definition, for each $s \in S$, $i, j \in \mathcal{N}$ and $t, t' \in \mathcal{T}$, we have $a^s_{jt'it} = 0$ if $t' > t$. We make this stronger by assuming that $a^s_{jt'it} = 0$ if $t' \geq t$ or $j = i$.
- By definition, for each $s \in S$, $i, j \in \mathcal{N}$ and $t \in \mathcal{T}$, we have $\sum_{t\in\mathcal{T}} a^s_{jt'it} \leq 1$.
- In practice, if a vehicle is expected to arrive at a particular zone at time period t, then it will most likely arrive about the same time, maybe a few time periods earlier or a few time period later. So, it is probably unreasonable to generate the scenarios completely randomly.

In light of the above, we first set all entries of $\mathcal{A}^s$ for each $s \in \mathcal{S}$ equal to zero, and then modify the scenarios. First we explain how we modify the first scenario, i.e., $\mathcal{A}^1$. For each $i \in \mathcal{N}$, $j \in \mathcal{N}\backslash\{i\}$ and $t \in \mathcal{T}$, we randomly selected a $t' \in \{1, \dots, t\}$ (if possible) and set $a^1_{jt'it} = 1$. We modified the other scenarios recursively. Specifically, for each $i, j \in \mathcal{N}$ and $t, t' \in \mathcal{T}$ with $a^{s-1}_{jt'it} = 1$, we randomly set either $a^s_{jt'i,t-1} = 1$ or $a^s_{jt'i,t+1} = 1$. Note that if we select to set $a^s_{jt'i,t-1} = 1$ but we realize that $t - 1 \leq t'$ then we set $a^s_{jt'it} = 1$ instead. Similarly, if we select to set $a^s_{jt'i,t+1} = 1$ but we realize that $t + 1 > T$ then we set $a^s_{jt'it} = 1$ instead.

Table 3 reports the numerical results for all 20 instances under S10, S15 and S100 settings. For each instance, the first three columns, i.e., 'Ins', 'M', and 'T', show the instance number, the number of zones, and the number of time periods, respectively. There are three columns under each setting including '#NDP', '#IP', and 'Time (Sec.)' that show the number of nondominated points in the nondominated frontier, the number of single-objective integer linear programs solved by BBM to compute the entire nondominated frontier, and the solution time of BBM, respectively. The solution time contains two numbers labeled by 'P6' and '$\frac{\text{P5}}{\text{P6}}$'. The first one is basically the solution time of BBM (in seconds) when solving P6, but the latter is the ratio of the solution time

Table 3: Numerical results obtained by running the BBM to solve P5 and P6.

Ins	N	T	S10				S15				S100			
			#NDP	#IP	Time (Sec.)		#NDP	#IP	Time (Sec.)		#NDP	#IP	Time (Sec.)	
					P6	$\frac{\text{P5}}{\text{P6}}$			P6	$\frac{\text{P5}}{\text{P6}}$			P6	$\frac{\text{P5}}{\text{P6}}$
1	7	7	190	416	907.1	2.8	190	420	389.2	-	190	420	303.5	-
2	5	8	155	339	799.4	1.0	148	312	472.3	2.1	148	312	267.1	-
3	8	8	-	-	-	-	-	-	-	-	289	617	2,206.5	-
4	5	6	77	161	25.5	3.1	85	183	75.0	2.8	77	161	26.1	131.9
5	7	7	226	486	1,737.0	-	221	479	743.6	-	221	479	606.3	-
6	5	6	93	203	49.3	2.4	93	203	33.4	8.5	93	203	27.4	-
7	6	6	151	331	218.4	3.7	138	296	159.6	5.1	138	296	90.3	-
8	5	8	55	121	25.2	2.1	55	121	23.0	5.2	55	121	16.3	139.0
9	7	7	-	-	-	-	201	443	632.6	-	201	443	389.9	-
10	8	5	272	574	1,000.7	-	267	579	642.6	-	267	579	423.4	-
11	5	7	133	287	642.7	1.1	120	258	154.5	4.4	120	258	74.1	-
12	5	7	147	313	164.1	4.1	147	313	435.5	2.4	147	313	151.4	-
13	6	6	167	359	140.6	5.4	161	341	172.3	8.0	161	341	125.5	-
14	7	8	162	344	673.6	2.4	155	327	481.9	5.2	155	327	293.7	-
15	6	5	115	255	49.1	5.5	115	255	59.5	10.1	115	255	51.9	-
16	6	6	187	405	339.2	3.8	173	377	326.2	3.8	173	377	250.2	-
17	6	5	138	302	93.0	3.9	135	295	82.5	6.3	135	295	78.0	-
18	6	8	153	337	670.4	1.5	153	337	343.3	5.8	153	337	239.1	-
19	7	7	-	-	-	-	267	563	1,142.9	-	267	563	840.1	-
20	5	7	117	257	120.9	2.2	117	257	136.4	3.5	117	257	80.2	-

of BBM when solving P5 to the solution time of BBM when solving P6. We impose a time limit of 3,600 seconds to solve each instance, and so the symbol '-' is used whenever BBM fails to solve P5 within the time limit. Next, we make a few observations from Table 3:

- For each instance, the number of single-objective integer linear programs solved by BBM to compute the entire nondominated frontier is no more than three times larger than the number of nondominated points. This perfectly follows the theory of BBM (Boland et al., 2015).

- Overall, it seems that for each instance, the number of nondominated points converges (to some number) as we increase the number of scenarios. This can be better understood by considering P6 and the way that we created the scenarios. By construction, all the scenarios of S10 exist in S15 and all the scenarios of S15 exist in S100. Also, all the scenarios are created recursively starting from the first scenario, i.e., $\mathcal{A}^1$. So, roughly speaking, it is expected that, in P6, the value of $\bar{a}_{jt'it}$ converges to $\frac{a^1_{jt'it}}{2}$ for each $i, j \in \mathcal{N}$ and $t, t' \in \mathcal{T}$ as we increase the number of scenarios. Consequently, this can be a reason for the convergence of the number of nondominated points.

- The solution time of BBM when solving P6 varies as the number of scenarios increases. However, overall, increasing the number of scenarios seems to help BBM solve the problem faster. This is justifiable by the fact that the size of P6 does not depend on the number of scenarios and only the structure of its uncertain constraints changes by increasing the number of scenarios, i.e., the value of $\bar{a}_{jt'it}$ changes for each $i, j \in \mathcal{N}$ and $t, t' \in \mathcal{T}$.

- By increasing the number of scenarios, the solution time of BBM significantly increases when solving P5 since the size of this reformulation depends on the number of scenarios. We see that the solution time of BBM when solving P5 is more than the solution time of BBM when solving P6 by a factor of up to (around) 6, 10 and 139 for S10, S15, and S100, respectively (for the instances solved within the imposed time limit). It is also worth mentioning, in our computational experiments, we observed that Instance 9 under the setting S10 is the only case that is solved by P5 in about 3,200 seconds but it is not solved by P6 within the imposed time limit.

6. Final remarks

Repositioning vehicles between stations to serve the demand in future time periods is a typical problem that arises in carsharing systems. When the carsharing business model combines with the technology of autonomous vehicles, a responsive, fast paced and time sensitive repositioning operation becomes crucial. In this paper, we studied a bi-objective version of this problem when the goal is to minimize both the unsatisfied demand and the total operating cost. The most important contribution of our study is that we considered the travel time uncertainty and developed a BOMILP to handle it in a way that the size of the formulation is independent of the number of scenarios. We showed that BBM can compute all nondominated points of the proposed BOMILP. These points can in turn be provided to decision makers for selecting their most desirable nondominated point. Improving the proposed BOMILP, for example by developing valid inequalities or preprocessing techniques, to be able to solve larger instances can be one further research direction building on this study. Incorporating the demand uncertainty in the formulation is another research direction. Finally, incorporating the option of ride sharing for the autonomous vehicles also requires further research.

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