On types of degenerate critical points of real polynomial functions☆

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Abstract

In this paper, we consider the problem of identifying the type (local minimizer, maximizer or saddle point) of a given isolated real critical point c, which is degenerate, of a multivariate polynomial function f. To this end, we introduce the definition of faithful radius of c by means of the curve of tangency of f. We show that the type of c can be determined by the global extrema of f over the Euclidean ball centered at c with a faithful radius. We propose algorithms to compute a faithful radius of c and determine its type.

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1. Introduction

Let $f \in \mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n]$, the polynomial ring over the real number field $\mathbb{R}$ with n variables. Throughout this paper, we denote upper case letters (like X, Y) as variables and lower case letters (like x, y) as points in the ambient spaces. Denote $\mathbf{0}$ as the origin or the vector of zeros. Given $c \in \mathbb{R}^n$ such that the gradient $\nabla f(c) = \mathbf{0}$ and the Hessian matrix $\nabla^2 f(c)$ is singular, i.e. c is a degenerate real critical point of f. An interesting problem is to identify the type of c, i.e. is c a local minimizer, maximizer or saddle point of f? To solve it, it is intuitive to consider the higher order partial derivatives of f at c. However, to the best of our knowledge, it is difficult to obtain a straightforward and simple method, which takes into account the higher order derivatives of f, to systematically solve this problem. When f is a sufficiently smooth function (not necessarily a polynomial), some partial answers to this problem were given in [6, 8] under certain assumptions on its Taylor expansion at c. When f is a multivariate real polynomial, Qi investigated its critical points and extrema structures in [22] without giving a computable method to determine their types. Nie gave a numerical method in [20] to compute all H-minimizers (critical points at which the Hessian matrices are positive semidefinite) of a polynomial by semidefinite relaxations. However, there is no completed procedure in [20] to verified that a H-minimizer is a saddle point.

Without loss of generality, we suppose that $c = \mathbf{0}$ and $f(\mathbf{0}) = 0$. In this paper, we consider the case when

$$\mathbf{0} \textit{ is an } \textbf{\textit{isolated}} \textit{ real critical point of } f,$$

i.e., there exists a neighborhood $\mathcal{O} \subseteq \mathbb{R}^n$ of $\mathbf{0}$ such that $\mathbf{0}$ is the only real critical point of f in $\mathcal{O}$. It is well known that any small changes of the coefficients of f may render $\mathbf{0}$ nondegenerate. Hence, we aim to present a computable and symbolic method to determine the type of $\mathbf{0}$.

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Now let us briefly introduce the basic idea we use to deal with this problem and the contribution made in this paper. Denote $\mathbb{R}_+$ as the set of positive real numbers and $\|x\|_2$ as the Euclidean norm of $x \in \mathbb{R}^n$. For any $r \in \mathbb{R}_+$, let

$$\mathsf{B}_r := \{x \in \mathbb{R}^n \mid \|x\|_2 \leq r\} \quad \text{and} \quad \mathsf{S}_r := \{x \in \mathbb{R}^n \mid \|x\|_2 = r\}. \tag{1.1}$$

Define

$$f_r^{\min} := \min\{f(x) \mid x \in \mathsf{B}_r\} \quad \text{and} \quad f_r^{\max} := \max\{f(x) \mid x \in \mathsf{B}_r\}. \tag{1.2}$$

Obviously, it holds that

(i) if $\mathbf{0}$ is a local minimizer, then $f_r^{\max} > 0$ and $f_r^{\min} = 0$ for some $r \in \mathbb{R}_+$;

(ii) if $\mathbf{0}$ is a local maximizer, then $f_r^{\max} = 0$ and $f_r^{\min} < 0$ for some $r \in \mathbb{R}_+$;

(iii) if $\mathbf{0}$ is a saddle point, then $f_r^{\max} > 0$ and $f_r^{\min} < 0$ for any $r \in \mathbb{R}_+$.

Now we consider the above statements the other way around. That is, can we classify the degenerate critical point $\mathbf{0}$ by the signs of $f_r^{\min}$ and $f_r^{\max}$? Two issues have to be addressed.

(1) If $\mathbf{0}$ is a local minimizer or maximizer, it can be certified by giving a radius r such that $f_r^{\min} = 0$ or $f_r^{\max} = 0$. The difficulty is that if $f_r^{\min} < 0 < f_r^{\max}$ for some $r \in \mathbb{R}_+$, then what is the type of $\mathbf{0}$? Since we do not know if the radius r is sufficiently small, we can not claim that $\mathbf{0}$ is a saddle point. For example, consider the polynomial $f(X_1, X_2) = X_1^2 + (1 - X_1)X_2^4$ (Example 4.18) with $\mathbf{0}$ being an isolated real critical point. Notice that $\mathbf{0}$ is degenerate. If we choose $r = 2\sqrt{2}$, then $f_r^{\min} \leq f(2, 2) = -12 < 0 < 52 = f(-2, 2) \leq f_r^{\max}$. However, if $r < 1$, we have $f(x_1, x_2) > 0$ for any $x \in \mathsf{B}_r \backslash \{\mathbf{0}\}$ and $f_r^{\min} = 0$. Hence, $\mathbf{0}$ is not a saddle point but a strict local minimizer. Therefore, in order to determine the type of $\mathbf{0}$ by the global extrema of f over some ball B_r, we need to ensure that the radius r is sufficiently small.

(2) Note that the optimization problems in (1.2) are themselves NP-hard. In particular, maximizing a cubic polynomial over a unit ball is NP-hard [19]. Numerically, approximation methods for polynomial optimization problems based on semidefinite relaxations have been extensively studied [13, 15, 16, 17]. However, since we need to certify that $f_r^{\min}$ and $f_r^{\max}$ in cases (i) and (ii) are exact 0, any numerical errors in the output of approximation methods for (1.2) may mislead us to the wrong case (iii). Symbolically, a univariate polynomial whose roots contain $f_r^{\min}$ and $f_r^{\max}$ may be obtained by means of the KKT system of (1.2) and some elimination computations as in [28]. However, to determine the signs of $f_r^{\min}$ and $f_r^{\max}$, extra symbolic computations are needed to find one point in some real algebraic set or to certify its emptiness [24, 25].

In this paper, we aim to tackle the above issues and classify isolated critical point $\mathbf{0}$ of f by its global extrema over some Euclidean balls. For the first issue, we define the so-called *faithful radius* (Definition 3.6) of $\mathbf{0}$, via the curve of tangency of f at $\mathbf{0}$ which is studied in [9, 12], such that the type of $\mathbf{0}$ can be determined by the signs of $f_r^{\max}$ and $f_r^{\min}$ for any faithful radius r of $\mathbf{0}$ (Theorem 3.7). Provided that an isolation radius (Definition 3.5) of $\mathbf{0}$ is known, we propose an algorithm (Algorithm 4.16) to compute a faithful radius of $\mathbf{0}$. We also discuss some strategies to compute an isolation radius of $\mathbf{0}$. For the second issue, instead of computing the extrema $f_r^{\max}$ and $f_r^{\min}$ in (1.2), we present an algorithm (Algorithm 5.4) to identify the type of $\mathbf{0}$ by computing isolating intervals for each real root of a zero-dimensional polynomial system, which can be done by, for example, the Rational Univariate Representations (RUR) [23] for multivariate polynomial systems.

The paper is organized as follows. Some notation and preliminaries used in this paper are given in Section 2. We define the faithful radius of $\mathbf{0}$ in Section 3. An algorithm for computing a faithful radius of $\mathbf{0}$ is presented in Section 4. We show that how to determine the type of $\mathbf{0}$ in a symbolic way in Section 5. Some conclusion is made in Section 6.

2. Notation and preliminaries

The symbol $\mathbb{R}$ (resp., $\mathbb{C}$) denotes the set of real (resp., complex) numbers. Denote $\mathbb{R}^{n\times n}$ (resp., $\mathbb{C}^{n\times n}$) as the set of $n \times n$ matrices with real (resp. complex) number entries. $\mathbb{R}[X] = \mathbb{R}[X_1, \ldots, X_n]$ denotes the ring of polynomials in variables $X = (X_1, \ldots, X_n)$ with real coefficients. Denote $\|X\|_2^2 \in \mathbb{R}[X]$ as the polynomial $X_1^2 + \cdots + X_n^2$ in variables X while $\|x\|_2$ as the Euclidean norm of $x \in \mathbb{R}^n$. If f, g are two functions with suitably chosen domains and codomains, then $f \circ g$ denotes the composite function of f and g.

A subset $I \subseteq \mathbb{R}[X]$ is called an ideal if $0 \in I$, $I + I \subseteq I$ and $p \cdot q \in I$ for all $p \in I$ and $q \in \mathbb{R}[X]$. The product of two ideals I and J in $\mathbb{R}[X]$, denoted by $I \cdot J$, is the ideal generated by all products $f \cdot g$ where $f \in I$ and $g \in J$. For $g_1, \ldots, g_s \in \mathbb{R}[X]$, denote $\langle g_1, \ldots, g_s\rangle$ as the ideal in $\mathbb{R}[X]$ generated by g_i's, i.e., the set $g_1\mathbb{R}[X] + \cdots + g_s\mathbb{R}[X]$. An ideal is radical if $f^m \in I$ for some integer $m \geq 1$ implies that $f \in I$. The radical of an ideal $I \subseteq \mathbb{R}[X]$, denoted $\sqrt{I}$, is the set $\{f \in \mathbb{R}[X] \mid f^m \in I$ for some integer $m \geq 1\}$. An (resp. real) affine variety is a subset of $\mathbb{C}^n$ (resp. $\mathbb{R}^n$) that consists of common zeros of a set of polynomials. For an ideal $I \subseteq \mathbb{R}[X]$, denote $\mathbf{V}_{\mathbb{C}}(I)$ and $\mathbf{V}_{\mathbb{R}}(I)$ as the affine varieties defined by I in $\mathbb{C}^n$ and $\mathbb{R}^n$, respectively. For a polynomial $g \in \mathbb{R}[X]$, respectively replace $\mathbf{V}_{\mathbb{C}}(\langle g\rangle)$ and $\mathbf{V}_{\mathbb{R}}(\langle g\rangle)$ by $\mathbf{V}_{\mathbb{C}}(g)$ and $\mathbf{V}_{\mathbb{R}}(g)$ for simplicity. Given a set $V \subseteq \mathbb{C}^n$, denote $\mathbf{I}(V) \subseteq \mathbb{R}[X]$ as the vanishing ideal of V in $\mathbb{R}[X]$, i.e., the set of all polynomials in $\mathbb{R}[X]$ which equal zero at every point in V. For an ideal $I \subseteq \mathbb{R}[X]$, denote $\dim(I)$ as the Hilbert dimension of I, i.e., the degree of the affine Hilbert polynomial of I. For an ideal $I \subseteq \mathbb{R}[X]$, the decomposition $I = I_1 \cap \cdots \cap I_s$ is called the equidimensional decomposition of I if each ideal I_i is pure dimensional, i.e., all its associated primes have the same dimension. For an affine variety $V \subseteq \mathbb{C}^n$, denote $\dim(V) = \dim(\mathbf{I}(V))$ as its dimension. When $\mathbf{V}_{\mathbb{C}}(I)$ is finite, the ideal I is called to be zero-dimensional. For any subset $S \subseteq \mathbb{C}^n$, denote $\overline{S}^{\mathcal{Z}}$ as the Zariski closure of S in $\mathbb{C}^n$, i.e., $\overline{S}^{\mathcal{Z}} = \mathbf{V}_{\mathbb{C}}(\mathbf{I}(S))$. The l-th elimination ideal I_l of an ideal $I \in \mathbb{R}[X]$ is the ideal of $\mathbb{R}[X_{l+1}, \ldots, X_n]$ defined by $I_l = I \cap \mathbb{R}[X_{l+1}, \ldots, X_n]$ which can be computed by the Groebner basis of I with respect to an elimination order of X. For more basic concepts from algebraic geometry, we refer to [7, 10]. The following procedures are considered as black boxes in this paper (c.f. [4, 10]):

- Compute the Hilbert dimension of a given ideal $I \subseteq \mathbb{R}[X]$;
- Compute the equidimensional decomposition of a given ideal $I \subseteq \mathbb{R}[X]$;
- Test whether a given ideal $I \subset \mathbb{R}[X]$ is radical and compute $\sqrt{I}$ if it is not;
- Compute the vanishing ideal $\mathbf{I}\left(\overline{\mathbf{V}_{\mathbb{C}}(I)\backslash\mathbf{V}_{\mathbb{C}}(J)}^{\mathcal{Z}}\right) \subseteq \mathbb{R}[X]$ for some ideals $I, J \subseteq \mathbb{R}[X]$;
- Compute isolating intervals for each real root of a zero-dimensional polynomial system. This can be done by, for example, the Rational Univariate Representations (RUR) [23] for multivariate polynomial systems.

We recall some background in real algebraic geometry and refer to [5] for more details. A semi-algebraic subset of $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ satisfying a boolean combination of polynomial equations and inequalities with real coefficients. In this paper, $\mathbb{R}^n$ will always be considered with its Euclidean topology, unless stated otherwise. Let $S_1 \subseteq \mathbb{R}^m$ and $S_2 \subseteq \mathbb{R}^n$ be two semi-algebraic sets. A mapping $\psi : S_1 \to S_2$ is semi-algebraic if its graph is semi-algebraic in $\mathbb{R}^{m+n}$.

Theorem 2.1. *[5, Theorem 2.3.6] Every semi-algebraic subset S of $\mathbb{R}^n$ is the disjoint union of a finite number of semi-algebraic sets $\cup_{i=1}^s S_i$. Each S_i is semi-algebraically homeomorphic to an open hypercube $(0,1)^{d_i} \subseteq \mathbb{R}^{d_i}$ for some $d_i \in \mathbb{N}$.*

The dimension $\dim(S)$ of a semi-algebraic set $S \subseteq \mathbb{R}^n$ is the maximum of d_i as in Theorem 2.1. A subset $S \subseteq \mathbb{R}^n$ is connected if for every pair of sets S_1 and S_2 closed in S, disjoint and satisfying $S_1 \cup S_2 = S$, one has $S_1 = S$ or $S_2 = S$. Given points p_1 and p_2 of a subset $S \subseteq \mathbb{R}^n$, a path in S from p_1 to p_2 is a continuous map $\varphi : [a, b] \to S$ of some closed interval in the real line into S, such that $\varphi(a) = p_1$ and $\varphi(b) = p_2$. A subset $S \subseteq \mathbb{R}^n$ is said to be path connected if every pair of points of S can be jointed by a path in S.

Combining Theorems 2.4.4, 2.4.5 and Proposition 2.5.13 in [5], it follows that

Proposition 2.2. *Let S be a semi-algebraic set of $\mathbb{R}^n$. Then,*

(i) *S has a finite number of connected components which are closed in S;*

(ii) *S is connected if and only if it is path connected.*

Hence, the rest of this paper, by saying that a semi-algebraic subset of $\mathbb{R}^n$ is connected, we also mean that it is path connected.

Theorem 2.3. *[18, Curve selection lemma] Let S be a semi-algebraic subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$ a point belonging to the closure of S. Then there exists an analytic semi-algebraic mapping $\varphi : [0, \epsilon] \to \mathbb{R}^n$ such that $\varphi(0) = x$ and $\varphi((0, \epsilon]) \subset S$.*

3. Faithful radius and types of degenerate critical points

In the rest of this paper, we always denote f as the considered polynomial in $\mathbb{R}[X]$ with $\mathbf{0}$ being an isolated real critical point.

Denote $\mathsf{Crit}_{\mathbb{R}}(f)$ and $\mathsf{Crit}_{\mathbb{C}}(f)$ as the sets of real and complex critical points of f, respectively. Define

$$\Gamma_{\mathbb{R}}(f) := \{x \in \mathbb{R}^n \mid \exists \lambda \in \mathbb{R} \quad \text{s.t. } \nabla f(x) = \lambda x\}. \tag{3.1}$$

Since $\mathbf{0} \in \mathsf{Crit}_{\mathbb{R}}(f)$, we have

$$\Gamma_{\mathbb{R}}(f) = \left\{ x \in \mathbb{R}^n \;\middle|\; \frac{\partial f}{\partial x_i} x_j = \frac{\partial f}{\partial x_j} x_i,\ 1 \le i < j \le n \right\}.$$

The real variety $\Gamma_{\mathbb{R}}(f)$ is called the *tangency variety* at the origin [9, 12]. Geometrically, the tangency variety $\Gamma_{\mathbb{R}}(f)$ consists of all points x in $\mathbb{R}^n$ at which the level set of f is tangent to the sphere in $\mathbb{R}^n$ centered at the origin with radius $\|x\|_2$.

Proposition 3.1. *For any $r \in \mathbb{R}_+$ and $u \in \mathsf{B}_r$, there exists a point $v \in \Gamma_{\mathbb{R}}(f)$ with $\|v\|_2 \le \|u\|_2$ such that $f(v) = f(u)$.*

Proof. Consider the following optimization problem

$$\min_{x \in \mathbb{R}^n} \|x\|_2^2 \quad \text{s.t. } f(x) = f(u).$$

Since u is a feasible point, there exists a minimizer v with $\|v\|_2 \le \|u\|_2$. If $\nabla f(v) = \mathbf{0}$, then clearly $v \in \Gamma_{\mathbb{R}}(f)$. Otherwise, the linear independence constraint qualification condition holds at v and therefore v satisfies the Karush–Kuhn–Tucker optimality condition. It implies that $v \in \Gamma_{\mathbb{R}}(f)$. □

Remark 3.2. *For any $r \in \mathbb{R}_+$, the semi-algebraic set $\Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_r$ has finitely many connected components K_i with $\mathbf{0}$ belonging to their closures by Proposition 2.2. For each i, by the curve selection lemma, there exists an analytic curve $\varphi_i : [0, \epsilon] \to \mathbb{R}^n$ such that $\varphi_i(0) = \mathbf{0}$ and $\varphi_i(t) \in K_i$ for $t \neq 0$. By Proposition 3.1, it can be shown that the behavior of f along the curves φ_i captures all information of f near $\mathbf{0}$. That is, we can identify the type of the critical point $\mathbf{0}$ of f by extremal test of the univariate functions $f \circ \varphi_i$ at 0. This approach was studied in [2, 3, 12] which, however, provide no general procedures to compute the expressions of the analytic functions φ_i.*

For any $r \in \mathbb{R}_+$, recall the definition of $f_r^{\min}$ and $f_r^{\max}$ in (1.2).

Corollary 3.3. *For any $r \in \mathbb{R}_+$, we have*

$$f_r^{\min} = \min\{f(x) \mid x \in \Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_r\} \quad \textit{and} \quad f_r^{\max} = \max\{f(x) \mid x \in \Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_r\}.$$

Proof. Since $f_r^{\min}$ and $f_r^{\max}$ can be reached, the conclusion follows from Proposition 3.1. □

Corollary 3.4. $\mathbf{0}$ *is not isolated in* $\Gamma_{\mathbb{R}}(f)$ *and* $\dim(\Gamma_{\mathbb{R}}(f)\backslash\mathsf{Crit}_{\mathbb{R}}(f)) \geq 1$.

Proof. Since f is not a zero polynomial, either $f_r^{\max}$ or $f_r^{\min}$ is nonzero for any $r \in \mathbb{R}_+$. Hence by Proposition 3.1, there exists a nonzero $u_r \in \Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_r$ such that either $f(u_r) = f_r^{\max} \neq 0$ or $f(u_r) = f_r^{\min} \neq 0$. Thus, $\mathbf{0}$ is not isolated in $\Gamma_{\mathbb{R}}(f)$ since $\lim_{r\to 0} u_r = \mathbf{0}$.

Note that $\{f(x) \mid x \in \mathsf{Crit}_{\mathbb{R}}(f)\}$ is a finite set by Sard's theorem. Because $f(u_r) \neq 0$ for each r and $\lim_{r\to 0} f(u_r) = 0$, there must be infinitely many $u_r \in \Gamma_{\mathbb{R}}(f)\backslash\mathsf{Crit}_{\mathbb{R}}(f)$. Then we have $\dim(\Gamma_{\mathbb{R}}(f)\backslash\mathsf{Crit}_{\mathbb{R}}(f)) \geq 1$ by Theorem 2.1. □

Definition 3.5. *We call a* $R \in \mathbb{R}_+$ *an* ***isolation radius*** *of* $\mathbf{0}$ *if* $\mathsf{Crit}_{\mathbb{R}}(f) \cap \mathsf{B}_R = \{\mathbf{0}\}$.

Definition 3.6. *We call an isolation radius* R *of* $\mathbf{0}$ *a* ***faithful radius*** *if the following conditions hold:* (i) $\Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_R$ *is connected;* (ii) $\Gamma_{\mathbb{R}}(f) \cap \mathbf{V}_{\mathbb{R}}(f) \cap \mathsf{B}_R = \{\mathbf{0}\}$.

Note that $\Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_R$ is also path connected if R is a faithful radius by Proposition 2.2. The following result shows that if R is a faithful radius, then we can classify the degenerate real critical point $\mathbf{0}$ of f by the signs of its global extrema over the ball $\mathbf{B}_R$.

Theorem 3.7. *Suppose* $R \in \mathbb{R}_+$ *is a faithful radius, then*

(1) $\mathbf{0}$ *is a local minimizer if and only if* $f_R^{\max} > 0$ *and* $f_R^{\min} = 0$*;*

(2) $\mathbf{0}$ *is a local maximizer if and only if* $f_R^{\max} = 0$ *and* $f_R^{\min} < 0$*;*

(3) $\mathbf{0}$ *is a saddle point if and only if* $f_R^{\max} > 0$ *and* $f_R^{\min} < 0$.

Proof. (1) and (2) are clear if we can prove (3). Since $\mathbf{0}$ is an isolated real critical point, we only need to prove the "if" part. By Corollary 3.3, there exists a $u \in \Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_R$ such that $f(u) = f_R^{\min}$. Since R is a faithful radius, $\mathbf{0}$ and u are path connected, i.e. there exists a continuous mapping $\phi(t) : [a, b] \to \Gamma_{\mathbb{R}}(f)$ such that $\mathbf{0} \notin \phi((a, b))$, $\phi(a) = \mathbf{0}$ and $\phi(b) = u$. We have $f(\phi(t)) < 0$ for all $t \in (a, b]$. Otherwise, by the continuity, there exists $\bar{t} \in (a, b)$ such that $f(\phi(\bar{t})) = 0$. Since R is faithful, we have $\phi(\bar{t}) = \mathbf{0}$ by the definition, a contradiction. Similarly, let $f_R^{\max} > 0$ be reached at $v \in \Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_R$, then there exists a continuous mapping $\varphi(t) : [a, b] \to \Gamma_{\mathbb{R}}(f)$ such that $\mathbf{0} \notin \varphi((a, b))$, $\varphi(a) = \mathbf{0}$, $\varphi(b) = v$ and $f(\varphi(t)) > 0$ for all $t \in (a, b]$. Therefore, $\mathbf{0}$ is a saddle point of f. □

There always exists a faithful radius of $\mathbf{0}$. In fact,

Theorem 3.8. $\mathbf{0}$ *is an isolated real critical point of* f *if and only if there is a faithful radius of* $\mathbf{0}$.

Proof. We only need to prove the "only if" part and assume that $\mathbf{0}$ is an isolated real critical point of f.

(i) By the assumption, there is an isolation radius $R_1 \in \mathbb{R}_+$ such that $\mathsf{Crit}_{\mathbb{R}}(f) \cap \mathsf{B}_{R_1} = \{\mathbf{0}\}$.

(ii) Since $\Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_R$ is a closed semi-algebraic set, by Proposition 2.2, it has finitely many connected components $\mathcal{C}_1, \ldots, \mathcal{C}_s$ which are closed in $\mathbb{R}^n$. Assume that the components $\mathcal{C}_i$, $2 \leq i \leq s$, do not contain $\{\mathbf{0}\}$. For each $2 \leq i \leq s$, since the component $\mathcal{C}_i$ is closed and bounded, the function $\sum_{i=1}^n X_i^2$ reaches its minimum on $\mathcal{C}_i$ at a minimizer $u^{(i)} \in \mathcal{C}_i$. Fix a $R_2 \in \mathbb{R}_+$ such that $0 < R_2 < \min_{2\leq i\leq s} \|u^{(i)}\|_2$, then $\Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_{R_2}$ is connected.

(iii) We claim that there exists $R_3 \in \mathbb{R}_+$ such that $\Gamma_{\mathbb{R}}(f) \cap \mathbf{V}_{\mathbb{R}}(f) \cap \mathsf{B}_{R_3} = \{\mathbf{0}\}$. Suppose to the contrary that such R_3 does not exist. By the Curve Selection Lemma, we can find an analytic curve $\phi\colon [0, \epsilon] \to \mathbb{R}^n$ such that $\phi(0) = \mathbf{0}, f(\phi(t)) = 0$ and $\phi(t) \in \Gamma_{\mathbb{R}}(f)\backslash\{\mathbf{0}\}$ for all $t \in (0, \epsilon]$. For each t, by the definition, $\nabla f(\phi(t)) = \lambda(t)\phi(t)$ for some $\lambda(t) \in \mathbb{R}$ and furthermore,

$$0 = \frac{\mathrm{d}(f \circ \phi)(t)}{\mathrm{d}t} = \left\langle \nabla f(\phi(t)), \frac{\mathrm{d}\phi(t)}{\mathrm{d}t} \right\rangle = \lambda(t)\frac{\mathrm{d}\|\phi(t)\|_2^2}{2\mathrm{d}t}.$$

By the monotonicity lemma, $\lambda(t) = 0$ and hence $\nabla f(\phi(t)) = \mathbf{0}$ for $0 \le t \ll 1$, which is a contradiction since $\mathbf{0}$ is an isolated real critical point of f.

Clearly, $R := \min\{R_1, R_2, R_3\}$ is a faithful radius of f. □

4. Computational aspects of faithful radius

In this section, we present some computational criteria and an algorithm for computing a faithful radius of the isolated real critical point $\mathbf{0}$ of the polynomial f.

4.1. Curve of tangency

We now recall some background about the tangency variety at a general point which is studied in [9, 12]. For any $a \in \mathbb{R}^n$, let

$$\Gamma_{\mathbb{R}}(f, a) = \{x \in \mathbb{R}^n \mid \exists \lambda \in \mathbb{R} \quad \text{s.t. } \nabla f(x) = \lambda(x - a)\}.$$

In particular, $\Gamma_{\mathbb{R}}(f) = \Gamma_{\mathbb{R}}(f, \mathbf{0})$. Geometrically, the tangency variety $\Gamma_{\mathbb{R}}(f, a)$ consists of all points in $\mathbb{R}^n$ at which the level set of f is tangent to the sphere in $\mathbb{R}^n$ centered in a with radius $\|x - a\|_2$.

Proposition 4.1. *[12, Lemma 2.1] It holds that*

(i) $\Gamma_{\mathbb{R}}(f, a)$ *is a nonempty, unbounded and semi-algebraic set;*

(ii) *There exists a proper algebraic set* $\Omega \subseteq \mathbb{R}^n$ *such that for each* $a \in \mathbb{R}\backslash\Omega$*, the set* $\Gamma_{\mathbb{R}}(f, a)\backslash\mathsf{Crit}_{\mathbb{R}}(f)$ *is a one-dimensional submanifold of* $\mathbb{R}^n$*.*

Therefore, $\Gamma_{\mathbb{R}}(f, a)$ is also called *curve of tangency.* Note that for the given $f \in \mathbb{R}[X]$, $\mathbf{0}$ might not belong to Ω as in Proposition 4.1 and then the statement (ii) in Proposition 4.1 is not necessarily true for $\Gamma_{\mathbb{R}}(f) = \Gamma_{\mathbb{R}}(f, \mathbf{0})$. However, in the following we will show that $\Gamma_{\mathbb{R}}(f)\backslash\mathsf{Crit}_{\mathbb{R}}(f)$ is indeed a one-dimensional semi-algebraic set of $\mathbb{R}^n$ after a generic linear change of the coordinates of f.

For $f \in \mathbb{R}[X]$ and an invertible matrix $A \in \mathbb{R}^{n\times n}$, denote $f^A = f(Ax)$ the polynomial obtained by applying the change of variables A to f. Denote $\mathcal{S}^{n\times n} \subset \mathbb{R}^{n\times n}$ (resp. $\mathcal{S}^{n\times n}_{++} \subset \mathbb{R}^{n\times n}$) as the set of symmetric (resp. positive definite) matrices with real number entries. For any matrix $P \in \mathbb{R}^{n\times n}$, define

$$\Gamma_{\mathbb{R}}(f, P) = \{x \in \mathbb{R}^n \mid \exists \lambda \in \mathbb{R}, \quad \text{s.t. } \nabla f(x) = \lambda P x\}.$$

Given an invertible matrix $A \in \mathbb{R}^{n\times n}$ and a subset $S \subseteq \mathbb{R}^n$, let

$$A(S) = \{Ax \mid x \in S\}.$$

Lemma 4.2. *Given an invertible matrix* $A \in \mathbb{R}^{n\times n}$*, let* $P = A^{-T}A^{-1}$*, then we have* $\Gamma_{\mathbb{R}}(f^A) = A^{-1}(\Gamma_{\mathbb{R}}(f, P))$ *and* $\mathsf{Crit}_{\mathbb{R}}(f^A) = A^{-1}(\mathsf{Crit}_{\mathbb{R}}(f))$*.*

Proof. By the definition, we have

$$\begin{aligned}\Gamma_{\mathbb{R}}(f^A) &= \{x \in \mathbb{R}^n \mid \exists \lambda \in \mathbb{R} \quad \text{s.t. } \nabla f^A(x) = \lambda x\} \\ &= \{x \in \mathbb{R}^n \mid \exists \lambda \in \mathbb{R} \quad \text{s.t. } A^T \nabla f(Ax) = \lambda x\} \\ &= \{A^{-1}y \in \mathbb{R}^n \mid \exists \lambda \in \mathbb{R} \quad \text{s.t. } \nabla f(y) = \lambda A^{-T}A^{-1}y\} \\ &= A^{-1}(\Gamma_{\mathbb{R}}(f, P)).\end{aligned}$$

Similarly, it holds that $\mathsf{Crit}_{\mathbb{R}}(f^A) = A^{-1}(\mathsf{Crit}_{\mathbb{R}}(f))$. □

Let $\mathcal{I}^{n\times n}$ be the set of all invertible $n \times n$ matrices in $\mathbb{R}^{n\times n}$.

Theorem 4.3. *There exists an open and dense semi-algebraic set $\mathcal{U} \subset \mathcal{I}^{n\times n}$ such that for all $A \in \mathcal{U}$, the set $\Gamma_{\mathbb{R}}(f^A)\backslash\mathsf{Crit}_{\mathbb{R}}(f^A)$ is a one-dimensional semi-algebraic set of $\mathbb{R}^n$.*

Proof. Clearly, $\mathcal{S}^{n\times n}_{++}$ is an open semi-algebraic subset of $\mathcal{S}^{n\times n} \equiv \mathbb{R}^{\frac{n(n+1)}{2}}$, where we identify $P := (p_{ij})_{n\times n} \in \mathcal{S}^{n\times n}$ with

$$(p_{11}, \ldots, p_{1n}, p_{22}, \ldots, p_{2n}, \ldots, p_{nn}) \in \mathbb{R}^{\frac{n(n+1)}{2}}.$$

We first show that $\Gamma_{\mathbb{R}}(f, P)\backslash\mathsf{Crit}_{\mathbb{R}}(f)$ is a semi-algebraic set of dimension ≤ 1 for almost every $P \in \mathcal{S}^{n\times n}_{++}$. To do this, we consider the semi-algebraic map

$$\begin{array}{rcl} F:\ (\mathbb{R}^n\backslash\mathsf{Crit}_{\mathbb{R}}(f)) \times \mathbb{R} \times \mathcal{S}^{n\times n}_{++} & \to & \mathbb{R}^n \\ (x, \lambda, P) & \mapsto & \nabla f(x) - \lambda P x. \end{array}$$

We will show that $\mathbf{0} \in \mathbb{R}^n$ is a regular value of the map F. Take any $(x, \lambda, P) \in F^{-1}(\mathbf{0})$, then $x \neq \mathbf{0}$ and $\lambda \neq 0$. Otherwise, we have $\nabla f(x) = \mathbf{0}$ and hence $x \in \mathsf{Crit}_{\mathbb{R}}(f)$, a contradiction. Without loss of generality, we assume that $x_1 \neq 0$. Note that $p_{ij} = p_{ji}$. Then, a direct computation shows that the Jacobian matrix $\mathsf{Jac}(F)$ of the map F contains the following columns

$$-\lambda \cdot \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ 0 & x_1 & 0 & \cdots & 0 \\ 0 & 0 & x_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & x_1 \end{bmatrix},$$

which correspond to the partial derivatives of F with respect to the variables p_{1j} for $j = 1, \ldots, n$. Therefore, for all $(x, \lambda, P) \in F^{-1}(\mathbf{0})$, we have $\mathsf{rank}(\mathsf{Jac}(F)) = n$ and hence $\mathbf{0}$ is a regular value of F. By Thom's weak transversality theorem ([11], [13, Theorem 1.10]), there exists a semi-algebraic set $\Sigma \subset \mathcal{S}^{n\times n}_{++}$ of dimension $< \frac{n(n+1)}{2}$ such that for all $P \in \mathcal{S}^{n\times n}_{++} \setminus \Sigma$, $\mathbf{0}$ is a regular value of the map

$$\begin{array}{rcl} F_P:\ (\mathbb{R}^n\backslash\mathsf{Crit}_{\mathbb{R}}(f)) \times \mathbb{R} & \to & \mathbb{R}^n \\ (x, \lambda) & \mapsto & F(x, \lambda, P). \end{array}$$

Thus, $F_P^{-1}(\mathbf{0})$ is either empty or a one-dimensional submanifold of $\mathbb{R}^n$. Since $\Gamma_{\mathbb{R}}(f, P)\backslash\mathsf{Crit}_{\mathbb{R}}(f)$ is the projection of $F_P^{-1}(\mathbf{0})$ on the first n coordinates, by [5, Proposition 2.8.6], we have $\dim(\Gamma_{\mathbb{R}}(f, P)\backslash\mathsf{Crit}_{\mathbb{R}}(f)) \leq 1$.

Next, it is easy to see that

$$\mathcal{S}^{n\times n}_{++} \to \mathcal{S}^{n\times n}_{++}, \quad P \mapsto P^{-1},$$

is a semi-algebraic homeomorphism. Hence,

$$\Sigma^{-1} := \{P^{-1} \in \mathcal{S}^{n\times n}_{++} \mid P \in \Sigma\} \subset \mathbb{R}^{\frac{n(n+1)}{2}}$$

is a semi-algebraic set of dimension $< \frac{n(n+1)}{2}$. Consequently, by [13, Lemma 1.4], there exists a non-constant polynomial $\mathcal{F}\colon \mathbb{R}^{\frac{n(n+1)}{2}} \to \mathbb{R}$ such that

$$\Sigma^{-1} \subset \{Q \in \mathcal{S}^{n\times n}_{++} \mid \mathcal{F}(Q) = 0\}.$$

Note that the corresponding

$$\mathcal{I}^{n\times n} \to \mathcal{S}^{n\times n}_{++}, \quad A \mapsto AA^T,$$

is a polynomial map. Thus,

$$\{A \in \mathcal{I}^{n\times n} \mid \mathcal{F}(AA^T) = 0\}$$

is an algebraic set. It follows that $\mathcal{U} := \{A \in \mathcal{I}^{n\times n} \mid \mathcal{F}(AA^T) \neq 0\}$ is an open and dense semi-algebraic subset of $\mathcal{I}^{n\times n}$. Furthermore, by the definition, for all $A \in \mathcal{U}$, we have $P := (AA^T)^{-1} \in \mathcal{S}^{n\times n}_{++} \setminus \Sigma$ and hence, $\dim(\Gamma_{\mathbb{R}}(f^A)\backslash\mathsf{Crit}_{\mathbb{R}}(f^A)) \leq 1$ by Lemma 4.2. Since $A \in \mathcal{U}$ is invertible, we have $\dim(\Gamma_{\mathbb{R}}(f^A)\backslash\mathsf{Crit}_{\mathbb{R}}(f^A)) \geq 1$ by Corollary 3.4 and then the conclusion follows. □

Remark 4.4. (i) The technique of a generic linear change of variables was also used in [25] to show the dimension of polar varieties; (ii) We may also use the new inner product $\langle x, x'\rangle_P := \langle Px, Px'\rangle$ (and the corresponding norm $\sqrt{\langle x, x'\rangle_P}$) for some generic $P \in \mathcal{S}_{++}^{n\times n}$ instead of using a generic linear change of variables $y = Ax$. In fact, it is not hard to see that with this new inner product, $\Gamma_{\mathbb{R}}(f)\backslash \mathsf{Crit}_{\mathbb{R}}(f)$ is also a one-dimensional semi-algebraic set of $\mathbb{R}^n$.

We illustrate the result in Theorem 4.3 by the following simple example.

Example 4.5. *Consider the polynomial* $f(X_1, X_2) = X_1^2 + X_2^2$. *We have* $\Gamma_{\mathbb{R}}(f) = \mathbb{R}^2$. *However, if we make a linear change of variables and let* $f^A = (a_{1,1}X_1 + a_{1,2}X_2)^2 + (a_{2,1}X_1 + a_{2,2}X_2)^2$, *then* $\Gamma_{\mathbb{R}}(f^A) = \{(x_1, x_2) \in \mathbb{R}^2 \mid (a_{1,1}a_{1,2} + a_{2,1}a_{2,2})(x_2^2 - x_1^2) + (a_{1,1}^2 + a_{2,1}^2 - a_{1,2}^2 - a_{2,2}^2)x_1x_2 = 0\}$. *Clearly,* $\dim(\Gamma_{\mathbb{R}}(f^A)\backslash \mathsf{Crit}_{\mathbb{R}}(f^A)) = 1$ *whenever* $a_{1,1}a_{1,2} + a_{2,1}a_{2,2} \neq 0$ *or* $a_{1,1}^2 + a_{2,1}^2 - a_{1,2}^2 - a_{2,2}^2 \neq 0$.

For $f \in \mathbb{R}[X]$ and any matrix $P \in \mathbb{R}^{n\times n}$, let

$$\begin{aligned} \Gamma_{\mathbb{C}}(f) &:= \{x \in \mathbb{C}^n \mid \exists \lambda \in \mathbb{C} \quad \text{s.t. } \nabla f(x) = \lambda x\}, \\ \Gamma_{\mathbb{C}}(f, P) &:= \{x \in \mathbb{C}^n \mid \exists \lambda \in \mathbb{C} \quad \text{s.t. } \nabla f(x) = \lambda P x\}. \end{aligned} \tag{4.1}$$

Recall that $\mathsf{Crit}_{\mathbb{C}}(f)$ and $\mathsf{Crit}_{\mathbb{C}}(f^A)$ denote the sets of complex critical points of f and f^A, respectively. As in Lemma 4.2, we still have

Lemma 4.6. *Given an invertible matrix* $A \in \mathbb{R}^{n\times n}$, *let* $P = A^{-T}A^{-1}$, *then we have* $\Gamma_{\mathbb{C}}(f^A) = A^{-1}(\Gamma_{\mathbb{C}}(f, P))$ *and* $\mathsf{Crit}_{\mathbb{C}}(f^A) = A^{-1}(\mathsf{Crit}_{\mathbb{C}}(f))$.

Corollary 4.7. *There exists an open and dense semi-algebraic set* $\mathcal{U} \subset \mathcal{I}^{n\times n}$ *such that for all* $A \in \mathcal{U}$, *the Zariski closure* $\overline{\Gamma_{\mathbb{C}}(f^A)\backslash \mathsf{Crit}_{\mathbb{C}}(f^A)}^{\mathcal{Z}}$ *is a one-dimensional algebraic variety in* $\mathbb{C}^n$.

Proof. Denote $\mathcal{S}_{\mathbb{C}}^{n\times n}$ as the set of symmetric matrices in $\mathbb{C}^{n\times n}$, which can be identified with the space $\mathbb{C}^{\frac{n(n+1)}{2}}$. By similar arguments as in Theorem 4.3, it is easy to see that $\mathbf{0}$ is a regular value of the map

$$\begin{aligned} F:\ (\mathbb{C}^n\backslash \mathsf{Crit}_{\mathbb{C}}(f)) \times \mathbb{C} \times \mathcal{S}_{\mathbb{C}}^{n\times n} &\rightarrow \quad \mathbb{C}^n \\ (x, \lambda, P) &\mapsto \quad \nabla f(x) - \lambda P x. \end{aligned}$$

Then according to Thom's weak transversality theorem (cf. [11], [13, Theorem 1.10], and [26, Proposition B.3]), there exists a Zariski closed subset $\Sigma_{\mathbb{C}} \subset \mathcal{S}_{\mathbb{C}}^{n\times n}$ such that for all $P \in \mathcal{S}_{\mathbb{C}}^{n\times n}\backslash\Sigma_{\mathbb{C}}$, $\mathbf{0}$ is a regular value of the map

$$\begin{aligned} F_P:\ (\mathbb{C}^n\backslash \mathsf{Crit}_{\mathbb{C}}(f)) \times \mathbb{C} &\rightarrow \quad \mathbb{C}^n \\ (x, \lambda) &\rightarrow \quad F(x, \lambda, P). \end{aligned}$$

It follows that $F_P^{-1}(\mathbf{0})$ is either empty or a one-dimensional quasi-affine set of $\mathbb{C}^{n+1}$. Since $\overline{\Gamma_{\mathbb{C}}(f,P)\backslash \mathsf{Crit}_{\mathbb{C}}(f)}^{\mathcal{Z}}$ is the Zariski closure of the projection of $F_P^{-1}(\mathbf{0})$ on the first n coordinates, we have $\dim(\overline{\Gamma_{\mathbb{C}}(f,P)\backslash \mathsf{Crit}_{\mathbb{C}}(f)}^{\mathcal{Z}}) \leq 1$ for all $P \in \mathcal{S}_{\mathbb{C}}^{n\times n}\backslash\Sigma_{\mathbb{C}}$. Let $\Sigma = \Sigma_{\mathbb{C}} \cap \mathcal{S}_{++}^{n\times n}$, then it is clear that $\Sigma \subset \mathcal{S}_{++}^{n\times n}$ is a semi-algebraic set of dimension $< \frac{n(n+1)}{2}$ and $\dim(\overline{\Gamma_{\mathbb{C}}(f,P)\backslash \mathsf{Crit}_{\mathbb{C}}(f)}^{\mathcal{Z}}) \leq 1$ for all $P \in \mathcal{S}_{++}^{n\times n}\backslash\Sigma$. Hence, the conclusion follows by similar arguments as in the proof of Theorem 4.3. □

4.2. Sufficient criteria for faithful radius

For a given $\mathscr{R} \in \mathbb{R}_+$, we consider the following condition

Condition 4.8. *For any* $\mathbf{0} \neq u \in \Gamma_{\mathbb{R}}(f)$ *with* $\|u\|_2 < \mathscr{R}$, *there exist a neighborhood* $\mathcal{O}_u \subset \mathsf{B}_{\mathscr{R}}$ *of* u, *a differentiable map* $\phi(t) : (a, b) \rightarrow \mathbb{R}^n$ *and* $\bar{t} \in (a, b)$ *such that* $\phi((a, b)) = \Gamma_{\mathbb{R}}(f) \cap \mathcal{O}_u$, $\phi(\bar{t}) = u$ *and*

$$\frac{\mathrm{d}\left(\sum_{i=1}^n \phi_i^2\right)}{\mathrm{d}t}(\bar{t}) \neq 0. \tag{4.2}$$

Theorem 4.9. *Suppose that $\mathscr{R} \in \mathbb{R}_+$ satisfies Condition 4.8 and $0 < R < \mathscr{R}$, then $\Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_R$ is connected. Moreover, if R is an isolation radius, then $\Gamma_{\mathbb{R}}(f) \cap \mathbf{V}_{\mathbb{R}}(f) \cap \mathsf{B}_R = \{\mathbf{0}\}$ and hence R is a faithful radius.*

Proof. Suppose that $\Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_R$ is not connected, then it has a connected component $\mathcal{C}$ such that $\mathbf{0} \notin \mathcal{C}$. Since $\Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_R$ is closed, $\mathcal{C}$ is closed by Proposition 2.2. Then, the function $\|X\|_2^2$ reaches its minimum on $\mathcal{C}$ at a minimizer $u \in \mathcal{C}$. By the assumption, there exist a neighborhood $\mathcal{O}_u$ of u and a differentiable mapping $\phi(t) : (a, b) \to \mathbb{R}^n$ and $\bar{t} \in (a, b)$ such that $\phi((a, b)) = \Gamma_{\mathbb{R}}(f) \cap \mathcal{O}_u$ and $\phi(\bar{t}) = u$. By choosing a, b near enough to $\bar{t}$, we may assume that $\phi((a, b)) \subseteq \mathcal{C} \cap \mathcal{O}_u$. Then, the function $\sum_{i=1}^n \phi_i^2$ reaches its local minimum at $\bar{t}$, which contradicts (4.2). Hence, $\Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_R$ is connected.

Now suppose that R is also an isolation radius. Assume to the contrary that there exists $\mathbf{0} \neq v \in \Gamma_{\mathbb{R}}(f) \cap \mathbf{V}_{\mathbb{R}}(f) \cap \mathsf{B}_R$. Since $\Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_R$ is connected, there exists a path connecting $\mathbf{0}$ and v. Then, f has a local extremum on a relative interior of this path, say u. By the assumption, there exists a differentiable mapping $\phi(t)$ on (a, b) and $\bar{t} \in (a, b)$ as described in the statement. Then the differentiable function $f(\phi(t))$ reaches a local extremum at $\bar{t}$. By the mean value theorem,

$$0 = \frac{\mathrm{d}f(\phi)}{\mathrm{d}t}(\bar{t}).$$

On the other hand, since R is an isolation radius, $\phi(\bar{t}) \in \Gamma_{\mathbb{R}}(f) \setminus \mathsf{Crit}_{\mathbb{R}}(f)$ and hence there exists $\lambda \neq 0$ such that

$$\frac{\partial f}{\partial x_i}(\phi(\bar{t})) = \lambda \phi_i(\bar{t}), \quad \text{for } i = 1, \dots, n.$$

Therefore,

$$0 = \frac{\mathrm{d}f(\phi)}{\mathrm{d}t}(\bar{t}) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\phi(\bar{t})) \frac{\mathrm{d}\phi_i}{\mathrm{d}t}(\bar{t}) = \sum_{i=1}^n \lambda \phi_i(\bar{t}) \frac{\mathrm{d}\phi_i}{\mathrm{d}t}(\bar{t}) = \lambda \frac{\mathrm{d}\left(\sum_{i=1}^n \phi_i^2\right)}{2\mathrm{d}t}(\bar{t}),$$

which contradicts (4.2). Therefore $\Gamma_{\mathbb{R}}(f) \cap \mathbf{V}_{\mathbb{R}}(f) \cap \mathsf{B}_R = \{\mathbf{0}\}$. □

According to Theorem 4.9, if we can compute a $\mathscr{R} \in \mathbb{R}_+$ satisfying Condition 4.8 and an isolation radius R of $\mathbf{0}$ is given, then any $r \in \mathbb{R}_+$ with $r < \min\{R, \mathscr{R}\}$ is a faithful radius of $\mathbf{0}$. Hence, we next show that how to compute such a $\mathscr{R}$.

For a given ideal $I \subseteq \mathbb{R}[X]$ with $\dim(I) = 1$, compute its equidimensional decomposition $I = I^{(0)} \cap I^{(1)}$ where $\dim(I^{(i)}) = i$ for $i = 0, 1$. Compute the radical ideal $\sqrt{I^{(1)}} = \langle g_1, \dots, g_s \rangle$ with generators $g_1, \dots, g_s \in \mathbb{R}[X]$. Note that there are efficient algorithms for the equidimensional decomposition of I such that $I^{(0)}$ and $I^{(1)}$ are themselves radical (c.f. [1, Section 3] and [10, Algorithm 4.4.9]). Recall that $\|X\|_2^2 = X_1^2 + \cdots + X_n^2$. Denote $\mathscr{D}$ as the set of the determinants of the Jacobian matrices $\mathsf{Jac}\left(g_{i_1}, \dots, g_{i_{n-1}}, \|X\|_2^2\right)$ for all $\{i_1, \dots, i_{n-1}\} \subset \{1, \dots, s\}$. (Note that $s \geq n - 1$ because $\dim(I^{(1)}) = 1$.) Define

$$\begin{aligned} \mathscr{R}_{I^{(0)}} &:= \min\{r \in \mathbb{R}\backslash\{0\} \mid \exists x \in \mathbf{V}_{\mathbb{C}}(I^{(0)}), \text{s.t. } x_1^2 + \cdots + x_n^2 = r^2\}, \\ \Delta_{I^{(1)}} &:= \{g_1, \dots, g_s\} \cup \mathscr{D}, \\ \mathscr{R}_{I^{(1)}} &:= \inf\{r \in \mathbb{R}\backslash\{0\} \mid \exists x \in \mathbf{V}_{\mathbb{C}}(\Delta_{I^{(1)}}), \text{s.t. } x_1^2 + \cdots + x_n^2 = r^2\}, \\ \mathscr{R}_I &:= \min\{\mathscr{R}_{I^{(0)}}, \mathscr{R}_{I^{(1)}}\}. \end{aligned} \tag{4.3}$$

We have (see also [18, Corollary 2.8]).

Lemma 4.10. *If $\dim(I) = 1$, then $\mathscr{R}_I > 0$.*

Proof. Since $I^{(0)}$ is zero-dimensional, we have $\mathscr{R}_{I^{(0)}} > 0$. Now we show that $\mathscr{R}_{I^{(1)}} > 0$. Consider the map

$$\begin{aligned} \Phi: \ \mathbf{V}_{\mathbb{C}}(I^{(1)}) &\to \mathbb{C} \\ x &\mapsto x_1^2 + \cdots + x_n^2. \end{aligned}$$

Since $\langle g_1, \dots, g_s \rangle$ is radical and equidimensional one, $\mathbf{V}_{\mathbb{C}}(\Delta_{I^{(1)}})$ consists of the singular locus of $\mathbf{V}_{\mathbb{C}}(I^{(1)})$ and the set of critical points of Φ. Since $\dim(\mathbf{V}_{\mathbb{C}}(I^{(1)})) = 1$, its singular locus is zero-dimensional. By the algebraic Sard's theorem [27], there are only finitely many critical values of the map Φ (note that if Φ is not dominant, the conclusion is clearly true). Hence, the set $\Phi(\mathbf{V}_{\mathbb{C}}(\Delta_{I^{(1)}}))$ is finite and $\mathscr{R}_{I^{(1)}} > 0$. □

Theorem 4.11. *Suppose that an ideal $I \subseteq \mathbb{R}[X]$ and a $\mathscr{R} \in \mathbb{R}_+$ satisfy*: (i) $\dim(I) = 1$; (ii) $\mathscr{R} < \mathscr{R}_I$; (iii) $\Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_{\mathscr{R}} = \mathbf{V}_{\mathbb{R}}(I) \cap \mathsf{B}_{\mathscr{R}}$, *then Condition 4.8 holds for $\mathscr{R}$.*

Proof. Fix a $\mathbf{0} \neq u \in \Gamma_{\mathbb{R}}(f)$ with $\|u\|_2 < \mathscr{R}$. Since $\mathscr{R} < \mathscr{R}_I$ and $\Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_{\mathscr{R}} = \mathbf{V}_{\mathbb{R}}(I) \cap \mathsf{B}_{\mathscr{R}}$, by Corollary 3.4 and the definition of $\mathscr{R}_I$, we have $\mathbf{V}_{\mathbb{R}}(I) \cap \mathsf{B}_{\mathscr{R}} = \mathbf{V}_{\mathbb{R}}(I^{(1)}) \cap \mathsf{B}_{\mathscr{R}}$ and hence $u \in \mathbf{V}_{\mathbb{R}}(I^{(1)})$. Because $u \notin \mathbf{V}_{\mathbb{R}}(\Delta_{I^{(1)}})$, there is a Jacobian matrix of the form

$$M = \begin{bmatrix} \frac{\partial g_{i_1}}{\partial x_1}(u) & \cdots & \frac{\partial g_{i_1}}{\partial x_n}(u) \\ \vdots & \vdots & \vdots \\ \frac{\partial g_{i_{n-1}}}{\partial x_1}(u) & \cdots & \frac{\partial g_{i_{n-1}}}{\partial x_n}(u) \\ u_1 & \cdots & u_n \end{bmatrix} \tag{4.4}$$

with full rank.

Let $\tilde{g}(X) = \sum_{i=1}^n u_i X_i - \sum_{i=1}^n u_i^2$ and $\tilde{I}^{(1)} = \langle g_{i_1}, \ldots, g_{i_{n-1}} \rangle$. Define a function $Y = G(X) := (g_{i_1}(X), \ldots, g_{i_{n-1}}(X), \tilde{g}(X))$, then we have $G(u) = \mathbf{0}$ and the Jacobian of $G(X)$ at u is nonsingular. Hence, by the inverse function theorem, $G(X)$ is an invertible function in a neighborhood $\mathcal{O}_u$ of u. Without loss of generality, we can assume that $\mathcal{O}_u \subseteq \mathsf{B}_{\mathscr{R}}$. Thus, an invertible funtion $X = G^{-1}(Y) = (G_1^{-1}(Y), \ldots, G_n^{-1}(Y))$ exists in some neighborhood $\mathcal{O}_{\mathbf{0}}$ of $\mathbf{0}$. Moreover, $X = G^{-1}(Y)$ is differentiable in $\mathcal{O}_{\mathbf{0}}$. Define $\phi(t) = (\phi_i(t))$ with $\phi_i(t) = G_i^{-1}(0, \ldots, 0, t)$. Then, there is an interval (a, b) such that $\phi((a, b)) = \mathbf{V}_{\mathbb{R}}(\tilde{I}^{(1)}) \cap \mathcal{O}_u$. Moreover, we have $0 \in (a, b)$ and $\phi(0) = u$. Then we have

$$M \cdot \begin{bmatrix} \frac{\partial G_1^{-1}}{\partial Y_n}(\mathbf{0}) \\ \vdots \\ \frac{\partial G_{n-1}^{-1}}{\partial Y_n}(\mathbf{0}) \\ \frac{\partial G_n^{-1}}{\partial Y_n}(\mathbf{0}) \end{bmatrix} = \begin{bmatrix} \frac{\partial Y_1}{\partial Y_n}(\mathbf{0}) \\ \vdots \\ \frac{\partial Y_{n-1}}{\partial Y_n}(\mathbf{0}) \\ \frac{\partial \sum_{i=1}^n (G_i^{-1})^2}{2\partial Y_n}(\mathbf{0}) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{\mathrm{d} \sum_{i=1}^n \phi_i^2}{2\mathrm{d}t}(0) \end{bmatrix}$$

By the implicit function theorem, there is an i_0 such that $\frac{\partial G_{i_0}^{-1}}{\partial Y_n}(\mathbf{0}) \neq 0$. Since the matrix M in (4.4) is nonsingular, it implies $\frac{\mathrm{d} \sum_{i=1}^n \phi_i^2}{\mathrm{d}t}(0) \neq 0$. Recalling Condition 4.8, since $\Gamma_{\mathbb{R}}(f) \cap \mathcal{O}_u = \mathbf{V}_{\mathbb{R}}(I^{(1)}) \cap \mathcal{O}_u$, it remains to prove that $\mathbf{V}_{\mathbb{R}}(I^{(1)}) \cap \mathcal{O}_u = \mathbf{V}_{\mathbb{R}}(\tilde{I}^{(1)}) \cap \mathcal{O}_u$.

It suffices to prove that $\mathbf{V}_{\mathbb{C}}(I^{(1)}) \cap \mathcal{O} = \mathbf{V}_{\mathbb{C}}(\tilde{I}^{(1)}) \cap \mathcal{O}$ for some Zariski open set $\mathcal{O} \subseteq \mathbb{C}^n$ containing u. Let $\mathbf{V}_{\mathbb{C}}(I^{(1)}) = V_1 \cup \cdots \cup V_s$ and $\mathbf{V}_{\mathbb{C}}(\tilde{I}^{(1)}) = \widetilde{V}_1 \cup \cdots \cup \widetilde{V}_t$ be the irreducible decompositions of $\mathbf{V}_{\mathbb{C}}(I^{(1)})$ and $\mathbf{V}_{\mathbb{C}}(\tilde{I}^{(1)})$, respectively. Since the first $n-1$ rows of M is linear independent, there is a unique irreducible component, say $\widetilde{V}_1$, of $\mathbf{V}_{\mathbb{C}}(\tilde{I}^{(1)})$ containing u and $\widetilde{V}_1$ is smooth of dimension one at u. Let V_1 be an irreducible component of $\mathbf{V}_{\mathbb{C}}(I^{(1)})$ containing u. Since $I^{(1)} \supseteq \tilde{I}^{(1)}$, we have $\mathbf{V}_{\mathbb{C}}(I^{(1)}) \subseteq \mathbf{V}_{\mathbb{C}}(\tilde{I}^{(1)})$ and hence $V_1 \subseteq \widetilde{V}_1$. Because $\dim(V_1) = 1$, it follows that $V_1 = \widetilde{V}_1$ which also implies that V_1 is the unique irreducible component of $\mathbf{V}_{\mathbb{C}}(I^{(1)})$ containing u. Let $\mathcal{O} = \mathbb{C}^n \backslash \left(\bigcup_{i=2}^s V_i \cup \bigcup_{i=2}^t \widetilde{V}_i \right)$, then we have $\mathbf{V}_{\mathbb{C}}(I^{(1)}) \cap \mathcal{O} = \mathbf{V}_{\mathbb{C}}(\tilde{I}^{(1)}) \cap \mathcal{O}$ which ends the proof. □

Combining Theorems 4.9 and 4.11, we obtain

Theorem 4.12. *Suppose that an ideal $I \subseteq \mathbb{R}[X]$ and a $\mathscr{R} \in \mathbb{R}_+$ satisfy*: (i) $\dim(I) = 1$; (ii) $\mathscr{R} < \mathscr{R}_I$; (iii) $\Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_{\mathscr{R}} = \mathbf{V}_{\mathbb{R}}(I) \cap \mathsf{B}_{\mathscr{R}}$, *then any isolation radius $R \in \mathbb{R}_+$ of $\mathbf{0}$ with $R < \mathscr{R}$ is a faithful radius of $\mathbf{0}$.*

Let $\boldsymbol{\gamma} = \{\gamma_{i,j} \mid i, j = 1, \ldots, n\}$ where $\gamma_{i,j} := \frac{\partial f}{\partial X_i} X_j - \frac{\partial f}{\partial X_j} X_i$, then $\Gamma_{\mathbb{R}}(f) = \mathbf{V}_{\mathbb{R}}(\langle \boldsymbol{\gamma} \rangle)$. If $\dim(\langle \boldsymbol{\gamma} \rangle) = 1$, let $\mathscr{R}_{\langle \boldsymbol{\gamma} \rangle}$ be defined as in (4.3).

Corollary 4.13. *If* $\dim(\langle \gamma \rangle) = 1$*, then any isolation radius* R *with* $R < \mathscr{R}_{\langle \gamma \rangle}$ *is a faithful radius of* $\mathbf{0}$*.*

Note that the ideal $\langle \gamma \rangle$ may not be one-dimensional. Let $\mathcal{G} := \mathbf{I}\left(\overline{\Gamma_{\mathbb{C}}(f) \backslash \mathsf{Crit}_{\mathbb{C}}(f)}^{\mathcal{Z}}\right)$, i.e., the vanishing ideal of $\overline{\Gamma_{\mathbb{C}}(f) \backslash \mathsf{Crit}_{\mathbb{C}}(f)}^{\mathcal{Z}}$ in $\mathbb{R}[X]$. According to Corollary 4.7, we have $\dim(\mathcal{G}) = 1$ up to a generic linear change of coordinates. Moreover,

Proposition 4.14. $\Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_R = \mathbf{V}_{\mathbb{R}}(\mathcal{G}) \cap \mathsf{B}_R$ *holds for any isolation radius* R*.*

Proof. Let $V_1 \cup \cdots \cup V_s \cup V_{s+1} \cup \cdots \cup V_t$ be the decomposition of $\Gamma_{\mathbb{C}}(f)$ as a union of irreducible components. Assume that $V_i \not\subseteq \mathsf{Crit}_{\mathbb{C}}(f)$ for $1 \leq i \leq s$ and $V_j \subseteq \mathsf{Crit}_{\mathbb{C}}(f)$ for $s+1 \leq j \leq t$. Let $V^{(1)} = V_1 \cup \cdots \cup V_s$ and $V^{(2)} = V_{s+1} \cup \cdots \cup V_t$, then $\mathbf{V}_{\mathbb{C}}(\mathcal{G}) = V^{(1)}$ and $V^{(2)} \cap \mathsf{B}_R \subseteq \{\mathbf{0}\}$. We have $\Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_R = \Gamma_{\mathbb{C}}(f) \cap \mathsf{B}_R = (V^{(1)} \cap \mathsf{B}_R) \cup (V^{(2)} \cap \mathsf{B}_R) \subseteq (\mathbf{V}_{\mathbb{R}}(\mathcal{G}) \cap \mathsf{B}_R) \cup \{\mathbf{0}\}$. Since $\Gamma_{\mathbb{R}}(f) \backslash \mathsf{Crit}_{\mathbb{R}}(f) \subset \mathbf{V}_{\mathbb{R}}(\mathcal{G})$, by Corollary 3.4, $\mathbf{0} \in \mathbf{V}_{\mathbb{R}}(\mathcal{G})$ and hence $\Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_R \subseteq \mathbf{V}_{\mathbb{R}}(\mathcal{G}) \cap \mathsf{B}_R$. It is clear that $\Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_R \supseteq \mathbf{V}_{\mathbb{R}}(\mathcal{G}) \cap \mathsf{B}_R$ and thus the conclusion follows. □

Corollary 4.15. *Suppose that* $\dim(\mathcal{G}) = 1$ *and* R *is an isolation radius of* f*. Then, any* $r \in \mathbb{R}_+$ *with* $r < \min\{R, \mathscr{R}_{\mathcal{G}}\}$ *is a faithful radius of* $\mathbf{0}$*.*

Provided that an isolation radius of $\mathbf{0}$ is known, we now present an algorithm to compute a faithful radius of $\mathbf{0}$.

Algorithm 4.16. FaithfulRadius(f, R_{iso})
Input: A polynomial $f \in \mathbb{R}[X]$ with $\mathbf{0}$ as an isolated real critical point and an isolation radius R_{iso} of $\mathbf{0}$.
Output: $R \in \mathbb{R}_+$ such that any $0 < r < R$ is a faithful radius of $\mathbf{0}$.

1. If $\dim(\langle \gamma \rangle) = 1$, then let $I = \langle \gamma \rangle$; otherwise, make a linear change of coordinates of f such that $\dim(\mathcal{G}) = 1$ and let $I = \mathcal{G}$;
2. Compute the equidimensional decomposition $I = I^{(0)} \cap I^{(1)}$ and the set $\Delta_{I^{(1)}}$ as defined in (4.3);
3. Compute elimination ideals $I_n^{(0)} := (I^{(0)} + \langle \|X\|_2^2 - X_{n+1} \rangle) \cap \mathbb{R}[X_{n+1}]$ and $I_n^{(1)} := (\langle \Delta_{I^{(1)}} \rangle + \langle \|X\|_2^2 - X_{n+1} \rangle) \cap \mathbb{R}[X_{n+1}]$;
4. Compute the isolation intervals $\{[a_i, b_i] \mid i = 1, \ldots, t\}$ of $\mathbf{V}_{\mathbb{R}}(I_n^{(0)} \cdot I_n^{(1)})$;
5. Let $\mathscr{R} = \min\{\sqrt{a_i} \mid a_i > 0, i = 1, \ldots, t\}$ and return $R = \min\{\mathscr{R}, R_{\mathsf{iso}}\}$.

Theorem 4.17. *Algorithm 4.16 runs successfully and is correct.*

Proof. According to the proof of Proposition 4.14, we have $\mathbf{0} \in \mathbf{V}_{\mathbb{R}}(I^{(1)})$ and hence $0 \in \mathbf{V}_{\mathbb{R}}(I_n^{(0)} \cdot I_n^{(1)})$ by the definition of $\Delta_{I^{(1)}}$. Then, we have $\mathscr{R} < \mathscr{R}_I$ in Step 5 since $[a_i, b_i]$'s are isolation intervals of $\mathbf{V}_{\mathbb{R}}(I_n^{(0)} \cdot I_n^{(1)})$. Then, by Corollary 4.7 and the proof of Lemma 4.10, the algorithm runs successfully. Its correctness can be seen by combining Theorem 4.12 and Corollary 4.15. □

Example 4.18. *Consider the polynomial* $f = X_1^2 + (1 - X_1)X_2^4$ *discussed in the introduction. The origin* $\mathbf{0}$ *is an isolated real critical point and degenerate. The graphs of* f *are shown in Figure 1. On the left hand side, the graph is drawn with the variables* X_1, X_2 *varying in the range* $[-2, 2]$*. It seems from this graph that* $\mathbf{0}$ *is a saddle point. However, if we zoom in, then we get the graph on the right hand side which indicates that* $\mathbf{0}$ *is in fact a strict local minimizer. Now we use Algorithm 4.16 to obtain a faithful radius of* $\mathbf{0}$*.*

Figure 1: The graphs of f in Example 4.18.

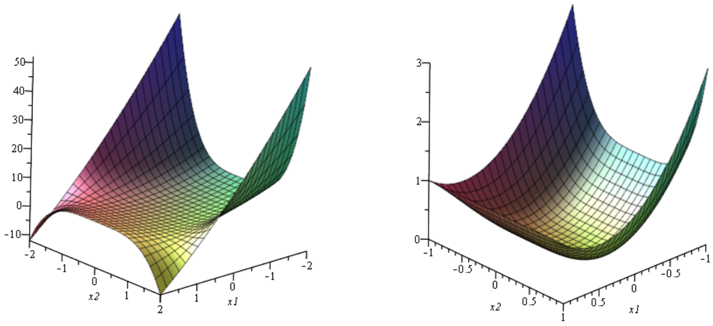
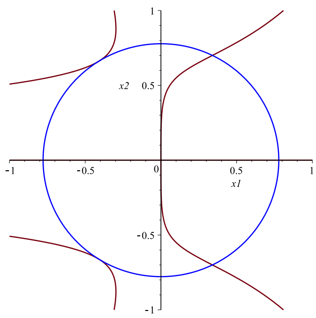


Figure 2: The curve of tangency of f in Example 4.18.



It is easy to check that $R_{\mathsf{iso}} = 1$ *is an isolation radius of* $\mathbf{0}$ *and* $\boldsymbol{\gamma} = \{4X_1^2X_2^3 - X_2^5 - 4X_1X_2^3 + 2X_1X_2\}$. *The curve of tangency* $\Gamma_{\mathbb{R}}(f) = \mathbf{V}_{\mathbb{R}}(\langle\boldsymbol{\gamma}\rangle)$ *is shown (red) in Figure 2. Since* $\dim(\langle\boldsymbol{\gamma}\rangle) = 1$, *we let* $I = \langle\boldsymbol{\gamma}\rangle$. *We implement Algorithm 4.16 in the software Maple. With inputs* f *and* R_{iso}, *we get the return* $R = \frac{\sqrt{7674514669980086316063001398 61}}{1125899906842624} \approx 0.778 < 1$. *The circle centered at* $\mathbf{0}$ *with radius* R *is shown (blue) in Figure 2. Hence, any* $r < R$ *is a faithful radius of* $\mathbf{0}$.

4.3. On the computation of isolation radius

As we have seen, if an isolation radius of $\mathbf{0}$ is available, a faithful radius of $\mathbf{0}$ can be obtained by Algorithm 4.16. To end this section, we propose some strategies to compute an isolation radius of $\mathbf{0}$.

Let $\mathscr{C} = \langle \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n} \rangle$. If $\dim(\mathscr{C}) = 0$, then an isolation radius of $\mathbf{0}$ can be computed by the RUR method [23] for zero-dimensional systems.

Assume that $\dim(\mathscr{C}) > 0$. We now borrow the idea from [1] which aims to compute one point on each semi-algebraically connected component of a real algebraic variety by the critical point method of a distance function. Compute the equidimensional decomposition $\mathscr{C} = \mathscr{C}^{(0)} \cap \mathscr{C}^{(1)} \cap \cdots \cap \mathscr{C}^{(t)}$ where $\mathscr{C}^{(k)}$ is radical and of dimension k for each $k = 0, \ldots, t$. For an efficient algorithm of such decomposition, see [1, Section 3]. Then, we can compute the minimal distance $d^{(k)}$ of $\mathbf{V}_{\mathbb{R}}(\mathscr{C}^{(k)}) \backslash \{\mathbf{0}\}$ to $\mathbf{0}$ for each k and choose a positive number less than the smallest one as an isolation radius. Suppose that $\mathscr{C}^{(k)} = \langle h_1, \ldots, h_l \rangle$. Let $\mathscr{M}(\mathscr{C}^{(k)})$ be the set of $h_1, \ldots, h_l$ and all the $(n-k+1, n-k+1)$ minors of the Jacobian matrix $\mathsf{Jac}(h_{i_1}, \ldots, h_{i_{n-k}}, \|X\|_2^2)$ for all $\{i_1, \ldots, i_{n-k}\} \subset \{1, \ldots, l\}$. Consider the map

$$\begin{array}{rccl} \Phi: & \mathbf{V}_{\mathbb{C}}(\mathscr{C}^{(k)}) & \rightarrow & \mathbb{C} \\ & x & \mapsto & x_1^2 + \cdots + x_n^2. \end{array}$$

Then, $\mathbf{V}_{\mathbb{C}}(\mathscr{M}(\mathscr{C}^{(k)}))$ consists of the singular locus $\mathsf{Sing}(\mathscr{C}^{(k)})$ of $\mathbf{V}_{\mathbb{C}}(\mathscr{C}^{(k)})$ and the set of critical points of Φ. By the first part in the proof of [1, Theorem 2.3], the point of $\mathbf{V}_{\mathbb{R}}(\mathscr{C}^{(k)}) \backslash \{\mathbf{0}\}$ at the minimal distance to $\mathbf{0}$ is contained in $\mathbf{V}_{\mathbb{R}}(\mathscr{M}(\mathscr{C}^{(k)}))$. Compute the elimination ideal $\mathscr{M}_n(\mathscr{C}^{(k)}) = \langle \mathscr{M}(\mathscr{C}^{(k)}), \|X\|_2^2 - X_{n+1}\rangle \cap \mathbb{R}[X_{n+1}]$ and then we have $(d^{(k)})^2 \in \mathbf{V}_{\mathbb{R}}(\mathscr{M}_n(\mathscr{C}^{(k)}))$. If $\mathscr{M}_n(\mathscr{C}^{(k)}) \neq \langle 0\rangle$, then the smallest positive real root in $\mathbf{V}_{\mathbb{R}}(\mathscr{M}_n(\mathscr{C}^{(k)}))$, which can be obtained by any real root isolation algorithm for univariate polynomials, is a lower bound of $(d^{(k)})^2$. If $\mathscr{M}_n(\mathscr{C}^{(k)}) = \langle 0\rangle$, by Sard's theorem, it happens if and only if the set $\Phi(\mathsf{Sing}(\mathscr{C}^{(k)}))$ is infinite. In this case, we can replace $\mathscr{C}$ by $\mathscr{M}(\mathscr{C}^{(k)})$ and repeat the above procedure recursively. Since $\dim(\mathsf{Sing}(\mathscr{C}^{(k)})) < k$, this process will finitely terminate and return an isolation radius of $\mathbf{0}$.

Alternatively, when $\dim(\mathscr{C}) > 0$, we can compute an isolation radius of $\mathbf{0}$ by testing the emptyness of a real algebraic variety. Adding two new variables X_{n+1} and X_{n+2}, a $R \in \mathbb{R}_+$ is an isolation radius of $\mathbf{0}$ if and only if the following polynomial system has no real root

$$\frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n}, \|X\|_2^2 + X_{n+1}^2 - R^2, \|X\|_2^2 \cdot X_{n+2} - 1.$$

Hence, we can set an initial R and test the emptyness of the real algebraic variety generated by the above polynomials. If it is empty, then R is an isolation radius; otherwise, try $R/2$ and repeat. For algorithms of such tests, see [1, Section 4] and [25].

5. Certificates of types of degenerate critical points

If $R \in \mathbb{R}_+$ is a faithful radius of the isolated real critical point $\mathbf{0}$ of f, then we can compute the extrema $f_R^{\min}$ and $f_R^{\max}$ in (1.2) to classify the type of $\mathbf{0}$ by Theorem 3.7. To deal with the issues when computing $f_R^{\min}$ and $f_R^{\max}$ as mentioned in the introduction, we next show that how to decide the type of $\mathbf{0}$ by means of real root isolation of zero-dimensional polynomial systems. Recall the notation S_r in (1.1).

Proposition 5.1. *Suppose that $\mathscr{R} \in \mathbb{R}_+$ satisfies Condition 4.8 and $0 < R < \mathscr{R}$. Then for any $\mathbf{0} \neq u \in \Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_R$, there exists a continuous map $\varphi(t) : [a, b] \to \Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_R$ with $\varphi(a) = u$, $\varphi(b) \in \mathsf{S}_R$ and $\mathbf{0} \notin \varphi([a, b])$.*

Proof. Consider the following semi-algebraic set

$$S := \Gamma_{\mathbb{R}}(f) \cap \{x \in \mathbb{R}^n \mid \|u\|_2^2/2 \leq \|x\|_2^2 \leq R^2\}.$$

Let $\mathcal{C}$ be the connected component of S containing u. If $\mathcal{C} \cap \mathsf{S}_R \neq \emptyset$, then the conclusion follows since $\mathcal{C}$ is path connected. Otherwise, the function $\|X\|_2^2$ reaches its maximum on $\mathcal{C}$ at a maximizer in $\mathcal{C}$. Then we can get a contradiction using arguments similar to the first part of the proof of Theorem 4.9. □

For any $r \in \mathbb{R}_+$, comparing with Corollary 3.3, define

$$f_r^- := \min\{f(x) \mid x \in \Gamma_{\mathbb{R}}(f) \cap \mathsf{S}_r\} \quad \text{and} \quad f_r^+ := \max\{f(x) \mid x \in \Gamma_{\mathbb{R}}(f) \cap \mathsf{S}_r\}. \tag{5.1}$$

Theorem 5.2. *Suppose that $\mathscr{R} \in \mathbb{R}_+$ satisfies Condition 4.8. Then for any isolation radius $R < \mathscr{R}$, it holds that*

(i) $\mathbf{0}$ *is a local minimizer if and only if* $f_R^- > 0$*;*

(ii) $\mathbf{0}$ *is a local maximizer if and only if* $f_R^+ < 0$*;*

(iii) $\mathbf{0}$ *is a saddle point if and only if* $f_R^+ > 0 > f_R^-$*.*

Proof. By Theorem 4.9, R is a faithful radius of $\mathbf{0}$. According to Theorem 3.7 and Definition 3.6 (ii), the "only if" parts in (i), (ii) and the "if" part in (iii) are clear.

(i). "if" part. Suppose that $f_R^- > 0$, then we have $f_R^{\min} = 0$. Otherwise, by Corollary 3.3, there exists $\mathbf{0} \neq u \in \Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_R \backslash \mathsf{S}_R$ such that $f(u) < 0$. By Proposition 5.1, there exists a continuous map $\varphi(t) : [a, b] \to \Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_R$ with $\varphi(a) = u$ and $\varphi(b) \in \mathsf{S}_R$. Then we have a continuous function $g(t) :=$

$f(\varphi(t)) : [a,b] \to \mathbb{R}$ such that $g(a) < 0$ and $g(b) > 0$. By the mean value theorem, there exists $\bar{t} \in (a,b)$ such that $g(\bar{t}) = f(\varphi(\bar{t})) = 0$. Since that $\mathbf{0} \neq \varphi(\bar{t}) \in \Gamma_{\mathbb{R}}(f) \cap \mathsf{B}_R$ by Proposition 5.1 and R is a faithful radius, we get a contradiction;

Similarly, we can prove (ii) and then (iii) follows. □

For any $r \in \mathbb{R}_+$, let $\mathsf{S}_{r,\mathbb{C}} = \{x \in \mathbb{C}^n \mid \sum_{i=1}^n x_i^2 = r^2\}$. Recall the definition $\mathscr{R}_I$ for an ideal I in (4.3).

Proposition 5.3. *Given an ideal $I \subseteq \mathbb{R}[X]$ with $\dim(I) = 1$, the system $\mathbf{V}_{\mathbb{C}}(I) \cap \mathsf{S}_{R,\mathbb{C}}$ is zero-dimensional for any $0 < R < \mathscr{R}_I$,*

Proof. It only needs to prove that $\mathbf{V}_{\mathbb{C}}(I^{(1)}) \cap \mathsf{S}_{R,\mathbb{C}}$ is zero-dimensional. Let $V_1 \cup \cdots \cup V_s$ be the decomposition of $\mathbf{V}_{\mathbb{C}}(I^{(1)})$ as a union of irreducible components. Fix an $1 \leq i \leq s$. If $V_i \cap \mathsf{S}_{R,\mathbb{C}} \neq \emptyset$, then we show that $V_i \not\subseteq \mathsf{S}_{R,\mathbb{C}}$. To the contrary, assume that $V_i \subseteq \mathsf{S}_{R,\mathbb{C}}$. By the definition of $\mathscr{R}_I$ and the proof of Lemma 4.10, we have $V_i \cap V_j = \emptyset$ for each $j \neq i$ since $V_i \cap V_j$ is contained in the singular locus of $\mathbf{V}_{\mathbb{C}}(I^{(1)})$. Then for any point p in the nonsingular part of V_i, it holds that $T_p(V_i) = T_p(\mathbf{V}_{\mathbb{C}}(I^{(1)}))$ where $T_p(V_i)$ and $T_p(\mathbf{V}_{\mathbb{C}}(I^{(1)}))$ denote the tangent spaces of V_i and $\mathbf{V}_{\mathbb{C}}(I^{(1)})$ at p, respectively. Then, for any $h \in \mathbf{I}(V_i)$, the differiential of h at p can be expressed as a linear combination of the differientials of $g_1, \ldots, g_s$ (the generators of $\sqrt{I^{(1)}}$) at p. In particular, by the assumption that $V_i \subseteq \mathsf{S}_{R,\mathbb{C}}$, it holds for $h := \|X\|_2^2 - R^2 \in \mathbf{I}(V_i)$. Hence, all the determinants in the set $\mathscr{D}$ in (4.3) vanish at any $p \in V_i$ since $\dim(I^{(1)}) = 1$. Consequently, we have $p \in \mathbf{V}_{\mathbb{C}}(\Delta_{I^{(1)}})$. By the definition, it implies that $R \geq \mathscr{R}_I$ which is a contradiction. Therefore, if $V_i \cap \mathsf{S}_{R,\mathbb{C}} \neq \emptyset$, then $V_i \not\subseteq \mathsf{S}_{R,\mathbb{C}}$ and $\dim(V_i \cap \mathsf{S}_{R,\mathbb{C}}) = \dim(V_i) - 1 = 0$ by Krull's Principal Ideal Theorem [14, Chap. V, Corollary 3.2]. The conclusion follows. □

Recall the definition of $\boldsymbol{\gamma}$ and Algorithm 4.16. We now give an algorithm to decide the type of the isolated real critical point $\mathbf{0}$ of f.

Algorithm 5.4. $\mathsf{Type}(f, R_{\mathsf{iso}})$
Input: A polynomial $f \in \mathbb{R}[X]$ with $\mathbf{0}$ as an isolated real critical point and an isolation radius R_{iso} of $\mathbf{0}$.
Output: The type of $\mathbf{0}$ as a critical point of f.

1. If $\dim(\langle \boldsymbol{\gamma} \rangle) = 1$, then let $I = \langle \boldsymbol{\gamma} \rangle$; otherwise, make a linear change of coordinates of f such that $\dim(\mathcal{G}) = 1$ and let $I = \mathcal{G}$;
2. Let $R = \mathsf{FaithfulRadius}(f, R_{\mathsf{iso}})$ and fix a radius $0 < r < R$;
3. Let $\bar{I} = I + \langle \|X\|_2^2 - r^2, f - X_{n+1} \rangle \subseteq \mathbb{R}[X, X_{n+1}]$;
4. Compute intervals $\{[a_i, b_i] \mid i = 1, \ldots, s\}$ such that $0 \notin [a_i, b_i]$ for each i and the coordinate X_{n+1} of every point in $\mathbf{V}_{\mathbb{R}}(\bar{I}) \subseteq \mathbb{R}^{n+1}$ lies in some unique $[a_i, b_i]$;
5. Let $m = \min\{a_i, i = 1, \ldots, s\}$ and $M = \max\{b_i, i = 1, \ldots, s\}$;
6. If $m > 0$, return "local minimizer"; if $M < 0$, return "local maximizer"; if $m < 0 < M$, return "saddle point".

Theorem 5.5. *Algorithm 5.4 runs successfully and is correct.*

Proof. By Proposition 4.14 and Algorithm 4.16, f_r^- and f_r^+ respectively equal the minimal and maximal coordinates X_{n+1} of the points in $\mathbf{V}_{\mathbb{R}}(\bar{I})$. Since $\bar{I}$ is zero-dimensional by Proposition 5.3 and f_r^-, f_r^+ are nonzero by Theorem 5.2, the isolation intervals $[a_i, b_i]$'s in step 5 can be obtained. Again, by Theorem 5.2, the outputs of Algorithm is correct. □

Example 4.18 continued. *We have shown that any $0 < r < R \approx 0.778 < 1$ is a faithful radius of $\mathbf{0}$. We set $r = \frac{7}{18}$ in Step 2 of Algorithm 5.4. In Step 4, by the command* Isolate *in Maple which uses RUR method [23] for zero-dimensional system, we obtain that $m = \frac{76810939241945}{562949953421312}$ and $M = \frac{437849963772149}{1125899906842624}$ in Step 5. Therefore, we can claim that $\mathbf{0}$ is a local minimizer of f by Step 6 of Algorithm 5.4.*

Example 5.6. *Consider the following polynomial (cf. [20, 21])*

$$f(X_1, X_2, X_3) = 47X_1^5 + 5X_1X_2^4 + 33X_3^5 - 95X_1^4 - 47X_1X_3^3 + 51X_2^2X_3^2 - 92X_1X_3^2 - 70X_2^2X_3 + 21X_2^2.$$

It can be checked that $\mathbf{0}$ is a degenerate critical point and moreover the set $\mathsf{Crit}_{\mathbb{C}}(f)$ is zero-dimensional. Hence, $\mathbf{0}$ is isolated real critical point. Using the command Isolate *in Maple, we get an isolation radius $R_{\mathsf{iso}} = \frac{70375577207295}{140737488355328} \approx 0.50$. Running Algorithm 4.16 with inputs f and R_{iso}, we get the output $R = \frac{459690419250099}{4503599627370496} \approx 0.102$. Setting $r = \frac{2}{39}$ in Step 2 of Algorithm 5.4, we obtain $m = -\frac{428092208351331}{2251799813685248}$ and $M = \frac{13589527442797}{70368744177664}$. Thus, $\mathbf{0}$ is a saddle point of f. In fact, it can be certified by letting $X_1 = X_2 = 0$ in f.*

Example 5.7. *Consider the polynomial $f(X_1, X_2, X_3) = X_1^2 + X_2^4 + X_3^4 - 4X_1X_2X_3$ with $\mathbf{0}$ as a degenerate critical point. It is shown in [8] that the method proposed therein fails to test the type of $\mathbf{0}$. For any $\varepsilon > 0$, we have $f(\varepsilon^2, \varepsilon, \varepsilon) < 0$ and $f(-\varepsilon, \varepsilon, \varepsilon) > 0$. Letting $\varepsilon \to 0$, we get that $\mathbf{0}$ is a saddle point of f. Since the set $\mathsf{Crit}_{\mathbb{C}}(f) = \{\mathbf{0}\}$, we set an isolation radius $R_{\mathsf{iso}} = 1$. Running Algorithm 4.16 with inputs f and R_{iso}, we get the output $R = \frac{1}{2}$. Setting $r = \frac{1}{4}$ in Step 2 of Algorithm 5.4, we obtain $m = -\frac{90700979328567}{9007199254740992}$ and $M = \frac{5629499534213}{35184372088832}$. Thus, we can detect that $\mathbf{0}$ is a saddle point of f.*

To conclude this section, we would like to point out that the cost of running Algorithms 4.16 and 5.4 can be reduced if some factor decomposition of f is available. The following propositions show that the problem of classifying the isolated real critical point $\mathbf{0}$ of f reduces to the case when f is square-free and each of its factors vanishes at $\mathbf{0}$.

Proposition 5.8. *Suppose $f(X) = g(X)h(X)^2$ where $g(X), h(X) \in \mathbb{R}[X]$. Then,*

Case 1. $g(\mathbf{0}) \neq 0$. *If $g_0 > 0$, then $\mathbf{0}$ is a local minimizer of $f(X)$; otherwise, $\mathbf{0}$ is a local maximizer. Here, g_0 denotes the constant term of $g(X)$.*

Case 2. $g(\mathbf{0}) = 0$. *If $\nabla g(\mathbf{0}) \neq \mathbf{0}$, then $\mathbf{0}$ is a saddle point of $f(X)$; otherwise, $\mathbf{0}$ is of the same type as a common critical point of $f(X)$ and $g(X)$.*

Proof. If $g(\mathbf{0}) \neq 0$ then there exists an open neighbourhood $U \subset \mathbb{R}^n$ of $\mathbf{0}$ such that either $g > 0$ or $g < 0$ on U. This implies easily that $\mathbf{0}$ is a local minimizer or maximizer of f.

Assume that $g(\mathbf{0}) = 0$ and $\nabla g(\mathbf{0}) \neq \mathbf{0}$. We have that $\mathbf{0}$ is not a local extremal point of g. Consequently, for any open neighbourhood $U \subset \mathbb{R}^n$ of $\mathbf{0}$, there exist points $u, v \in U$ such that $g(u) < 0 < g(v)$. On the other hand, the algebraic set $h^{-1}(0)$ has dimension $< n$. By the continuity of g, we may assume that $u, v \notin h^{-1}(0)$. Therefore, $f(u) < 0 < f(v)$, and so $\mathbf{0}$ is a saddle point of f.

Finally, suppose that $g(\mathbf{0}) = 0$ and $\nabla g(\mathbf{0}) = \mathbf{0}$. Because for any $x \in \mathbb{R}^n$ with $f(x) \neq 0$, $f(x)$ and $g(x)$ have the same sign, $\mathbf{0}$ is of the same type as a common critical point of f and g. □

Proposition 5.9. *Suppose that $\mathbf{0}$ is a critical point of $f \in \mathbb{R}[X]$ and $f = f_1 \cdot f_2$ where $f_1, f_2 \in \mathbb{R}[X]$. If $f_1(\mathbf{0}) \neq 0$, then*

(i) *$\mathbf{0}$ is a saddle point of f if and only if $\mathbf{0}$ is a saddle point of f_2;*

(ii) *If $f_1(\mathbf{0}) > 0$ ($f_1(\mathbf{0}) < 0$, resp.), then $\mathbf{0}$ is a minimizer of f if and only if $\mathbf{0}$ is a minimizer (maximizer, resp.) of f_2.*

Proof. Since $f_2(\mathbf{0}) = 0$ and $\nabla f_2(\mathbf{0}) = \mathbf{0}$, the conclusion is clear. □

6. Conclusions

We proposed a computable and symbolic method to determine the type of a given isolated real critical point, which is degenerate, of a multivariate polynomial function. Given an isolation radius of the critical point, the tangency variety of the polynomial function at the critical point is used to define and compute its faithful radius. Elimination ideals and root isolation of univariate polynomials are computed in finding a faithful radius. Once a faithful radius of the critical point is known, its type can be determined by isolating the real roots of a zero-dimensional polynomial system.

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