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Measuring axial symmetry in convex cones

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Abstract. The problem of measuring the degree of central symmetry of a convex body has been treated by various authors since the early twentieth century. This work addresses the issue of measuring the degree of axial symmetry of a convex cone. Passing from central symmetry in convex bodies to axial symmetry in convex cones is not a mere routine. Although it is possible to push the parallelism between both issues at a great level of generality, there are a number of substantial differences as well.

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1 Introduction

A convex body is a convex compact set with nonempty interior. There is a rich literature devoted to the problem of measuring the degree of central symmetry of a convex body C in the Euclidian space $\mathbb{R}^n$. Such a fundamental issue was raised from the early days of the theory of convex bodies. A popular choice as central symmetry measure of C is the Minkowski coefficient

$$\mathrm{cs}(C) := \max_{z\in\mathbb{R}^n} \Phi(C, z), \tag{1}$$

where the number

$$\Phi(C, z) := \max\{\beta \in \mathbb{R} : \beta(z - C) \subseteq C - z\}$$

measures the degree of symmetry of C relative to a particular point $z \in \mathbb{R}^n$. Clearly, the maximum in (1) can equally well be taken with z just on the interior of C. Geometrically speaking, a solution to the maximization problem (1) corresponds to a point with respect to which C is least asymmetric. Points of least asymmetry in a convex body always exist, but they are not necessarily unique. A wealth of information concerning the function $\Phi(C, \cdot\,)$ and the Minkowski coefficient (1) can be found in the seminal work of Grünbaum [11] and in a more recent paper of Belloni and Freund [3]. For an axiomatic introduction of the concept of central symmetry measure, see [11]. Measures of axial symmetry for convex bodies have also been considered in the literature, see for instance Devalcourt [7] and the recent work of Lassak and Nowicka [17]. Surprisingly enough, there is almost no published material on quantification of axial symmetry in convex cones. The purpose of this work is to partially fill this gap.

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Throughout this work we assume that n is at least two. That a linear subspace L of $\mathbb{R}^n$ is nontrivial means that $1 \leq \dim L \leq n-1$. Let $\mathcal{L}(\mathbb{R}^n)$ be the set of nontrivial linear subspaces of $\mathbb{R}^n$. A closed convex cone is called proper if it is pointed (i.e., it contains no line) and has nonempty interior. Let Π_n be the set of proper cones in $\mathbb{R}^n$. Following Barker [2], we say that $K \in \Pi_n$ is symmetric with respect to a linear subspace L if $(2P_L - I_n)(K) \subseteq K$, where P_L is the projection matrix onto L and I_n is the identity matrix of order n. Of course, $2P_L - I_n$ is nothing but the reflection matrix onto L. Axial symmetry means symmetry with respect to a one-dimensional linear subspace. A natural way of measuring the degree of symmetry of an arbitrary $K \in \Pi_n$ with respect to a linear subspace L is by using the generalized reflection matrix

$$R(\beta, L) := P_L + \beta(P_L - I_n),$$

where β is a parameter called reflection factor. Typically, β is considered to be a positive real, but the use of a nonpositive β also makes sense. For instance, $R(-1, L) = I_n$ and $R(0, L) = P_L$. The usual reflection matrix onto L corresponds to $R(1, L)$.

Definition 1. *Let $K \in \Pi_n$ and $L \in \mathcal{L}(\mathbb{R}^n)$. We call*

$$\sigma(K, L) := \sup\{\beta \in \mathbb{R} : R(\beta, L)(K) \subseteq K\}$$

the degree of symmetry of K relative to L. The axial symmetry degree of K is defined by

$$\mathrm{as}(K) := \sup_{L \in \mathcal{L}_1} \sigma(K, L), \tag{2}$$

where $\mathcal{L}_1$ stands for the set of one-dimensional linear subspaces of $\mathbb{R}^n$.

The optimization problems (1) and (2) can be cast into a unified framework in which a key role is played by generalized reflections onto affine subspaces. Having said this, we would like to stress from the outset of the discussion that the functions $\mathrm{cs}(\cdot)$ and $\mathrm{as}(\cdot)$ are not entirely analogous. Passing from central symmetry in convex bodies to axial symmetry in proper cones is not a mere routine. A first major difference between the optimization problems (1) and (2) concerns the issue of uniqueness of solutions.

Theorem 1. *Let $K \in \Pi_n$. Then the optimization problem* (2) *admits exactly one solution.*

The proof of Theorem 1 will be given in Section 2. The unique solution to (2), which we denote by L_K, has a clear geometric interpretation. We may view L_K as the line with respect to which the proper cone K is least asymmetric. So, we call L_K the least asymmetry axis of K. Calling L_K the symmetry axis of K would certainly be an abuse of language because such a terminology may suggest that K is symmetric with respect to L_K, which is not the case in general.

2 Proof of Theorem 1 and related results

We begin with a few notes on stability with respect to generalized reflections onto a given linear subspace $L \in \mathcal{L}(\mathbb{R}^n)$. If the stability condition

$$R(\beta, L)(K) \subseteq K \tag{3}$$

holds with a large reflection factor β, then the proper cone K has a high degree of symmetry with respect to L. As shown below, the scalar β cannot exceed a certain uniform bound.

Proposition 1. *Let $K \in \Pi_n$ and $L \in \mathcal{L}(\mathbb{R}^n)$. If* (3) *holds true, then $\beta \leq 1$.*

Proof. Suppose that (3) holds true with $\beta > 1$. Pick $x_0 \in K \backslash L$ and consider the sequence $\{x_k\}_{k \in \mathbb{N}}$ in K defined recursively by $x_{k+1} = R(\beta, L)x_k$. A direct computation shows that

$$x_k \;=\; [R(\beta, L)]^k x_0 \;=\; P_L x_0 + (-1)^{k+1}\beta^k(P_L x_0 - x_0)$$

for all $k \geq 1$. Since each line segment $\mathrm{co}\{x_k, x_{k+1}\}$ is contained in K, so does the whole line

$$\cup_{k \in \mathbb{N}} \mathrm{co}\{x_k, x_{k+1}\} \;=\; \{P_L x_0 + t(P_L x_0 - x_0) : t \in \mathbb{R}\},$$

contradicting the fact that K is pointed. □

The next proposition gives three alternative characterizations of the stability condition (3). In what follows we use the notation

$$D_K := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \in K \cap \mathbb{S}_n,\; y \in K^* \cap \mathbb{S}_n\},$$

where $\mathbb{S}_n$ is the unit sphere of $\mathbb{R}^n$ and K^* stands for the dual cone of K. Recall that the dual cone of a proper cone is yet another proper cone.

Proposition 2. *Let $K \in \Pi_n$ and $L \in \mathcal{L}(\mathbb{R}^n)$. Then, for each $\beta \in \mathbb{R}$, the stability condition* (3) *is equivalent to each of the following statements:*

(a) $\langle y, R(\beta, L)x \rangle \geq 0$ for all $(x, y) \in D_K$.

(b) $\beta\langle y, x - P_L x \rangle \leq \langle y, P_L x \rangle$ for all $(x, y) \in D_K$.

Furthermore, $\Gamma(K, L) := \{\beta \in \mathbb{R} : R(\beta, L)(K) \subseteq K\}$ is a nonempty closed interval.

Proof. The statements (a) and (b) are clearly equivalent. On the other hand, (a) is equivalent to (3) because, for any matrix A of order n, we have

$$\begin{aligned} A(K) \subseteq K \quad &\Leftrightarrow \quad \langle y, Ax \rangle \geq 0 \quad \text{for all } x \in K, y \in K^* \\ &\Leftrightarrow \quad \langle y, Ax \rangle \geq 0 \quad \text{for all } (x, y) \in D_K. \end{aligned}$$

That $\Gamma(K, L)$ is a closed interval can be seen directly from (b). Indeed,

$$\Gamma(K, L) = \bigcap_{(x,y) \in D_K} \{\beta \in \mathbb{R} : \beta\langle y, x - P_L x \rangle \leq \langle y, P_L x \rangle\}$$

is an intersection of closed intervals. Note that $-1 \in \Gamma(K, L)$ because $R(-1, L)$ is the identity matrix. □

From Propositions 1 and 2 we deduce that $\sigma(K, L)$ is in fact a maximum and it belongs to the interval $[-1, 1]$. Another direct consequence of Proposition 2 is the next duality result.

Corollary 1. *Let $K \in \Pi_n$ and $L \in \mathcal{L}(\mathbb{R}^n)$. Then the equivalence*

$$R(\beta, L)(K) \subseteq K \quad \Leftrightarrow \quad R(\beta, L)(K^*) \subseteq K^*$$

holds for each $\beta \in \mathbb{R}$. In particular, $\sigma(K^*, L) = \sigma(K, L)$.

Proof. The corollary follows from Proposition 2 (a) and the fact that $R(\beta, L)$ is symmetric. □

Remark 1. If C is a convex body containing the origin in its interior, then its polar set C° is another convex body containing the origin in its interior and

$$\Phi(C, 0) = \Phi(C^\circ, 0), \tag{4}$$

cf. [3, Proposition 3]. The last formula in Corollary 1 can be viewed as a conic version of (4), but the emphasis is now in symmetry relative to a subspace and not in symmetry relative to the origin.

2.1 Existence of a least asymmetry axis

The existential part of Theorem 1 can be stated in a broader context. Without extra effort we prove the following general existence result. The notation $\mathcal{L}_d$ refers to the set of d-dimensional linear subspaces of $\mathbb{R}^n$.

Proposition 3. *Let $K \in \Pi_n$ and $d \in \{1, \ldots, n-1\}$. Then the optimization problem*

$$\Sigma_d(K) := \sup_{L \in \mathcal{L}_d} \sigma(K, L) \tag{5}$$

admits at least one solution.

Proof. Let $\mathbb{M}_{n,d}$ be the linear space of real matrices of size $n \times d$ and

$$\mathbb{O}(n, d) := \{V \in \mathbb{M}_{n,d} : V^\top V = I_d\},$$

where the superscript "$\top$" stands for transposition. Each d-dimensional linear subspace of $\mathbb{R}^n$ can be represented in the form $\mathrm{Im}V := \{Vx : x \in \mathbb{R}^n\}$ with $V \in \mathbb{O}(n, d)$. Thus, the optimization problem (5) can be written as

$$\Sigma_d(K) = \sup_{V \in \mathbb{O}(n,d)} \sigma(K, \mathrm{Im}V). \tag{6}$$

By using Berge's maximum theorem (cf. [1, Theorem 1.4.6]) we see that

$$\sigma(K, \mathrm{Im}V) = \max_{\beta \in \Gamma(K, \mathrm{Im}V)} \beta$$

is upper-semicontinuous as function of $V \in \mathbb{O}(n, d)$. Hence, the problem (6) amounts to maximizing an upper-semicontinuous function on a compact set. This proves the proposition. □

We now state some stability properties of the function $\Sigma_d : \Pi_n \to \mathbb{R}$ defined by (5). All these properties apply of course to the function $\mathrm{as} : \Pi_n \to \mathbb{R}$ corresponding to the particular case $d = 1$. In what follows $\mathbb{O}(n)$ denotes the set of orthogonal matrices of order n. Furthermore, we equip the set Π_n with the truncated Pompeiu-Hausdorff metric

$$\rho(P, Q) := \mathrm{haus}(P \cap \mathbb{B}_n, Q \cap \mathbb{B}_n),$$

where $\mathbb{B}_n$ stands for the closed unit ball of $\mathbb{R}^n$ and

$$\mathrm{haus}(C, D) := \max\left\{\max_{z \in C} \mathrm{dist}(z, D), \max_{z \in D} \mathrm{dist}(z, C)\right\}$$

is the classical Pompeiu-Hausdorff metric on the collection of nonempty compact subsets of $\mathbb{R}^n$. As it is well known, the truncated Pompeiu-Hausdorff metric is Lipschitz equivalent to the spherical metric

$$\delta(P,Q) := \mathrm{haus}(P \cap \mathbb{S}_n, Q \cap \mathbb{S}_n).$$

In addition, convergence with respect to the truncated Pompeiu-Hausdorff metric is equivalent to convergence in the Painlevé-Kuratowski sense, cf. [16, 18].

Proposition 4. *Let $d \in \{1, \ldots, n-1\}$. Then $\Sigma_d : \Pi_n \to \mathbb{R}$ is upper-semicontinuous and*

$$\Sigma_d(K^*) = \Sigma_d(K) \tag{7}$$

$$\Sigma_d(U(K)) = \Sigma_d(K) \tag{8}$$

for all $K \in \Pi_n$ and $U \in \mathbb{O}(n)$.

Proof. The upper-semicontinuity of Σ_d is obtained by applying Berge's maximum theorem to the parametric optimization problem

$$\Sigma_d(K) = \max\{\beta : \, (\beta, V) \, \in N(K)\}\,,$$

where

$$N(K) := \{(\beta, V) \, \in \Omega : R(\beta, \mathrm{Im}V)(K) \subseteq K\}$$

and $\Omega := [-1,1] \times \mathbb{O}(n,d)$. We need to check that the graph of the multivalued map $N : \Pi_n \rightrightarrows \Omega$ is a closed set in the product space $\Pi_n \times \Omega$. Let $\{K_\nu\}_{\nu \in \mathbb{N}}$ be a sequence in Π_n converging to $K_\infty \in \Pi_n$ and $\{(\beta_\nu, V_\nu)\}_{\nu \in \mathbb{N}}$ be a sequence in Ω converging to $(\beta_\infty, V_\infty) \in \Omega$. Suppose that the inclusion $R(\beta_\nu, \mathrm{Im}V_\nu)(K_\nu) \subseteq K_\nu$ holds for every $\nu \in \mathbb{N}$. We must prove that

$$R(\beta_\infty, \mathrm{Im}V_\infty)(K_\infty) \subseteq K_\infty. \tag{9}$$

We know that $K_\infty \subseteq \liminf_{\nu \to \infty} K_\nu$ and $\limsup_{\nu \to \infty} K_\nu \subseteq K_\infty$, where the lower and upper limits are understood in the Painlevé-Kuratowski sense. Pick any $x_\infty \in K_\infty$. We have $x_\infty = \lim_{\nu \to \infty} x_\nu$ for some sequence $\{x_\nu\}_{\nu \in \mathbb{N}}$ with $x_\nu \in K_\nu$. Since

$$R(\beta_\nu, \mathrm{Im}V_\nu)x_\nu = \Big[(1+\beta_\nu)V_\nu V_\nu^\top - \beta_\nu I_n\Big]\, x_\nu$$

belongs to K_ν, the limit

$$R(\beta_\infty, \mathrm{Im}V_\infty)x_\infty = \Big[(1+\beta_\infty)V_\infty V_\infty^\top - \beta_\infty I_n\Big]\, x_\infty$$

belongs to K_∞. This confirms (9) and proves that the graph of N is a closed set. Formula (7) follows from Corollary 1. For proving (8), we first observe that $P_{U(L)} = UP_L U^\top$, then we get

$$\begin{aligned} R(\beta, U(L)) &= UR(\beta, L)U^\top \\ \sigma(U(K), U(L)) &= \sigma(K, L), \end{aligned}$$

and, finally, we write

$$\begin{aligned} \Sigma_d(U(K)) &= \max_{M \in \mathcal{L}_d} \sigma(U(K), M) \;=\; \max_{L \in \mathcal{L}_d} \sigma(U(K), U(L)) \\ &= \max_{L \in \mathcal{L}_d} \sigma(K, L) \;=\; \Sigma_d(K). \end{aligned}$$

This completes the proof of the proposition. □

Remark 2. Concerning the uniqueness of solutions to the problem (5), the case $d = 1$ is somewhat special. It turns out that, for $d \in \{2, \dots, n-1\}$, the solution set

$$S_d(K) := \{L \in \mathcal{L}_d : \sigma(K, L) = \Sigma_d(K)\}$$

is not necessarily a singleton. In fact, the cardinality of $S_d(K)$ may even be uncountable.

2.2 Uniqueness of the least asymmetry axis

The proof of the uniqueness part of Theorem 1 is subtle and requires some preliminary lemmas. Since a one-dimensional linear subspace of $\mathbb{R}^n$ is nothing but a line $\vec{\xi} := \{t\xi : t \in \mathbb{R}\}$ generated by a unit vector $\xi \in \mathbb{R}^n$, the optimization problem (2) can be written as

$$\mathrm{as}(K) = \sup_{\xi \in \mathbb{S}_n} f_K(\xi)\,, \tag{10}$$

where $f_K : \mathbb{R}^n \to \mathbb{R}$ is given by

$$f_K(\xi) := \max\{\beta \in \mathbb{R} : R(\beta, \vec{\xi}\,)(K) \subseteq K\}. \tag{11}$$

Clearly, f_K is homogeneous of degree 0. Note that f_K is not a continuous function. Depending on the orientation of the line $\vec{\xi}$ relative to the set $K \cap K^*$, the term $f_K(\xi)$ may be positive, equal to 0, or equal to -1. Since K is assumed to be a proper cone, the intersection $K \cap K^*$ is also a proper cone. In what follows "int" and "bd" are abbreviations for interior and boundary, respectively.

Lemma 1. *Let $K \in \Pi_n$ and $\xi \in \mathbb{R}^n$ be a nonzero vector. Then*

$$\begin{aligned} f_K(\xi) > 0 \quad &\Leftrightarrow \quad \vec{\xi} \cap \mathrm{int}(K \cap K^*) \neq \emptyset, \\ f_K(\xi) = 0 \quad &\Leftrightarrow \quad \vec{\xi} \cap \mathrm{bd}(K \cap K^*) \neq \{0\}, \\ f_K(\xi) = -1 \quad &\Leftrightarrow \quad \vec{\xi} \cap K \cap K^* = \{0\}. \end{aligned}$$

Proof. By homogeneity, we may assume that $\|\xi\| = 1$. Let $\mathcal{S}_n$ be the linear space of symmetric matrices of order n. We equip $\mathcal{S}_n$ with the usual Frobenius inner product. Since K is a proper cone in $\mathbb{R}^n$, the set

$$\Xi(K) := \{A \in \mathcal{S}_n : A(K) \subseteq K\}$$

is a proper cone in $\mathcal{S}_n$. Clearly, I_n belongs to the boundary of $\Xi(K)$. Due to Proposition 2 we can write

$$f_K(\xi) = \max\{\beta \in \mathbb{R} : P_{\vec{\xi}} + \beta(P_{\vec{\xi}} - I_n) \in \Xi(K)\}, \tag{12}$$

where $P_{\vec{\xi}} = \xi\xi^\top$ is the projection matrix onto $\vec{\xi}$. We distinguish between three mutually exclusive cases.

Case 1: $\vec{\xi} \cap \mathrm{int}(K \cap K^*) \neq \emptyset$. This condition amounts to saying that $P_{\vec{\xi}}$ belongs to the interior of $\Xi(K)$. Hence,

$$P_{\vec{\xi}} + \beta(P_{\vec{\xi}} - I_n) \in \Xi(K) \tag{13}$$

for all β near 0. This implies that (12) is positive.

Case 2: $\vec{\xi} \cap \mathrm{bd}(K \cap K^*) \neq \{0\}$. This condition amounts to saying that $P_{\vec{\xi}}$ belongs to the boundary of $\Xi(K)$. In particular, (13) holds with $\beta = 0$. We claim that (13) cannot occur with $\beta > 0$. At least one of the following subcases holds true:

$$\xi \in \mathrm{bd}(K), \quad -\xi \in \mathrm{bd}(K), \quad \xi \in \mathrm{bd}(K^*), \quad -\xi \in \mathrm{bd}(K^*). \tag{14}$$

Consider for instance the first subcase in (14), the other subcases can be treated mutatis mutandis. Pick any $x_0 \in \mathrm{int}(K)$ such that $\langle \xi, x_0 \rangle > 0$ and let $\tilde{x} := \langle \xi, x_0 \rangle^{-1} x_0$. Then $\tilde{x} \in \mathrm{int}(K)$ and $P_{\vec{\xi}}\tilde{x} = \xi$. Note that

$$R(\beta, \vec{\xi}\,)\tilde{x} = \left[P_{\vec{\xi}} + \beta(P_{\vec{\xi}} - I_n)\right]\tilde{x} \;=\; \xi + \beta(\xi - \tilde{x})$$

belongs to the exterior of K for any positive β. This proves our claim and confirms that $f_K(\xi) = 0$. *Case 3* : $\vec{\xi} \cap K \cap K^* = \{0\}$. This condition amounts to saying that $P_{\vec{\xi}}$ does not belong to $\Xi(K)$. If (13) were true for some $\beta > -1$, then

$$P_{\vec{\xi}} \;=\; \frac{1}{1+\beta}\left[P_{\vec{\xi}} \;+\; \beta(P_{\vec{\xi}} - I_n)\right] + \frac{\beta}{1+\beta} I_n$$

would be a convex combination of two elements of $\Xi(K)$, contradicting the fact that $P_{\vec{\xi}} \notin \Xi(K)$. Hence, the condition (13) cannot occur with $\beta > -1$. Thus, $f_K(\xi) = -1$. □

In connection with (10), observe that neither the feasible set $\mathbb{S}_n$ is convex, nor the objective function f_K is concave. However, by using a suitable nonlinear invertible transformation, it is possible to convert the maximization problem (10) involving the pair $(\mathbb{S}_n, f_K)$ into a convex minimization problem

$$\left\{ \begin{array}{l} \text{minimize } \; G_K(\xi) \\ \xi \in \mathbb{B}_n \end{array} \right. \tag{15}$$

for $\mathbb{B}_n$ and a certain extended-real-valued convex function $G_K : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$. By a reason that will be clear from the next lemma, we take G_K of the composite form

$$G_K(\xi) := \left\{ \begin{array}{ll} \log g_K(\xi) & \text{if } \; \xi \in \mathrm{int}(K \cap K^*), \\ \infty & \text{otherwise,} \end{array} \right.$$

where $g_K : \mathrm{int}(K \cap K^*) \to \mathbb{R}$ is given by

$$g_K(\xi) := \max_{(x,y) \in D_K} \frac{\langle x, y \rangle}{\langle \xi, x \rangle \langle \xi, y \rangle}\,. \tag{16}$$

Note that the denominator in (16) is positive for all $(x, y) \in D_K$. The function g_K plays an important role in this work. We call g_K the companion function associated to K.

Lemma 2. *Let $K \in \Pi_n$. Then*

(a) $g_K : \mathrm{int}(K \cap K^*) \to \mathbb{R}$ *is log-convex and positively homogeneous of degree* -2.

(b) For all $\xi \in \mathrm{int}(K \cap K^*)$, *we have*

$$\frac{1}{\|\xi\|^2} \;<\; g_K(\xi) \;\leq\; \frac{1}{\mathrm{dist}\,[\xi, \mathrm{bd}(K)]}\,\frac{1}{\mathrm{dist}\,[\xi, \mathrm{bd}(K^*)]} \tag{17}$$

and the pair of invertible nonlinear relations

$$g_K(\xi) \;=\; \frac{1}{\|\xi\|^2}\left[1 + \frac{1}{f_K(\xi)}\right], \tag{18}$$

$$f_K(\xi) \;=\; \frac{1}{\|\xi\|^2 g_K(\xi) - 1}\,. \tag{19}$$

Proof. Let $\xi \in \mathrm{int}(K \cap K^*)$. Clearly, $g_K(t\xi) = t^{-2} g_K(\xi)$ for all $t > 0$. To obtain the second inequality in (17), we just need to observe that

$$\begin{aligned} 1 &\geq \langle x, y \rangle \\ \langle \xi, x \rangle &\geq \mathrm{dist}\,[\xi, \mathrm{bd}(K^*)] \\ \langle \xi, y \rangle &\geq \mathrm{dist}\,[\xi, \mathrm{bd}(K)] \end{aligned}$$

for all $(x, y) \in D_K$. We now prove (18) and (19). Proposition 2 implies that

$$R(\beta, \vec{\xi}\,)(K) \subseteq K \quad \Leftrightarrow \quad \|\xi\|^{-2}(1+\beta)\,\langle y, \xi\xi^\top x\rangle \geq \beta\langle y, x\rangle \ \text{ for all } (x, y) \in D_K.$$

The maximum in (11) is attained by a positive β. Hence, $f_K(\xi)$ is equal to the supremum of all positive β satisfying

$$\|\xi\|^{-2}\,[1 + (1/\beta)] \ \geq\ g_K(\xi).$$

This leads directly to (18) and (19). The first inequality in (17) follows from (18) and the positivity of $f_K(\xi)$. Finally, we show the log-convexity result. Since $g_K(\xi)$ is positive, the optimization problem (16) is not affected by the addition of the constraint $\langle x, y\rangle > 0$. Note that

$$\log g_K(\xi) = \max_{\substack{(x,y) \in D_K \\ \langle x, y\rangle > 0}} \{\log\langle x, y\rangle - \log\langle \xi, x\rangle - \log\langle \xi, y\rangle\}$$

is convex as function of $\xi \in \mathrm{int}(K \cap K^*)$, because it is a pointwise maximum of convex functions. □

The first inequality in (17) ensures that the denominator in (19) is different from zero. The next lemma provides a sharpening of such inequality. In what follows,

$$\mathrm{e}(P, Q) := \min_{y \in P \cap \mathbb{S}_n} \ \max_{x \in Q \cap \mathbb{S}_n} \langle x, y\rangle \tag{20}$$

stands for the angular excess of $P \in \Pi_n$ over $Q \in \Pi_n$. The geometric interpretation of the min-max problem (20) is clear: we search for a unit vector in the cone P such that its smallest angle with respect to the cone Q is as large as possible. It is worthwhile mentioning that the angular excesses $\mathrm{e}(P, Q)$ and $\mathrm{e}(Q, P)$ can be combined in order to produce the spherical distance between P and Q. Indeed, by using the identity

$$\langle u, v\rangle = 1 - (1/2)\,\|u - v\|^2 \quad \text{ for all } u, v \in \mathbb{S}_n,$$

we readily get the formula

$$\min\{\mathrm{e}(P, Q), \mathrm{e}(Q, P)\} = 1 - (1/2)\,[\delta(P, Q)]^2.$$

It is not difficult to see that both angular excesses $\mathrm{e}(K^*, K)$ and $\mathrm{e}(K, K^*)$ are positive.

Lemma 3. *Let $K \in \Pi_n$ and $\xi \in \mathrm{int}(K \cap K^*)$. Then*

$$g_K(\xi) \ \geq \ \frac{1}{\|\xi\|} \max\left\{\frac{\mathrm{e}(K^*, K)}{\mathrm{dist}\,[\xi, \mathrm{bd}(K)]}, \frac{\mathrm{e}(K, K^*)}{\mathrm{dist}\,[\xi, \mathrm{bd}(K^*)]}\right\}.$$

Proof. Let $\bar{y}$ be any minimizer of $\langle \xi, \cdot \rangle$ on $K^* \cap \mathbb{S}_n$. Hence,

$$\langle \xi, \bar{y} \rangle \; = \; \min_{y \in K^* \cap \mathbb{S}_n} \langle \xi, y \rangle \; = \; \mathrm{dist}[\xi, \mathrm{bd}(K)].$$

Consider next any maximizer $\bar{x}$ of $\langle \cdot, \bar{y} \rangle$ on $K \cap \mathbb{S}_n$. We now have

$$\langle \bar{x}, \bar{y} \rangle \; = \; \max_{x \in K \cap \mathbb{S}_n} \langle x, \bar{y} \rangle \; \geq \; \mathrm{e}(K^*, K).$$

Since $\langle \xi, \bar{x} \rangle \leq \|\xi\|$, we get in this way

$$g_K(\xi) \; \geq \; \frac{\langle \bar{x}, \bar{y} \rangle}{\langle \xi, \bar{x} \rangle \langle \xi, \bar{y} \rangle} \; \geq \; \frac{1}{\|\xi\|} \frac{\mathrm{e}(K^*, K)}{\mathrm{dist}\,[\xi, \mathrm{bd}(K)]}\,.$$

The analogous inequality involving the angular excess $\mathrm{e}(K, K^*)$ is obtained mutatis mutandis. □

The next lemma concerns the behavior of $g_K(\xi)$ as the argument ξ approaches $\mathrm{bd}(K \cap K^*)$. In a sense, we may consider g_K as a barrier function for the set $K \cap K^*$.

Lemma 4. *Let $K \in \Pi_n$ and $\{\xi_k\}_{k \in \mathbb{N}}$ be a bounded sequence in $\mathrm{int}(K \cap K^*)$. Then*

$$\lim_{k \to \infty} \mathrm{dist}\,[\xi_k, \mathrm{bd}(K \cap K^*)] = 0 \quad \Leftrightarrow \quad \lim_{k \to \infty} g_K(\xi_k) = \infty. \tag{21}$$

Proof. For all $\xi \in \mathrm{int}(K \cap K^*)$, we have

$$\mathrm{dist}\,[\xi, \mathrm{bd}(K \cap K^*)] = \min\left\{\mathrm{dist}\,[\xi, \mathrm{bd}(K)]\,, \mathrm{dist}\,[\xi, \mathrm{bd}(K^*)]\right\}. \tag{22}$$

This and the second inequality in (17) yield

$$g_K(\xi) \; \leq \; \left(\frac{1}{\mathrm{dist}\,[\xi, \mathrm{bd}(K \cap K^*)]} \right)^2, \tag{23}$$

which proves the "if" part of (21). On the other hand, (22) and Lemma 3 yield

$$g_K(\xi) \quad \geq \quad \frac{1}{\|\xi\|} \frac{\min\{\mathrm{e}(K^*, K), \mathrm{e}(K, K^*)\}}{\mathrm{dist}\,[\xi, \mathrm{bd}(K \cap K^*)]}\,. \tag{24}$$

This proves the "only if" part of (21), because the numerator in (24) is positive and $\{\xi_k\}_{k \in \mathbb{N}}$ is bounded. □

The next proposition not only takes care of the uniqueness part of Theorem 1, but also it explains how to identify the least asymmetry axis of a proper cone.

Proposition 5. *Let $K \in \Pi_n$. Then*

(a) $\mathrm{as}(K)$ *is equal to the optimal value of the maximization problem*

$$\begin{cases} \textit{maximize} \;\; f_K(\xi) \\ \xi \in \mathbb{S}_n \cap \mathrm{int}(K \cap K^*). \end{cases} \tag{25}$$

This problem admits exactly one solution, which we denote by $\varrho(K)$.

(b) $\varrho(K)$ *coincides with the unique solution to the minimization problem*

$$\begin{cases} \text{minimize } g_K(\xi) \\ \xi \in \mathbb{S}_n \cap \operatorname{int}(K \cap K^*). \end{cases} \tag{26}$$

(c) Problem (10) *admits exactly two solutions, namely,* $\varrho(K)$ *and* $-\varrho(K)$*. Problem* (2) *admits the line generated by* $\varrho(K)$ *as unique solution.*

Proof. We start by proving that (15) admits exactly one solution. Let $\{\xi_k\}_{k\in\mathbb{N}} \subseteq \mathbb{B}_n$ be such that

$$\lim_{k\to\infty} G_K(\xi_k) = \inf_{\xi\in\mathbb{B}_n} G_K(\xi).$$

Taking a subsequence if necessary, we may assume that $\{\xi_k\}_{k\in\mathbb{N}}$ lies on $\operatorname{int}(K \cap K^*)$ and converges to some $\xi_\infty \in \mathbb{B}_n$. By Lemma 4, the limit ξ_∞ remains in the interior of $K \cap K^*$. By a continuity argument, we get

$$\log g_K(\xi_\infty) = \min_{\substack{\xi\in\mathbb{B}_n \\ \xi\in\operatorname{int}(K\cap K^*)}} \log g_K(\xi)$$

and, a posteriori,

$$g_K(\xi_\infty) = \min_{\substack{\xi\in\mathbb{B}_n \\ \xi\in\operatorname{int}(K\cap K^*)}} g_K(\xi). \tag{27}$$

Hence, the problems (15) and (27) have the same solution set. Furthermore, such a solution set, say S, is convex and nonempty. By an homogeneity argument, every solution to (27) must be on the sphere $\mathbb{S}_n$. Hence, $\|\xi_\infty\| = 1$ and $S = \{\xi_\infty\}$ is a singleton. In view of the relation (18), also (25) and (26) have the singleton S as solution set. The remaining items of the proposition follow afterward from Lemma 1 and the homogeneity of degree 0 of f_K. The details are omitted. □

3 Evaluation of the companion function g_K

For computational reasons, it is sometimes preferable to focus the attention on the unit vector $\varrho(K)$, rather than on the line L_K generated by such a vector. The following definition is given for the sake of future reference.

Definition 2. *Let* $K \in \Pi_n$*. The vector* $\varrho(K)$ *is called the least asymmetry direction of* K*.*

In practice, a convenient way of computing the axial symmetry degree $\operatorname{as}(K)$, as well as the least asymmetry direction $\varrho(K)$, is by handling the minimization problem (26). Thanks to (19), we have

$$\operatorname{as}(K) = \frac{1}{\mu(K) - 1}, \tag{28}$$

where $\mu(K)$ stands for the optimal value of (26). From a numerical point of view, an efficient evaluation of the companion function g_K is of crucial importance for a successful resolution of (26). Sections 3.1 to 3.3 provide explicit formulas for the companion function of some specially structured proper cones. We start by stating the following simple and intuitive result. Roughly speaking, Proposition 6 asserts that $g_K(\xi)$ is related to the degree of symmetry with respect to the point ξ of the compact convex set

$$s(K, \xi) := \{x \in K : \langle \xi, x\rangle = 1\}. \tag{29}$$

Note that (29) corresponds to a section of the proper cone K.

Proposition 6. *Let $K \in \Pi_n$ and ξ be a unit vector in* $\operatorname{int}(K \cap K^*)$ *. Then $s(K,\xi)$ is a convex body in the hyperplane $\xi^\diamond := \{x \in \mathbb{R}^n : \langle \xi, x\rangle = 1\}$. Furthermore,*

$$f_K(\xi) \quad = \quad \Phi(s(K,\xi),\xi). \tag{30}$$

Proof. The closed convex set $s(K,\xi)$ is bounded because $\xi \in \operatorname{int}(K^*)$. Indeed, we have

$$s(K,\xi) \subseteq \frac{1}{\operatorname{dist}[\xi, \operatorname{bd}(K^*)]}\, \mathbb{B}_n.$$

We also know that ξ is a unit vector in $\operatorname{int}(K)$. Hence, ξ belongs to the interior of $s(K,\xi)$ relative to the hyperplane $\xi^\diamond$ and, therefore, the maximum

$$\Phi(s(K,\xi),\xi) = \max\{\beta \in \mathbb{R} : \beta(\xi - s(K,\xi)) \subseteq s(K,\xi) - \xi\}$$

is attained by some positive β. But, for each positive β, the stability condition

$$\beta(\xi - s(K,\xi)) \subseteq s(K,\xi) - \xi$$

amounts to saying that $\xi + \beta\,(\xi - u) \in K$ for all $u \in s(K,\xi)$, i.e.,

$$\left\langle \xi + \beta\left(\xi - \frac{x}{\langle \xi, x\rangle}\right), y\right\rangle \geq 0 \quad \text{for all } x \in K\backslash\{0\} \text{ and } \; y \in K^*\backslash\{0\}.$$

The above inequality can be written in the more compact form $(1/\beta) + 1 \geq g_K(\xi)$. In this way, it has been shown that

$$g_K(\xi) = 1 + \frac{1}{\Phi(s(K,\xi),\xi)}\,.$$

This and (19) complete the proof of the announced formula (30). □

Remark 3. If C is a compact convex set with empty interior in $\mathbb{R}^n$, then we see C as a convex body in the affine hull of C and, consequently, we take the maximum in (1) with respect to z in $\operatorname{aff}(C)$.

3.1 Self-dual cones

Following Henrion and Seeger [12, Definition 2.1], we define

$$r(K) := \max_{\xi \in K \cap \mathbb{S}_n} \operatorname{dist}[\xi, \operatorname{bd}(K)] \tag{31}$$

as the inradius of $K \in \Pi_n$. The coefficient (31) arises in various branches of mathematics and it is known under different names, cf. [4, 8, 9, 10, 15, 23]. The unique solution to the maximization problem (31) is called the incenter of K and it is denoted by $\varrho_{\rm inc}(K)$. Geometrically speaking, $\varrho_{\rm inc}(K)$ is the "most interior" among all the unit vectors in the interior of K. Calculus rules for computing inradiuses and incenters can be found in [13, 14]. The next technical lemma provides lower and upper estimates for $\operatorname{as}(K)$ in terms of $r(K \cap K^*)$ and $r(K + K^*)$. Parenthetically, the sum $K + K^*$ and the intersection $K \cap K^*$ are mutually dual proper cones.

Lemma 5. *Let $K \in \Pi_n$. Then, for all $\xi \in \operatorname{int}(K \cap K^*)$, we have*

$$\operatorname{dist}[\xi, \operatorname{bd}(K \cap K^*)] \;\leq\; [g_K(\xi)]^{-1/2} \;\leq\; \operatorname{dist}[\xi, \operatorname{bd}(K + K^*)]. \tag{32}$$

In particular,

$$r(K \cap K^*) \;\leq\; \left[1 + \frac{1}{\operatorname{as}(K)}\right]^{-1/2} \;\leq\; r(K + K^*)\,. \tag{33}$$

Proof. Let $\xi \in \mathrm{int}(K \cap K^*)$. The first inequality in (32) is stated already in (23). For proving the second inequality in (32), we observe that

$$g_K(\xi) \;\geq\; \max_{x,\, y\, \in K\cap K^*\cap \mathbb{S}_n} \frac{\langle x, y\rangle}{\langle \xi, x\rangle\langle \xi, y\rangle}\,.$$

The above maximum is attained by the pair $(x, y) = (\eta_0, \eta_0)$, where η_0 is a minimizer of $\langle \xi, \cdot\rangle$ on $K \cap K^* \cap \mathbb{S}_n$. Hence, $\langle \xi, \eta_0\rangle^2 g_K(\xi) \geq 1$ and, a posteriori,

$$[g_K(\xi)]^{-1/2} \;\leq\; \langle \xi, \eta_0\rangle \;=\; \min_{\eta\in K\cap K^*\cap\mathbb{S}_n} \langle \xi, \eta\rangle \;=\; \mathrm{dist}[\xi, \mathrm{bd}(K + K^*)].$$

We obtain (33) by passing to the supremum in (32) with respect to all unit vectors ξ in the interior of $K \cap K^*$. □

Note that (32) becomes a chain of equalities if K is self-dual. The next proposition is then a straightforward consequence of Lemma 5.

Proposition 7. *Let K be a self-dual cone in $\mathbb{R}^n$. Then, for all $\xi \in \mathrm{int}(K)$, we have*

$$g_K(\xi) = \left(\frac{1}{\mathrm{dist}[\xi, \mathrm{bd}(K)]}\right)^2 .$$

In particular, $\varrho(K)$ is equal to the incenter of K and

$$\mathrm{as}(K) = [r(K)]^2 / \left(1 - [r(K)]^2\right) .$$

That $\varrho(K)$ coincides with $\varrho_{\mathrm{inc}}(K)$ is something rather unusual. What we mean is that, for a vast majority of proper cones, the least asymmetry direction is different from the incenter. After all, the vectors $\varrho(K)$ and $\varrho_{\mathrm{inc}}(K)$ reflect entirely different concepts.

3.2 Polyhedral cones

A polyhedral cone is an intersection of finitely many half-spaces, so it can be described by means of a finite collection of nonzero vectors. The dual of a polyhedral cone is yet another polyhedral cone, but the number of half-spaces in the representation of the dual cone may change. In what follows we use the notation $\mathbb{N}_q := \{1, \ldots, q\}$.

Proposition 8. *Let K be a polyhedral proper cone such that*

$$K \;=\; \{x \in \mathbb{R}^n : \langle a_i, x\rangle \geq 0 \;\textit{for}\; i \in \mathbb{N}_q\} \tag{34}$$

$$K^* \;=\; \{y \in \mathbb{R}^n : \langle b_j, y\rangle \geq 0 \;\textit{for}\; j \in \mathbb{N}_p\}, \tag{35}$$

where the a_i's and the b_j's are nonzero vectors in $\mathbb{R}^n$. Then

$$g_K(\xi) = \max_{\substack{i\in\mathbb{N}_q\\ j\in\mathbb{N}_p}} \frac{\langle a_i, b_j\rangle}{\langle a_i, \xi\rangle\langle b_j, \xi\rangle} \tag{36}$$

for all $\xi \in \mathrm{int}(K \cap K^*)$.

Proof. By passing to dual cones in (34) and (35), we get respectively

$$K^* = \{Av : v \in \mathbb{R}^q_+\}\,, \quad K = \{Bu : u \in \mathbb{R}^p_+\},$$

where $A = [a_1, \ldots, a_q]$ and $B = [b_1, \ldots, b_p]$. Let $\xi \in \mathrm{int}(K \cap K^*)$. The term $f_K(\xi)$ is equal to the supremum of all positive β such that

$$\langle y, R(\beta, \vec{\xi}\,)x\rangle \geq 0 \quad \text{for all} \;\; x \in K,\, y \in K^*$$

or, equivalently, such that

$$\langle v, A^\top R(\beta, \vec{\xi}\,)Bu\rangle \geq 0 \quad \text{for all} \;\; u \in \mathbb{R}^p_+,\; v \in \mathbb{R}^q_+.$$

The above inequality amounts to saying that all the entries of the rectangular matrix

$$A^\top R(\beta, \vec{\xi}\,)B \;=\; \frac{(1+\beta)}{\|\xi\|^2} A^\top \xi\xi^\top B - \beta A^\top B$$

are nonnegative, i.e.,

$$\frac{(1+\beta)}{\|\xi\|^2}\langle a_i, \xi\rangle\langle b_j, \xi\rangle - \beta\langle a_i, b_j\rangle \;\geq 0 \quad \text{for all} \;\; i \in \mathbb{N}_q,\; j \in \mathbb{N}_p.$$

The above system of inequalities can be written as

$$\frac{1}{\|\xi\|^2}\left[1 + \frac{1}{\beta}\right] \;\geq\; \max_{\substack{i\in\mathbb{N}_q\\ j\in\mathbb{N}_p}} \frac{\langle a_i, b_j\rangle}{\langle a_i, \xi\rangle\langle b_j, \xi\rangle}\,.$$

By passing to the supremum with respect to $\beta > 0$ and using the identity (18), we arrive at the announced formula (36). □

Clearly, $\langle a_i, b_j\rangle \geq 0$ for all $(i,j) \in \mathbb{N}_q \times \mathbb{N}_p$. The function $g_K : \mathrm{int}(K \cap K^*) \to \mathbb{R}$ being positive-valued, the maximum in (36) could equally well be taken just over the index set

$$\mathcal{M} := \{(i,j) \in \mathbb{N}_q \times \mathbb{N}_p : \langle a_i, b_j\rangle > 0\}\,.$$

So, in order to find the least asymmetry direction of a polyhedral cone K like in Proposition 8, we can solve the convex optimization problem

$$\begin{cases} \text{minimize} \;\; \max_{(i,j)\in\mathcal{M}} \frac{\langle a_i, b_j\rangle}{\langle a_i, \xi\rangle\langle b_j, \xi\rangle} \\ \|\xi\|^2 \leq 1 \\ \langle a_i, \xi\rangle > 0 \;\; \text{for } i \in \mathbb{N}_q \\ \langle b_j, \xi\rangle > 0 \;\; \text{for } j \in \mathbb{N}_p. \end{cases} \tag{37}$$

Since the objective function in (37) is nondifferentiable, we reformulate this problem in the equivalent form

$$\begin{cases} \text{minimize} \;\; t \\ 1 - \|\xi\|^2 \geq 0 \\ \log t + \log\langle a_i, \xi\rangle + \log\langle b_j, \xi\rangle - \log\langle a_i, b_j\rangle \geq 0 \;\; \text{for } (i,j) \in \mathcal{M} \\ t > 0 \\ \langle a_i, \xi\rangle > 0 \;\; \text{for } i \in \mathbb{N}_q \\ \langle b_j, \xi\rangle > 0 \;\; \text{for } j \in \mathbb{N}_p. \end{cases} \tag{38}$$

Note that (38) is a smooth convex optimization problem in the decision vector $(\xi, t) \in \mathbb{R}^{n+1}$. Such a problem can be handled with any available smooth optimization solver. A major drawback of Proposition 8 is the need of knowing both the a_i's and the b_j's. Sometimes such information is not available or it is costly to obtain.

3.3 Simplicial cones

A substantial simplification occurs in (36) if the polyhedral cone K is simplicial. In what follows the symbol $\mathbb{GL}(n)$ stands for the set of invertible matrices of order n and $B^{-\top}$ refers to the transpose of the inverse of $B \in \mathbb{GL}(n)$.

Proposition 9. *Consider a simplicial cone $K = \text{cone}\{b_1, \ldots, b_n\}$ generated by the columns of a certain matrix $B \in \mathbb{GL}(n)$. Let a_i denote the i-th column of $A = B^{-\top}$. Then*

$$g_K(\xi) = \left[\min_{i\in\mathbb{N}_n} \langle a_i, \xi\rangle\langle b_i, \xi\rangle\right]^{-1} \tag{39}$$

for all $\xi \in \text{int}(K \cap K^*)$.

Proof. We just need to apply Proposition 8 with $p = q = n$ and observe that $\langle a_i, b_j\rangle = \delta_{i,j}$ is the Kronecker delta. □

The next proposition is a nontrivial consequence of formula (39). Proposition 10 states two rather surprising results: firstly, the axial symmetry degree of a simplicial cone K depends only on the dimension n of the underlying space, i.e., the spatial configuration of the generators of K is irrelevant for the determination of $\text{as}(K)$. And, secondly, for a simplicial cone K, the least asymmetry direction $\varrho(K)$ coincides with the so-called volumetric center of K, which is a unit vector in $\text{int}(K \cap K^*)$ denoted by $\varrho_{\text{vol}}(K)$. The notion of volumetric center of a proper cone has been introduced in Seeger and Torki [20, Definition 1.2]. Rules for computing volumetric centers can be found in [20, 21]. Without further ado, we state:

Proposition 10. *Let K be a simplicial cone in $\mathbb{R}^n$. Then $\text{as}(K) = (n-1)^{-1}$ and $\varrho(K)$ coincides with the volumetric center of K.*

Proof. Let K be a simplicial cone as in Proposition 9. In view of (28), we must prove that $\mu(K) = n$. For each $i \in \mathbb{N}_n$ and $\xi \in \text{int}(K \cap K^*)$, the term $\kappa_i(\xi) := \langle a_i, \xi\rangle\langle b_i, \xi\rangle$ is positive and

$$\sum_{i=1}^{n} \kappa_i(\xi) \;=\; \sum_{i=1}^{n}\langle \xi, a_i b_i^\top \xi\rangle \;=\; \langle \xi, AB^\top \xi\rangle \;=\; \|\xi\|^2.$$

Hence, $n \min_{i\in\mathbb{N}_n} \kappa_i(\xi) \leq \|\xi\|^2$ and

$$g_K(\xi) = \left[\min_{i\in\mathbb{N}_n} \kappa_i(\xi)\right]^{-1} \geq \|\xi\|^{-2} n.$$

In this way we have proven that

$$\mu(K) = g_K(\varrho(K)) \geq n. \tag{40}$$

Next, consider a unit vector $\tilde{\xi} \in \text{int}(K \cap K^*)$ such that $\kappa_1(\tilde{\xi}) = \ldots = \kappa_n(\tilde{\xi}) = 1/n$. As shown in [21, Proposition 2.1], such a vector $\tilde{\xi}$ exists and it is unique. In fact, $\tilde{\xi} = \varrho_{\text{vol}}(K)$. By combining (40) and

$$g_K(\tilde{\xi}) = \left[\min_{i\in\mathbb{N}_n} \kappa_i(\tilde{\xi})\right]^{-1} \;=\; n,$$

we get the chain of equalities

$$\mu(K) = g_K(\varrho(K)) = g_K(\tilde{\xi}) = n.$$

Since the least asymmetry direction of K is unique, it follows that $\varrho(K) = \tilde{\xi}$. □

In view of formula (39), the least asymmetry direction of a simplicial cone K as in Proposition 9 can be found by solving the nonsmooth convex optimization problem

$$\begin{cases} \text{maximize} \;\; \min_{i\in\mathbb{N}_n} \langle a_i,\xi\rangle\langle b_i,\xi\rangle \\ \|\xi\|^2 \leq 1 \\ \langle a_i,\xi\rangle > 0 \text{ for } i\in\mathbb{N}_n \\ \langle b_i,\xi\rangle > 0 \text{ for } i\in\mathbb{N}_n. \end{cases} \tag{41}$$

Thanks to Proposition 10, the problem (41) is equivalent to the simpler problem

$$\begin{cases} \text{maximize} \;\; \sum_{i=1}^n \log\langle b_i,\xi\rangle \\ \|\xi\|^2 \leq 1 \\ \langle b_i,\xi\rangle > 0 \text{ for } i\in\mathbb{N}_n, \end{cases} \tag{42}$$

where only the b_i's enter into the picture. Indeed, the unique solution to (42) is known to be the vector $\varrho_{\rm vol}(K)$, cf. [21]. Solving the smooth convex optimization problem (42) offers no difficulty, even if the dimension n is large.

Remark 4. For a non-simplicial proper cone K, the vectors $\varrho(K)$ and $\varrho_{\rm vol}(K)$ may be different. Consider for instance the polyhedral cone K in $\mathbb{R}^3$ generated by the columns of

$$B = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

A matter of numerical computation shows that

$$\begin{aligned} \varrho(K) &= (0.1005,\, 0.0000,\, 0.9949)^\top \\ \varrho_{\rm vol}(K) &= (0.1346,\, 0.0000,\, 0.9909)^\top. \end{aligned}$$

In this example, the angle between $\varrho(K)$ and $\varrho_{\rm vol}(K)$ is roughly 2.03 degrees.

4 Further material concerning $\mathrm{as}(K)$ and $\varrho(K)$

This section gathers all sort of information concerning the axial symmetry degree $\mathrm{as}(K)$ and the least asymmetry direction $\varrho(K)$. We start by stating an easy result.

Proposition 11. *For all $K\in\Pi_n$ and $U\in\mathbb{O}(n)$, we have*

$$\mathrm{as}(K^*) = \mathrm{as}(K) \tag{43}$$
$$\varrho(K^*) = \varrho(K) \tag{44}$$
$$\mathrm{as}(U(K)) = \mathrm{as}(K) \tag{45}$$
$$\varrho(U(K)) = U\varrho(K). \tag{46}$$

Proof. The dual-invariance property (44) is because $f_K = f_{K^*}$, see Corollary 1. The orthogonality rule (46) is obtained by relying on the identities

$$\begin{aligned} \mathrm{int}\,[U(K)\cap(U(K))^*] &= U\,[\mathrm{int}(K\cap K^*)] \\ f_{U(K)}(U\xi) &= f_K(\xi). \end{aligned}$$

The dual-invariance property (43) and the orthogonal-invariance property (45) are contained already in Proposition 4. □

The orthogonality rule (46) is consistent with geometric intuition and has a number of useful consequences, among which we mention the next result.

Corollary 2. *Let $K \in \Pi_n$ and L be a linear subspace with respect to which K is symmetric. Then $\varrho(K) \in L$.*

Proof. The corollary is obtained by combining (46) and [19, Theorem 2.4]. □

Natural questions to ask are: how large and how small can be the axial symmetry degree of a proper cone? The next theorem is an important result of this section. It has to do with the extremal values taken by the function as : $\Pi_n \to \mathbb{R}$. Theorem 2 is somewhat reminiscent of a result in the same vein concerning the central symmetry degree of a convex body.

Theorem 2. *Let $K \in \Pi_n$. Then*

$$(n-1)^{-1} \leq \mathrm{as}(K) \leq 1. \tag{47}$$

Furthermore,

(a) The lower bound in (47) becomes an equality if and only if K is simplicial.

(b) The upper bound in (47) becomes an equality if and only if K is axially symmetric.

Proof. The upper bound in (47) follows from Proposition 1. Part (b) is obvious. The "if" part of (a) is contained in Proposition 10. The proof of the lower bound in (47) runs as follows. Our first observation is that

$$\mathrm{as}(K) = f_K(\varrho(K)) \geq f_K(e^{\mathrm{inc}}(K)), \tag{48}$$

where $e^{\mathrm{inc}}(K)$ is a unit vector in the interior of $K \cap K^*$ called the elliptic incenter of K, cf. Seeger and Torki [22, Definition 4.3]. Proposition 6 yields

$$f_K(e^{\mathrm{inc}}(K)) = \Phi(C_K, e^{\mathrm{inc}}(K)), \tag{49}$$

where

$$C_K := \{x \in K : \langle e^{\mathrm{inc}}(K), x\rangle = 1\}$$

is a convex body in the hyperplane

$$H_K := \{x \in \mathbb{R}^n : \langle e^{\mathrm{inc}}(K), x\rangle = 1\}.$$

From the theory of elliptic incenters developed in [22], we know that

$$e^{\mathrm{inc}}(K) = e^{\mathrm{John}}(C_K), \tag{50}$$

where the term on the right-hand side of (50) is the John center of C_K, i.e., the center of the maximal volume ellipsoid contained in C_K. Note that C_K is viewed as a convex body in H_K, and not as a convex body in $\mathbb{R}^n$. Since H_K is an affine space of dimension $n-1$, by a general result on the symmetry degree of a convex body relative to its John center (cf. [11, p. 248] or [5, Corollary 3.8]), we have

$$\Phi(C_K, e^{\mathrm{John}}(C_K)) \geq (n-1)^{-1}. \tag{51}$$

The lower bound in (47) is then obtained by combining (48), (49), (50), and (51). Furthermore, we know that (51) becomes an equality only if C_K is a simplex in H_K. But

$$K \;=\; \mathrm{cone}(C_K) \;=\; \{tu : t \geq 0, u \in C_K\}.$$

Hence, the lower bound in (47) becomes an equality only if K is a simplicial cone. □

4.1 Characterization of $\varrho(K)$ via optimality conditions

We know that $\varrho(K)$ is a unit vector in the interior of $K \cap K^*$. The next proposition provides a complete characterization of $\varrho(K)$. For $\xi \in \mathrm{int}(K \cap K^*)$, the notation

$$V_K(\xi) := \left\{ (x,y) \in D_K : \frac{\langle x, y\rangle}{\langle \xi, x\rangle\langle \xi, y\rangle} = g_K(\xi) \right\}$$

refers to the solution set to (16).

Proposition 12. *Let $K \in \Pi_n$. Then $\varrho(K)$ is the unique vector ξ of $\mathbb{R}^n$ that satisfies the system*

$$\xi \in \mathbb{S}_n \cap \mathrm{int}(K \cap K^*)\,, \tag{52}$$

$$\xi \in \mathrm{co}\left\{ \frac{1}{2}\left[\frac{x}{\langle \xi, x\rangle} + \frac{y}{\langle \xi, y\rangle} \right] : (x,y) \in V_K(\xi) \right\}, \tag{53}$$

where "co" stands for convex hull.

Proof. Recall that (15) is a convex minimization problem admitting a unique solution. For convenience, we write the constraint $\xi \in \mathbb{B}_n$ in the inequality form $\|\xi\|^2 \leq 1$ and introduce the Lagrangian function

$$L(\xi, \lambda) = G_K(\xi) + \lambda(\|\xi\|^2 - 1).$$

A vector $\xi \in \mathbb{R}^n$ solves (15) if and only if (52) holds and there exists a nonnegative Karush-Kuhn-Tucker multiplier $\lambda_0 \in \mathbb{R}$ such that

$$0 \in \partial\, G_K(\xi) + 2\lambda_0 \xi, \tag{54}$$

where $\partial\, G_K(\xi)$ denotes the subdifferential of G_K at ξ. But

$$\partial\, G_K(\xi) = \frac{1}{g_K(\xi)}\, \partial\, g_K(\xi)$$

for all ξ in the interior of $K \cap K^*$. An explicit formula for $\partial\, g_K(\xi)$ can be obtained by applying Valadier's calculus rule on the subdifferential of a pointwise maximum, cf. [24]. Alternatively, we can apply Danskin's directional differentiability theorem, cf. [6]. In one way or in another, we get

$$\partial\, g_K(\xi) = -2\, g_K(\xi)\, \mathrm{co}[\Gamma_K(\xi)] \tag{55}$$

with

$$\Gamma_K(\xi) := \left\{ \frac{1}{2}\left[\frac{x}{\langle \xi, x\rangle} + \frac{y}{\langle \xi, y\rangle} \right] : (x,y) \in V_K(\xi) \right\}.$$

Hence, the stationarity condition (54) can be written as $\lambda_0 \xi \in \mathrm{co}[\Gamma_K(\xi)]$. Since $\Gamma_K(\xi)$ is contained in the hyperplane $\xi^\diamond = \{\eta \in \mathbb{R}^n : \langle \xi, \eta\rangle = 1\}$, the multiplier λ_0 must be equal to 1. This completes the proof of the proposition. □

Directly from (55) we see that g_K is differentiable at a given point $\xi \in \mathrm{int}(K \cap K^*)$ if and only if $\Gamma_K(\xi)$ is a singleton. Very often in practice, g_K fails to be differentiable at $\varrho(K)$. In others words, the optimality condition (53) occurs usually as an inclusion and not as an equality. The condition (53) amounts to saying that

$$\xi = \sum_{k=1}^{m} \frac{t_k}{2}\left[\frac{x_k}{\langle \xi, x_k\rangle} + \frac{y_k}{\langle \xi, y_k\rangle} \right], \tag{56}$$

where m is a positive integer, the t_k's are positive reals summing up to 1, and the (x_k, y_k)'s are pairs in $V_K(\xi)$. Observe that (56) is a convex combination of vectors in a $(n-1)$-dimensional affine space. By Caratheodory's convex hull theorem, there is no loss of generality in assuming that m is less than or equal to n.

Remark 5. If K is a polyhedral proper cone as in Proposition 8, then we can write the optimality condition (53) in the special form

$$\xi \in \mathrm{co}\left\{ \frac{1}{2}\left[\frac{a_i}{\langle a_i, \xi\rangle} + \frac{b_j}{\langle b_j, \xi\rangle} \right] : \, (i,j) \in \mathcal{M}(\xi) \right\},$$

where

$$\mathcal{M}(\xi) := \left\{ (i,j) \in \mathcal{M} : \, \frac{\langle a_i, b_j\rangle}{\langle a_i, \xi\rangle \langle b_j, \xi\rangle} \; = \; g_K(\xi) \right\}.$$

4.2 Continuity issues

A natural question to ask is whether $\mathrm{as}(K)$ and $\varrho(K)$ are stable with respect to perturbations in the argument K. The next theorem gives an affirmative answer. For convenience, we recall first the Walkup-Wets Isometry Formula, as well as a metric lemma concerning the distance between the boundaries of a pair of proper cones.

Lemma 6. *For all $P, Q \in \Pi_n$, we have*

$$\rho\left(P^*, Q^*\right) \;=\; \rho(P,Q) \tag{57}$$

$$\mathrm{haus}\left[\mathbb{S}_n \cap \mathrm{bd}(P), \mathbb{S}_n \cap \mathrm{bd}(Q)\right] \;\leq\; 2\rho(P,Q). \tag{58}$$

The isometry formula (57) and the inequality (58) can be found in [25, Theorem 1] and [20, Lemma 5.6], respectively. We now are ready to prove:

Theorem 3. *The functions* $\mathrm{as} : \Pi_n \to \mathbb{R}$ *and* $\varrho : \Pi_n \to \mathbb{R}^n$ *are continuous.*

Proof. The upper-semicontinuity of $\mathrm{as}(\cdot)$ is taken care by Proposition 4. For proving the lower-semicontinuity of $\mathrm{as}(\cdot)$ we pick $K_\infty \in \Pi_n$ and check that

$$\mu(K_\infty) \geq \gamma := \limsup_{K \to K_\infty} \mu(K).$$

Let $\{K_\nu\}_{\nu \in \mathbb{N}}$ be a sequence converging to K_∞ and such that $\lim_{\nu\to\infty} \mu(K_\nu) = \gamma$. We claim that

$$\lim_{\nu \to \infty} \rho\left(K_\nu \cap K_\nu^*, K_\infty \cap K_\infty^*\right) = 0, \tag{59}$$

where $K_\infty^* := (K_\infty)^*$. By assumption, $\lim_{\nu\to\infty} \rho\left(K_\nu, K_\infty\right) = 0$. By using formula (57) we get $\lim_{\nu\to\infty} \rho\left(K_\nu^*, K_\infty^*\right) = 0$. The theory of Painlevé-Kuratowski limits shows that

$$\limsup_{\nu \to \infty} \left(K_\nu \cap K_\nu^*\right) \subseteq K_\infty \cap K_\infty^* \,.$$

Since K_∞ and K_∞^* have a common interior point, the lower counterpart

$$\liminf_{\nu \to \infty} \left(K_\nu \cap K_\nu^*\right) \supseteq K_\infty \cap K_\infty^*$$

is also true. This proves (59). Since $\xi_\infty := \varrho(K_\infty)$ belongs to the interior of $K_\infty \cap K_\infty^*$, Proposition 4.15 in [18] allows us to write $\xi_\infty = \lim_{\nu\to\infty} \xi_\nu$ with $\xi_\nu \in \mathrm{int}(K_\nu \cap K_\nu^*)$. Since $\|\xi_\infty\| = 1$, we may suppose that each ξ_ν is a unit vector. Now, let $(x_\nu, y_\nu) \in V_K(\xi_\nu)$, i.e.,

$$x_\nu \in K_\nu \cap \mathbb{S}_n, \quad y_\nu \in K_\nu^* \cap \mathbb{S}_n, \quad g_{K_\nu}(\xi_\nu) = \frac{\langle x_\nu, y_\nu\rangle}{\langle \xi_\nu, x_\nu\rangle\langle \xi_\nu, y_\nu\rangle}\,.$$

Taking subsequences if necessary, we may assume that $x_\infty := \lim_{\nu\to\infty} x_\nu$ exists and belongs to $K_\infty \cap \mathbb{S}_n$ and, similarly, $y_\infty := \lim_{\nu\to\infty} y_\nu$ exists and belongs to $K_\infty^* \cap \mathbb{S}_n$. By passing to the limit in the inequality

$$\mu(K_\nu) \leq g_{K_\nu}(\xi_\nu),$$

we get

$$\gamma \leq \frac{\langle x_\infty, y_\infty\rangle}{\langle \xi_\infty, x_\infty\rangle\langle \xi_\infty, y_\infty\rangle} \leq g_{K_\infty}(\xi_\infty) = \mu(K_\infty),$$

as desired. We now prove the continuity of $\varrho : \Pi_n \to \mathbb{R}^n$ at a given $K_\infty \in \Pi_n$. Consider a sequence $\{K_\nu\}_{\nu\in\mathbb{N}}$ converging to K_∞, not necessarily the same sequence as in the previous part. Let $\tilde{\xi}_\nu := \varrho(K_\nu)$. Without loss of generality, we assume that $\tilde{\xi}_\infty := \lim_{\nu\to\infty} \tilde{\xi}_\nu$ exists. Clearly, $\tilde{\xi}_\infty$ is a unit vector in $K_\infty \cap K_\infty^*$. We claim that

$$\tilde{\xi}_\infty \in \mathrm{int}(K_\infty \cap K_\infty^*). \tag{60}$$

As a by-product of (24) we get

$$\begin{aligned}
\mathrm{dist}\left[\tilde{\xi}_\nu, \mathrm{bd}(K_\nu \cap K_\nu^*)\right] &\geq \frac{\min\{\mathrm{e}(K_\nu^*, K_\nu), \mathrm{e}(K_\nu, K_\nu^*)\}}{g_{K_\nu}(\tilde{\xi}_\nu)}\\
&\geq \frac{\min\{r(K_\nu), r(K_\nu^*)\}}{g_{K_\nu}(\tilde{\xi}_\nu)}\\
&\geq \frac{1}{n}\min\{r(K_\nu), r(K_\nu^*)\}.
\end{aligned}$$

The last inequality is because $g_{K_\nu}(\tilde{\xi}_\nu) = \mu(K_\nu) \leq n$, as shown in Theorem 2. By using (58) and the continuity of the inradius function $r : \Pi_n \to \mathbb{R}$ (cf. [15]), we obtain

$$\mathrm{dist}\left[\tilde{\xi}_\infty, \mathrm{bd}(K_\infty \cap K_\infty^*)\right] \geq \frac{1}{n}\min\{r(K_\infty), r(K_\infty^*)\} > 0.$$

This proves the claim (60). We have shown insofar that $\tilde{\xi}_\infty$ is a feasible solution to (25). Since $\mathrm{as} : \Pi_n \to \mathbb{R}$ is continuous, we have

$$\mathrm{as}(K_\infty) = \lim_{\nu\to\infty} \mathrm{as}(K_\nu) = \lim_{\nu\to\infty} f_{K_\nu}(\tilde{\xi}_\nu). \tag{61}$$

By using Berge's maximum theorem, we can easily check that

$$f_K(\xi) = \max\{\beta : \beta \in \Gamma(K, \vec{\xi}\,)\}$$

is upper-semicontinuous as function of $(K, \xi) \in \Pi_n \times \mathbb{S}_n$. Hence,

$$\lim_{\nu\to\infty} f_{K_\nu}(\tilde{\xi}_\nu) \leq f_{K_\infty}(\tilde{\xi}_\infty) \leq \mathrm{as}(K_\infty).$$

This and (61) yield $\mathrm{as}(K_\infty) = f_{K_\infty}(\tilde{\xi}_\infty)$. The least asymmetry direction of K_∞ being unique, it follows that $\tilde{\xi}_\infty = \varrho(K_\infty)$, which is the desired conclusion. □

It is tempting to conjecture that as : $\Pi_n \to \mathbb{R}$ is Lipschitzian on each compact subset of Π_n. We shall not indulge on this lateral issue, but we mention at least the next negative result.

Proposition 13. *For $n \geq 3$, the function* as : $\Pi_n \to \mathbb{R}$ *is not Lipschitzian, i.e.,*

$$\sup_{\substack{P,Q\in\Pi_n \\ P\neq Q}} \frac{|\mathrm{as}(P) - \mathrm{as}(Q)|}{\rho(P,Q)} \;=\; \infty.$$

Proof. Let $b \in \mathbb{S}_n$. For each integer $k \geq 1$, consider the revolution cone

$$P_k = \{x \in \mathbb{R}^n : (\cos\theta_k)\|x\| \leq \langle b, x\rangle\}$$

with half-aperture angle $\theta_k = 1/k$. On the other hand, let Q_k be any simplicial cone contained in P_k. In such a case, we have $\rho(P_k, Q_k) \leq \sin(2\theta_k)$, $\mathrm{as}(P_k) = 1$, and $\mathrm{as}(Q_k) = (n-1)^{-1}$. The dimension n being fixed, we get

$$\lim_{k\to\infty} \frac{\mathrm{as}(P_k) - \mathrm{as}(Q_k)}{\rho(P_k, Q_k)} \;=\; \infty.$$

This example shows that as$(\cdot)$ is not Lipschitzian as function on the metric space (Π_n, ρ). □

4.3 Cartesian product of proper cones

The next theorem explains how to compute the axial symmetry degree and the least asymmetry direction of a Cartesian product of finitely many proper cones. To be more precise, we consider the Cartesian product of p proper cones in spaces of different dimensions. All our results have been stated for proper cones of dimension at least two, but the formal definitions of axial symmetry degree and least asymmetry direction make sense also for proper cones in $\mathbb{R}$. We have

$$\mathrm{as}(\mathbb{R}_+) = \infty,\ \mathrm{as}(\mathbb{R}_-) = \infty,\ \varrho(\mathbb{R}_+) = 1,\ \varrho(\mathbb{R}_-) = -1. \tag{62}$$

The first two equalities in (62) are not in conflict with (47), because Theorem 2 is stated in a space of dimension bigger than one.

Theorem 4. *Let $n := n_1 + \ldots + n_p$ with $p \geq 2$ and $n_1, \ldots, n_p \geq 1$. Let $K_1 \in \Pi_{n_1}, \ldots, K_p \in \Pi_{n_p}$. Then $K := K_1 \times \ldots \times K_p$ belongs to Π_n and*

$$1 + \frac{1}{\mathrm{as}(K)} \;=\; \sum_{i=1}^{p} \left[1 + \frac{1}{\mathrm{as}(K_i)}\right], \tag{63}$$

$$\varrho(K) \;=\; \left(\sqrt{t_1}\,\varrho(K_1), \ldots, \sqrt{t_p}\,\varrho(K_p)\right), \tag{64}$$

where

$$t_i := \left[1 + \frac{1}{\mathrm{as}(K)}\right]^{-1}\left[1 + \frac{1}{\mathrm{as}(K_i)}\right].$$

Proof. We consider only the particular case $p = 2$, the general case being obtained by an induction argument. Formulas (63) and (64) become

$$\mu(K) \;=\; \mu(K_1) + \mu(K_2), \tag{65}$$

$$\sqrt{\mu(K)}\,\varrho(K) \;=\; \left(\sqrt{\mu(K_1)}\,\varrho(K_1), \sqrt{\mu(K_2)}\,\varrho(K_2)\right), \tag{66}$$

respectively. Suppose first that K_1 and K_2 are polyhedral. By relying on Proposition 8, it is not difficult to check that

$$g_K(\xi) = \max\{g_{K_1}(\xi_1), g_{K_2}(\xi_2)\}$$

for all pair $\xi = (\xi_1, \xi_2)$ in the set

$$\mathrm{int}(K \cap K^*) = \underbrace{\mathrm{int}(K_1 \cap K_1^*)}_{\Omega_1} \times \underbrace{\mathrm{int}(K_2 \cap K_2^*)}_{\Omega_2}.$$

Hence,

$$\mu(K) = \min_{\substack{\xi_1 \in \Omega_1,\, \xi_2 \in \Omega_2 \\ \|\xi_1\|^2 + \|\xi_2\|^2 = 1}} \max\{g_{K_1}(\xi_1), g_{K_2}(\xi_2)\}\,. \tag{67}$$

By using homogeneity arguments, we get

$$\begin{aligned} \mu(K) &= \min_{t \in]0,1[} \min_{\substack{\xi_1 \in \Omega_1 \cap t\,\mathbb{S}_{n_1} \\ \xi_2 \in \Omega_2 \cap (1-t^2)^{1/2}\,\mathbb{S}_{n_2}}} \max\{g_{K_1}(\xi_1), g_{K_2}(\xi_2)\} \\ &= \min_{t \in]0,1[} \min_{\substack{\xi_1 \in \Omega_1 \cap \mathbb{S}_{n_1} \\ \xi_2 \in \Omega_2 \cap \mathbb{S}_{n_2}}} \max\{t^{-2} g_{K_1}(\xi_1), (1-t^2)^{-1} g_{K_2}(\xi_2)\} \\ &= \min_{t \in]0,1[} \max\left\{t^{-2}\,\mu(K_1), (1-t^2)^{-1}\mu(K_2)\right\}. \end{aligned}$$

The above minimum is attained by

$$\tilde{t} = \left[\frac{\mu(K_1)}{\mu(K_1) + \mu(K_2)}\right]^{1/2}$$

and $\mu(K) = \tilde{t}^{\,-2}\,\mu(K_1)$. This proves (65). The least asymmetry direction of K is equal to

$$(\tilde{\xi}_1, \tilde{\xi}_2) = \left(\tilde{t}\,\varrho(K_1), (1-\tilde{t}^2)^{1/2}\varrho(K_2)\right),$$

because such a vector achieves the minimum in (67). This proves (66). Finally, we drop the polyhedrality assumption made on K_1 and K_2. The non-polyhedral case can be treated by combining a continuity argument, namely Theorem 3, and a density argument, namely, every proper cone can be written as limit of a sequence of polyhedral proper cones. The details are omitted. □

The upper bound in (47) can be sharpened if $K \in \Pi_n$ is a Cartesian product. Indeed, Theorem 4 yields straightforwardly the following result.

Corollary 3. *Let $n \geq 3$. Suppose that $p \geq 2$ and $q \in \{0, 1, \ldots, p\}$. Let $K \in \Pi_n$ be a Cartesian product of p proper cones, q of which are of dimension bigger than 1. Then*

$$\mathrm{as}(K) \;\leq\; \frac{1}{p+q-1} \;\leq\; \frac{1}{2}\,. \tag{68}$$

Proof. Suppose that the first q factors of K are of dimension bigger than 1 and that the last $p-q$ factors of K are of dimension one. We have

$$\begin{aligned} 1 + \frac{1}{\mathrm{as}(K_i)} &\geq 2 \qquad \text{for } i \in \mathbb{N}_q, \\ 1 + \frac{1}{\mathrm{as}(K_i)} &= 1 \qquad \text{for } i \in \mathbb{N}_p \backslash \mathbb{N}_q. \end{aligned}$$

By passing to the sum with $i \in \mathbb{N}_p$ and using the formula (63), we get

$$1 + \frac{1}{\mathrm{as}(K)} \;\geq\; 2\,q + 1(p-q) \;=\; p+q.$$

This proves the first inequality in (68). The second inequality in (68) is because $p+q \geq 3$. Indeed, the case $p+q=2$ must be ruled out, because it implies $(p,q)=(2,0)$, which contradicts the hypothesis $n \geq 3$. □

5 Axial symmetrization of a convex cone

A convex body C in $\mathbb{R}^n$ can be transformed into an origin-symmetric convex body by passing to the balanced convex hull

$$\mathrm{bco}(C) := \mathrm{co}(C \cup -C). \tag{69}$$

Is there a natural way of transforming a proper cone into an axially symmetric proper cone? We propose to consider the following definition.

Definition 3. *Let $K \in \Pi_n$. The axial symmetrization of K is defined as the set*

$$\nabla_1 K := K + R(1, L_K)(K). \tag{70}$$

To some extent, the expression (70) corresponds to a conic version of (69). Since L_K is the line generated by $\varrho(K)$, we have

$$R(1, L_K) = 2\,\varrho(K)\,[\varrho(K)]^\top - I_n.$$

Computing the axial symmetrization of a proper cone K is relatively easy, provided the vector $\varrho(K)$ is already known.

Example 1. Let K be a polyhedral proper cone in $\mathbb{R}^n$ generated by nonzero vectors $\{b_1, \ldots, b_p\}$. The reflection of K onto L_K is given by

$$R(1, L_K)(K) = \mathrm{cone}\{c_1, \ldots, c_p\},$$

where $c_k := R(1, L_K) b_k$ is the reflection onto L_K of b_k. Hence,

$$\nabla_1 K = \mathrm{cone}\{b_1, \ldots, b_p, c_1, \ldots, c_p\}. \tag{71}$$

As a particular instance of formula (71), we see that the axial symmetrization of the nonnegative orthant $\mathbb{R}^3_+$ is the polyhedral cone generated by the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}.$$

The axial symmetrization of $\mathbb{R}^n_+$ admits a similar representation, but involving $2n$ vectors.

Among other results, the next theorem explains why $\nabla_1 : \Pi_n \to \Pi_n$ behaves as an axial symmetrization operator on the class of proper cones. For the sake of convenience, we introduce the notation

$$\nabla_\beta K := K + R(\beta, L_K)(K)$$

for an arbitrary $\beta \in [0,1]$.

Theorem 5. *Let $K \in \Pi_n$. Then*

(a) $\beta \mapsto \nabla_\beta K$ is continuous as function from $[0,1]$ to Π_n.

(b) $0 \leq \beta_1 \leq \beta_2 \leq 1$ implies $\nabla_{\beta_1} K \subseteq \nabla_{\beta_2} K$.

(c) $\nabla_1 K$ is symmetric with respect to the line L_K. Furthermore, K is axially symmetric if and only if $\nabla_1 K = K$.

(d) The axial symmetry degree of K admits the characterization

$$\mathrm{as}(K) = \max\{\beta \in [0,1] : \nabla_\beta K = K\}. \tag{72}$$

Proof. Part (a). Let $\beta \in [0,1]$. Clearly, $\nabla_\beta K$ is a convex cone containing K. In particular, $\nabla_\beta K$ has nonempty interior. In order to prove that $\nabla_\beta K$ is closed, it suffices to check that

$$u, v \in K,\ u + R(\beta, L_K)v = 0 \quad \Rightarrow \quad u = 0,\ v = 0. \tag{73}$$

Let u and v be as in the left-hand side of the implication (73). If $v = 0$, then also $u = 0$ and we are done. Suppose that $v \neq 0$. We must arrive to a contradiction. The equation $u + R(\beta, L_K)v = 0$ says that

$$u = \beta v - (1+\beta)\langle \varrho(K), v\rangle\, \varrho(K).$$

By taking the inner product with respect to the vector $\langle \varrho(K), v\rangle^{-1} \varrho(K)$, we get

$$\frac{\langle \varrho(K), u\rangle}{\langle \varrho(K), v\rangle} \;=\; -1,$$

contradicting the fact that $\langle \varrho(K), u\rangle$ and $\langle \varrho(K), v\rangle$ are positive. Hence, (73) holds and $\nabla_\beta K$ is closed. The implication (73) serves also to prove that $\nabla_\beta K$ is pointed. The details are omitted. It remains to check that $\nabla_\beta K$ behaves continuously as function of $\beta \in [0,1]$. Let $\{\beta_\nu\}_{\nu \in \mathbb{N}}$ be a sequence in $[0,1]$ converging to β_∞. Since

$$R(\beta_\nu, L_K) - R(\beta_\infty, L_K) = (\beta_\nu - \beta_\infty)(P_{L_K} - I_n),$$

it is clear that

$$\lim_{\nu \to \infty} \|R(\beta_\nu, L_K) - R(\beta_\infty, L_K)\| \;=\; 0.$$

By applying general calculus rules on Painlevé-Kuratowski limits of convex sets (cf. [18, Chapter 4]), we get

$$\begin{aligned} \lim_{\nu \to \infty} [K + R(\beta_\nu, L_K)(K)] &= K + \lim_{\nu \to \infty} [R(\beta_\nu, L_K)(K)] \\ &= K + R(\beta_\infty, L_K)(K). \end{aligned}$$

This shows that $\beta \mapsto \nabla_\beta K$ is continuous at $\beta_\infty \in [0,1]$. *Part* (b). Let $0 \leq \beta_1 \leq \beta_2 \leq 1$. Then,

$$R(\beta_1, L_K) = (1-t)I_n + tR(\beta_2, L_K)$$

with $t := (1+\beta_1)/(1+\beta_2) \in [0,1]$. It follows that

$$R(\beta_1, L_K)(K) \subseteq K + R(\beta_2, L_K)(K)$$

and, a posteriori, $\nabla_{\beta_1} K \subseteq \nabla_{\beta_2} K$. *Part* (c). Since the reflection matrix $R(1, L_K)$ is an involution, we have

$$\begin{aligned} R(1, L_K)(\nabla_1 K) &= \{R(1, L_K)(u + R(1, L_K)v) : \ u, v \in K\} \\ &= \{R(1, L_K)u + v : \ u, v \in K\} \\ &= \nabla_1 K. \end{aligned}$$

This shows that $\nabla_1 K$ is symmetric with respect to L_K. The second statement in (c) is now obvious. *Part* (d). For $\beta \in [0, 1]$, it is not difficult to see that

$$\beta \leq \mathrm{as}(K) \ \Leftrightarrow \ R(\beta, L_K)(K) \subseteq K \ \Leftrightarrow \ \nabla_\beta K \subseteq K \ \Leftrightarrow \ \nabla_\beta K = K.$$

This proves the announced formula (72). □

Theorem 5 has many useful consequences. For instance, it shows that $\beta \mapsto \nabla_\beta K$ is a continuous path in Π_n joining K to its axial symmetrization $\nabla_1 K$. Hence, as : $\Pi_n \to \mathbb{R}$ attains all the intermediate values in (47).

Corollary 4. *Let* $(n-1)^{-1} \leq c \leq 1$. *Then there exists* $K \in \Pi_n$ *such that* $\mathrm{as}(K) = c$.

Proof. Let $K_0 \in \Pi_n$ be simplicial and $h : [0, 1] \to \mathbb{R}$ be given by $h(\beta) := \mathrm{as}\left(\nabla_\beta K_0\right)$. Note that h is continuous and

$$\begin{aligned} h(0) &= \mathrm{as}\left(\nabla_0 K_0\right) = \mathrm{as}\left(K_0\right) = (n-1)^{-1} \\ h(1) &= \mathrm{as}\left(\nabla_1 K_0\right) = 1. \end{aligned}$$

By the Intermediate Value Theorem, there exists $\tilde{\beta} \in [0, 1]$ such that $\nabla_{\tilde{\beta}} K_0$ has axial symmetry degree equal to c. □

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