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Robust Combinatorial Optimization under Budgeted-Ellipsoidal Uncertainty

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Abstract In the field of robust optimization uncertain data is modeled by uncertainty sets which contain all relevant outcomes of the uncertain problem parameters. The complexity of the related robust problem depends strongly on the shape of the chosen set. Two popular classes of uncertainty are budgeted uncertainty and ellipsoidal uncertainty. In this paper we introduce a new uncertainty class which is a combination of both. More precisely we consider ellipsoidal uncertainty sets with the additional restriction that at most a certain number of ellipsoid axes can be used at the same time to describe a scenario. We define a discrete and a convex variant of the latter set and prove that in both cases the corresponding robust min-max problem is *NP*-hard for several combinatorial problems. Furthermore we prove that for uncorrelated budgeted-ellipsoidal uncertainty in both cases the min-max problem can be solved in polynomial time for several combinatorial problems if the number of axes which can be used at the same time is fixed. We derive exact solution methods and formulations for the problem which we test on random instances of the knapsack problem and of the shortest path problem.

Keywords Robust Optimization · Combinatorial Optimization · Budgeted Uncertainty · Ellipsoidal Uncertainty · Complexity

1 Introduction

In the field of combinatorial optimization we consider problems of the form

$$\min_{x\in X} c^\top x \tag{1}$$

where $X \subseteq \{0,1\}^n$ is the set of feasible solutions, given by their incidence vectors, and $c \in \mathbb{R}^n$ is a given cost vector. Many well-studied problems can be modeled by (1), e.g. the shortest path problem, the spanning-tree problem or the knapsack problem.

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In nearly every practical application uncertainty can occur in the problem parameters. For example in transportation problems we normally do not know the exact travel times due to changes in the traffic situations, i.e. we do not know the exact cost vector c. In this paper we assume that uncertainty is only given in the objective function and therefore only the parameters of the cost vector c are uncertain. One way to implement uncertainty in combinatorial optimization problems is given by the robust optimization approach. Here, for a given uncertainty set $U \subset \mathbb{R}^n$ we optimize the worst objective value over all scenarios in U. This idea leads to the min-max problem

$$\min_{x\in X} \max_{c\in U} c^\top x. \tag{2}$$

In contrast to stochastic optimization approaches, where the uncertain parameters are assumed to follow a given probability distribution, in the robust optimization approach no probability distribution is needed. Therefore this approach can be useful for real-world applications where we can observe a set of arising scenarios but specifying a probability distribution is too difficult or may be too inaccurate since the set of observed scenarios is too small.

The idea of robust optimization was first mentioned in [33] and was studied later in several seminal works for different classes of uncertainty sets [6,25,8,9,17,16,13,12]. After the approach received increasing attention several surveys about this topic were published; see e.g. [4,11,24,10]. A survey which studies the different uncertainty classes and the related robust counterparts can be found in [26]. Later several new robust paradigms where presented and studied [7,27,1,18].

The complexity of Problem (2) depends on the set X and on the set U. It has been shown that even if the set X is the feasible set of several tractable problems, e.g. the shortest path problem, Problem (2) can become *NP*-hard. Therefore the complexity for tractable problems mainly depends on the shape of U. In the literature different classes of uncertainty sets U are considered. On the one hand discrete sets $U = \{c_1, \ldots, c_m\}$ have been studied. It has been proven that even if U only contains two scenarios, Problem (2) becomes *NP*-hard for several tractable combinatorial problems and even becomes strongly *NP*-hard if the number of scenarios is part of the input [25,2,3,5]. Since in most real-world applications it is only possible to observe a discrete set of scenarios, e.g. a set of traffic situations, discrete uncertainty sets are easy to be modeled. Nevertheless if we consider a transportation problem and we observe different travel times on an edge it is reasonable to assume that each travel time in the interval between the longest and the shortest travel time is possible as well.

The latter situation motivates the class of interval uncertainty, where

$$U = \{c \in \mathbb{R}^n \mid l \le c \le u\} =: [l, u]$$

is a box. Since in this case the worst-case scenario is always the vector u, Problem (2) reduces to Problem (1) and is therefore solvable in polynomial time if the underlying deterministic problem is solvable in polynomial time. Despite the positive complexity of the robust counterpart, interval uncertainty sets have two main drawbacks: first, any uncertain parameter is allowed to vary in its interval independently of all other parameters. Considering transportation problems, it is more realistic to model correlations between certain edges since if a traffic scenario appears on one edge it is very likely that this scenario affects the traffic on some adjacent edges as well. To model correlations, an appropriate way is to use ellipsoidal uncertainty sets, i.e. U is defined by

$$U = \left\{ c \in \mathbb{R}^n \mid (c - c_0)^\top \Sigma (c - c_0) \leq 1 \right\}$$

where c_0 is the given center of U and $\Sigma \in \mathbb{R}^{n \times n}$ a given positive definite correlation matrix. Here besides the correlations another advantage is that a smooth closed form expression of the objective function can be derived; see Section 2. Furthermore compared to interval uncertainty even uncorrelated ellipsoids, i.e. ellipsoids which are axis-parallel, model a kind of correlation since, considering the boundary of the set, if one uncertain parameter increases another one has to decrease. There are several applications which justify the use of ellipsoidal uncertainty sets. E.g. if a set of discrete scenarios is observed in a real-world application the ellipsoid with minimal volume which contains all observed scenario can be used as uncertainty set. Furthermore considering the value-at-risk and assuming a normal distribution on the uncertain profits, we obtain a robust min-max problem with ellipsoidal uncertainty, where the correlation of the normal distribution is modeled by the correlation matrix above [15].

The class of ellipsoidal uncertainty sets has been intensively studied in [8,9]. It has been proved that for general ellipsoidal uncertainty sets Problem (2) is *NP*-hard for several tractable combinatorial problems [14]. The min-max version of the unconstrained binary problem under ellipsoidal uncertainty is even strongly *NP*-hard [5]. On the other hand if we consider uncorrelated ellipsoidal uncertainty sets then Problem (2) is solvable in polynomial time if X is a matroid [23, 29]. Therefore as a special case the latter result holds for the unconstrained binary problem and the spanning tree problem. However for most of the classical combinatorial problems, e.g. for the shortest path problem, the complexity is not known for uncorrelated ellipsoidal uncertainty.

The second drawback of interval uncertainty is that it leads to very conservative solutions since the worst-case scenario is always attained in u. However it is very unlikely that all uncertain parameters differ from their mean value at the same time, as it is the case in scenario u. Considering a transportation problem again it is very unlikely that on each edge the worst possible traffic situation is attained at the same time. Therefore Bertsimas and Sim proposed to add a budget constraint to the box U which limits the number of uncertain parameters which can differ from their mean value at the same time [13]. For a given $\Gamma \in \mathbb{N}$ the so called budgeted uncertainty set can be defined in a discrete version

$$U_\Gamma^d := \left\{ c = c_0 + \sum_{i=1}^{n} \delta_i d_i e_i \mid \sum_{i=1}^{n} \delta_i \leq \Gamma, \delta_i \in \{0, 1\} \right\},$$

or a convex version

$$U_\Gamma^c = \left\{ c = c_0 + \sum_{i=1}^{n} \delta_i e_i \mid \sum_{i=1}^{n} \delta_i \leq \Gamma, \delta_i \in [0, d_i] \right\}$$

where e_i is the i-th unit vector, c_0 is a mean vector and d is a deviation vector. In [12] Bertsimas and Sim prove that Problem (2) with discrete budgeted uncertainty can be solved by solving at most $n + 1$ times a deterministic problem (1).

For the convex variant it is even sufficient to solve two deterministic problems. Therefore Problem (2) under budgeted uncertainty remains solvable in polynomial time if the underlying problem is solvable in polynomial time. Similar results hold for an extension of the budgeted uncertainty where the parameter Γ is not fixed but depends on the actual solution [14,31].

However budgeted uncertainty sets are not able to model correlations between uncertain parameters as it is possible in the case of ellipsoidal uncertainty. On the other hand for a scenario in an ellipsoidal uncertainty set it is still possible that all uncertain parameters differ from their mean value, i.e. the center of the ellipsoid, at the same time. To obtain an uncertainty set which can be used to model correlations and at the same limits the number of parameters which may differ from their mean value simultaneously, we combine the concepts of budgeted uncertainty and ellipsoidal uncertainty to derive a new uncertainty class which we call *budgeted-ellipsoidal uncertainty.* In this paper we will define a convex and a discrete variant of the latter set and analyze the complexity of the corresponding robust min-max problem. The discrete variant is motivated by the discussion above and is in contrast to most of the classical uncertainty sets not a convex set. The convex variant is defined as the intersection of a budgeted uncertainty set and an ellipsoidal uncertainty set. Note that in the literature the intersection of interval uncertainty and ellipsoidal uncertainty has been studied (see [6]) while, to the best of our knowledge, the intersection of budgeted uncertainty and ellipsoidal uncertainty has not been studied yet.

In section 2 we will consider the discrete variant while in section 3 the convex variant is studied. In both sections we will prove that the robust problem is *NP*-hard for general budgeted-ellipsoidal uncertainty for any fixed Γ. Furthermore we prove that it can be solved in polynomial time for several combinatorial problems if we consider uncorrelated budgeted-ellipsoidal uncertainty and if Γ is fixed. In Section 4 we derive an exact cutting-plane algorithm for the discrete variant and a quadratic formulation for the convex variant. Finally in Section 5 we test the derived exact methods on random instances for the knapsack problem and the shortest path problem.

2 Discrete Budgeted-Ellipsoidal Uncertainty

Let E be an ellipsoid given by

$$E = \left\{ c \in \mathbb{R}^n \mid (c - c_0)^\top \Sigma (c - c_0) \leq 1 \right\}$$

where c_0 is the given center of E and $\Sigma \in \mathbb{R}^{n \times n}$ the given positive definite correlation matrix. Then Σ can be transformed to

$$\Sigma = \sum_{i=1}^{n} \lambda_i v_i v_i^\top$$

where $\lambda_1, \ldots, \lambda_n > 0$ are the eigenvalues of Σ, and $v_1, \ldots, v_n$ are the corresponding eigenvectors which define an orthonormal basis. Geometrically v_i is the direction of the i-th axis of the ellipsoid and therefore we call v_i the axis-vectors of the ellipsoid. We define l_i as the axis-length of the axis with direction v_i such that $\lambda_i = \left(\frac{1}{l_i}\right)^2$ holds. We can equivalently write

$$E = \left\{ c = c_0 + \sum_{i=1}^{n} \mu_i v_i \mid \sum_{i=1}^{n} \left(\frac{\mu_i}{l_i}\right)^2 \leq 1,\ \mu_i \in \mathbb{R} \right\}.$$

The corresponding robust counterpart (2) with uncertainty set E can be rewritten as

$$\min_{x \in X} c_0^\top x + \sqrt{\sum_{i=1}^{n} (l_i v_i^\top x)^2}.$$

In this section we consider the discrete variant of budgeted-ellipsoidal uncertainty which we define by

$$E_\Gamma^d := \left\{ c_0 + \sum_{i=1}^{n} \mu_i v_i \middle| \sum_{i=1}^{n} \frac{\mu_i^2}{l_i^2} \leq 1, \sum_{i=1}^{n} y_i \leq \Gamma, |\mu_i| \leq l_i y_i, \mu_i \in \mathbb{R}, y_i \in \{0,1\} \right\} \quad (3)$$

for a given $\Gamma \in \{1, \ldots, n\}$. The idea behind the definition is the same as for the classical budgeted uncertainty sets. It is very unlikely that all parameters differ from their nominal value at the same time, or in the correlated ellipsoidal case, that all axis-vectors have to be considered at the same time to describe a scenario. Therefore we want to limit the number of axis-vectors which can be used by Γ. In the uncorrelated case it follows that at most Γ many uncertain parameters can differ from the ellipsoid center at the same time. Depending on the application the parameter Γ can be used to adjust the amount of uncertainty which is considered in the problem. Geometrically each scenario $c \in E_\Gamma^d$ can be described by at most Γ many axis-vectors, i.e. it lies on an at most Γ-dimensional sub-ellipsoid (see Figure 2). Note that the latter set is not convex in general. Therefore it is not surprising that the corresponding robust min-max problem is *NP*-hard for any fixed Γ as we will prove in the following. The problem is even strongly *NP*-hard if we fix the distance between Γ and the dimension of the problem.

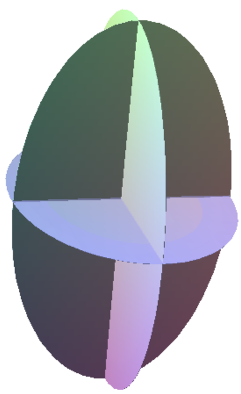

Fig. 1 Example of E_Γ^d in $\mathbb{R}^3$ with $\Gamma = 2$.

We now consider the robust min-max problem

$$\min_{x\in X}\max_{c\in E^d_\Gamma} c^\top x \tag{4}$$

where $X\subseteq\{0,1\}^n$. Obviously it holds

$$\min_{x\in X}\max_{c\in E^d_\Gamma} c^\top x \le \min_{x\in X}\max_{c\in E} c^\top x.$$

Furthermore for any scenario $c\in U^d_\Gamma$ with $c=c_0+\sum_{i=1}^n \delta_i d_i e_i$ where $\delta_i\in\{0,1\}$ and $\sum_{i=1}^n \delta_i\le\Gamma$, consider the corresponding lower-dimensional axis-parallel ellipsoid with center point c_0 and axis-lengths $\delta_i d_i$. Then the north-east part of the ellipsoid is contained in the box $[c_0, c_0+\sum_{i=1}^n \delta_i d_i e_i]$ and therefore it holds

$$\min_{x\in X}\max_{c\in E^d_\Gamma} c^\top x \le \min_{x\in X}\max_{c\in U^d_\Gamma} c^\top x.$$

Note that for $\Gamma=n$ Problem (4) is the classical robust min-max problem with ellipsoidal uncertainty. Since the latter is known to be *NP*-hard for most of the classical combinatorial problems, Problem (4) is *NP*-hard for the same problems if we assume the parameter Γ to be part of the input. In the following we will prove that if we fix Γ to a certain positive integer the problem remains *NP*-hard. To this end we first prove the following lemma.

Lemma 1 *Problem* (4) *is equivalent to*

$$\min_{x\in X}\ c_0^\top x+\sqrt{\sum_{i\in I(x,\Gamma)}\left(l_i v_i^\top x\right)^2} \tag{5}$$

where $I(x,\Gamma)$ contains the indices of the Γ largest values $l_i|v_i^\top x|$.

Proof For any given $x\in X$ using the definition of E^d_Γ and by substituting $\frac{\mu_i}{l_i}$ by a new variable α_i we can reformulate the objective function $\max_{c\in E^d_\Gamma} c^\top x$ of Problem (4) as

$$\begin{aligned}
\max_{y,\alpha}\ & c_0^\top x+\sum_{i=1}^n \alpha_i l_i v_i^\top x\\
s.t.\ & \sum_{i=1}^n \alpha_i^2\le 1\\
& |\alpha_i|\le y_i \quad \forall\, i=1,\dots,n\\
& \sum_{i=1}^n y_i\le\Gamma\\
& y_i\in\{0,1\},\ \alpha_i\in[-1,1] \quad \forall\, i=1,\dots,n.
\end{aligned} \tag{6}$$

The latter problem can be solved by solving the following problem

$$\begin{aligned}
\max_{\alpha}\ & c_0^\top x+\sum_{j=1}^r \alpha_{i_j}(l_{i_j} v_{i_j}^\top x)\\
s.t.\ & \sum_{j=1}^r \alpha_{i_j}^2\le 1\\
& \alpha_{i_j}\in\mathbb{R} \quad \forall\, j=1,\dots,r
\end{aligned}$$

for each combination of at most Γ many indices $i_1, \dots, i_r \in \{1, \dots, n\}$ with $r \le \Gamma$. The latter problem is a linear optimization problem over the unit ball in dimension r. The optimal solution is $a^* = \frac{w}{\|w\|_2}$ where w is the vector with entries

$$w_{i_j} = l_{i_j} v_{i_j}^\top x$$

for each $j = 1, \dots, r$. The optimal value is $c_0^\top x + \|w\|_2$. Therefore the optimal value of Problem (6) is attained for the indices $i_1, \dots, i_r$ which correspond to the Γ largest values of $\left\{l_1|v_1^\top x|, \dots, l_n|v_n^\top x|\right\}$. Hence to calculate the objective value for the given solution x we only have to solve problem

$$\begin{aligned} \max_{\alpha}\ & c_0^\top x + \sum_{i \in I(x,\Gamma)} \alpha_i l_i v_i^\top x \\ s.t.\ & \sum_{i \in I(x,\Gamma)} \alpha_i^2 \le 1 \\ & \alpha_i \in \mathbb{R} \quad \forall\, i \in I(x,\Gamma). \end{aligned}$$

By the observations above the latter problem has the optimal value

$$c_0^\top x + \sqrt{\sum_{i \in I(x,\Gamma)} \left(l_i v_i^\top x\right)^2}$$

which proves the result.

Proposition 1 *For any fixed $\Gamma \in \mathbb{N}$, Problem (4) is* NP*-hard for the shortest path problem, the spanning-tree problem, the assignment problem, the min s-t cut problem and the unconstrained binary problem.*

Proof It has been proved that the min-max problem with two-scenarios is *NP*-hard for all the problems in the theorem [25,3,5]. In the following we reduce the latter problems to (4) for any fixed $\Gamma \in \mathbb{N}$. The main idea of the following proof has been already used to prove the *NP*-hardness of (2) for full-dimensional ellipsoids in [23].

Assume a given instance of the two-scenario min-max problem, i.e. a set of feasible solutions $X \subseteq \{0,1\}^n$ and scenarios $c_1, c_2 \in \mathbb{R}^n$. For an instance of the budgeted-ellipsoidal uncertainty we define the center $c_0 := \frac{1}{2}(c_1 + c_2)$ and the first axis-vector $v_1 := c_1 - c_2$. Then we extend v_1 to an orthogonal basis $v_1, \dots, v_n$, e.g. by using the gram-schmidt procedure. Furthermore we define axis-lengths $l_1 := 1$ and

$$l_i := \frac{1}{2n^2}$$

for all $i = 2, \dots, n$. Then analogously to the proof in [23] we can show that the instance of Problem (3) which we defined above and the min-max problem with two scenarios $c_1, c_2 \in \mathbb{R}$ are equivalent which proves the result.

We now prove that Problem (4) is even strongly *NP*-hard for any fixed distance between Γ and the dimension of the problem.

Proposition 2 *For any fixed $k := n - \Gamma$, Problem (4) is strongly* NP*-hard even if $X = \{0,1\}^n$.*

Proof We will show a reduction from Problem (2) with $X = \{0,1\}^n$ and ellipsoidal uncertainty. This problem is known to be strongly *NP*-hard which can be shown by a reduction from binary quadratic programming [5]. The idea of the proof is to set Γ to the dimension of the given instance and extend the dimension by k variables. Then by shifting the center of the ellipsoid by a large value in the new dimensions we ensure that an optimal solution only uses variables of the original instance and is hence optimal for it.

For a given instance of the latter problem, i.e. $X = \{0,1\}^n$ and a full-dimensional ellipsoid $E \subset \mathbb{R}^n$ with axis-vectors $v_1, \ldots, v_n$ and axis-lengths $l_1, \ldots, l_n$, we create an instance of Problem (4) as follows: we define $\Gamma := n$ and consider the problem

$$\min_{x\in\{0,1\}^{n+k}} \max_{c\in E_\Gamma^d} c^\top x \tag{7}$$

where E_Γ^d has axis-vectors $\tilde{v}_1, \ldots, \tilde{v}_{n+k}$ and axis-lengths $\tilde{l}_1, \ldots, \tilde{l}_{n+k}$ which are defined in the following. For $i = 1, \ldots, n$ we define

$$\tilde{v}_i = (v_i^\top, 0 \ldots, 0)^\top,$$

i.e. we extend the axis-vectors v_i by 0 in the new dimensions. For $i = 1, \ldots, k$ we define $\tilde{v}_{n+i} = e_{n+i}$ where e_j is the j-th unit vector. Note that $\tilde{v}_1, \ldots, \tilde{v}_{n+k}$ is an orthonormal-basis in $\mathbb{R}^{n+k}$. For the axis-lengths we set $\tilde{l}_i := l_i$ for $i = 1, \ldots, n$ and

$$\tilde{l}_{n+i} := 1$$

for each $i = 1, \ldots, k$. The new center point is defined by

$$\tilde{c}_0 = (c_0^\top, M \ldots, M)$$

where M can be chosen as

$$M := \sum_{i=1}^n l_i^2 \left(\sum_{j=1}^n |(v_i)_j|\right)^2 + \sum_{j=1}^n |(c_0)_j|.$$

By the definition of the center $\tilde{c}_0$ follows that each optimal solution x of Problem (7) is zero in the new dimensions, i.e. $x_{n+i} = 0$ for all $i = 1, \ldots, k$. Hence it holds

$$\tilde{l}_{n+i}|\tilde{v}_{n+i}^\top x| = x_{n+i} = 0 \le \tilde{l}_j|\tilde{v}_j^\top x|$$

for each $i = 1, \ldots, k$ and $j = 1, \ldots, n$. Furthermore $\tilde{l}_j|\tilde{v}_j^\top x| = l_j|v_j^\top x|$. Therefore

$$\tilde{c}_0^\top x + \sqrt{\sum_{i\in I(x,\Gamma)} \left(\tilde{l}_i \tilde{v}_i^\top x\right)^2} = c_0^\top x_N + \sqrt{\sum_{i=1}^n \left(l_i v_i^\top x_N\right)^2}$$

where x_N is the vector x restricted to the first n dimensions. The right-hand side of the latter equation is the objective function of (2) which proves the result.

Corollary 1 *For any fixed $k := n - \Gamma$, Problem* (4) *is* NP*-hard for the shortest path problem and the spanning-tree problem.*

Proof The respective min-max problems with ellipsoidal uncertainty are known to be *NP*-hard [32]. For both problems it is easy to see that we can extend the given graph by k edges without changing any of the solutions in the original graph. Using this construction we can define an instance in dimension $n + k$ analogously to the proof of Proposition 2 which proves the result.

2.1 Uncorrelated Budgeted-Ellipsoidal Uncertainty

We now analyze the discrete budgeted-ellipsoidal uncertainty for uncorrelated ellipsoids, i.e. the given ellipsoid E is axis-parallel. In this case we have $v_i = e_i$ for each $i = 1, \dots, n$. In the proof of Theorem 1 we have to solve the underlying deterministic problem with fixations i.e. we have to solve

$$\min_{\substack{x\in X \\ x_i=\pi_i \ \forall i\in I}} \quad c^\top x \tag{8}$$

for a given index set $I \subset \{1, \dots, n\}$ and given fixation values $\pi \in \{0, 1\}^{|I|}$ or decide that no feasible solution for the given fixations exists. We call (8) the *fixation problem*. Using the result of Lemma 1 we can prove the following theorem.

Theorem 1 *For uncorrelated budgeted-ellipsoidal uncertainty, Problem* (4) *can be solved in polynomial time for each fixed Γ if the underlying deterministic problem can be solved in polynomial time for any cost-vector $c \in \mathbb{R}^n$.*

Proof We show that we can solve the equivalent problem given in Lemma 1 in polynomial time if Γ is fixed. For uncorrelated budgeted-ellipsoidal uncertainty we obtain

$$\min_{x\in X} \ c_0^\top x + \sqrt{\sum_{i\in I(x,\Gamma)} l_i^2 x_i} \tag{9}$$

where $I(x, \Gamma)$ contains the indices of the Γ largest values $l_i x_i$. Without loss of generality we may assume that $l_1 \le \ldots \le l_n$. The algorithm works as follows: for each subset of at most Γ many indices, i.e. $i_1, \dots, i_r \in \{1, \dots, n\}$ with $r \le \Gamma$, we consider the fixation vector π where

$$\pi_i = \begin{cases} 1 & \text{if } i \in \{i_1, \dots, i_r\} \\ 0 & \text{if } l_i \ge \min_{j\in\{i_1,\dots,i_r\}} l_j \end{cases} .$$

We now solve Problem (8) with the defined fixation π and objective function c_0. Let $opt_\pi \in \mathbb{R}$ be the optimal value. Then in Problem (9) the optimal solution of Problem (8) has an objective value of $v := opt_\pi + \sqrt{\sum_{j=1}^r l_{i_j}}$. After iterating over all possible sets of at most Γ many indices, the solution with the best value v is the optimal solution of Problem (4). Since the number of combinations of indices we have to consider is bounded by n^Γ, the algorithm has polynomial runtime if Γ is fixed. Note that the fixation problem is equivalent to the deterministic problem since we can add M or $-M$ to the respective costs c_i, where M is a large enough value. Then the corresponding variables must be 1 or 0 in any optimal solution.

3 Convex Budgeted-Ellipsoidal Uncertainty

In this chapter, for a given ellipsoid E, we define the convex variant of the budgeted-ellipsoidal uncertainty set by

$$E_\Gamma^c = \left\{ c = c_0 + \sum_{i=1}^n \mu_i v_i \mid \sum_{i=1}^n \left(\frac{\mu_i}{l_i}\right)^2 \le 1, \ \sum_{i=1}^n |\mu_i| \le \Gamma, \ \mu_i \in \mathbb{R} \right\}. \tag{10}$$

The idea behind the definition was derived directly from the definition of the convex variant of the classical budgeted uncertainty, where not the number of deviations is bounded by Γ but the absolute deviation. Similarly here for the given ellipsoid the absolute deviation on the axes is bounded by Γ and not the number of axes which may be used as in the previous section.

We first show that both sets, the discrete and the convex variant, can be different.

Proposition 3 *In general $E_\Gamma^d \nsubseteq E_\Gamma^c$ and $E_\Gamma^c \nsubseteq E_\Gamma^d$.*

Proof Let $\Gamma < n$ and let E be an ellipsoid such that for at least one $j' \in \{1, \ldots, n\}$ we have $l_{j'} > \Gamma$ and for each i we have $l_i \geq \frac{\Gamma}{n}$. Then define $c := c_0 + \sum_{i=1}^n \mu_i v_i$ where $\mu_{j'} = l_{j'}$ and $\mu_j = 0$ for $j \neq j'$. Then $c \in E_\Gamma^d$ but $\sum_{i=1}^n |\mu_i| > \Gamma$ and therefore $c \notin E_\Gamma^c$.

On the other hand define $\hat{c} = c_0 + \sum_{i=1}^n \mu_i v_i$ where $\mu_j = \frac{\Gamma}{n}$. Then $\hat{c} \in E_\Gamma^c$ but $\hat{c} \notin E_\Gamma^d$ since $\Gamma < n$.

Furthermore the min-max problem with E_Γ^d defined in the last chapter is in general not equivalent to the min-max problem with E_Γ^c as the following example shows.

Example 1 Consider an ellipsoid E with axis-vectors $e_1, \ldots, e_n$ where e_i is the i-th unit vector and axis-lengths $l_i = \Gamma + i$ for a given $\Gamma \in \mathbb{N}$. We define $c_0 = 0$. We consider the feasible set

$$X := \{e_1, \ldots, e_n, \mathbf{1}\}$$

where $\mathbf{1}$ is the all-one vector. Then for uncertainty set E_Γ^c all solutions in X have objective value Γ and are therefore optimal. If we consider E_Γ^d then solution e_i has objective value $\Gamma + i$ and solution $\mathbf{1}$ has an objective value greater or equal to $\Gamma + n$. Therefore only solution e_1 is optimal.

In the following we analyze the complexity of the robust min-max problem with convex uncertainty set E_Γ^c, i.e.

$$\min_{x \in X} \max_{c \in E_\Gamma^c} c^\top x. \tag{11}$$

Note that from the definition of E_Γ^c we directly obtain

$$\min_{x \in X} \max_{c \in E_\Gamma^c} c^\top x \leq \min_{x \in X} \max_{c \in E} c^\top x$$

and

$$\min_{x \in X} \max_{c \in E_\Gamma^c} c^\top x \leq \min_{x \in X} \max_{c \in U_\Gamma^c} c^\top x.$$

Proposition 4 *For any fixed $\Gamma \in \mathbb{N}$, Problem* (11) *is* NP*-hard for the shortest path problem, the spanning-tree problem, the assignment problem, the min s-t cut problem and the unconstrained binary problem.*

Proof All the problems in the theorem are *NP*-hard in its min-max versions with general ellipsoidal uncertainty [23]. We reduce the latter problems to Problem (11). The idea of the proof is to scale the given ellipsoid E of the min-max instance such that the constraint $\sum_{i=1}^{n} |\mu_i| \leq \Gamma$ is redundant and therefore optimizing over E_Γ^c is equivalent to optimizing over E.

For a given instance of the min-max problem, i.e. a feasible set $X \subseteq \{0,1\}^n$ and an ellipsoidal uncertainty set $E \subset \mathbb{R}^n$ with axis-vectors $v_1, \ldots, v_n$ and axis-lengths $l_1, \ldots, l_n$ we define an instance of the convex budgeted-ellipsoidal uncertainty set E_Γ^c by scaling the axis-lengths and the center of the ellipsoid by the factor $\frac{l_i}{n l_{\max}}$ where $l_{\max} = \max\{l_i \mid i = 1, \ldots, n\}$. Then the constraint $\sum_{i=1}^{n} |\mu_i| \leq \Gamma$ is redundant and the optimal solutions of the min-max problem with uncertainty set E and Problem (10) are the same.

The following lemma shows an equivalent formulation for Problem (11) which will be used in the next subsection.

Lemma 2 *Problem* (11) *is equivalent to Problem*

$$\min_{x \in X, \lambda \geq 0} \; c_0^\top x + \lambda \Gamma + \sqrt{\sum_{i=1}^{n} \left(\max\left\{l_i(|v_i^\top x| - \lambda), 0\right\}\right)^2} \tag{12}$$

Proof The objective function $\max_{c \in E_\Gamma^c} c^\top x$ of Problem (11) is equivalent to

$$\begin{aligned} \max \; & c_0^\top x + \sum_{i=1}^{n} \alpha_i l_i |v_i^\top x| \\ s.t. \; & \sum_{i=1}^{n} (\alpha_i)^2 \leq 1 \\ & \sum_{i=1}^{n} \alpha_i l_i \leq \Gamma \\ & \alpha_i \geq 0 \quad \forall\, i = 1 \ldots, n. \end{aligned}$$

By relaxing the last constraint and using Lagrange-dualization, we obtain the equivalent problem

$$\min_{\lambda \geq 0} \max_{\substack{\sum_{i=1}^{n} (\alpha_i)^2 \leq 1 \\ \alpha_i \geq 0 \;\, i=1,\ldots,n}} c_0^\top x + \lambda \Gamma + \sum_{i=1}^{n} \alpha_i \left(l_i(|v_i^\top x| - \lambda) \right). \tag{13}$$

Because of the constraint $\alpha_i \geq 0$, in any optimal solution the variable α_i is only positive if the term $l_i(|v_i^\top x| - \lambda)$ is positive. For any $\lambda \geq 0$ an optimal solution α^* of the inner maximization problem is then given by $\alpha^* = \frac{\tilde{\alpha}}{\|\tilde{\alpha}\|_2}$ where

$$\tilde{\alpha}_i = \begin{cases} l_i(|v_i^\top x| - \lambda) & \text{if } l_i(|v_i^\top x| - \lambda) > 0 \\ 0 & \text{else.} \end{cases}$$

By substituting the latter solution into (13) we obtain the expression given in the lemma.

3.1 Uncorrelated Convex Budgeted-Ellipsoidal Uncertainty

We now analyze the convex budgeted-ellipsoidal uncertainty for uncorrelated ellipsoids, i.e. the given ellipsoid E is axis-parallel. We use Lemma 2 to prove the following result. The idea of the proof is based on the proof in [12].

Theorem 2 *Problem* (11) *with uncorrelated budgeted-ellipsoidal uncertainty can be solved by solving one deterministic problem and one uncorrelated ellipsoidal min-max problem over* X*.*

Proof For uncorrelated budgeted-ellipsoidal uncertainty we have $v_i = e_i$ where e_i is the i-th unit vector in $\mathbb{R}^n$. Therefore by using the equivalent problem in Lemma 2 we obtain

$$\min_{x\in X,\lambda\geq 0} \; c_0^\top x + \lambda\Gamma + \sqrt{\sum_{i=1}^{n} \max\left\{l_i(x_i-\lambda),0\right\}^2}. \tag{14}$$

Since $x_i \in \{0,1\}$ and $\lambda \geq 0$ we have

$$\max\left\{l_i(x_i-\lambda),0\right\}^2 = l_i^2 x_i \max\left\{1-\lambda,0\right\}^2.$$

Hence for $\lambda \in [0,1]$ Problem (14) can be written as

$$\min_{x\in X,\lambda\geq 0} \; c_0^\top x + \lambda\Gamma + (1-\lambda)\sqrt{\sum_{i=1}^{n} l_i^2 x_i}$$

and for $\lambda \geq 1$ we have

$$\min_{x\in X,\lambda\geq 0} \; c_0^\top x + \lambda\Gamma.$$

In both cases we have a linear problem in λ. Therefore and since $\Gamma \geq 0$ the optimum must be obtained at $\lambda = 0$ or $\lambda = 1$. Hence, to solve (14), we only have to solve problems

$$\min_{x\in X} \; c_0^\top x + \Gamma$$

and

$$\min_{x\in X} c_0^\top x + \sqrt{\sum_{i=1}^{n} l_i^2 x_i}$$

and return the solution with the better objective value. Note that the first problem is a linear problem and the second one is equivalent to a min-max problem with uncorrelated ellipsoidal uncertainty.

Corollary 2 *Problem* (11) *with uncorrelated budgeted-ellipsoidal uncertainty can be solved in polynomial time for the unconstrained problem and the spanning-tree problem.*

Proof Since the set of feasible solutions X for both problems is a matroid the min-max versions of both problems with uncorrelated ellipsoidal uncertainty can be solved in polynomial time [29,28,23]. The result for the unconstrained problem can also be derived by the fact that the objective function of (2) for uncorrelated ellipsoidal uncertainty is submodular [19].

4 Exact Methods

In this chapter we derive exact methods to solve Problems (4) and (11). For the former problem we derive a cutting-plane algorithm based on a quadratic integer formulation of the problem. Since the algorithm given in the proof of Theorem (1) iterates over all subsets of indices of cardinality at most Γ, the algorithm is not efficient for practical computations. Hence in Section 5 we will apply the latter cutting-plane algorithm for both, correlated and uncorrelated discrete budgeted-ellipsoidal uncertainty.

For Problem (11) we derive a quadratic formulation for general budgeted-ellipsoidal uncertainty. While this formulation is used in Section 5 to solve the correlated case, for the uncorrelated case we implemented the algorithm given in Theorem 2.

For the discrete budgeted-ellipsoidal case, in the following, we will derive an quadratic integer formulation for Problem (4). To this end we consider the equivalent problem (5). This problem can be equivalently written as

$$\min_{x\in X} \; c_0^\top x + \max_{\substack{y\in\{0,1\}^n \\ y^\top \mathbf{1}=\Gamma}} \sqrt{\sum_{i=1}^{n} y_i \left(l_i v_i^\top x\right)^2}$$

which is again equivalent to

$$\begin{aligned} \min_{x,z} \; & c_0^\top x + z \\ s.t. \; & \sum_{i\in I} \left(l_i v_i^\top x\right)^2 \leq z^2 \quad \forall\, I \subset \{1,\dots,n\} : |I| = \Gamma \\ & x \in X, \;\; z \in \mathbb{R}. \end{aligned} \tag{15}$$

The latter problem is a quadratic problem with $\binom{n}{\Gamma}$ quadratic constraints. Since the number of constraints is exponential in Γ we propose to solve the latter problem by the cutting-plane Algorithm 1. Here in each iteration Problem (15) for a subset of constraints is solved. For the calculated optimal solution (x^*, z^*) the index set I' of Γ many indices which most violates the constraint

$$\sum_{i\in I'} \left(l_i v_i^\top x^*\right)^2 \leq (z^*)^2$$

is calculated and the corresponding constraint is added to the problem. The index set is determined by choosing the indices of the Γ largest values $\left(l_i v_i^\top x^*\right)^2$. This is done until no index set exists which violates the constraint and therefore the last calculated solution must be feasible for all constraints of the original problem (15) and is therefore optimal.

For the convex case we derive a quadratic formulation for Problem (11). To this end we consider the equivalent formulation (12) and add variables y_i for each $i = 1 \dots, n$ to replace the maximum terms

$$\max\left\{l_i(|v_i^\top x| - \lambda), 0\right\}$$

Algorithm 1 Cutting-plane algorithm for Problem (4)

Input: E^d_Γ, $X \subseteq \{0,1\}^n$
Output: optimal solution of Problem (4)

$\mathcal{I} := \emptyset$
find ordering $l_{i_1} \geq \ldots \geq l_{i_n}$
repeat
  add $\{i_1, \ldots, i_\Gamma\}$ to $\mathcal{I}$
  calculate optimal solution (x^*, z^*) of

$$\begin{aligned} \min\ & c_0^\top x + z \\ s.t.\ & \sum_{i \in I} \left(l_i v_i^\top x \right)^2 \leq z^2 \quad \forall I \in \mathcal{I} \\ & x \in X \end{aligned}$$

  find ordering $\left(l_{i_1} v_{i_1}^\top x^* \right)^2 \geq \ldots \geq \left(l_{i_n} v_{i_n}^\top x^* \right)^2$
until $\sum_{j=1}^{\Gamma} \left(l_{i_j} v_{i_j}^\top x^* \right)^2 \leq (z^*)^2$
return x^*

under the square-root. Furthermore we add variables w_i for each $i = 1 \ldots, n$ to model the absolute value $|v_i^\top x|$. We obtain the equivalent formulation

$$\begin{aligned} \min\ & c_0^\top x + \lambda \Gamma + z \\ s.t.\ & \sum_{i=1}^{n} l_i^2 y_i^2 \leq z^2 \\ & w_i \geq v_i^\top x \quad \forall i = 1, \ldots, n \\ & w_i \geq -v_i^\top x \quad \forall i = 1, \ldots, n \\ & y_i \geq w_i - \lambda \quad \forall i = 1, \ldots, n \\ & x \in X, \lambda \geq 0 \\ & y \geq 0, w \geq 0. \end{aligned} \tag{16}$$

Both, Problems (15) and (16) can be solved by modern IP solvers as CPLEX [22] or Gurobi [20]. Note that all exact methods derived in this section and the algorithm given in the Theorem 2 can be easily implemented for several combinatorial problems.

5 Computational Results

In this section we apply the exact methods derived in Section 4 and the algorithm derived in the proof of Theorem 2 to Problems (4) and (11). We tested all algorithms on random budgeted-ellipsoidal instances for the knapsack problem and for the shortest path problem.

For the knapsack problem we created random instances similar to the uncorrelated and the weakly correlated instances in [30]. For any $n \in \{100, 200, 300, 400\}$ we created 10 random uncorrelated knapsack instances and for $n \in \{25, 50\}$ we created 10 correlated knapsack instances each with a random budgeted-ellipsoidal uncertainty set. The weights w_i of the knapsack were chosen randomly from the set $\{1,\dots,100\}$ and the capacity of the knapsack was set to 35% of the total sum of the weights. For the budgeted-ellipsoidal uncertainty sets the ellipsoid center c_0 was chosen randomly with entries $(c_0)_i \in \{1,\dots,100\}$ in the uncorrelated case and with $(c_0)_i \in [w_i - 10, w_i + 10] \cap \mathbb{N}$ for the correlated case. For general ellipsoids the axes were calculated as random orthonormal bases by the Gram-Schmidt process: for each $i = 1,\dots,n$ we create a vector v_i with random entries in $[-1,1]$. Then we project v_i to the complement of the space spanned by the vectors $v_1,\dots,v_{i-1}$ and normalize it. The axis-lengths l_i were chosen randomly between 0 and $\frac{c_i}{2}$ for each $i = 1,\dots,n$. Each instance was solved for the values of $\varGamma$ from the set $\{0.05n, 0.1n\}$, rounded down if fractional. We created a general and an axis-parallel version of the corresponding budgeted-ellipsoidal set for each instance. Furthermore for each instance we additionally scaled the axis-lengths by the factor $\varOmega = 2$. The results of the computations for the knapsack problem can be found in Tables 1 and 2 for the uncorrelated knapsack instances and in Tables 3 and 4 for the correlated knapsack instances. For each combination of n, $\varGamma$ and $\varOmega$ we show the average over all 10 instances of the following numbers (from left to right): the total computation time (in seconds); the number of generated cuts in Algorithm 1 in the discrete case or the computation time of the linear problem and the uncorrelated ellipsoidal problem (in seconds) in the convex uncorrelated case; the absolute value of the difference (in percent) of the optimal value of Problem (2) with budgeted-ellipsoidal uncertainty to the optimal value of Problem (2) with classical budgeted uncertainty; the absolute value of the difference (in percent) of the optimal value of Problem (2) with budgeted-ellipsoidal uncertainty to the optimal value of Problem (2) with classical ellipsoidal uncertainty. Note that the objective value for budgeted-ellipsoids is always at least as good as the one for the classical uncertainty sets and therefore the latter two values show the percental improvement of the optimal value for budgeted-uncertainty sets. All values are rounded to two decimal places.

For the shortest path problem we tested our algorithm for the instances used in [21]. The authors create graphs with $20, 25, \dots, 50$ nodes, corresponding to points in the Euclidean plane with random coordinates in $[0, 10]$. They choose a budgeted uncertainty set of the form

$$U := \left\{ c + \sum_{(i,j)\in E} \delta_{ij} d_{ij} e_{ij} \mid \sum_{(i,j)\in E} \delta_{ij} \leq \varGamma, \delta_{ij} \in \{0,1\} \right\},$$

where c_{ij} is the Euclidean distance of node i to node j and $d_{ij} = \frac{c_{ij}}{2}$. The parameter $\varGamma$ is chosen from $\{3, 6\}$. For the budgeted-ellipsoidal uncertainty sets we set $c_0 = c$, i.e. the center is the vector containing the euclidean distances given by the instances. For general ellipsoids again the axes of the ellipsoids were calculated as random orthonormal bases as above. The axis-lengths l_i were set to $\frac{(c_0)_i}{2}$ for each $i = 1,\dots,n$. For each instance we additionally scaled the axis-lengths by the factor $\varOmega = 2$. For each instance we created a general and an axis-parallel version of the corresponding set. The results of the computations for the shortest problem can be found in Tables 5 and 6. Here for each combination of n, $\varGamma$ and $\varOmega$ we show the average over all 100 instances created in [21].

For the discrete budgeted-ellipsoidal instances we solved Problem (4) by the cutting-plane algorithm 1. For the convex budgeted-ellipsoidal instances we solved

			General E				Axis-parallel E			
n	Γ	Ω	t	#cuts	Δ_Γ	Δ_{E}	t	#cuts	Δ_Γ	Δ_{E}
100	5	1	0.63	3.80	3.47	1.03	0.18	2.90	3.35	1.76
100	5	2	2.91	7.00	7.59	2.25	0.47	4.20	7.07	3.65
100	10	1	0.72	3.80	8.15	0.58	0.21	3.30	7.52	1.05
100	10	2	5.25	6.20	18.31	1.25	0.50	4.70	16.68	2.18
200	10	1	5.02	4.60	4.21	0.74	1.26	3.50	4.17	1.24
200	10	2	96.70	11.30	9.03	1.55	1.62	4.10	8.83	2.54
200	20	1	9.81	5.30	9.08	0.40	1.53	3.90	8.81	0.75
200	20	2	3719.73	14.40	20.16	0.86	4.62	7.20	19.44	1.54
300	15	1	15.35	5.30	4.58	0.56	2.94	2.70	4.55	1.00
300	15	2	150.25	9.80	9.80	1.16	4.90	4.10	9.63	2.04
300	30	1	16.08	5.40	9.44	0.32	4.81	4.10	9.19	0.60
300	30	2	1182.90	15.50	21.07	0.68	9.43	6.90	20.42	1.22
400	20	1	90.69	7.30	4.63	0.47	10.01	3.80	4.63	0.86
400	20	2	16200.10	20.80	9.87	0.97	19.52	6.60	9.81	1.74
400	40	1	48.37	6.90	9.78	0.29	11.98	4.50	9.54	0.50
400	40	2	13276.90	22.60	21.68	0.59	26.25	8.00	21.09	1.02

Table 1 Results for the uncorrelated knapsack instances with discrete budgeted-ellipsoidal uncertainty.

the instances for general ellipsoids by formulation (16) and the axis-parallel instances by the algorithm given in the proof of Theorem 2. The quadratic problem arising in the cutting-plane algorithm and Problem (16) have been solved by CPLEX 12.6. For the algorithm given in the proof of Theorem 2 we solved the axis-parallel ellipsoidal problem by CPLEX 12.6 as well. Note that the latter problem can be reformulated as a second-order-cone problem. The linear problem was solved by a dynamic programming algorithm for the knapsack problem. All computations were calculated on a cluster of 64-bit Intel(R) Xeon(R) E7340 processors running at 2.40 GHz with 4MB cache.

The results for the uncorrelated knapsack instances are shown in Tables 1 and 2. The cutting-plane algorithm for the discrete case runs very fast and generates only a few cuts for $\Gamma = 1$. Even for a dimension of 400 the instances for general ellipsoids could be solved in less than 2 minutes in average. Interestingly for the larger ellipsoids with $\Gamma = 2$ the number of cuts and the computation time increases significantly. For the largest instances the algorithm took up to 4.5 hours in average. Nevertheless for axis-parallel instances the computation time is much smaller. In average all instances could be solved in at most 27 seconds. The lower computation time comes along with a smaller number of generated cuts. Furthermore, compared to the general case, no mixed quadratic terms appear in the formulation of the quadratic problem. The percental gap of the optimal value to the optimal value of the problem with classical budgeted uncertainty (Δ_Γ) increases if the size of the ellipsoid increases. In general the gap is large and can be up to 20%. Compared to this the gap to the problem with classical ellipsoidal uncertainty (Δ_E) is very small; 2% at most. Nevertheless a strictly positive improvement of the objective value for all configurations is obtained compared to the classical uncertainty sets. For the axis-parallel ellipsoids the gaps Δ_Γ are slightly smaller while the gaps Δ_E are slightly larger.

In the convex case for general ellipsoids the computation time increases much fast with larger n. The instances with $n = 200$ and $n = 300$ take more time than the corresponding instances for discrete budgeted-ellipsoidal uncertainty. On the other hand the instances for $n = 400$ could be solved faster. The differences Δ_Γ and Δ_E are slightly larger than in the discrete case. The gap Δ_E can be up to 6%

			General E			Axis-parallel E				
n	Γ	Ω	t	Δ_Γ	Δ_{E}	t	t_{lin}	t_{SOCP}	Δ_Γ	Δ_{E}
100	5	1	0.09	5.77	3.28	0.03	0.00	0.03	5.93	4.30
100	5	2	1.01	12.54	6.95	0.12	0.00	0.11	12.72	9.11
100	10	1	0.16	10.45	2.72	0.03	0.00	0.02	10.77	4.10
100	10	2	0.73	23.96	6.07	0.07	0.00	0.07	24.31	8.85
200	10	1	199.42	5.77	2.25	0.07	0.01	0.05	5.97	2.99
200	10	2	150.92	12.59	4.87	0.22	0.01	0.21	12.82	6.29
200	20	1	137.03	10.73	1.91	0.14	0.01	0.13	11.12	2.88
200	20	2	265.29	24.49	4.50	0.81	0.01	0.81	24.97	6.23
300	15	1	857.85	5.80	1.73	0.15	0.02	0.12	5.99	2.39
300	15	2	766.26	12.65	3.79	0.87	0.02	0.86	12.86	5.04
300	30	1	2167.52	10.62	1.40	0.12	0.02	0.10	10.96	2.23
300	30	2	1909.48	24.29	3.36	1.37	0.02	1.35	24.77	4.87
400	20	1	3414.94	5.60	1.40	0.31	0.04	0.26	5.83	2.01
400	20	2	866.48	12.25	3.17	3.84	0.03	3.80	12.51	4.24
400	40	1	4466.91	10.72	1.14	0.21	0.04	0.17	11.05	1.89
400	40	2	6429.20	24.41	2.84	1.80	0.04	1.76	24.85	4.16

Table 2 Results for the uncorrelated knapsack instances with convex budgeted-ellipsoidal uncertainty.

			General E				Axis-parallel E			
n	Γ	Ω	t	*#cuts*	Δ_Γ	Δ_{E}	t	*#cuts*	Δ_Γ	Δ_{E}
25	1	1	0.27	5.60	1.92	3.99	0.13	7.80	0.00	3.76
25	1	2	0.69	7.00	3.35	7.69	0.19	10.50	0.00	6.95
25	2	1	0.68	7.40	3.27	2.68	0.26	8.00	2.17	1.95
25	2	2	3.17	9.60	5.79	5.26	0.64	12.50	3.64	3.24
50	2	1	10.74	9.10	1.81	2.75	1.27	14.10	1.44	2.22
50	2	2	88.10	15.30	3.00	5.07	3.01	20.70	2.36	4.25
50	5	1	215.04	21.20	6.29	1.45	28.84	22.20	5.24	0.89
50	5	2	3805.19	36.90	11.42	2.94	106.37	37.10	9.21	1.79

Table 3 Results for the correlated knapsack instances with discrete budgeted-ellipsoidal uncertainty.

here. Otherwise for axis-parallel ellipsoids all instances could be solved in at most 4 seconds in average. Interestingly here both gaps Δ_Γ and Δ_E are larger than in the case of general ellipsoids.

The correlated knapsack instances (see Tables 3 and 4) are much harder to solve in the discrete case. We were not even able to solve instances with a dimension of $n = 75$ in appropriate time. Here the computation time and the number of generated cuts increases significantly with growing n. The gaps Δ_Γ are much smaller than for the uncorrelated knapsack instances. Therefore the gaps Δ_{E} are larger. For axis-parallel ellipsoids the computation time is much better while the number of cuts is similar. Here in contrast to the uncorrelated instances the gaps Δ_Γ and Δ_{E} are smaller than the gaps for general ellipsoids. In the convex case the correlated knapsack instances could be solved very fast. The instances with $n = 50$ which were very hard to solve in the discrete case could be solved in at most a tenth of a second. Furthermore the gaps Δ_Γ and Δ_{E} are even larger than for the discrete case for general and for axis-parallel ellipsoids. Surprisingly here the instances for axis-parallel ellipsoids have larger computation times than the instances with general ellipsoids which is mainly caused by the second-order cone problem.

For the shortest path problem the computation times are slower and the number of cuts is larger compared to the knapsack instances of similar dimension. On the other hand the gaps Δ_Γ and Δ_{E} are significantly larger for general ellipsoids.

			General E			Axis-parallel E				
n	Γ	Ω	t	Δ_Γ	Δ_{E}	t	t_{lin}	t_{SOCP}	Δ_Γ	Δ_{E}
25	1	1	0.03	6.60	8.76	0.07	0.00	0.07	6.68	10.72
25	1	2	0.02	11.59	16.27	0.19	0.00	0.19	11.68	19.49
25	2	1	0.03	8.60	8.00	0.06	0.00	0.06	8.78	8.54
25	2	2	0.03	15.82	15.25	0.16	0.00	0.16	16.03	15.58
50	2	1	0.04	5.63	6.61	0.32	0.00	0.32	5.77	6.59
50	2	2	0.03	10.16	12.38	14.45	0.00	14.45	10.31	12.34
50	5	1	0.06	10.38	5.36	0.81	0.00	0.81	10.70	6.12
50	5	2	0.04	19.57	10.50	13.13	0.00	13.13	19.97	11.81

Table 4 Results for the correlated knapsack instances with convex budgeted-ellipsoidal uncertainty.

			General E				Axis-parallel E			
n	Γ	Ω	t	*#cuts*	Δ_Γ	Δ_{E}	t	*#cuts*	Δ_Γ	Δ_{E}
57	3	1	0.28	4.57	39.76	7.77	0.34	9.09	10.58	1.90
57	3	2	0.76	6.53	46.61	13.32	0.86	14.16	16.22	3.55
57	6	1	0.38	4.78	43.87	5.30	0.52	9.96	17.35	0.02
57	6	2	1.30	7.15	52.40	9.04	2.29	17.04	26.29	0.11
90	3	1	1.69	7.13	25.98	9.15	1.22	13.57	10.21	2.16
90	3	2	5.43	10.66	34.79	15.63	5.04	23.61	15.77	3.93
90	6	1	2.82	7.99	31.84	6.79	2.92	17.50	17.62	0.05
90	6	2	12.44	13.06	42.77	11.62	17.18	31.15	26.65	0.19
131	3	1	10.01	9.78	25.55	10.06	5.66	20.54	9.90	2.22
131	3	2	40.30	15.68	34.44	17.17	25.24	37.11	15.24	4.19
131	6	1	25.67	12.15	31.61	7.87	15.98	26.49	17.50	0.10
131	6	2	126.85	20.50	42.81	13.47	115.76	53.21	26.36	0.34
179	3	1	22.69	11.45	19.98	10.74	18.13	27.61	9.63	2.36
179	3	2	96.74	19.31	29.40	18.29	103.56	52.17	14.63	4.51
179	6	1	45.17	14.26	26.91	8.75	60.00	37.60	17.42	0.13
179	6	2	302.83	26.92	38.97	14.90	521.97	77.96	26.14	0.47
234	3	1	53.94	12.71	21.77	11.45	40.17	32.91	9.48	2.37
234	3	2	241.09	22.37	31.29	19.45	267.91	62.73	14.65	4.69
234	6	1	128.07	17.17	28.83	9.55	127.26	45.82	17.53	0.12
234	6	2	832.22	32.49	40.86	16.27	3370.22	99.69	26.25	0.57

Table 5 Results for the shortest path problem with discrete budgeted-ellipsoidal uncertainty.

Interestingly here the axis-parallel case is harder to solve for some configurations. Furthermore the gaps Δ_Γ and Δ_{E} are much smaller than for general ellipsoids. Especially for $\Gamma = 6$ the gaps Δ_{E} are never larger than 0.57 percent.

In the convex case the computation times are very small, both for general and axis-parallel ellipsoids, in contrast to the knapsack instances. Here all configurations could be solved in at most 3 seconds in average. Furthermore the gaps Δ_Γ and Δ_{E} are very large. Even the gaps Δ_{E} can be up to 23% in contrast to the very small gaps for the knapsack instances. For axis-parallel ellipsoids the gaps are significantly smaller than for general ellipsoids. Especially for $\Gamma = 6$ the gaps Δ_{E} are not larger than 0.46%.

6 Conclusion

The complexity results for budgeted-ellipsoidal uncertainty sets are not surprising. For the correlated case we could show that the corresponding min-max problem is *NP*-hard as it is the case for general ellipsoidal uncertainty. Note that the discrete variant of the budgeted-ellipsoidal uncertainty is not even convex. On the other hand positive complexity results could be achieved for the uncorrelated case if the underlying problem can be solved in polynomial time. This is an interesting

			General E			Axis-parallel E				
n	Γ	Ω	t	Δ_Γ	Δ_{E}	t	t_{lin}	t_{SOCP}	Δ_Γ	Δ_{E}
57	3	1	0.11	15.95	6.18	0.03	0.00	0.03	9.16	0.36
57	3	2	0.09	31.42	18.88	0.07	0.00	0.07	25.95	14.80
57	6	1	0.16	19.58	0.94	0.03	0.00	0.03	17.33	0.00
57	6	2	0.15	35.81	10.52	0.07	0.00	0.07	26.56	0.46
90	3	1	0.21	16.69	7.95	0.11	0.00	0.11	8.42	0.22
90	3	2	0.17	31.81	20.68	0.29	0.00	0.28	25.06	14.56
90	6	1	0.35	20.71	2.43	0.11	0.00	0.11	17.58	0.00
90	6	2	0.30	37.70	13.46	0.29	0.00	0.28	26.64	0.18
131	3	1	0.41	17.11	8.99	0.36	0.00	0.36	8.00	0.17
131	3	2	0.34	31.91	21.82	0.92	0.00	0.92	23.99	14.12
131	6	1	0.76	21.39	3.65	0.36	0.00	0.36	17.42	0.00
131	6	2	0.65	38.43	15.31	0.92	0.00	0.91	26.25	0.19
179	3	1	0.77	17.46	9.81	0.90	0.00	0.89	7.53	0.10
179	3	2	0.62	31.65	22.49	2.31	0.00	2.30	22.87	13.78
179	6	1	1.86	22.19	4.80	0.90	0.00	0.90	17.31	0.00
179	6	2	1.32	39.02	16.67	2.29	0.00	2.28	25.84	0.07
234	3	1	1.22	17.75	10.68	1.40	0.00	1.40	7.36	0.09
234	3	2	1.03	31.93	23.43	4.20	0.00	4.20	22.47	13.45
234	6	1	2.59	22.83	5.91	1.41	0.00	1.41	17.43	0.00
234	6	2	2.30	39.69	18.10	4.22	0.00	4.21	25.87	0.07

Table 6 Results for the shortest path problem with convex budgeted-ellipsoidal uncertainty.

result as for the classical robust counterpart with ellipsoidal uncertainty a similar result has only been proved for the spanning tree problem and the unconstrained problem. Besides the complexity results we presented algorithms to solve the robust counterpart for all versions of budgeted-ellipsoidal uncertainty which can be applied to many combinatorial problems. We could solve knapsack instances with up to 400 variables and shortest-path instances with up to 234 variables. It turned out that the knapsack instances for general ellipsoids and the shortest path instances for the discrete version were the hardest to solve. Here the computation time increased very fast with growing n. Nevertheless for axis-parallel ellipsoids and for the shortest path problem with convex budgeted-ellipsoidal uncertainty all instances could be solved very fast. Furthermore the price of robustness, i.e. the percental difference of the optimal value to the optimal values of the classical problems with budgeted and ellipsoidal uncertainty is often larger than 10%. Therefore an improvement of the algorithm for the discrete version would lead to an applicable and reasonable model for optimization problems under uncertainty.

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