The Traveling Salesperson Problem with Forbidden Neighborhoods on Regular 3D Grids

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Abstract

We study the traveling salesperson problem with forbidden neighborhoods (TSPFN) on regular three-dimensional (3D) grids. The TSPFN asks for a shortest tour over all grid points such that successive points along a tour have at least some given distance. We present optimal solutions and explicit construction schemes for the Euclidean TSP and the TSPFN, where edges of length one are forbidden, on regular 3D grids.

Keywords: Traveling salesman problem, forbidden neighborhood.

1 Introduction

In this paper we study the Traveling Salesperson Problem with Forbidden Neighborhoods (TSPFN) on regular three-dimensional (3D) grids. The task of the TSPFN is to determine a shortest Hamiltonian cycle over given points in the Euclidean plane, such that the distance between consecutive points along the tour is larger than some given $r \in \mathbb{R}_+$.

The TSPFN was originally motivated by an application in laser beam melting, where a workpiece is built in several layers. By excluding the heating of positions that are too close during this process, one hopes to reduce the internal stresses of the workpiece, see [3] and the references therein for further details. Furthermore, the TSPFN has connections to the maximum scatter TSP (msTSP), where the length z of a shortest edge in a tour is maximized. Clearly for optimal z, there exists a TSPFN tour for all $r < z$. For details on the msTSP see, e. g., [1] for the general case and [4] for a version on 2D grids.

In this work we determine optimal TSPFN tours on regular 3D grids with arbitrary grid sizes for the smallest reasonable forbidden neighborhoods $r = 0$ and $r = 1$. We consider $m \times n \times \ell$ grids, where m is the number of rows, n is the number of columns and ℓ is the number of layers. So each cell/vertex of the grid is represented by three coordinates. For a visualization we refer to Fig. 1. In the 3D case we always assume $m, n, \ell \geq 2$. If $1 \in \{m, n, \ell\}$, we are in the 2D case of the TSPFN. Results for this case can be found in [3].

The TSPFN on grids can easily be described as an integer linear program (ILP). In comparison to the classic formulation of the TSP by Dantzig et al. [2], we additionally forbid connections that

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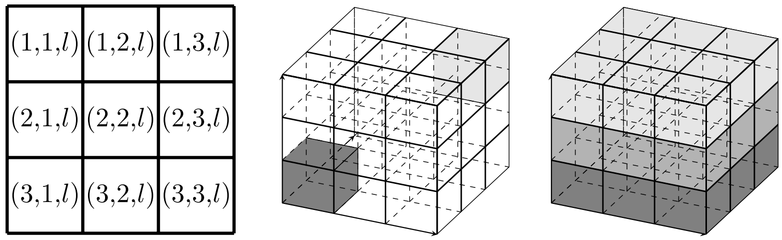


Figure 1: Visualizations of the $3 \times 3 \times 3$ grid. The left picture shows the numbering of layers $l \in \{1, 2, 3\}$. In the middle cube, the dark gray cell has the coordinates $(3, 1, 1)$ and the light gray one has the coordinates $(1, 3, 3)$. In the right cube, each layer has one color, where layers with larger numbers are lighter.

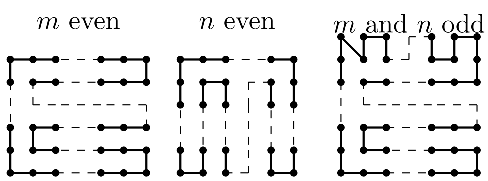


Figure 2: Optimal TSP tours on $m \times n$ grids for m even, n even and both m and n odd [3]. The two tours on the left are denoted as "rook tours".

are too short by setting the respective variables to zero, see [3] for the ILP formulation of the TSPFN in the 2D case.

2 Results for $r = 0$

If $r = 0$, then the TSPFN is equivalent to the Euclidean TSP. Since our constructions in the 3D case are an extension of the ones for the 2D case, first we recall some construction schemes of optimal TSP tours on regular 2D grids in Fig. 2.

Considering now the 3D case, we divide the grid cells into *odd* and *even vertices*, where a vertex is *odd (even)*, if the sum of its indices is odd (even). We denote the subgrid of the odd (even) vertices as o-grid (e-grid).

For $r = 0$, the shortest connections have length 1, running between an even and an odd vertex. Hence a trivial lower bound on the optimal tour length is $mn\ell$. If m, n, ℓ are odd, there is one more odd than even vertex. Thus in this case $mn\ell - 1 + \sqrt{2}$ is a lower bound. Next we show that these lower bounds are in fact the optimal lengths of TSPFN tours with $r = 0$ on regular 3D grids.

Theorem 1. *An optimal TSP tour on an $m \times n \times \ell$ grid has length $mn\ell$, if m or n or ℓ is even, and length $(mn\ell - 1) + \sqrt{2}$, if m, n, ℓ are odd.*

Proof. We prove this by presenting tour construction schemes, where the tour lengths equal the lower bounds given above. The construction always starts in $(1, 1, 1)$ and we go with steps of length 1 to $(1, 1, 2)$, $(1, 1, 3)$, ..., $(1, 1, \ell)$. When we reach Layer ℓ, we apply the appropriate construction schemes from the 2D case, see Fig. 2, of course not yet closing the tour. We distinguish two cases:

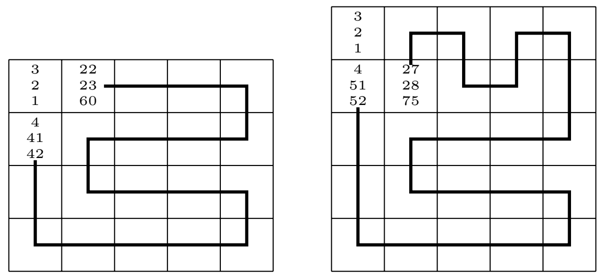


Figure 3: Optimal TSP tours on the $4 \times 5 \times 3$ grid and the $5 \times 5 \times 3$ grid, respectively. The three numbers in the cells correspond to the three layers with Layer 1 at the bottom.

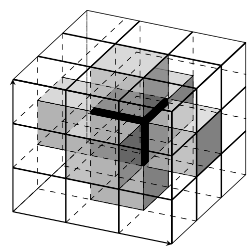
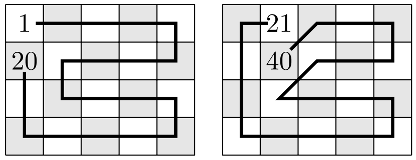


Figure 4: Illustration of the forbidden neighborhood for the TSPFN with $r = 1$ on regular 3D grids. The current grid cell is the black one in the middle, the forbidden cells are the gray ones around it and the cells that we are allowed to visit next are white.

- **m or n or ℓ is even:** W. l. o. g. we assume that m or n is even. Hence we can apply one of the rook tours that end in $(1, 2, \ell)$. Now we change the layer by taking a step of length 1 to $(1, 2, \ell - 1)$. Here we apply the open rook tour again, but in the other direction, such that it ends in $(2, 1, \ell - 1)$. We repeat using these construction schemes until we reach Layer 1. Depending on the parity of ℓ, the last rook tour ends in $(2, 1, 1)$ or $(1, 2, 1)$. Both cells have distance 1 to the start vertex $(1, 1, 1)$.
- **m, n, ℓ are odd:** We proceed in a similar way as above, applying the construction scheme of the 2D case with both m, n odd. The alternating start and end cells of these open rook tours are $(2, 1, i)$ and $(2, 2, i)$, $i \in \{1, \dots, \ell\}$. Finally, in Layer 1 we end in $(2, 2, 1)$ that has a distance of $\sqrt{2}$ to $(1, 1, 1)$.

To further clarify the construction schemes of optimal TSP tours on regular 3D grids, we depict some optimal tours on the $4 \times 5 \times 3$ and $5 \times 5 \times 3$ grids in Fig. 3.



1	21	3	38	5
23	9	34	7	36
19	33	11	31	13
25	17	27	15	29

22	2	39	4	37
20	40	8	35	6
24	10	32	12	30
18	26	16	28	14

Figure 5: Illustration of an optimal TSPFN tour with $r = 1$ on the $4 \times 5 \times 2$ grid. The tour consists of two Hamiltonian paths on the o-grid (left) and the e-grid (Picture 2) that are connected by steps of length $\sqrt{3}$. The two right pictures display the tour explicitly (Level 1 left).

3 Results for $r = 1$

For the TSPFN with $r = 1$ the shortest possible step has length $\sqrt{2}$, see Fig. 4 for a visualization of the corresponding forbidden neighborhood. A step of length $\sqrt{2}$ is either possible between two even or between two odd vertices. For determining a lower bound we assume the existence of Hamiltonian paths on the o-grid and the e-grid that consist only of steps of length $\sqrt{2}$. The shortest feasible step for connecting the paths on the subgrids has length $\sqrt{3}$. Hence $(mn\ell - 2)\sqrt{2} + 2\sqrt{3}$ is a lower bound for the length of optimal TSPFN tours with $r = 1$ on an $m \times n \times \ell$ grid. In the following we present construction schemes for tours whose lengths coincide with this lower bound. We start with $\ell = 2$, then we consider $\ell = 3$ and eventually we extend our construction schemes to an arbitrary number of layers.

Construction scheme for $\ell = 2$: We construct tours on each subgrid and connect them with steps of length $\sqrt{3}$. Putting two layers of a subgrid upon each other, we obtain a full 2D grid, where every step is allowed, because vertices belonging to the same subgrid have a distance of at least $\sqrt{2}$. Now we can again apply the construction schemes for $r = 0$ of the 2D case, this time for determining tours over two layers, where every step of length 1 in the original construction scheme corresponds to a step of length $\sqrt{2}$. Slightly adapting the 2D solution by using steps of length $\sqrt{2}$ within one layer, we can choose different start- and end-cells and derive a tour of the desired length. To further clarify this construction scheme we depict the optimal TSPFN tour with $r = 1$ on the $4 \times 5 \times 2$ grid in Fig. 5.

Construction scheme for $\ell = 3$: As above we put layers of the same subgrid upon each other so that they look like a full 2D grid and connect the two subgrids by steps of length 3. Note that in this case vertices of the 2D visualization that are in Layer 1 or 3 have to be visited twice and vertices that are in Layer 2 have to be visited once. Our construction scheme starts on the o-grid in $(1, 1, 1)$. Initially we visit all cells in the first two columns and continue with covering the columns pairwise until three columns are left. Then we go to $(1, n - 3, 1)$ and visit the first two remaining rows. Again we cover the rows pairwise and stop when five rows are left. It remains a 5×3 grid on which we apply an explicitly determined subpath. With a step of length $\sqrt{3}$ we change to the e-grid. The construction scheme on the e-grid is a slightly adapted and rotated (by 180 degrees) version of the construction of the o-grid. A detailed description of the construction scheme over three layers is given in Fig. 6. To further clarify this construction scheme we depict an optimal TSPFN tour with $r = 1$ on the $7 \times 5 \times 3$ grid in Fig. 7.

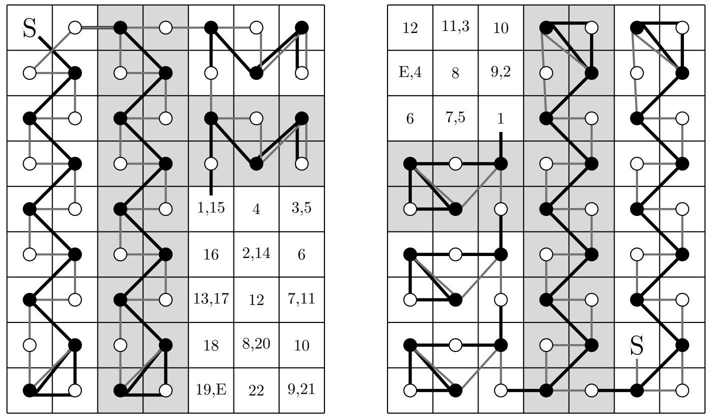


Figure 6: Illustration of drawing patterns for $\ell = 3$. The pictures show the construction scheme for the odd vertices (left) and the even vertices (right). For larger m, n the gray parts can be repeated. Black vertices correspond to Layers 1 and 3 and hence are visited twice. White vertices correspond to Layer 2 and hence are visited once. “S” and “E” indicate the start and end vertices.

1,20	21	22,29	27	24,26
19	2,18	30	23,28	25
3,16	17	31,45	34	33,35
15	4,14	46	32,44	36
5,12	13	43,47	42	37,41
11	6,10	48	38,50	40
7,9	8	49,53	52	39,51

104	103,95	102	68,65	67
105,96	100	101,94	64	69,66
98	99,97	93	70,63	62
90,87	91	92,85	60	71,61
88	89,86	84	72,59	58
81,78	82	83,76	54	73,57
79	80,77	75	74,55	56

Figure 7: Illustration of an optimal TSPFN tour with $r = 1$ on the $7 \times 5 \times 3$ grid. The odd (even) vertices are depicted in the left (right) picture. The gray cells belong to Layers 1 and 3 and hence are visited twice.

Construction scheme for an arbitrary ℓ: Finally in the proof of Theorem 2 we describe how the above construction schemes for $\ell = 2$ and $\ell = 3$ can be used to obtain a construction scheme for an arbitrary number of layers ℓ.

Theorem 2. *An optimal TSPFN tour on the $m \times n \times \ell$ grid with $r = 1$ has length $(mn\ell - 2)\sqrt{2} + 2\sqrt{3}$.*

Proof. We prove this by presenting an explicit construction scheme, where the length of the tours equals the lower bound derived above. The optimal solutions for the $3 \times 3 \times 3$ and the $3 \times 5 \times 3$ grids were derived explicitly by the ILP formulation discussed in the introduction. So let us consider the other cases.

First note that for all grid sizes there exists a construction scheme over two layers that visits all odd vertices such that $(1, 1, i)$ and $(1, 2, i + 1)$ for i odd are the start- and end-cells, see Fig. 8. We start in $(1, 1, 1)$, apply this construction scheme and hence terminate in $(1, 2, 2)$. Then we take a

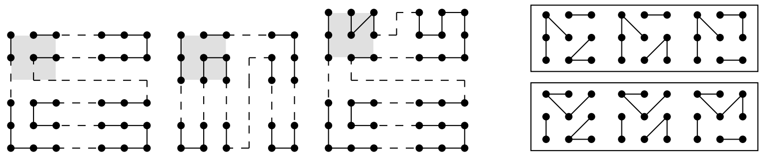


Figure 8: Construction schemes for tours on either the o-grid or the e-grid over two layers using only steps of length $\sqrt{2}$. The three left construction schemes are applied for subgrids with different parities, i. e. m even, n even or m and n odd (from left to right). For the e-grid over Layers 1 and 2 (Picture 4) as well as Layers $\ell-1$ and ℓ (Picture 5) slight adaptions are needed. The pictures show the interesting part of the upper 9 highlighted nodes, the remaining connections do not change.

step of length $\sqrt{2}$ to $(1,1,3)$ and apply again the same construction scheme. Now we distinguish two cases:

1. m **or** n **or** ℓ **is even:** We assume, w. l. o. g., that ℓ is even. Hence we can continue with the application of the construction scheme over two layers until we reach $(1,2,\ell)$. Now we have covered all odd vertices and need to take a step of length $\sqrt{3}$ to change to the e-grid. We go to $(2,1,\ell-1)$ and apply a slightly adapted construction scheme that is depicted in Fig. 8, Picture 5. Using this drawing pattern we visit all even vertices on the Layers ℓ and $\ell-1$ and terminate in $(1,2,\ell-1)$. For the even vertices on the remaining layers, except for Layers 1 and 2, we again iteratively apply the construction schemes illustrated in Pictures 1 to 3 of Fig. 8. Finally on Layers 1 and 2 we use a slightly adapted construction depicted in Fig. 8, Picture 4, in order to end in $(2,2,2)$. From $(2,2,2)$ we go to $(1,1,1)$ with a step of length $\sqrt{3}$.

2. m,n,ℓ **are odd:** We apply the construction scheme over two layers from Fig. 8 on the o-grid until we reach $(1,2,\ell-3)$. We make a step of length $\sqrt{2}$ to $(1,1,\ell-2)$ and then use the construction scheme from Fig. 6 on the last three layers and end in $(2,1,\ell-2)$. We make a step of length $\sqrt{2}$ to $(1,1,\ell-3)$. Finally, we visit the remaining even vertices as in Case 1.

It remains for future work to extend the presented results to larger values of r or to grids, where the cells are cuboids. Looking at the application in laser beam melting it is not only interesting to enforce some minimum distance between consecutive points in the tour, but also of points that are close in the tour so that, e. g., some areas of the workpiece can cool down over a longer time period.

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