Mathematical models for Multi Container Loading Problems with practical constraints

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Abstract

We address the multi container loading problem of a company that has to serve its customers by first putting the products on pallets and then loading the pallets into trucks. We approach the problem by developing and solving integer linear models. To be useful in practice, our models consider three types of constraints: geometric constraints, so that pallets lie completely inside the trucks and do not overlap; weight constraints, defining the maximum weights supported by a truck and by each axle, as well as the position of the centre of gravity of the cargo; and dynamic stability constraints. These last constraints forbid empty spaces between pallets to avoid cargo displacement when the truck is moving, and limit differences between the heights of adjacent pallets to prevent tall pallets tipping over short ones. We also consider extensions of the models to the case of heavy loads, requiring a special configuration of the pallets in the truck, and to the case in which the demands must be served over a set of time periods to meet delivery dates. The models have been tested on a large set of real instances involving up to 44 trucks, obtaining optimal solutions in most cases and very small gaps when optimality could not be proven.

Keywords: container loading, optimization, integer programming, cutting stock problem

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1. Introduction

The Multi Container Loading Problem (MCLP) consists in loading a set of products into the minimum number of containers, while satisfying different types of constraints. Among the many variants of the MCLP, the real case inspiring this study is the problem of a distribution company that has to serve the demand of its customers by first putting the required products on pallets and then loading the pallets into trucks. Achieving an optimal solution to this problem, that is, reducing the number of trucks to a minimum, has many economic and environmental benefits.

The loading problem consists of two interrelated phases. In the pallet building phase the items are grouped in layers and then the layers are stacked on the pallet base. A layer is an arrangement of items of the same product, composing a rectangle whose dimensions and number of items are known. A layer completely covers the pallet base in horizontal directions and other layers can be placed on top of it to form the pallet. The layer composition of each product has been previously decided, so the problem consists in stacking layers to build pallets. Once the pallets are built, they are placed into the trucks. We assume that there is an infinite supply of identical trucks.

The means of transport, in this case the use of trucks, introduces some constraints that have to be respected for safety reasons. There is a strict limit on the maximum weight that can be loaded into a truck. There are also limits on the maximum weight each axle can bear. Excesses over these weight limits represent a risk for traffic safety and can cause damage to the road. Therefore, they are strictly controlled and violations are severely punished. Moreover, the load has to be well distributed in the truck so that the center of gravity lies between the axles and as near as possible to the geometric center of the truck.

Another important concern in truck transportation is the stability of the cargo. Static stability (or vertical stability), related to the capacity of the loaded boxes to be in equilibrium when the truck is not moving, has to be ensured to avoid problems in the loading/unloading process. Dynamic stability (or horizontal stability), related to the capacity of the loaded boxes not to be displaced when the truck is moving and subjected to acceleration, braking, and lateral turns, also has to be ensured to prevent cargo damage. Any useful solution to the MCLP has to address these stability issues satisfactorily.

In the literature on Single and Multi Container Loading Problems, there is a increasing number of studies that address these practical constraints. As the review by Bortfeldt and Wäscher [1] indicates, weight and static stability have been considered by many authors in the single container loading problem, but dynamic stability has received less attention so far. The multi container case

has been less studied and more research is needed on problems with different types of containers and realistic constraints, as concluded in the survey by Zhao et al. [2]. In most cases, heuristic and metaheuristic approaches have been followed, while mathematical models and exact algorithms have usually addressed only basic problems.

In this paper, our proposal is to attempt to solve practical problems exactly, by using Integer Linear Programming (ILP) models. Apart from being fairly easy to implement for a practitioner, ILP models are flexible tools for adding or removing constraints so as to meet the requirements of the specific MCLP at hand, and in recent years they have acquired very good computational behavior, as also witnessed by our results below. Starting from a model we developed in a previous study (Alonso et al. [3]), already including weight constraints such as total weight, maximum weight supported by the axles, and position of the center of gravity, we now focus on dynamic stability constraints. First, we add constraints preventing empty spaces between pallets, to avoid cargo displacement when the truck is moving. Then, we consider several alternatives to limit excessive differences between the heights of adjacent pallets and prevent a tall pallet tipping over a short one, and discuss their relative advantages. We also study extensions of the models to the case of heavy cargoes, which requires a specific configuration of the pallets in the truck, and to the case, which arises frequently in practice in companies like the one in this study, in which the demands have to be served over a set of time periods to meet delivery dates. For this extension, two alternatives have been studied and compared in terms of flexibility and computational complexity.

The models have been tested on a set of 111 real instances provided by a distribution company and the results show that optimal solutions can be obtained in most cases with short computing times. In the cases in which optimality cannot be proven, the solutions usually have a very small gap, and therefore we can state that the models we have developed are able to obtain high quality solutions on our set of instances.

The remainder of the paper is organized as follows. An overview of related existing research is presented in Section 2. In Section 3, the problem is formally described. In Section 4, the test instances are analyzed and upper and lower bounds for each instance are calculated. In Section 5, we introduce the initial model, which includes the main characteristics of the problem as well as several types of weight constraints. In Section 6, we consider dynamic stability and propose alternative ways of dealing with this issue. Section 7 studies the special case of heavy loads in which a specific distribution of pallets is needed, while Section 8 extends the model to the case of demands to be served over a set of periods. Section 9 contains the conclusions.

2. Previous work

The Single Container Loading Problem (SCLP) has been extensively studied. Besides the geometric constraints, realistic constraints, similar to those we are facing in this study, have been receiving increasing attention for some time now. In 1995, Bischoff and Ratcliff [4] listed twelve conditions to be considered when solving practical problems for which feasible loading plans are to be constructed. More recently, Bortfeldt and Wäscher [1] have written an exhaustive review of the container loading problem and its relevant constraints. Zhao et al. [2] also review the use of these constraints, basically in the single container problem.

Some of these practical constraints are related to weight. There is a maximum weight that can be loaded into the container (Bortfeldt et al. [5], Egeblad et al. [6]), and in trucks with several axles the maximum weight each axle can support is also limited (Lim et al. [7], Pollaris et al. [8]). Indeed, when the cargo is heavy, weight becomes a very restrictive constraint, even more than the space occupied. Moreover, the weight of the cargo has to to be evenly spread over the container floor. To achieve a good weight distribution, the center of gravity of the load should be in the geometric mid-point of the container floor or must not be located more than a certain distance away from it (Bortfeldt and Gehring [9]).

Another very common constraint concerns the orientation of the items. Although in some cases the orientation is not restricted (Parreño et al. [10]), it is more usual that only one vertical orientation is permitted, while 90° rotations of items in the horizontal plane are allowed (Iori and Martello [11]). Sometimes, items cannot be rotated at all (Junqueira et al. [12]).

Stackability or load-bearing constraints have also been introduced to avoid damaging the items located at the bottom of the stacks. They can be defined by limiting the number of items that an item can bear above it (Bischoff and Ratcliff [4]), by prohibiting items of a particular type being placed on top of another type (Terno et al. [13]), or by limiting the maximum weight that can be applied to an item per unit area (Junqueira et al. [12], Alonso et al. [14]). Load-bearing constraints, as well as other problem-specific conditions, have also been considered by Toffolo et al. [15] and Correcher et al. [16] in the solution of a multi container problem arising in the distribution centers of a large automotive company.

Another family of constraints that is receiving increasing attention due to their practical importance is related to the stability of the load. So-called vertical or static stability prevents items from falling when the vehicle is not moving (Ramos et al. [17]). An item must be supported from below by a given percentage of the surface of its base. If this percentage is 100%, then full base support

is required (Araujo and Armentano [18], Fanslau and Bortfeldt [19]), whereas lower values indicate that only partial base support is needed (Jin et al. [20], Junqueira et al. [12]). So-called horizontal or dynamic stability on the other hand ensures that items will not move when the truck is moving. Preventing cargo displacements inside the truck during the journey is important to reduce damage to the transported products (Ramos et al. [21]).

There are not many papers that address all the issues studied here and consider pallet and truck loading together. Following the typology for cutting and packing problems introduced by Wäscher et al. [22], the two problems, pallet and truck loading, can both be classified as Single Stock Size Cutting Stock Problems. Morabito et al. [23] deal with this type of problem using a two-phase approach. In a first phase, they load the maximum number of products onto a pallet, using the 5-block algorithm proposed by Morabito and Morales [24]. When the pallets have been built, they use the same algorithm, in their second phase, to load the pallets into the trucks. Takahara [25] solves the problem using two lists: an ordered list of items and an ordered list of pallets and containers into which the items have to be loaded. These lists are handled by applying several metaheuristic procedures. Doerner et al. [26] deal with a particular vehicle routing problem in which the items are placed on pallets and stacked one above the other, producing piles. They propose two metaheuristic algorithms, a Tabu Search and an Ant Colony Optimization algorithm. Pollaris et al. [8] combine a capacitated vehicle routing problem with the loading of homogeneous pallets inside the vehicle, and propose a mixed ILP formulation. Pallets may be placed in two rows inside the vehicle but cannot be stacked on top of each other because of their weight, fragility, or customer preferences. Their model tries to minimize the transportation cost, respecting the axle weight constraints and the order in which customers have to be served along the route. Moura and Bortfeldt [27] also solve a combined vehicle routing and container loading problem. First, homogeneous pallets are built by adapting the GRASP algorithm proposed by Moura and Oliveira [28] and then pallets are loaded into trucks using a tree search procedure. Sheng et al. [29] load pallets into containers but allow the container to be filled up with individual boxes, using a tree search procedure to load the pallets and then a greedy algorithm for filling the residual spaces.

3. Problem description

The distribution company receives an order from a customer, composed of a list of products $j \in J$ that have to be shipped. Each product has a layer composition whose dimensions do not exceed those of the pallet base, a weight q_j, and a number of demanded layers n_j. The layers are placed

5

on pallet bases with dimensions (l^p, w^p, h^p) and weight q^p, forming pallets. A pallet is therefore composed of a base and a set of layers placed one on top of the other.

Pallets are loaded into trucks. A set of K trucks, large enough to accommodate all the products, is available. The trucks are identical and each has a total weight Q^e, a load space of dimensions (L, W, H), a maximum weight capacity Q, and maximum weight capacities on the front and rear axles equal to Q_1 and Q_2, respectively. The distances from the front of the truck to the two axles are δ_1 and δ_2. Figure 1 shows the main characteristics of a truck. The objective of the problem is to send all the products demanded by the customer with the minimum number of trucks.
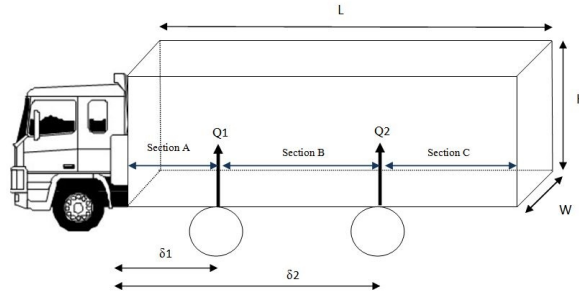

Figure 1: Dimensions and axles positions of the truck

As the dimensions of trucks and pallets are fixed, pallets are placed in the truck in fixed positions $i \in I$, where $|I| = \left\lfloor \frac{L}{w^p} \right\rfloor * \left\lfloor \frac{W}{l^p} \right\rfloor$, in accordance with a grid that has $\left\lfloor \frac{L}{w^p} \right\rfloor$ pallets along the truck's length and $\left\lfloor \frac{W}{l^p} \right\rfloor$ across the truck's width. The grid for the most common case in which two pallets fit into the truck width can be seen in Figure 2. In the following, let (G_x, G_y) denote the coordinates of the center of the truck and let (p_i^x, p_i^y) denote the coordinates of the center of position $i \in I$.

Figure 2: Pallet positions on the truck floor

The way in which the pallet weight is supported by the axles follows the law of levers. The load

supported by each axle depends on the position of the load in the truck. In Figure 1, the truck is divided into three sections, Section A, in front of the front axle, Section B, between the two axles, and Section C, behind the rear axle. If a pallet of weight q_i is placed in a position i in the truck, defined by its middle point p_i^x on the length dimension, the force applied on each axle is shown in Table 1. The forces have to be in equilibrium, not exceeding the maximum force supported by the front and rear axles (Q_1 and Q_2).

	Position	Force on front axle	Force on rear axle
Section A	$0 \leq p_i^x \leq \delta_1$	$q_i(\delta_2 - p_i^x)$	$-q_i(\delta_1 - p_i^x)$
Section B	$\delta_1 < p_i^x \leq \delta_2$	$q_i(\delta_2 - p_i^x)$	$q_i(p_i^x - \delta_1)$
Section C	$\delta_2 < p_i^x \leq L$	$-q_i(p_i^x - \delta_2)$	$q_i(p_i^x - \delta_1)$

Table 1: Axle forces per section

4. A real world benchmark dataset

The benchmark set used in this study is composed of 111 real instances taken from the everyday distribution activity of a large company. The set has been provided to us by ORTEC [30], a company developing planning and optimization solutions and services for manufacturing and logistics companies. The instances, which were already used by Alonso et al. [3] in a previous paper, show high variability. The distribution of the products ranges between 1 and 142 different types, whereas the customers' demands vary from 241 to 9537 layers.

We first calculated an initial lower bound on the number of trucks required for each instance, according to weight and number of pallets. The bound based on the weight is the sum of the weights of all layers divided by the weight capacity of the truck. The bound based on the number of pallets is calculated by dividing the sum of the heights of the layers by the truck height and by the number of positions in the truck. The maximum of these two values, given by

$$L_{init} = max \left\{ \left\lceil \frac{\sum_{j \in J} q_j n_j}{Q} \right\rceil, \left\lceil \frac{\sum_{j \in J} h_j n_j}{HI} \right\rceil \right\} \tag{1}$$

is a valid lower bound on the number of trucks required for each instance for all the problem configurations that we address in the following sections. The bound based on the weight tends to be larger, but this tendency is not uniform in all instances, and, in fact, there are cases in which the bound based on the number of pallets is the larger.

We also computed an upper bound on the number of trucks, called U_{init}, by using a constructive algorithm. The algorithm originates from the work in ([3]) but has been adapted to take into consideration all the constraints described in the following sections. It builds a solution by means

of an iterative process comprising two steps. The first step is to select a position in the truck in which to place the next pallet. As the weight has to be balanced between the two axles, we divide the truck into two parts, front and back, and start the loading process from the center. By placing a pallet on a different side each time, we can control the weight balance. Every time a pallet is completed, the weight supported by each axle is calculated. If the weight on the front axle is greater than that on the rear axle, the next space is selected at the back, otherwise it is chosen at the front. In the second step, a pallet is built at the selected position. The layers of products are ordered by density, with the most dense first. A layer is selected if the weight of the pallet, including this layer, does not exceed the maximum weight allowed for this position. The layer is added to the pallet, and the process continues until no more layers can be added and the pallet can be thus considered completed. The process continues until all the layers have been placed on pallets or no more pallets can be loaded into the truck because one of the constraints has been reached. If there are still products to be shipped, a new truck is opened and the process is repeated.

The heuristic algorithm has been adapted to take into consideration all the constraints described in the following sections, so it always returns a feasible solution for the corresponding configuration. For the initial model, the number of trucks provided by the upper bound ranges between 5 and 44 trucks. On the basis of the difference between these initial lower and upper bounds, the instances have been classified into four classes (A when the difference is 0, B when it is 1, C when it is 2, and D when it is 3 or more).

5. The initial MCLP model

In this section, we review the model we developed in a previous study (Alonso et al. [3]), because it will be the starting point of our new models to which we will progressively add realistic constraints.

We define variables:

x_{kij} = number of layers of product j packed in position i of truck k

$$y_k = \begin{cases} 1, & \text{if truck } k \text{ is used} \\ 0, & \text{otherwise} \end{cases}$$

$$z_{ki} = \begin{cases} 1, & \text{if a pallet is packed in position } i \text{ of truck } k \\ 0, & \text{otherwise} \end{cases}$$

The initial model is:

$$\text{(Initial)} \quad \min \sum_{k \in K} y_k \tag{2}$$

8

$$\sum_{k \in K} \sum_{i \in I} x_{kij} \geq n_j \qquad\qquad j \in J \quad (3)$$

$$\sum_{j \in J} h_j x_{kij} + h^p z_{ki} \leq H' y_k \qquad\qquad k \in K, i \in I \quad (4)$$

$$\sum_{i \in I} (q^p z_{ki} + \sum_{j \in J} q_j x_{kij}) \leq Q y_k \qquad\qquad k \in K \quad (5)$$

$$\sum_{i \in I} (q^p z_{ki} + \sum_{j \in J} q_j x_{kij})(\delta_2 - p_i^x) \leq Q_1 (\delta_2 - \delta_1) y_k \qquad\qquad k \in K \quad (6)$$

$$\sum_{i \in I} (q^p z_{ki} + \sum_{j \in J} q_j x_{kij})(p_i^x - \delta_1) \leq Q_2 (\delta_2 - \delta_1) y_k \qquad\qquad k \in K \quad (7)$$

$$\sum_{j \in J} h_j x_{kij} \leq (H - h^p) z_{ki} \qquad\qquad k \in K, i \in I \quad (8)$$

$$z_{ki} \leq \sum_{j \in J} x_{kij} \qquad\qquad k \in K, i \in I \quad (9)$$

$$Q^e G_x + \sum_{i \in I} p_i^x q^p z_{ki} + \sum_{i \in I} \sum_{j \in J} p_i^x q_j x_{kij} \leq (\sum_{i \in I} (q^p z_{ki} + \sum_{i \in I} \sum_{j \in J} q_j x_{kij} + Q^e)(G_x + \tau_1^x) \qquad k \in K \quad (10)$$

$$Q^e G_x + \sum_{i \in I} p_i^x q^p z_{ki} + \sum_{i \in I} \sum_{j \in J} p_i^x q_j x_{kij} \geq (\sum_{i \in I} (q^p z_{ki} + \sum_{i \in I} \sum_{j \in J} q_j x_{kij} + Q^e)(G_x - \tau_2^x) \qquad k \in K \quad (11)$$

$$Q^e G_y + \sum_{i \in I} p_i^y q^p z_{ki} + \sum_{i \in I} \sum_{j \in J} p_i^y q_j x_{kij} \leq (\sum_{i \in I} (q^p z_{ki} + \sum_{i \in I} \sum_{j \in J} q_j x_{kij} + Q^e)(G_y + \tau_1^y) \qquad k \in K \quad (12)$$

$$Q^e G_y + \sum_{i \in I} p_i^y q^p z_{ki} + \sum_{i \in I} \sum_{j \in J} p_i^y q_j x_{kij} \geq (\sum_{i \in I} (q^p z_{ki} + \sum_{i \in I} \sum_{j \in J} q_j x_{kij} + Q^e)(G_y - \tau_2^y) \qquad k \in K \quad (13)$$

$$y_k \geq y_{k+1} \qquad\qquad k \in K : k < |K| \quad (14)$$

$$y_k = 1 \qquad\qquad k \in K : k \leq L_{init} \quad (15)$$

$$x_{kij} \geq 0, \text{integer} \qquad\qquad k \in K, i \in I, j \in J \quad (16)$$

$$y_k \in \{0, 1\} \qquad\qquad k \in K \quad (17)$$

$$z_{ki} \in \{0, 1\} \qquad\qquad k \in K, i \in I \quad (18)$$

The objective function (2) minimizes the number of trucks required. Constraints (3) ensure that the demand for each product is met. The height of each pallet cannot exceed the truck height, according to constraints (4). On the right-hand side we could have put H, the truck inner height, but this value can be adjusted to H' by solving the following Subset Sum Problem (SSP):

$$H' = h^p + \max\{\sum_{j \in J} h_j \xi_j, \text{ subject to } \sum_{j \in J} h_j \xi_j \leq (H - h^p), 0 \leq \xi_j \leq n_j \text{ and integer for } j \in J\}$$

which gives the maximum height that a subset of layers, plus the pallet base, can attain without exceeding H. Similarly, constraints (5) do not allow the total weight of the pallets in a truck to

exceed the total weight Q. Constraints (6) and (7) limit the weight on the front and rear axles, respectively. Constraints (8) and (9) link layers and pallets bases: at each position of each truck, if there are some layers, then there must be a pallet base too, and viceversa.

For safety reasons, the center of gravity of the loaded truck has to be located as near as possible to its geometric center and never behind the rear axle. Recall that (G_x, G_y) are the coordinates of the center of the truck. We consider that a load is feasible if its center of gravity lies in the x-interval $[G_x - \tau_1^x, G_x + \tau_2^x]$ and in the y-interval $[G_y - \tau_1^y, G_y + \tau_2^y]$, where τ are input tolerance parameters fixed by the user. Constraints (10)–(13) ensure that the load is feasible with respect to these tolerances. Constraints (14) sort the trucks, so truck $k + 1$ can only be used if truck k is also used, whereas constraints (15) set to 1 variables associated with trucks whose index is lower than or equal to the initial lower bound. Finally, constraints (16)–(18) define the domain of the variables.

The model was tested on the benchmark previously described. In our tests, we set τ_1^x and τ_2^x so that lengthwise the center of gravity had to be between the front of the truck and the rear axle. We then set τ_1^y and τ_2^y to $\frac{W}{8}$, and Q^e to 3500 kg. The results obtained by the ILP model (2)–(18) are given in Table 2.

The table gives aggregate information on the behavior of the model, as each line presents average or total values obtained for the corresponding class of instances. Apart from the name of the class and the number of instances in the class (*inst.*), the columns have the following meanings: L and U give the average lower and upper bound returned by the model; *missed* gives the total number of instances not solved to proven optimality; *gap* gives the total absolute gap, that is, the total difference between U and L on all the instances in the line; *nodes* gives the average number of nodes explored by the model's enumeration tree, and *sec* the average computational effort in seconds; $nodes_{opt}$ and sec_{opt} provide, respectively, the same information given by *nodes* and *sec,* but refer only to the instances solved to proven optimality by the model. The model was coded in C++ and solved on a computer Intel Core i7-4790CPU (3.6GHz, 16GB) using CPLEX 12.51 with 4 threads. The time limit per instance was set to 3600 CPU seconds.

Classes A and C seem easier and the model is able to solve all their instances in less than one second on average. Classes B and D are more challenging. The five instances not solved to optimality are very different, with numbers of products ranging from 1 to 25 and numbers of layers between 1929 and 4043, and they are not the ones requiring the largest numbers of trucks. The difference between the upper and the lower bound is just one truck in all cases not solved to proven optimality. In summary, the initial model can be considered satisfactorily solved.

Table 2: Computational results of the initial model (2)–(18)

class	inst.	missed	gap	L	U	nodes	sec	nodes$_{opt}$	sec$_{opt}$
A	27	0	0	8.19	8.19	0	0.2	0	0.2
B	53	2	2	9.11	9.15	1169969	196.2	388138	62.8
C	20	0	0	9.45	9.45	198	1.0	198	1.0
D	11	3	3	18.27	18.55	2224105	988.8	721	9.4
avg/sum	111	5	5	9.86	9.90	779076	191.9	186837	31.1

6. Dynamic stability constraints

Stability is considered one of the most important issues when solving a container loading problem (CLP), because unstable loads would cause damage to the products being sent and could be dangerous for the people handling them. Loading strategies that do not take stability into account cannot be used in practice. This is why an increasing number of studies in the CLP literature consider stability constraints (37.3% of the papers reviewed by Bortfeldt and Wäscher [1]). Two types of stability can be distinguished: static and dynamic stability.

Static stability (also known as vertical stability) concerns the capacity of the loaded boxes to withstand the force of gravity acting on them, and deals with situations in which the truck is not moving. In all our models, static stability is always ensured because all pallets are placed on the floor of the truck and every layer is supported by the pallet base or by another layer.

Dynamic stability (also known as horizontal stability) is related to the capacity of the loaded boxes to withstand the inertia of their own bodies and not be displaced with respect to the x and y axes. This kind of stability encompasses situations where the truck is displaced horizontally and is exposed to speed variations during its journey. Bischoff et al. [4] proposed two metrics for evaluating dynamic cargo stability. The first metric (M1) is the average number of supporting boxes for each box that is not located on the truck floor, with an alternative (M1a) that does not consider contact areas of less than 5% of the base area of a box. The last metric (M2) is related to lateral support and measures the percentage of boxes that do not have at least three of their four sides in contact with other boxes or the walls of the truck. In our case, all pallets are located on the truck floor and all layers are supported by other layers or pallet bases, so M1 (and M1a) is always 1. It is more interesting to consider M2 for the pallets in a truck, to know whether a pallet has at least three of its sides in contact with other pallets or the walls of the truck, because that would be a good indicator of the stability of the cargo.

In the remaining part of this section, we consider two conditions that would increase the dynamic stability of the cargo, on the one hand, forbidding empty spaces between pallets to prevent horizontal displacements, and on the other hand, avoiding excessive differences in height between adjacent pallets, which could result in the taller pallet tipping over the shorter one.

6.1. Compactness

A pallet placed in a truck is dynamically unstable if it is displaced when the truck is moving due to acceleration and braking forces or when the truck turns to the right or to the left. Longitudinal stability is required when the truck is subjected to speeding-up and braking forces, applied in the lengthwise direction, whereas lateral stability is required when the truck turns and the forces are applied in a widthwise direction. If pallets are loaded as they appear in Figure 3, those numbered 1, 5, and 6 could be displaced towards the back of the truck due to acceleration. Pallets 2 and 6 could be displaced instead towards the front if the truck brakes. Similarly, if the pallets are placed in the truck as shown in the top view in Figure 4, pallets 2, 3, 4, and 6 could be displaced when the truck turns right or left.
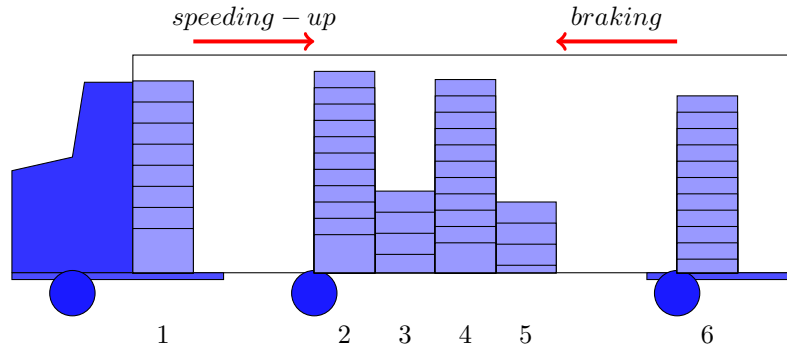

Figure 3: Longitudinal stability forces

To prevent longitudinal movements during the journey, we impose placing the pallets in consecutive positions along the truck, thus avoiding gaps among them. To achieve this, we make use of two new families of variables:

$$f_{ki} = \begin{cases} 1, & \text{if there is an empty space to the left of position } i \text{ of truck } k \\ 0, & \text{otherwise} \end{cases}$$

$$b_{ki} = \begin{cases} 1, & \text{if there is an empty space to the right of position } i \text{ of truck } k \\ 0, & \text{otherwise} \end{cases}$$
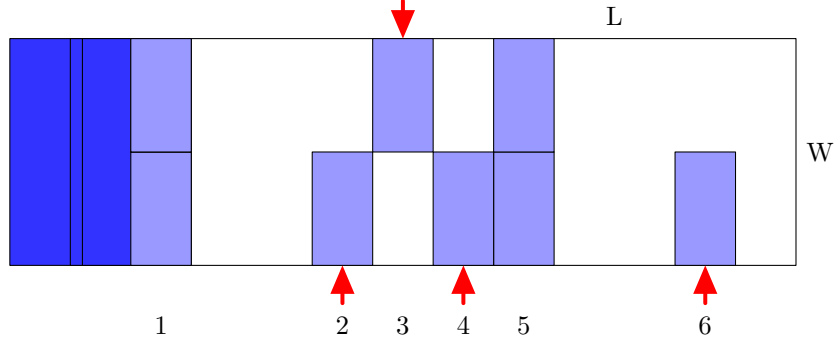
Figure 4: Lateral stability forces

Variables f_{ki} take value 1 when there is a pallet in position i but not in position $(i-1)$. Conversely, variables b_{ki} take value 1 if there is a pallet in position i but not in position $(i+1)$.

To compact the load longitudinally, we add the following constraints to the initial model:

$$f_{ki} \geq z_{ki} - z_{k,i-1} \quad k \in K, i \in \{2, ..., |I|\} \setminus \{|I|/2+1\} \tag{19}$$

$$f_{ki} \geq z_{ki} \quad\quad k \in K, i \in \{1, |I|/2+1\} \tag{20}$$

$$b_{ki} \geq z_{ki} - z_{k,i+1} \quad k \in K, i \in \{1, ..., |I|-1\} \setminus \{|I|/2\} \tag{21}$$

$$b_{ki} \geq z_{ki} \quad\quad k \in K, i \in \{|I|/2, |I|\} \tag{22}$$

$$\sum_{1 \leq i \leq |I|/2} f_{ki} \leq 1 \quad\quad k \in K \tag{23}$$

$$\sum_{|I|/2+1 \leq i \leq |I|} f_{ki} \leq 1 \quad\quad k \in K \tag{24}$$

$$\sum_{1 \leq i \leq |I|/2} b_{ki} \leq 1 \quad\quad k \in K \tag{25}$$

$$\sum_{|I|/2+1 \leq i \leq |I|} b_{ki} \leq 1 \quad\quad k \in K \tag{26}$$

$$f_{ki}, b_{ki} \in \{0, 1\} \quad\quad k \in K, i \in I \tag{27}$$

Constraints (19) force variables f_{ki} to take value 1 if there is a pallet in position i ($z_{ki} = 1$) but not in position $i-1$ ($z_{ki-1} = 0$). Positions 1 and $|I|/2+1$ are in the front row of the truck, so no pallet can be placed to their left. In these cases, considered in constraints (20), variables f_{ki} simply take the value of the corresponding variables z_{ki}. This is done to make a proper count of the number of empty spaces to the left of each column, which is then limited to 1 by constraints (23) and (24). In this way, there can be no more than one empty space to the left of all the pallets placed in each column and no empty space between two pallets. The same reasoning applies for variables b_{ki}: the

special cases correspond to the last positions in each column, $|I|/2$ and $|I|$, that are considered in constraints (22), whereas the number of empty spaces is limited by constraints (25) and (26).

Similarly, we enforce lateral stability by adding new variables and constraints to the model to avoid situations in which there are rows with a pallet in one position and an empty space in the other position. The new variables that we invoke are:

$$s_{ki} = \begin{cases} 1, & \text{if there is only one pallet in the row defined by position } i \text{ in truck } k, \ i \in \{1,...,|I|/2\} \\ 0, & \text{otherwise} \end{cases}$$

To prevent lateral displacements, the following constraints are added to the initial model:

$$s_{ki} \geq z_{ki} - z_{k,i+|I|/2} \qquad k \in K, i \in \{1,...,|I|/2\} \tag{28}$$

$$s_{ki} \geq z_{k,i+|I|/2} - z_{ki} \qquad k \in K, i \in \{1,...,|I|/2\} \tag{29}$$

$$\sum_{1 \leq i \leq |I|/2} s_{ki} \leq 1 \qquad k \in K \tag{30}$$

$$s_{ki} \in \{0,1\} \qquad k \in K, i \in \{1,...,|I|/2\} \tag{31}$$

Constraints (28) force variables s_{ki} to take value 1 if there is a pallet in a position i in the first column and not in the same row of the second column (that is, in position $i + |I|/2$). Conversely, constraints (29) force variables s_{ik} to be equal to 1 when there is a pallet in a position of the second column, $i \in |I|/2+1,...,|I|$, but not in the same row of the first column (position $i - |I|/2$). Constraints (30) limit the number of lateral empty spaces to 1, so as to allow solutions where an odd number of pallets is loaded into a truck.

In Figure 5 we can see an overhead view of the first truck in a solution with $|I| = 22$, showing variables f_{ki}, b_{ki}, and s_{ki} taking value 1. This solution does not satisfy the longitudinal and lateral stability constraints because there is more than one empty space to the left, to the right, and to the side of the pallets loaded. Figure 6 shows a solution in which these longitudinal and lateral stability constraints are satisfied.

The computational results obtained by adding both longitudinal and lateral stability constraints to the initial model are shown in Table 3. It can be observed that the resulting model is harder to solve than the previous one, and that the number of instances for which optimality is not proven increases from 5 to 13. However, the gap for each instance is at most one truck, and the computational effort required is still quite low on average. The effect of these stability constraints on the lengthwise and widthwise compactness of the load can be observed in Figure 7, which depicts a truck load taken from different solutions of the same benchmark instance. Figure 7(a) shows part

14

Figure 5: A solution that does not accomplish with constraints (19)-(31)

Figure 6: A solution satisfying constraints (19)-(31)

of the solution obtained by the initial model, whereas Figure 7(b) takes into account longitudinal constraints and Figure 7(c) longitudinal and lateral constraints. An undesired effect that can be seen in Figure 7(c) is that empty spaces can be filled with short pallets with very few layers. This produces an increment in the number of pallets in the solution and also large differences in the heights of adjacent pallets that can affect stability. The next subsection will address this question, by looking for pallets with similar heights.

Table 3: Computational results of the model with longitudinal and lateral stability constraints (2)–(31)

class	inst.	missed	gap	L	U	nodes	sec	nodes$_{opt}$	sec$_{opt}$
A	27	0	0	8.19	8.19	9	0.4	9	0.4
B	53	9	9	9.11	9.28	1755607	613.4	2977	2.5
C	20	1	1	9.45	9.50	131262	183.3	5208	3.5
D	11	3	3	18.27	18.55	2433382	1078.0	5298	132.1
avg/sum	111	13	13	9.86	9.97	1103062	432.8	2781	12.7

15

(a) Without stability constraints   (b) With longitudinal stability constraints

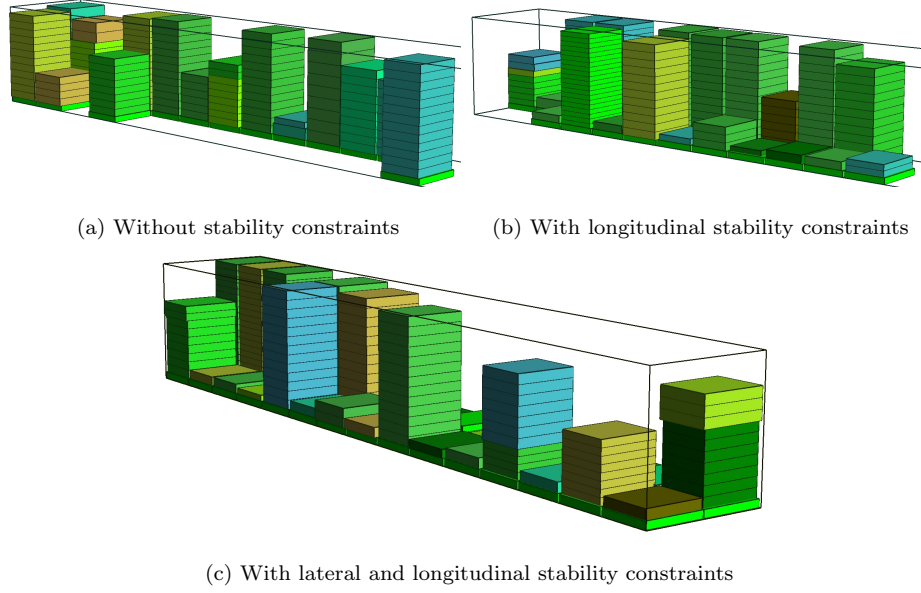(c) With lateral and longitudinal stability constraints

Figure 7: Solution without and with stability constraints

6.2. Pallet heights

There are several ways in which the difference between the heights of the pallets can be addressed. In this section we consider three alternatives and show the results obtained by each of them in terms of the solution difficulty of the resulting model and the number of pallets obtained.

- *Limiting the difference in height between consecutive pallets*

  A direct way of addressing the question is to limit the difference in height between consecutive adjacent pallets in the same row or column. We define three new parameters:

  – $\lambda_{speed-up}$, the maximum height difference between pallets in positions i and $i + 1$,

  – λ_{brake}, the maximum height difference between pallets in positions $i + 1$ and i;

  – λ_{side}, the maximum height difference between two pallets in the same row.

  By looking at Figure 3, we can see that $\lambda_{speed-up}$ aims at controlling the height difference between pallets 2 and 3, to prevent pallet 2 from tipping over pallet 3 when the truck accelerates, and λ_{brake} controls the difference between pallets 4 and 3 to prevent pallet 4 from tipping over pallet 3 when the truck brakes.

The following constraints are added to the model:

$$\sum_{j \in J} h_j x_{kij} - \sum_{j \in J} h_j x_{k,i+1,j} \leq \lambda_{speed-up} + H b_{ki} \qquad k \in K, i \in I \setminus \{|I|/2, |I|\} \tag{32}$$

$$\sum_{j \in J} h_j x_{k,i+1,j} - \sum_{j \in J} h_j x_{kij} \leq \lambda_{brake} + H f_{k,i+1} \qquad k \in K, i \in I \setminus \{|I|/2, |I|\} \tag{33}$$

$$\sum_{j \in J} h_j x_{kij} - \sum_{j \in J} h_j x_{k,i+|I|/2,j} \leq \lambda_{side} + H s_{ki} \qquad k \in K, i \in \{1, ..., |I|/2\} \tag{34}$$

$$\sum_{j \in J} h_j x_{k,i+|I|/2,j} - \sum_{j \in J} h_j x_{kij} \leq \lambda_{side} + H s_{ki} \qquad k \in K, i \in \{1, ..., |I|/2\} \tag{35}$$

Constraints (32)–(35) limit the difference between consecutive pallets in all directions, except for the cases in which there is an empty space, identified by variables f_{ki}, b_{ki}, and s_{ki}. As the heights of pallet bases are the same at all positions, it is not necessary to include them when calculating the difference in height between two pallets.

The results obtained by the resulting model are shown in Table 4. The parameters chosen for this run were $\lambda_{speed-up} = \lambda_{brake} = \lambda_{side} = \frac{H}{2}$. It can be observed that adding these constraints makes the model harder, increasing the number of instances not solved to optimality and including six large instances for which not even a feasible integer solution was found within the time limit. In these cases, the solution provided is the one obtained by the heuristic algorithm described in Section 4. This situation in which not even a feasible solution was found within the time limit also appeared in the other models in this section. In order to reduce these cases to a minimum, the CPLEX option of putting emphasis on feasibility was activated. In Figure 8 the effect of constraints (32)–(35) can be observed. They control the difference in height between adjacent pallets, but can produce a staircase effect.

Table 4: Computational results when limiting the height difference of consecutive pallets (model (2)–(31)+(32)–(35))

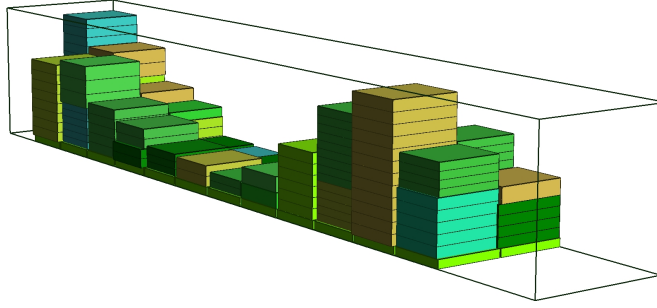| class	inst.	missed	gap	L	U	nodes	sec	nodes$_{opt}$	sec$_{opt}$
A	27	0	0	8.19	8.19	108	1.0	108	1.0
B	53	10	10	9.13	9.32	1152683	658.6	23223	42.9
C	20	2	3	9.40	9.55	46748	114.2	51943	114.2
D	11	6	10	18.00	18.91	1904291	1365.5	719	10.2
avg/sum	111	18	23	9.83	10.04	747543	449.3	20861	42.8

- *Uniform height of the pallets*

Figure 8: Limiting the difference between consecutive pallets

Another way of avoiding consecutive pallets with an excessive difference in height is to impose an overall balance between all the pallets in the solution or among the pallets in each truck. This can be done by invoking a continuous variable γ, changing the objective function from (2) to (36), and adding a new constraint (37) to the model with dynamic stability. In this way, the maximum pallet height is minimized and therefore all pallets have a similar height, although very short pallets are allowed.

$$\min \quad \sum_{k \in K} H y_k + \gamma \tag{36}$$

$$\sum_{j \in J} h_j x_{kij} + h^p z_{ki} \leq \gamma \qquad k \in K, i \in I \tag{37}$$

The results of this configuration appear in Table 5. Although the objective function has been changed, in order to compare with the other tables the *gap* column still shows the difference in the number of trucks between upper and lower bounds. There are 16 instances for which the number of trucks in the best solution obtained did not match the lower bound, as reported in the table, and for one of them no integer feasible solution was found. In addition, the objective function contains a continuous variable γ and its optimal value could only be found in 12 out of 111 instances, thus producing long average running times. The effect of this change in the model can be observed in Figure 9. The heights of the pallets are quite similar, but the balance is sometimes obtained by making short pallets and therefore wasting much of the truck's volume.

- *Imposing a minimum pallet height*

  The two alternatives just described try to control the difference in height between pallets, but they do not control the number of pallets in the solution. Sometimes, the constraints introduced

Table 5: Computational results when imposing uniform pallets heights (model (3)–(31)+(36)–(37))

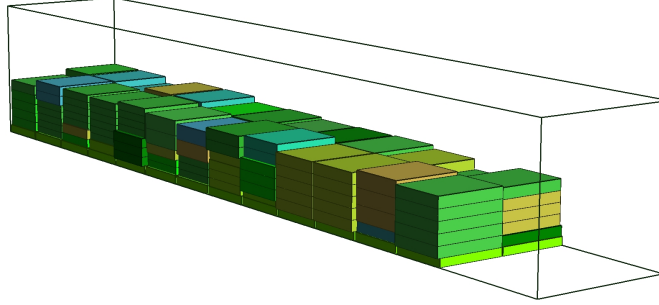| class	inst.	missed	gap	L	U	nodes	sec	nodes$_{opt}$	sec$_{opt}$
A	27	0	0	8.19	8.19	1109117	3466.8	1109117	3466.8
B	53	9	9	9.08	9.25	1342454	3396.4	1095052	3354.7
C	20	1	1	9.45	9.50	1027569	2701.5	1081432	2654.2
D	11	6	9	18.18	19.00	6607229	3273.0	650374	2880.4
avg/sum	111	16	19	11.22	11.48	1750695	3276.1	1072921	3221.5

Figure 9: Balancing the height of the pallets

for control produce the undesired effect of increasing the number of pallets, building some very short pallets that are stable but very inefficient in handling and transporting products. An alternative is to impose a minimum height for all the pallets, for instance $H/2$. If all pallets are taller than $H/2$, the differences in height cannot be very large and, at the same time, no short pallets are allowed, thus reducing the total number of pallets.

To impose this condition, a single new constraint must be added to the model:

$$\sum_{j \in J} h_j x_{kij} \geq (H/2) z_{ki} \qquad\qquad k \in K, i \in I \qquad\qquad (38)$$

The results, shown in Table 6, are slightly better than those in Table 5, both in terms of instances non-optimally solved, 9, and instances for which a feasible solution is not found, 4. It is worth noting that there is an instance ($D31$) in which an odd number of layers of a unique product has to be sent, but, in order to satisfy (38), pallets should be composed of at least two layers. In this quite singular case, the model cannot produce a feasible solution. The solution reported in Table 6 for this case was obtained by the constructive heuristic, and contains one pallet with just one layer whose height is lower than $H/2$. In Figure 10, the effect

of constraints (38) can be observed. Each truck contains a set of tall pallets placed without gaps. If the products are heavy, the weight limits can be reached with a number of pallets lower than the number of allowed positions in the truck, as can be seen in the figure.

Table 6: Computational results when imposing a minimum pallet height (model (2)–(31)+(38))

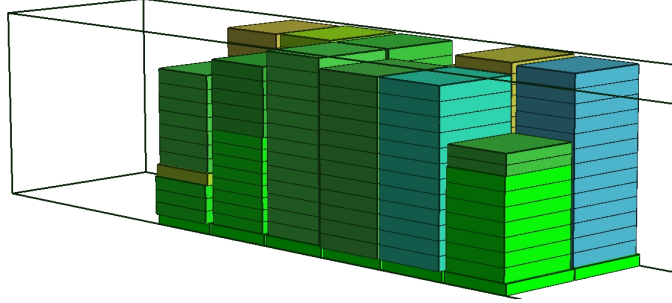| class	inst.	missed	gap	L	U	nodes	sec	nodes$_{opt}$	sec$_{opt}$
A	27	0	0	8.19	8.19	253	5.1	253	5.1
B	53	1	1	9.00	9.02	18375	107.5	1178	40.3
C	20	0	0	9.25	9.25	1819	187.4	1819	187.4
D	11	8	13	17.91	19.09	44702	2060.3	787	7.0
avg/sum	111	9	14	9.73	9.86	13593	224.3	1047	58.9

Figure 10: Imposing a minimum height for the pallets

A possible way of introducing some flexibility into the last model is to allow at most one pallet shorter than $H/2$ in every truck. This can be done by defining a new variable:

$$u_{ki} = \begin{cases} 1, & \text{if the pallet at position } i \text{ of truck } k \text{ is shorter than } H/2 \\ 0, & \text{otherwise} \end{cases}$$

and replacing constraints (38) with:

$$\sum_{j \in J} h_j x_{kij} \geq (H/2) z_{ki} - H u_{ki} \qquad k \in K, i \in I \qquad (39)$$

$$\sum_{i \in I} u_{ki} \leq 1 \qquad k \in K \qquad (40)$$

thus ensuring that at most one pallet per truck can be shorter than $H/2$.

The addition of these new variables and constraints makes the model more difficult to solve, as can be observed in Table 7. Instance $D31$ is now solved, but for other two instances no integer solution is found within the time limit, and the average computational effort is doubled.

Table 7: Computational results when imposing minimum height on all pallets but one (model (2)–(31)+(39)–(40))

class	inst.	missed	gap	L	U	nodes	sec	nodes$_{opt}$	sec$_{opt}$
A	27	0	0	8.19	8.19	966	23.7	966	23.7
B	53	8	8	9.09	9.25	233809	581.1	2983	111.5
C	20	0	0	9.45	9.45	1866	286.6	1866	286.6
D	11	7	12	18.00	19.09	43249	1546.7	1897	6.5
avg/sum	111	15	20	9.82	10.00	116496	447.3	2138	118.9

The three alternatives considered in this section are compared in Table 8. For each model, including the initial one, the table shows the number of instances optimally and non-optimally solved, the number of instances for which a feasible solution was not found, the sum of the lower bounds, the total number of trucks corresponding to the sum of the upper bounds, the total number of pallets, and the value of metric $M2$ (percentage of pallets not supported by other pallets or by the walls of the truck on at least three sides).

Table 8: Comparing alternative models for controlling height differences between pallets

Model	Missed			LB	Trucks	Pallets	M2 (%)
	Optimal	Non-opt	No Sol				
Initial model (Sec. 5, Table 2)	106	5	0	1094	1099	18459	31.0
Compactness (Sec. 6.1, Table 3)	98	13	0	1094	1107	29949	0.6
Limiting difference (Sec 6.2.a, Table 4)	93	12	6	1091	1114	26573	0.3
Uniform height (Sec. 6.2.b, Table 5)	95	15	1	1091	1110	29488	0.7
Minimum height (Sec. 6.2.c, Table 6)	98	9	4	1091	1108	18153	2.8
One short pallet (Sec 6.2.d, Table 7)	96	10	5	1090	1110	18204	2.6

The results in Table 8 show that imposing stability constraints only increases the number of trucks needed in a few instances. Models are harder to solve but can still produce optimal solutions for most instances. Most of the solutions found that are not proven to be optimal are only one truck away from the lower bound. For the very few cases in which models did not provide a feasible output, the solution provided by the heuristic can be considered acceptable.

Among the different alternatives that we tested, the one imposing a minimum pallet height seems to be the best, as it dramatically reduces the number of pallets built, at the expense of just a minor increase in M2. Indeed, for the initial model, in which stability was not included, M2 is very high, indicating very unstable loads. For all the other models, in contrast, M2 is very low. Considering, for instance, the model in which a minimum pallet height is imposed, the total number of pallets

not laterally supported on at least three sides is 506 in the 1108 trucks, corresponding to the cases in which a truck contains an odd number of pallets.

7. Middle positions

The grid proposed in Section 3 fixes the positions in which pallets can be placed on the truck floor. It is determined by dividing the truck width into as many columns as the dimensions of pallet and truck allow, but that may not be adequate in some cases, especially if there are very heavy products. A simple example would be locating a pallet of very large weight, say M, in a truck of capacity $Q = M$. The truck could support the pallet, but the pallet could not be placed in any of the positions defined by the grid, because this would cause the load to be too skewed to one side of the truck and hence would not satisfy the center of gravity constraint. The right position for such a pallet would be in the center of the truck floor. In order to make this option possible, we decided to extend our grid and include a new central column of "middle positions", overlapping with the other two existing columns, as shown in Figure 11.
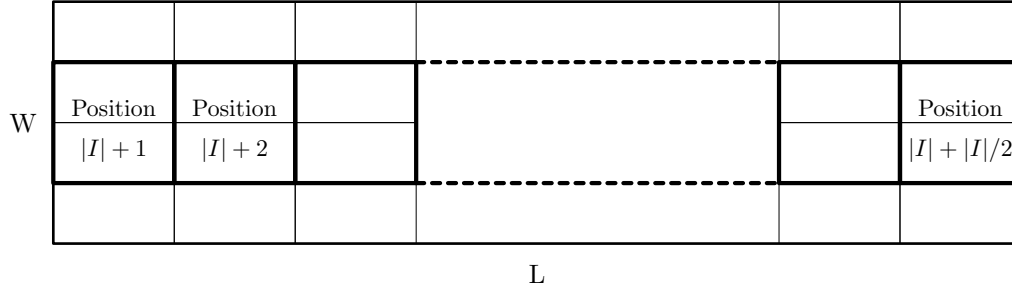

Figure 11: A grid with middle positions

Now, we consider a grid with three columns: Column 1, having positions from 1 to $|I|/2$; Column 2, with positions from $|I|/2+1$ to $|I|$; and a Middle column, with positions from $|I|+1$ to $|I|+|I|/2$.

If there is a pallet in the middle column, the positions in the same row in the other two columns cannot be occupied. This can be ensured by the following constraints:

$$z_{ki} + z_{k,i+|I|} \leq 1 \qquad k \in K, i \in \{1, ..., |I|/2\} \tag{41}$$

$$z_{ki} + z_{k,i+|I|/2} \leq 1 \qquad k \in K, i \in \{|I|/2, ..., |I|\} \tag{42}$$

22

The lateral and longitudinal stability constraints must also be adapted to the new positions. Constraints (43)–(50) adapt constraints (19)–(27). Constraints (51) ensure that if there is only one pallet in a row, it is in the middle column.

$$f_{ki} \geq (z_{ki} + z_{k,i+|I|}) - (z_{k,i-1} + z_{k,i-1+|I|}) \qquad k \in K, i \in \{2, ..., |I|/2\} \qquad (43)$$

$$f_{ki} \geq (z_{ki} + z_{k,i+|I|/2}) - (z_{k,i-1} + z_{k,i-1+|I|/2}) \qquad k \in K, i \in \{|I|/2+2, ..., |I|\} \qquad (44)$$

$$f_{k1} \geq (z_{k1} + z_{k,1+|I|}) \qquad k \in K, \qquad (45)$$

$$f_{k,|I|/2+1} \geq (z_{k,|I|/2+1} + z_{k,|I|+1}) \qquad k \in K \qquad (46)$$

$$b_{ki} \geq (z_{ki} + z_{k,i+|I|}) - (z_{k,i+1} + z_{k,i+1+|I|}) \qquad k \in K, i \in \{1, ..., |I|/2-1\} \qquad (47)$$

$$b_{ki} \geq (z_{ki} + z_{k,i+|I|/2}) - (z_{k,i+1} + z_{k,i+1+|I|/2}) \qquad k \in K, i \in \{|I|/2+1, ..., |I|-1\} \qquad (48)$$

$$b_{k,|I|/2} \geq z_{k,|I|/2} + z_{k,|I|/2+|I|} \qquad k \in K \qquad (49)$$

$$b_{k,|I|} \geq z_{k,|I|} + z_{k,|I|+|I|/2} \qquad k \in K \qquad (50)$$

$$\sum_{1 \leq i \leq |I|/2} s_{ki} = 0 \qquad k \in K \qquad (51)$$

The results of the initial model of Section 5 modified by including these middle positions appear in Table 9. All instances but eleven are solved to optimality. For instances in which optimality is not proven, the gaps are always of one truck. Middle positions are occupied mainly when heavy products are involved, but also in trucks with an odd number of pallets in which one pallet tends to occupy a middle position. Figure 12 shows the solution obtained using middle positions in an instance with 20 layers each weighing 1000 kg.

Table 9: Computational results of the model with middle positions (model (2)–(18)+(41)–(50))

class	# inst.	missed	gap	L	U	nodes	sec	nodes$_{opt}$	sec$_{opt}$
A	27	0	0	8.19	8.19	0	0.4	0	0.4
B	53	8	8	9.11	9.26	598685	544.7	125	1.4
C	20	0	0	9.45	9.45	38	2.1	38	2.1
D	11	3	3	18.27	18.55	1262418	1017.8	3275	49.3
avg/sum	111	11	11	9.89	9.95	410970	361.4	326	5.1

8. Demand over time

In some practical situations, the demand that a distribution company receives from a customer may involve different delivery dates. Some products have to be received on day 1, some others
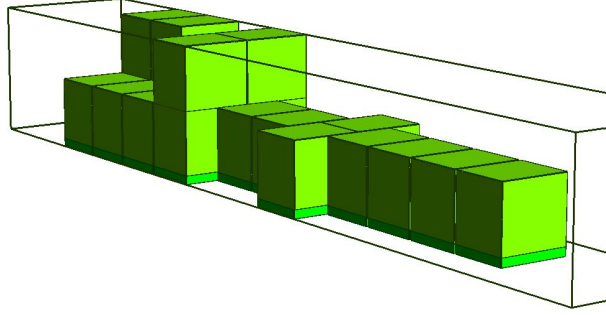
23

Figure 12: Solution for an instance with 20 layers of 1000 kg.

on day 2, and so on, within a planning horizon that usually corresponds to the working days of a week. In our study case, products can be sent before their delivery date, but not later. Therefore, a solution for the whole shipping problem, that is, for the entire time horizon, can be better than the combination of the solutions of each single day, because unused space in the trucks sent on a certain day can be used to ship in advance products required later on.

The delivery dates impose the order in which products are loaded into trucks: products for day 1 must be in the first trucks (1 to K_1), some products for day 2 may fill up some of these trucks and the remaining products for day 2 must be in the next trucks ($K_1 + 1$ to K_2), and so on for the remaining periods of the planning horizon. Using this type of solution, the company can send the first K_1 trucks on the first day. On the second day, it can send trucks from $K_1 + 1$ to K_2, but, if necessary, it can adjust its data to accommodate last minute orders or cancelations and solve the problem again for the remaining days. Besides being able to adjust to changes, the company can use the trucks in a more regular way over the days of the planning horizon.

We have developed two ways of taking account of the delivery dates:

- *Solving a single model with time constraints*

  We introduce the delivery dates into the model described in Section 5. Let D be the time horizon and recall that n_j is the total demand for product j. We denote the number of layers of product j with delivery date d by n_{dj}, for $d \in D$, so that $\sum_{d \in D} n_{dj} = n_j$. Let $N_d = \sum_j n_{dj}$ be the number of layers to be delivered on day d, and P_j be the set of products whose demand must be partially or entirely delivered on day d. The original variables x_{kij} are replaced by:

  x_{dkij} = number of layers of product j packed in position i of truck k and delivered on day d,

24

and a new set of variables is needed:

$$
y_{dk} = \begin{cases} 1, & \text{if truck } k \text{ contains products with delivery date } d \\ 0, & \text{otherwise} \end{cases}
$$

The modified model is:

$$
\text{(M-days)} \quad \min \sum_{k \in K} y_k \tag{52}
$$

$$
\sum_{k \in K} \sum_{i \in I} x_{dkij} \geq n_{dj} \qquad\qquad j \in J, d \in D \tag{53}
$$

$$
\sum_{d \in D} \sum_{j \in J} h_j x_{dkij} + h^p z_{ki} \leq H' y_k \qquad\qquad k \in K, i \in I \tag{54}
$$

$$
\sum_{i \in I} (q^p z_{ki} + \sum_{d \in D} \sum_{j \in J} q_j x_{dkij}) \leq Q y_k \qquad\qquad k \in K \tag{55}
$$

$$
\sum_{i \in I} (q^p z_{ki} + \sum_{d \in D} \sum_{j \in J} q_j x_{dkij})(\delta_2 - p_i^x) \leq Q_1(\delta_2 - \delta_1) y_k \qquad\qquad k \in K \tag{56}
$$

$$
\sum_{i \in I} (q^p z_{ki} + \sum_{d \in D} \sum_{j \in J} q_j x_{dkij})(p_i^x - \delta_1) \leq Q_2(\delta_2 - \delta_1) y_k \qquad\qquad k \in K \tag{57}
$$

$$
\sum_{d \in D} \sum_{j \in J} h_j x_{dkij} \leq (H - h^p) z_{ki} \qquad\qquad k \in K, i \in I \tag{58}
$$

$$
z_{ki} \leq \sum_{d \in D} \sum_{j \in J} x_{dkij} \qquad\qquad k \in K, i \in I \tag{59}
$$

$$
\sum_{i \in I} \sum_{j \in P_j} x_{dkij} \leq N_d y_{kd} \qquad\qquad k \in K, d \in D \tag{60}
$$

$$
\sum_{k > k'} \sum_{d < d'} y_{kd} \leq (K - k')(1 - y_{k'd'}) \qquad\qquad k' \in K, d' \in D, d' > 1 \tag{61}
$$

$$
Q^e G_x + \sum_{i \in I} p_i^x q^p z_{ki} + \sum_{d \in D} \sum_{i \in I} \sum_{j \in J} p_i^x q_j x_{dkij} \leq (\sum_{i \in I} (q^p z_{ki} + \sum_{d \in D} \sum_{i \in I} \sum_{j \in J} q_j x_{dkij} + Q^e)(G_x + \tau_1^x)
$$
$$
k \in K \tag{62}
$$

$$
Q^e G_x + \sum_{i \in I} p_i^x q^p z_{ki} + \sum_{d \in D} \sum_{i \in I} \sum_{j \in J} p_i^x q_j x_{dkij} \geq (\sum_{i \in I} (q^p z_{ki} + \sum_{d \in D} \sum_{i \in I} \sum_{j \in J} q_j x_{dkij} + Q^e)(G_x - \tau_2^x)
$$
$$
k \in K \tag{63}
$$

$$
Q^e G_y + \sum_{i \in I} p_i^y q^p z_{ki} + \sum_{d \in D} \sum_{i \in I} \sum_{j \in J} p_i^y q_j x_{dkij} \leq (\sum_{i \in I} (q^p z_{ki} + \sum_{d \in D} \sum_{i \in I} \sum_{j \in J} q_j x_{dkij} + Q^e)(G_y + \tau_1^y)
$$
$$
k \in K \tag{64}
$$

$$
Q^e G_y + \sum_{i \in I} p_i^y q^p z_{ki} + \sum_{d \in D} \sum_{i \in I} \sum_{j \in J} p_i^y q_j x_{dkij} \geq (\sum_{i \in I} (q^p z_{ki} + \sum_{d \in D} \sum_{i \in I} \sum_{j \in J} q_j x_{dkij} + Q^e)(G_y - \tau_2^y)
$$
$$
k \in K \tag{65}
$$

$$
y_k \geq y_{k+1} \qquad\qquad k \in K : k < |K| \tag{66}
$$

$$y_k = 1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k \le LB \quad (67)$$

$$\sum_{d \in D} y_{kd} \le |D| y_k \qquad\qquad\qquad\qquad\qquad\qquad\qquad k \in K \quad (68)$$

$$x_{dkij} \ge 0, \text{integer} \qquad\qquad\qquad\qquad d \in D, k \in K, i \in I, j \in J \quad (69)$$

$$y_k \in \{0,1\} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k \in K \quad (70)$$

$$y_{dk} \in \{0,1\} \qquad\qquad\qquad\qquad\qquad\qquad\qquad d \in D, k \in K \quad (71)$$

$$z_{ki} \in \{0,1\} \qquad\qquad\qquad\qquad\qquad\qquad\qquad k \in K, i \in I \quad (72)$$

The following main differences from the previous model can be noted. Constraints (60) link variables x_{dkij} with variables y_{dk}, so that if there are some products in truck k with delivery date d, then $y_{dk} = 1$. Constraints (61) impose an order of the products loaded into the trucks. Products with different delivery dates can be together in a truck, but if in truck k' there are products with delivery date d', in the following trucks $k > k'$ there cannot be products with earlier delivery dates, $d < d'$. For example, if products with delivery date $d = 1$ need more than two trucks, but they do not fill up a third truck, constraints (61) impose the condition that trucks 1 and 2 will only contain products with delivery date 1, and in truck 3 there will be the remaining products from day 1 plus some products from day 2. A truck with products with different delivery dates will only appear when the remaining products for a delivery date do not completely fill the truck and products for the next delivery date are used to fill it up. Then only trucks containing products for this delivery date have to be sent each day, and the part of the solution corresponding to the following days can be adjusted to accommodate last-minute changes. Constraints (68) link the new variables y_{kd} with the original variables y_k.

- *Solving a model for each day in the planning horizon*

  An alternative, more flexible, way of taking the delivery dates into account consists in solving a sequence of models: the first for products with delivery date $d = 1$, the second for products with delivery dates $d \in \{1, 2\}$, and so on. Each model, except the first, takes the solution of the previous model as a reference. In more detail:

  - *Model 1*: We only consider $D = \{1\}$, the products in P_1, and variables x_{1kij} and y_{1k}. Constraints (61) are deactivated. Let K_1 be the optimal solution to this model. These K_1 trucks will be sent on day 1 because they are needed to fulfill the demand for this day.

– *Model 2*: We now consider $D = \{1, 2\}$ and all the corresponding sets, parameters, variables, and constraints. Constraints (61) are still deactivated, but in order to ensure that all products with delivery date $d = 1$ are sent on day 1, we add

$$\sum_{k > K_1} y_{1k} = 0 \tag{73}$$

so all products for day 1 are loaded into the first K_1 trucks. Note that this alternative is more flexible, because it accepts solutions in which products with delivery date $d = 2$ are loaded into any of the trucks $k = 1, \ldots, K_1$, and not only in truck K_1 as in the previous single model. Suppose the solution of model 2 requires K_2 trucks, then $K_2 - K_1$ trucks are sent on day 2 to cover the demands for that day.

The procedure outlined for Model 2 is repeated for all the following models by appropriately adjusting constraints (73). In our tests this involves solving $|D| = 3$ models for each instance.

The results of these two alternative ways of addressing the demand over time are reported in Table 10. Both alternatives are slightly harder than the initial model, as was to be expected because of the increased numbers of variables and constraints. The second alternative is more flexible and could in theory have produced solutions with fewer trucks, but as it is harder to solve, its theoretical advantage is not achieved in practice, as shown by the results.

Table 10: Comparing alternative models for the case of demands over time

Model	Optimal	Non-opt	No Sol	LB	Trucks
Initial model	106	5	0	1094	1099
Solving a single model with time constraints	103	6	2	1093	1103
Solving a model for each day in the planning horizon	100	10	1	1093	1106

9. Conclusions and future work

The increasing need to produce high-quality solutions for complex loading problems, involving many containers or trucks and considering many realistic constraints, can be addressed in different ways. In this paper we have chosen to use mathematical models and explore the possibility of first modeling the relevant constraints and then solving the models in reasonable times to produce

optimal or quasi-optimal solutions. Starting from an initial model, already including constraints related to total weight, maximum weight supported by the axles, and position of the center of gravity, we have studied in detail the dynamic stability conditions required for a loading plan to be useful in practice. Several models have been proposed and the results show that optimal solutions are obtained in most cases. For the instances for which optimality is not proven, the distance to a known lower bound is at most one truck, and for the very few cases in which a feasible solution was not found by a model, the solution built by a heuristic algorithm was provided. Although heuristic and metaheuristic procedures can be faster, the quality obtained by the solution of the mathematical models is difficult to match and therefore this approach appears to be useful for solving complex multi container problems.

Some interesting extensions of our models have also been considered, namely, the possibility of using positions on the longitudinal axis of the truck, an especially useful option in the case of heavy loads, and the case of splitting demands over a time horizon according to their expected delivery dates. A future line of research concerns the study of other interesting extensions that appear frequently in practice. An example of a possible extension appears when considering the truck with the smallest load in a solution. Usually, managers do not want to send a truck containing just a few pallets, having a low percentage of its volume or weight filled. To deal with this, two options can be considered. If the truck is not sent, then constraints and/or penalties in the objective function could be considered to find the best subset of shipped products. If the truck is sent, then similar reasoning could be used to find the best subset of products whose delivery could be brought forward. The models that we developed could be adapted to address both options.

Acknowledgements

References

[1] A. Bortfeldt, G. Wäscher, Constraints in container loading. A state of the art review, European Journal of Operational Research 229 (1) (2013) 1 – 20.

[2] X. Zhao, J. Bennell, T. Betkas, K. Dowsland, A comparative review of 3D container loading algorithms, International Transactions in Operational Research 23 (2016) 287–320.

[3] M. Alonso, R. Alvarez-Valdes, M. Iori, F. Parreño, J.M.Tamarit, Mathematical models for multicontainer loading problems, Omega 66 (2017) 106–117.

[4] E. Bischoff, M. Ratcliff, Issues in the development of approaches to container loading, Omega 23 (4) (1995) 377 – 390.

[5] A. Bortfeldt, H. Gehring, D. Mack, A parallel tabu search algorithm for solving the container loading problem, Parallel Computing 29 (5) (2003) 641 – 662.

[6] J. Egeblad, C. Garavelli, S. Lisi, D. Pisinger, Heuristics for container loading of furniture, European Journal of Operational Research 200 (3) (2010) 881 – 892.

[7] A. Lim, H. Ma, C. Qiu, W. Zhu, The single container loading problem with axle weight constraints, International Journal of Production Economics 144 (1) (2013) 358 – 369.

[8] H. Pollaris, K. Braekers, A. Caris, G. Janssens, S. Limbourg, Capacitated vehicle routing problem with sequence-based pallet loading and axle weight constraints, EURO Journal on Transportation and Logistics 5 (2016) 231–255.

[9] A. Bortfeldt, H. Gehring, A hybrid genetic algorithm for the container loading problem, European Journal of Operational Research 131 (1) (2001) 143 – 161.

[10] F. Parreño, R. Alvarez-Valdes, J. Tamarit, J. Oliveira, A maximal-space algorithm for the container loading problem, INFORMS Journal on Computing 20 (3) (2008) 412–422.

[11] M. Iori, S. Martello, Routing problems with loading constraints, TOP 18 (1) (2010) 4 – 27.

[12] L. Junqueira, R. Morabito, D. Yamashita, Three-dimensional container loading models with cargo stability and load bearing constraints, Computers & Operations Research 39 (1) (2012) 74 – 85.

[13] J. Terno, G. Scheithauer, U. Sommerweiss, J. Riehme, An efficient approach for the multi-pallet loading problem, European Journal of Operational Research 123 (2) (2000) 372 – 381.

[14] M.Alonso, R.Alvarez-Valdes, F.Parreño, J.Tamarit, A reactive GRASP algorithm for the container loading problem with load-bearing constraints, European journal of industrial engeneering 8 (1) (2014) 669–694.

[15] T. Toffolo, E. Esprit, T. Wauters, G. Vanden Berghe, A two-dimensional heuristic decomposition approach to a three-dimensional multiple container problem, European Journal of Operational Research 257 (2) (2017) 526–538.

[16] J. Correcher, M. Alonso, F. Parreño, R. Alvarez-Valdes, Solving a large multicontainer loading problem in the car manufacturing industry, Computers & Operations Research 82 (1) (2017) 139–152.

[17] A. Ramos, J. Oliveira, J. Gonçalves, M. Lopes, A container loading algorithm wiht static mechanical equilibrium stability constraints, Transportation Research Part B 91 (2016) 565–581.

[18] O. Araujo, V. Armentano, A multi-start random constructive heuristic for the container loading problem, Pesquisa Operacional 27 (2) (2007) 311 – 331.

[19] T. Fanslau, A. Bortfeldt, A tree search algorithm for solving the container loading problem, INFORMS Journal on Computing 22 (2) (2010) 222–235.

[20] Z. Jin, K. Ohno, J. Du, An efficient approach for the three-dimensional container packing problem with practical constraints, Asia-Pacific Journal of Operational Research 21 (3) (2004) 279–295.

[21] A. G. Ramos, J. F. Oliveira, J. F. Gonçalves, M. P. Lopes, Dynamic stability metrics for the container loading problem, Transportation Research Part C: Emerging Technologies 60 (2015) 480 – 497.

[22] G. Wäscher, H. Haußner, H. Schumann, An improved typology of cutting and packing problems, European Journal of Operational Research 183 (3) (2007) 1109 – 1130.

[23] R. Morabito, S. Morales, J. Widmer, Loading optimization of palletized products on trucks, Transportation Research Part E: Logistics and Transportation Review 36 (4) (2000) 285 – 296.

[24] R. Morabito, S. Morales, A simple and effective recursive procedure for the manufacturer's pallet loading problem, Journal of the Operational Research Society 49 (8) (1998) 819 – 828.

[25] S. Takahara, Loading problem in multiple containers and pallets using strategic search method, in: Modeling Decisions for Artificial Intelligence, Vol. 3558 of Lecture Notes in Computer Science, Springer Berlin Heidelberg, 2005, pp. 448 – 456.

[26] K. Doerner, G. Fuellerer, M. Gronalt, R. Hartl, M. Iori, Metaheuristics for the vehicle routing problem with loading constraints, Networks 49 (4) (2007) 294 – 307.

[27] A. Moura, A. Bortfeldt, A two-stage paking problem procedure, International Transactions in Operational Research 24 (2017) 43–58.

[28] A. Moura, J. Oliveira, A GRASP approach to the container-loading problem, IEEE Intelligent Systems 20 (4) (2005) 50–57.

[29] L. Sheng, Z. Hongxia, D. Xisong, C. Changjian, A heuristic algorithm for container loading of pallets with infill boxes, European Journal of Operational Research 252 (2016) 728–736.

[30] ORTEC, company, `www.ortec.com`, [Online; accessed 20-May-2017] (2017).